Malinowski and Suszko on Many-valued Logics: On the Reduction of Many-valuedness to Two-valuedness

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Abstract

We first describe the main features of Malinowski's book *Many-Valued Logics* [22]: its orientation and its contents. The rest of our paper is devoted to the discussion of its original point, the presentation of Suszko's thesis, and questions directly related to it: What is many-valuedness? Are there only two truth-values?

We analyse and discuss the characterization of many-valuedness and the reduction of many-valuedness to two-valuedness presented by Malinowski. Then we argue against Suszko's thesis, taking examples of paraconsistent logic and of Malinowski's inferential many-valuedness. However, we also present some arguments to reject supplementary truth-values in the case of two topics discussed by Malinowski: modality and partiality.

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1 General presentation of Malinowski's study.

1.1 What kind of book did Malinowski write?

Grzegorz Malinowski (University of Lodz, Poland) wrote a very good book on many-valued logic [22]; combining clarity and precision, the book is extremely elegant and pleasant to read.

This is not the first monograph devoted to many-valued logic, but it is the first one written by a Polish logician and in the Polish style (we must however emphasize that fortunately the author does not use the so-called Polish notation). And this is not a detail if we remember that modern many-valued logic arose with Jan Łukasiewicz and is strongly connected with matrix theory, which has been developed especially by Polish logicians and is not well known outside Poland. (It is astonishing, for example, that the famous monograph by J. Łos (cf. [18]) has not yet been published in English.)

The book, thus based on a clear general framework, presents a synthetic view of many-valued logic, with a fine balance between technical aspects, philosophical questions and applications. Malinowski succeeds in drawing an overview of the topic in 130 pages without missing rigour or escaping technical difficulties and always being clear. It is obvious that at present there is no comparable published book on many-valued logic.

A supplementary positive feature is that Malinowski raises many fundamental questions about the very nature and significance of many-valued logic (and also about the nature of logic in general). Its original point is the presentation and discussion of Suszko's thesis. We shall focus accordingly our discussion on this topic.

1.2 What there is in the book.

The fourteen chapters of the book can be divided into four parts (this partition is our own):

I. Basic introduction.

The first part provides the basic tools for the correct understanding of many-valued logic, including rudiments of classical logic (Chapter 1) and Łukasiewicz's views (Chapter 2), matrix theory (Chapter 3), and consequence operator theory (Chapter 4).

II. Description of the main many-valued logics.

Łukasiewicz logics (Chapter 5), Post logics (Chapter 6), Kleene logic (Chapter 7, Section 1) and Bochvar logic (Chapter 7, Section 2) are considered in outline.

III. Further technical aspects.

Issues connected to the axiomatization (Chapter 8) and the characterization (Chapter 10) of many-valued logics as well as the theory of quantification (Chapter 11) are discussed within this part.

IV. Relations with other topics.

Some of the connections holding between many-valued logics and some other domains — such as partial logics (Chapter 7, Section 3), probability (Chapter 9), intuitionism and modal logic (Chapter 12), fuzzy logic (Chapter 13) — are presented. Finally some applications of many-valued logics are examined (Chapter 14).

I. Basic introduction. Chapter 1, which presents some basic features of classical logic, was written in order to meet two basic aims: it shall furnish a concise introduction to those not yet acquainted with the subject, and it is supposed to supply a clear framework organized in such a way that easy comparisons and distinctions between classical and many-valued logics might be articulated.

The classical propositional calculus (CPC) is presented in a standard way. Truth-tables and bivaluations are introduced, the set of classical tautologies is then defined and some remarks on the functional completeness of this calculus are made. In a brief section, some axiomatizations of CPC are mentioned and the question of their completeness and soundness is succinctly touched upon. The classical predicate calculus is also introduced, both semantically and syntactically; its undecidability is mentioned and also its completeness. The chapter concludes with some general considerations on algebraizations of classical logic. After defining the concept of Boolean algebras, it is explained in what sense CPC constitutes such an algebra. Given that similar connections can be found between the predicate calculus and some special Boolean algebras, Malinowski remarks that "the theory of Boolean algebra is, in a sense, an algebraic version of classical logic" [22, p. 15].

Having presented this framework, Malinowski, in Chapter 2, considers in broad outline the origin and some fundamental features of the first three-valued propositional system: Łukasiewicz three valued logic. Some philosophical moti vations are related to the Brentano-Twardowski-Meinong general theory of objects, which assumes the existence of objects having contradictory properties, as well as Russell's paradox in set theory, a fact acknowledged by Łukasiewicz and which led to his attack on the principle of contradiction (see [22, pp. 16–17]). Some additional motivations came from Łukasiewicz's concerns with problems connected with induction and the theory of probability, which drove him to adopt an alternative concept of logical value. But it would be only when studying issues related to determinism and modalities that he eventually would be led to the elaboration of a three-valued logic. Given his characterization of the first two problems (in terms of determined or non-determined statements), it was rather natural, while considering them, to propose the assignment of a third value (besides truth and falsity) to non-determined propositions, especially to those describing casual future events (*future contingents*). Łukasiewicz's three-valued logic (L3) is then briefly introduced through the tables of its basic connectives. The concepts of valuation and tautology are defined as well, and some comments are made in order to point out the radical differences between L3 and CPC: (*i*) some classical tautologies are not valid in L3 (such as the law of excluded middle and the principle of contradiction); moreover, (*ii*) within the latter, some classical contradictions are consistent (for instance, $p \leftrightarrow \neg p$). Malinowski also presents Łukasiewicz's proposals for formalizing the modal operators of possibility and necessity through the employment of this three-valued logic, and concludes the chapter with some remarks on the delicate problem of supplying an intuitive interpretation of Łukasiewicz's logic. He briefly considers

Gonseth's argument for the interpretation of the third value, either as possibility or as undetermination, and presents Słupecki's interesting "formal" interpretation of L3. Such an interpretation, incidentally, given that in a certain sense it holds only in De Morgan lattices and not in Boolean algebras (as Nowak has shown), has the following undeniably attractive feature: "three-valued logic (...) can be interpreted as a set of propositions describing events which form a non-classical algebra", and so, under this interpretation, Łukasiewicz's third value "is assigned to propositions concerning non-Boolean, undetermined events" [22, p. 23].

Chapter 3 is concerned with a more general issue, namely the introduction of some conceptual tools that are to be employed later in the book: logic algebras and matrices. Following a Polish tradition which can be traced as far back as some ideas of Lindenbaum presented at the end of the 1920s, Malinowski presents a propositional language in terms of an algebraic structure (an absolute free algebra); then he defines, within the verifunctional semantic framework, algebraic interpretation structures that are supposed to supply a meaning (or a semantic correlate) to the objects of the absolute free algebra (formulas). He also discusses functional completeness, which that he considers as "the property of finite logic algebras which warrants the biggest expressive power of the corresponding bunch of connectives", [22, p. 24]. Since few of the known logic algebras have this feature, Malinowski presents two criteria (one due to Słupecki, the other, to Picard) which are elaborated in order to decide whether or not in some particular algebras such a property is found. Moreover, in connection with this issue, he also shows that Łukasiewicz's algebra is (functionally) incomplete. The chapter concludes with the presentation of logical matrices, "interpretation structures equipped with a distinguished subset of the set of semantic correlates corresponding to propositions of a specified kind (e.g. true propositions)" [22, p. 28]. Within the matrix framework, the notion of *consequence* is defined (*matrix consequence*) as is the transposition of some operations on algebras (such as, taking subalgebras and direct products, forming quotient algebras, and so on) into corresponding operations on matrices.

Given this matrix setting, in Chapter 4 an interesting inquiry is developed in order to characterize logical many-valuedness. Two criteria, formulated in terms of logical matrices, are then tentatively formulated, one concerned with the content of the matrix ($K1$), the other one with the consequence induced by the matrix ($K2$). Malinowski insists on the fact that these two criteria are not equivalent: $K1$ implies $K2$ but the converse is false, as Malinowski proves by giving a counterexample. Then Malinowski presents some fundamental results linking matrices and structural consequence operations (in the sense of [19]). The first result is a famous theorem of Lindenbaum (every structural set of propositions can be determined by a matrix) extended by Wójcicki (every structural consequence operator can be determined by a class of matrices — its *Lindenbaum bundle*). Another result, also proved by Wójcicki, is referred to: a necessary and sufficient condition for a structural consequence operation to be characterized by a single matrix. These developments constitute an entertaining, though very concise, introduction to matrix theory. However, it is possible to have some doubts to what extent the interpretation of these results is relevant for providing a good account of many-valuedness (we will discuss this issue in more detail in section 2.1). The chapter ends with consideration of the interconnections between finite consequence operations and deduction systems involving rules. Rules are defined as sets of pairs of type (X, α) where X is a set of formulas and α is a formula (following some interpretations of Gentzen's ideas, Malinowski calls such a pair a *sequent*). Malinowski presents the result of Wójcicki stating that every strongly finite consequence (i.e. consequence which can be characterized by a finite class of finite matrices) is finite, and then recalls the obvious fact that consequences induced by a finite set of finite rules are finite.

II. Description of the main many-valued logics. The next three chapters [22, Chapters 5–7], which form the second part of the book, describe in outline the chief many-valued logics: Łukasiewicz's, Post's, and Kleene's and Bochvar's logics. We shall regard them in turn.

Łukasiewicz's logics — roughly speaking, a family of many-valued matrices, both finite and infinite-valued, which were obtained through generalization from Łukasiewicz's three-valued construction (considered by Malinowski in Chapter 2) — are examined in Chapter 5 from a fourfold perspective: (*i*) Some of their basic concepts and properties are initially presented, including: Łukasiewicz n-valued ($n \in N - \{0, 1\}$ or $n = \aleph_0$ or $n = \aleph_1$) matrices, the celebrated Lindenbaum condition (on the relations between the contents of finite matrices), as well as the fact that infinite Łukasiewicz matrices have a common content. (*ii*) The important question of the definability of certain functions in Łukasiewicz matrices is then considered, and its connection with the issue of functional completeness is mentioned. In this context, McNaughton's general definability test for Łukasiewicz matrices is formulated in two versions, the non-effective criterion (which depends on the axiom of choice) as well as the partial, but effective one. (*iii*) Some possible axiomatizations of distinct versions of these logics are also pointed out, including: the $(\neg, \rightarrow)$-fragment of Łukasiewicz's three-valued (Wajsberg), the $\aleph_0$-valued propositional calculi (Łukasiewicz – Wajsberg – Rose – Rosser – Chang), the functionally complete three-valued logic (Słupecki), as well as n-valued logics ($3 < n < \aleph_0$) (Grigolia and Tokarz). Finally, (*iv*) two algebraic interpretations of Łukasiewicz's logics are discussed in connection to Moisil and Chang algebras, representing in fact, as Malinowski stresses, "the attempts to obtain algebras which would play the same role for Łukasiewicz calculi as Boolean algebras do for classical logic" [22, p. 40]).

Also from a fourfold viewpoint, Malinowski examines, in Chapter 6, "a class of finite-valued, functionally complete propositional logics", [22, p. 44]: Post logics. (*i*) He introduces n-valued ($n \in N$, $n \geq 2$) Post algebras and their associated n-valued Post matrices. Some interesting remarks are made in terms of these in order to contrast the laws of some n-valued logics determined by these matrices with many-valued counterparts of some significant classical tautologies (such as, for instance, the generalized law of excluded middle). (*ii*) Malinowski then analyses Post's semantical interpretation for his own matrices, and argues for its adequacy with regard to Post's intentions. (*iii*) Algebraization issues of Post logics are considered in turn, elucidating, as Malinowski emphasizes, "basic intuitions underlying the abstract theory of Post algebras" [22, p. 47]. In conclusion, (*iv*) some axiomatizations of functionally complete systems of n-valued logics ($n \in N$, $n > 2$) are shown, based mainly on Słupecki's work, which has in fact provided a general method. As a corollary, Post's original systems are also axiomatizable, though, as Malinowski notes, "the problem of providing axioms for their original version still remains open" [22, p. 49].

Chapter 7 is concerned with Kleene's and Bochvar's three-valued logics. Their systems were initially motivated by the indeterminacy of certain propositions in particular levels of scientific development. Kleene's proposal aimed at the construction of a logic that allows the "analysis of partially defined predicates (propositional functions)" [22, p. 51]. In order to do so, Kleene considers, besides true and false propositions, some of a third category as well, "i.e. such as whose logical value (of truth or falsity) is undefined, undetermined by means of accessible algorithms, or not essential for actual considerations" [22, p. 51]. Kleene then presents the truth-tables for the connectives of such a logic, which, just as Łukasiewicz's, preserves classical logic truth conditions, extending them to the "undetermined" case. Körner's interesting interpretation of Kleene's account, based on the concept of "inexact classes", is then introduced, as well as Kleene's own system of weaker connectives. Bochvar (internal and external) logics are briefly considered in turn; in particular their connection with Kleene's views as well as with classical logic.

III. Further technical aspects. The third part of Malinowski's [22] book (Chapters 8–10–11) is concerned with the development of additional technical features of many-valued constructions. Chapter 8 is based on the work of Rosser and Turquette, who have "determined the conditions that make finitely valued propositional logics resemble more the CPC, and hence simplified the problem of axiomatization and also the question of their extension to predicate logics" [22, p. 60]. After presenting the "standard conditions" and stressing that Łukasiewicz and Post matrices are standard, Malinowski outlines the general method of axiomatization of Rosser and Turquette. The chapter ends with the adaptation of the standard conditions to consequence operations.

Chapter 10 discusses three accounts of many-valued logics, developed in the 1970s and articulated in terms of zero-one valuations: Suszko's proposals and his associated thesis (which we shall consider in Sections 3 and 4), Scott's method and Urquhart's interpretation. Each of these descriptions generates a particular interpretation of many-valuedness which is also specifically examined by Malinowski.

In Chapter 11, Malinowski analyses the role of quantifiers in many-valued logic. The ordinary (many-valued) predicate calculi are concisely introduced, and the delicate problems of supplying a semantic interpretation of their quantifiers, as well as of formulating axiomatic systems for those calculi (when such axiomatizations are possible at all!), are also addressed. The significant problem of the foundation of set theory for many-valued logic is then briefly discussed. Concluding the chapter, Rosser and Turquette's generalized quantifiers are defined, and a method of axiomatization of the resulting system (also due to Rosser and Turquette) is considered as well.

IV. Relations with other topics. The last part of Malinowski's [22] book (Chapters 9–12–13–14 and part of Chapter 7) is devoted to some connections holding between many-valued logics and other related topics. At the end of Chapter 7, partial logics are discussed from the viewpoint of van Fraassen's supervaluations and within the framework of matrix theory (in this case the truth-value gap is represented by an explicit object).

In Chapter 9, the selected subject of comparison is probability theory. As it is often pointed out, there are striking similarities between probability and many-valuedness; it is worth remembering that the former was responsible for important motivations that would lead to Łukasiewicz's development of the later. As Malinowski stresses, Łukasiewicz has proposed a theory of "logical probability" in which undetermined propositions are linked to a fraction proportional to "the number of variable values verifying the proposition and the number of all values of a given finite domain." [22, p. 66]. Within this setting, it is obvious that probability, as opposed to the usual mathematical presentations, is assigned to propositions and not to events. As obvious drawback of Łukasiewicz's view on probability, however, consists in the fact that infinite sets of individuals cannot be accepted within his framework and thus his proposals "cannot be taken seriously within the theory of probability" [22, p. 17]. After Łukasiewicz, nevertheless, many researchers (among them, Zawirski and Reichenbach) tried to reconcile the two subjects, but as far as Malinowski formulates it, it would be only in the 1970's that Giles would present an "operationalistic conception of subjective probability, interpreted unquestionably in denumerable Łukasiewicz logic" [22, p. 66]. The chapter is thus divided into two sections. The first one deals with the formulation of logical probability (in Reichenbach-Zawirski's perspective), stressing especially its connection to many-valuedness, as well as the intensional character of its semantics. The second section, on the other hand, presents Giles's ingenious operationalistic interpretation.

Chapter 12 constitutes, to a certain extent, a brief introduction to the axiomatic approach to intuitionistic and modal logics (not only at the propositional, but also at the predicate levels). Some of these nonclassical logics, as Malinowski points out, though not constructed with the underlying intent of introducing additional logical values, turned out to be characterizable "exclusively by means of infinite-valued matrices" [22, p. 87], baring thus some striking connections with many-valued logic. After a succinct historical summary, Malinowski presents Heyting's axiomatization of the intuitionistic propositional calculus (INT), as well as his celebrated interpretation of logical constants and quantifiers of intuitionistic logic, which identifies the validity of particular propositions to their provability. The connections between INT and CPC are also investigated, as are Gödel's and Jaśkowski's results about matric characterization of INT. With regard to modal logic, Lewis's proposals are concisely considered, Gödel's formalization of $S4$ and $S5$ are then presented, and finally Kripke's, as well as other algebraic semantics for modal logics, are advanced. The chapter concludes with some remarks on the introduction of quantifiers to both intuitionistic and modal logics, and the analysis of their role within these logics.

Rudiments of fuzzy logic, as one of the main views motivated by logical many-valuedness, are expounded in Chapter 13. After the formulation of the basic conceptual tools, some applications of fuzzy logics to the modelling of inexact predicates and imprecise reasonings are indicated.

Finally, concluding the book, in Chapter 14, Malinowski offers some interesting applications of many-valued logic — thereby also conveying some aspects of the significance of the topic. These logics are useful not only as a mathematical device to demonstrate the independence of particular axioms (Bernays-Lukasiewicz), or, from a philosophical perspective, to formalize some intensional functions (Łos), but also to be applied to such areas as switching theory and computer science.

1.3 What there is not in the book.

Many-valuedness is a wide subject, and it is a difficult challenge to deal with all its aspects. Malinowski quite succeeds in this respect. However, we may mention three important topics which do not appear in his book.

Firstly, the application of many-valued logic to quantum physics. For instance, P. Destouches-Février has done important work in this regard, using a third value to give an account of Heisenberg's indeterminacy principle. We shall speak about this again later.

Secondly, the application of three-valued logic to sequent calculus in the question of cut-elimination (see e.g. [12]).

Thirdly, the connections between the method of tableaux and three-valued logic (see e.g. [8]).

2 What is many-valuedness?

Undoubtedly, a fundamental problem concerning many-valuedness is to know what it really is. This may seem a triviality; however, despite the fact that many-valued logic is a wide and prolific field of modern logic, it seems that the question of its very nature has not yet been completely elucidated. The merit of Malinowski's book is to contribute to the clarification of this issue.

2.1 Matrix characterization.

Malinowski does not explicitly define the concept of many-valued logic; but founding his approach on matrix theory, he defines *when a matrix M determines a many-valued logic*, on the basis of an absolutely free algebra (language) of similar type as the underlying algebra of the matrix: "whenever the content of M or the consequence determined by M cannot be described by any (two-valued) matrix" [22, p. 30].

This leads to two formal criteria depending on whether one takes into account the content of the matrix or the consequence determined by it; we will focus on the second criteria ($K2$), which is the following [22, p. 30]:

> ($K2$) M *determines a many-valued logic* if and only if for no two-element matrix N for a language L, $Cn_M = Cn_N$.

Let us turn this into an explicit definition. First, a terminological remark: We will call a *Polish logic*, or simply a *logic* when there will be no confusion, any structural consequence operation (this is what is done by many Polish logicians, see e.g., [23]).

First attempt at an explicit definition:

A logic is *many-valued* iff there is no two-element matrix whose consequence operation is identical to this logic.

Note the negative character of this definition: a many-valued logic is what is out of the scope of two-element single matrices.

In order to have a good idea of what is not in the scope of two-element matrix theory it is good to know exactly what falls into its scope.

This has been determined by da Costa, drawing in fact the consequences of the two fundamental theorems of Post about classical propositional logic on functional and on axiomatic completeness.

Roughly speaking, *da Costa's theorem* states that any two-valued system (i.e. a Hilbert-type system) is a subsystem of a convenient version of the classical system (see e.g. [14]). The converse of course is false. This result means that any proper extension of classical logic cannot be two-valued. Thus, from da Costa's theorem we can deduce, using the Gödel translation of classical logic into intuitionistic logic, that the latter cannot be determined by a two-valued matrix (this result was originally obtained by Gödel using other techniques), as well as some similar results concerning some modal and paraconsistent logics.

This first attempt may be criticized because generally a many-valued logic is a logic which is determined by a matrix or a class of matrices. However, adding this condition may seem rather superfluous given that, as Wójcicki has shown (cf. [22, pp. 32–33]), any Polish logic can be determined by a class of matrices. ([22, p. 33] also presents the results of Wójcicki characterizing Polish logics which can be determined by a single matrix.)

But on the other hand, logics which cannot be characterized by a finite matrix or a finite class of finite matrices are usually not considered as many-valued logics. This is typically the case of intuitionistic logic (which can be determined by an infinite class of finite matrices, as Jaśkowski has proved (cf. [22, p. 90]), but cannot be characterized by a finite matrix or a finite class of finite matrices, as Gödel has shown (cf. [22, p. 89]); Wronski has also proved that it cannot be characterized by a single infinite matrix).

G. Moisil [24] introduced the name "non-Chrysippean" logics to denote many-valued logics, and has considered that Heyting's, Johannson's, Kolmogorov's and Lewis's logics are all non-Chrysippean even if they are not "Lukasiewiczean". However, nowadays, it seems that only those finite Lukasiewiczean logics are really classified under the label "many-valued" logics, and in a second attempt at characterization, we will isolate them. Nevertheless, we must note that the situation is rather confused and that [22] has devoted a chapter to "Intuitionism and the modal logics of Lewis" (Chapter 12).

Remark. Speaking about Moisil, we shall recall that the Roumanian logician has devoted all his life to many-valued logics and that his book *Essais sur les logiques non-chrysipiennes* is a compilation of nearly one thousand pages of all his papers (including unedited papers). From the viewpoint of its scope and style, the book of Moisil cannot be compared to Malinowski's. Moisil's book is an exciting and enormous work for further reading on the subject.

Second attempt:

A logic is *many-valued* iff it can be determined by a n-element matrix ($2 < n < \omega$) or a finite class of finite matrices but not by a single two-element matrix.

However, this characterization is purely matrical and there are several reasons to wish not to be blocked within matrix theory. Firstly, because the intuitive idea of many-valuedness does not necessarily depend on matrix theory, and secondly, because the rise of two-valued non-matrical semantics has shed a new light on the problem, a problem that we will discuss in connection with Suszko's thesis.

2.2 Transgression of the principle of bivalence.

There is a much more intuitive definition of many-valued logic, but it is rather fuzzy and in fact does not in general coincide with the matrical one, although it would be interesting to try to systematize their interconnections. Although Malinowski does not face the problem directly, various elements in his book are a collateral treatment of it.

The intuitive definition we are mentioning can be expressed as follows:

A logic is many-valued iff it violates the principle of bivalence.

In order to avoid confusion, we shall call such kinds of logics, following a suggestion of Lukasiewicz (cf. [20]) and Moisil, non-Chrysippean. Employing the same pattern used for the matrix characterization, we shall mean by a non-Chrysippean logic, a logic which *cannot* be constructed without violating the principle of bivalence.

Remark. The expression "non-Chrysippean" is due to the fact that Chrysippus seems to be the first to have explicitly stated and admitted the principle of bivalence, which is not the case of Aristotle, as it is known.

But what is the principle of bivalence?

There are several formulations of the principle which are not necessarily equivalent, and there is in general a confusion between at least two of them. We will first present here one version ($PB1$); the other one will be discussed later. (Malinowski does not avoid the confusion; compare the formulations of the principle of bivalence presented on p. 1 and on p. 7 of [22].)

First formulation of the principle of bivalence:

$PB1$ **Every proposition is either true or false but not both.**

We will be pedantic to avoid any confusion; this principle means that:

(A) To any proposition only one truth-value is associated;

(B) there are only two possibilities for the choice of the truth-value.

That means that we have a *function* (A) from the set of propositions into a set of *two elements* (B).

In fact, if we consider only truth-functional semantics, all non-Chrysippean logics (according to $PB1$) are many-valued logics in the sense of our first attempt at a matrix-type definition (unpacking of Malinowski's definition).

Leaving aside truth-functionality, everything turns out to be much more confused and complex.

There are at least two problems:

- The question of leaving aside functionality presupposed in (A);
- the question of considering non-truth-functional semantics.

We will not discuss the first of these problems, given that until now only a few studies have been carried out in this direction.

3 Suszko's thesis.

3.1 What is Suszko's thesis?

In Chapter 10 (Section 10.1) of his [22], Malinowski explains Suszko's thesis; his discussion is very interesting and is strongly connected with the understanding of many-valued logic. Unfortunately he does not go very far in this direction (probably because such kinds of inquiry lead to some problems which are outside the scope of an introductory book on many-valued logic). In fact, this shows the difficulty of isolating a subject like many-valued logic: if we want to discuss the very nature of many-valuedness we are driven far away from many-valuedness, or at least of what is generally considered as such.

Suszko's thesis is connected with the reduction of many-valuedness to two-valuedness; as Malinowski writes: "In the 1970s the investigations of logical formalizations bore several descriptions of many-valued constructions in terms of zero-one valuations" [22, p. 72].

At first sight, this seems to be an absurdity according to the definition of many-valuedness. Such an absurdity however disappears if we explicitly distinguish two kinds of semantics, as Suszko did.

Suszko's thesis is presented in a paper of his entitled "The Fregean Axiom and Polish mathematics in the 1920s" ([27]). In this extremely dense and very short paper, opened by a quotation from R. Adrey defining the human being as the only animal capable of lying to himself, Suszko speaks concisely of a great quantity of intricate fundamental problems of logic. The paper is not easy to understand, not only because it is the summary of a talk, but also because it is a kind of synthesis, in four pages, of some deep reflections carried out by Suszko over forty years (this paper is in fact the last published paper of R. Suszko before his death in 1979).

Until now, Suszko's titanic work has not received the attention it deserves. Thus it is one of the best features of Malinowski's [22] book to mention it and to throw some light on it.

Strangely enough, neither the quoted paper of Suszko nor the book of Malinowski state explicitly *Suszko's thesis*. Nevertheless in another paper, Malinowski writes "Suszko's thesis (...) states that each logic, i.e., a structural consequence operation conforming Tarski's conditions, is logically two-valued" [23].

In his paper, Suszko explicitly states that any inference relation is logically two-valued; thus, at first sight, it seems rather strange to speak of a *thesis* instead of a *theorem*.

The proper elucidation of this point seems to be the following:

- *Suszko's theorem* shows that any logic is *logically two-valued*.

- *Suszko's thesis* states that there are only two *logical* values.

According to Suszko's thesis, "Any multiplication of logical values is a mad idea and, in fact, Łukasiewicz did not actualize it" [27, p. 378]. Suszko does not consider the elements of Łukasiewicz's matrix, 0, 1/2, 1, as *logical* values, but as *algebraic* values. For him, an *algebraic valuation* is a morphism from the free algebra of formulas into an algebra of similar type, and a (two-valued) *logical valuation* is simply a function which associates one value to each formula. A semantics based on *logical valuation* is called a *valuation semantics*, in opposition to *referential semantics* based on algebraic valuations.

There are several ways to show that any logic is logically two-valued, and we will describe them in order to explain in detail the significance of this result and its import for many-valuedness. Let us begin by noting, to insist on the importance of the problem, that according to this result, there is no non-Chrysippean logic, if we adopt the principle of bivalence in its first formulation ($PB1$).

3.2 Reductions to logical two-valuedness.

3.2.1 Three ways to two-valuedness.

Suszko's reduction

The reduction of Suszko, as presented in [27], is far from being explicit. In his book, [22, pp. 72–73], Malinowski gives a more detailed description; we will suppose that it is a faithful exposition of Suszko's result. (Nevertheless this is not absolutely clear, for example D. Batens gives a different interpretation of it in [2].)

Malinowski shows how to construct from a matrix a set of logical bivaluations which is adequate for the logic determined by the matrix.

Thus, he concludes that, due to the fact that any logic can be characterized by a class of matrices, "each (structural) *propositional logic* (L, C) can be determined by a class of logical valuations of the language $\mathcal{L}$ or, in other words, it is *logically two-valued*" ([27]), [22, p. 73].

Da Costa's reduction

The reduction to two-valuedness has been discovered by N. C. A. da Costa, independently of Suszko, and from a different point of view (see e.g. [15]).

Da Costa's starting idea is that of a generalized Hilbertian calculus (rules may have an infinite number of premisses). The set of all bivaluations which preserve the rules appears to be an adequate logically two-valued semantics for the logic induced by this kind of calculus.

In fact, we should note that the notion of a calculus *à la da Costa* is equivalent to the one of consequence operation. The point is that da Costa stays at the *abstract* level without taking into account the underlying structure of the set of formulas.

Remark. We must recall that the theory of consequence was originally presented by [28] at the abstract level. This theory has been transformed into a structural theory of consequence by Los and Suszko. However, Suszko has used the expression "Abstract Logic" in a misleading way, and we shall use it here in its natural way.

What is clear from da Costa's result is that:

- Structurality does not play any role in the reduction,
- it is not necessary to make a detour by matrices in order to get the reduction.

Béziau's reduction

J.-Y. Béziau (cf. [4]) has presented a result which is a kind of middle term between Suszko's reduction and da Costa's reduction.

Following Tarski and da Costa, Béziau starts at the abstract level considering an *abstract logic* as a pair $L = \langle L; \vdash \rangle$, with $\vdash$ obeying the usual consequence laws.

Remark. At this point, we must emphasize that *da Costa's calculus*, *consequence operation*, and *abstract logic* are equivalent notions.

Then Béziau gives the following very general definition of semantics. A semantics on a given set L is a pair $\langle M; \mathrm{mod} \rangle$, where M is a set and mod is a function from L to the power set of M. The logic induced by the semantics is defined naturally: $T \vdash_G a$ iff $\mathrm{mod}\, T \subseteq \mathrm{mod}\, a$.

Béziau then shows that any semantics can be reduced to a bivalent semantics, in the sense that given any semantics on a set L, we can find a bivalent semantics on L which induces the same logic.

But what is a bivalent semantics?

Béziau states that a *bivalent semantics* on a set L is a semantics where M is a set of functions from L to $\{0, 1\}$ (*bivaluations*) and mod is defined as follows: $\beta \in \mathrm{mod}\, a$ iff $\beta(a) = 1$.

Of course, using Suszko's terminology, these bivaluations are logical and not algebraic, this definition being carried out at the abstract level. And Béziau goes on to define the notion of n-valent logical semantics ($n > 2$): a n-valent semantics on a set L is a semantics where M is a set of functions from L to $\{0, \ldots, n\}$ and mod is defined as follows: $\nu \in \mathrm{mod}\, a$ iff $\nu(a) \in D$ being a proper nonempty subset of $\{0, \ldots, n\}$, the set of *designated* values.

We may wonder if such kinds of many-valued semantics have any sense (a "mad idea" according to Suszko) or any use (because of the reduction theorem); we will nevertheless show that it can be rationally defended.

3.2.2 Significance of the reduction.

Some people may think that these kinds of reduction results constitute the burial of many-valued logic. Malinowski insists on the fact that referential many-valuedness still plays a fundamental role, that we need to use it in order to describe logical bivaluations: "It seems that giving a general method for recursive description of these valuations without knowing precisely the structure of the class K of matrices adequate for C is hardly possible" [22, p. 73].

Malinowski's feeling is that the method of logical valuations is not workable: "Even for simple relations of inference the conditions defining valuations are illegible" [22, p. 73]. And he gives the example of the logical bivalent semantics for Łukasiewicz's logic, presented by Suszko in [26]. Drawing the conclusion of Malinowski's arguments, referential many-valuedness appears as a useful mathematical device, but one which must not be misinterpreted: it does not contradict Suszko's thesis.

In fact, in what follows, we will present a diametrically opposed point of view, showing firstly that logical bivaluations can really be a systematic and practical tool, and secondly that, even in these conditions, there are no good reasons to admit Suszko's thesis.

The *Theory of Valuations* has been developed by da Costa. It consists in a systematic use of the method of logical bivalent semantics (for a general perspective, see [9], [6]). This theory was originally developed by da Costa and his Brazilian group in order to give semantics to his paraconsistent systems. These logics cannot be characterized by finite matrices. But da Costa and his school succeeded in providing a zero-one semantics for them which is quite intuitive and practicable. In particular they succeeded in generalizing the usual method of truth-tables which can be employed to prove the decidability of some of these logics.

Following the same pattern of research, J.-Y. Béziau has recently provided a systematic connection between logical bivaluations and structurally standard systems of sequents [5]. With this general result it is very easy to jump from sequent rules to bivaluation conditions, and thus to provide axiomatization and completeness.

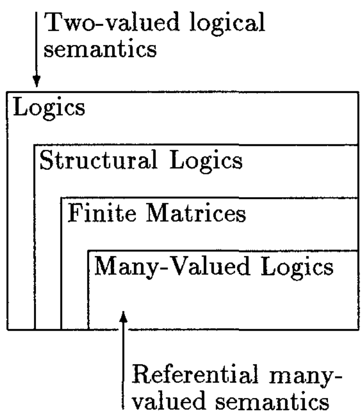
Béziau has also presented a systematic study of the class of bivaluation semantics which is adequate for a given abstract logic (see e.g.

[4]). In [27], Suszko, speaking about adequate sets of bivaluations for a given inference relation, says: "The adequate sets V form an interval ($V_1 \subseteq V \subseteq V_2$) between the smallest adequate set V_1 and the largest one V_2. Some adequate sets are better, some other are worse." This description of the state of affairs is, however, erroneous or, in the best case, rather imprecise. The class of adequate bivalent semantics does not form, in general, a linear order structure. Moreover, if there is a largest semantics, there is, in general, no smallest one. Nevertheless, Béziau has shown that under certain conditions there is a minimal bivalent semantics.

From the perspective of the *Theory of Valuations*, we can see that the reduction theorem is not just a vacant result without any application, but that it is supported by a strongly effective and practicable theory. However, the rivalry between *referential semantics* and *valuation semantics* is left open: valuation semantics have been used especially for logics which have no (finite) referential semantics, and it is not yet clear if a valuation semantics will be used with profit in case where a referential semantics can be provided, e.g. in the case of Lukasiewicz's logic.

But certainly the essential merit of the *Theory of Valuations* is that it is a general framework, wider than (finite) matrix semantics or Kripke semantics, which allows us to give a systematic account of logic.

We have now the conditions to present a complete picture of the situation:



4 Criticizing Suszko's thesis.

We shall now explain why, in our opinion, Suszko's thesis, according to which there are only two logical truth-values, seems unfounded.

4.1 Paraconsistency and many-valuedness.

According to the logical bivalent semantics of the paraconsistent logic $C1$, given a bivaluation β, propositions can be divided into three classes: A *true* proposition is a proposition such that $\beta(p) = 1$ and $\beta(\neg p) = 0$, a *false* proposition is a proposition such that $\beta(p) = 0$ and $\beta(\neg p) = 1$, and a *quasi-true* proposition is a proposition such that $\beta(p) = 1$ and $\beta(\neg p) = 1$.

Following this intuition, Béziau has transformed the logical bivalent semantics of $C1$ into a *logical trivalent semantics*, that is to say, into a *non-referential trivalent semantics* or a *non-truth-functional trivalent semantics* (see [3]). Under these conditions, the "mad idea" of the multiplication of logical valuations was actually realized.

The technique of such a reduction is based on Béziau's reduction, but employed in a reverse way. The main advantage of this semantics is that the value of a compound proposition depends (not functionally) on the value of its components, contrarily to the standard bivalent semantics for $C1$, where it depends furthermore on negations of some components.

This semantics of $C1$ clearly violates the principle of bivalence ($PB1$) and, adapting the terminology, we can say that it is a non-Chrysippean semantics, although it is not many-valued in the sense of matrix theory; and it is not a referential semantics.

But it is clear that, if we break the limits of matrix theory, a paraconsistent logic is not necessarily non-Chrysippean: the logic $C1$ constitutes a good counterexample.

On the other hand, within the framework of referential semantics, it seems possible to develop a bivalent semantics for paraconsistent logic if we modify the morphological basis and get rid of algebra. For instance, we may consider the negation as a relation, rather than as a function. The corresponding referential semantics will be a generalization of the notion of matrix, taking as underlying structures not only algebras, but any kind of structures, e.g., relational structures.

4.2 PB = PC + EM?

Just as we can construct paraconsistent logics without violating the principle of bivaluation, it is also possible to construct paracomplete logics (logics which violate the principle of excluded middle) without doing any harm to the principle of bivalence.

The situation is in general not clear, because a confusion is made between the principle of bivalence and the principles of contradiction and excluded middle, the principle of bivalance being regarded as the "sum" of the other two.

This point of view is exposed by many authors, including Malinowski himself [22, p. 7]:

> (T1) $p \vee \neg p$ (*law of excluded middle*)
> (T2) $\neg(p \wedge \neg p)$ (*principle of contradiction*)
> (...)
> (T1) and (T2) are usually 'read': given two contradictory propositions p, $\neg p$:
> * at least one of them is true; (T1)
> * at least one of them is false; (T2)
>
> Observe that, thus interpreted, (T1) and (T2) together represent the logical principle of bivalence.

Transposing Malinowski, we just give the

second formulation of the principle of bivalence:

*PB*2 **Given two contradictory propositions p, $\neg p$, at least one of them is true, at least one of them is false.**

This version of the principle of bivalence is not equivalent to the first one (*PB*1), tacitly presented by [22, p. 1].

Strictly speaking, *PB*2 does not imply part (A) of *PB*1: a proposition can have more than one truth-value without contradicting *PB*2; and if this is the case, part (B) of *PB*1 may also be falsified without infringing *PB*2.

On the other hand, *PB*1 can hold even if *PB*2 is violated, this being typically the case for the semantics of paracomplete and paraconsistent da Costa logics.

This confusion between *PB*1 and *PB*2 is particularly vivid in connection with the analysis of Vasil'ev's work. Some people consider him as a precursor of many-valued logic, others as a forerunner of paraconsistent logics, and even of non-alethic logic (logics which are at the same time paraconsistent and paracomplete; see [25]).

The problem is essentially due to the fact that (T1) is confused with (B), and (T2) with (A). Many authors called (A), the principle of contradiction, and (B), the principle of excluded middle.

We give just two examples:

G. Asser in [1, p. 1] presents the "Satz der zweiwertigkeit" (principle of bivalence) as follows: *"Jeder Aussage ist entweder wahr oder falsch, d.h., jede Aussage ist wahr oder falsch (Prinzip von ausgeschlossen Dritten) und es gibt keine Aussage, die sowohl wahr als auch falsch ist (Prinzip vom ausgeschlossenen Widerspruch)".*

G. Moisil writes: *"Aucune proposition n'est en même temps vraie et fausse, c'est le principe de contradiction; toute proposition est vraie ou fausse et il n'y a pas une troisième possibilité: c'est le principe du tiers-exclu. Nous dirons qu'une proposition peut avoir l'une des* *deux valeurs logiques: le vrai et le faux. Cette affirmation constitue le principe de bivalence"* ([24, p. 34]).

Finally, we must insist also on the fact that the statements "$\neg(p \wedge \neg p)$" and "among two contradictory propositions, p and $\neg p$, one of them is false" are not necessarily equivalent.

Bearing all this in mind, it is interesting to consider Tarski's truth condition. Intuitively, the introduction of a third value implies a violation of Tarski's formal condition, since if a proposition p is neither false nor true, then it is false that it is true, and thus p is not equivalent to the proposition stating that p is true.

But a paraconsistent logic, or a paracomplete logic (provided with a bivalent semantics), is not necessarily in conflict with Tarski's condition. If p is true and $\neg p$ is also true, we can consider that it is true that p is true.

In fact, the systematic reduction to two-valuedness permits one to preserve in all cases Tarski's principle.

Nevertheless this is not something altogether desirable, for it is possible to defend the claim that the multiplication of truth-values allows us to express exactly the idea of "degrees of correspondence".

4.3 Inferential many-valuedness.

Malinowski insists also on another very important point, namely the fact that many-valued logic based on matrix theories still retains, but at a deeper fundamental level, the idea of bivalence: "The matrix method inspired by truth-tables embodies a distinct shadow of two-valuedness in the division of the matrix universe into two subsets of designated and undesignated elements" ([22, p. 72]).

This more fundamental aspect of bivalence is also preserved in Béziau's non-referential many-valued semantics.

In fact, it is also possible to get rid of this aspect of bivalance. For example, considering a three-valued matrix, it is possible to define three kinds of logical "modalities":

- Logical truth: a proposition a is *logically true* iff for all homomorphisms h, $h(a) = 1$;
- Logical falsity: $\forall h$, $h(a) = 0$;
- Logical indetermination: $\forall h$, $h(a) = 1/2$.

The problem may then be put as follows:

Can logical truth also be multivalent?

It seems that *a priori* there is no good philosophical argument to reject this possibility, and this is another reason why we can reject Suszko's thesis.

Moreover, taking into account this possibility, the reduction theorem will obviously fail. Indeed the reduction is allowed when we take as a fundamental concept a bivalency, such as a set of tautologies, or a consequence operation; but if we change the paradigm, the situation will be quite different.

Along these lines Malinowski, in [23] (see also [21]), formulated the very interesting notion of "inferential many-valuedness".

After presenting Suszko's reduction, Malinowski writes: "One may then naturally ask whether logical many-valuedness is possible at all." He then presents his notion of inferential many-valuedness as a possible solution to this problem.

This notion of course does not invalidate Suszko's reduction, but it seems to invalidate Suszko's thesis, at least if such inferences are not reducible to two-valued inferences.

In our opinion, it seems that the "true" way to many-valuedness, taking into account the reduction result, is reached exactly through this notion of inferential many-valuedness. The real n-dimensional logics ($n > 2$) have to be developed by breaking down the deepest root of the principle of bivalence. We can easily imagine, for instance, a rule of deduction with three poles or more.

Thus, we can only regret that Malinowski did not include in his book his most recent researches on the subject.

5 When many-valuedness is not necessary.

5.1 The third value as possibility.

According to Suszko, "Lukasiwiecz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to present day" [27, p. 377].

Even someone who does not support Suszko's thesis will probably agree that referential many-valuedness is not a good basis for the study of standard modalities.

As is well known, the development of three-valued logic by Lukasiewicz was strongly tied to the problem of *future contingents*.

Malinowski recalls that, in relation to this problem, Aristotle can be considered as the founder of three-valued logic, writing [22, p. 1]:

> The roots of many-valued logics can be traced back to Aristotle (4th century BC) who considered, within the modal framework, *future contingents* sentences. In Chapter IX of his treatise *De Interpretatione* Aristotle provides the time-honoured sentence-example representing this category: 'There will be a sea-battle tomorrow'. The philosopher from Stagira emphasizes the fact that *future contingents* are neither actually true nor actually false, which suggests the existence of a 'third' logical status of propositions.

Malinowski reminds us also that Łukasiewicz started investigating many-valued logics motivated by these problems, writing [22, p. 17]: "The studies that finally led Łukasiewicz to the construction of three-valued logic touch upon determinism, indeterminism and some related problems like the causality principle and modality (i.e. possibility and necessity)".

According to Łukasiewicz, the relations for Aristotle and his work regarding the third value and modality are clearly stated in the following passage:

> Quant à l'interprétation intuitive de sa logique, M. Łukasiewicz indique qu'il faut considérer celle-ci comme une logique modale dans laquelle la valeur 1/2 représenterait le possible. Aristote déjà remarquait que les propositions relatives à des événements futurs peuvent être aujourd'hui ni vraies, ni fausses" [13, p. 101].

According to Suszko, it is absurd to consider that there is a third logical value, and that this third value is possibility: Łukasiewicz "did not, as he could not, create any new logical value besides truth and falsity. To be sure, *POSSIBILITY* is our only hope and the headspring of all our failures. It is, however, neither a logical value nor what formulas may refer to" [27, p. 379].

As is well-known, modal logics have been successfully developed using other techniques, and Kripkean semantics appears to give a far better account of modalities than does ordinary matrix semantics. Valuation semantics has also been developed for modal logics, especially by A. Loparic (see [16] and [17]).

However, we would like to recall here Wittgenstein's account of modalities, which is an elegant treatment of this issue within standard two-valued semantics and which also may give an account of the problem of determinism.

Wittgenstein's conception is articulated against the idea of modality as a logical operator:

> 5.525 Gewißheit, Möglichkeit oder Unmöglichkeit einer Sachlage wird nicht durch einen Satz ausgedrückt, son-

> dern dadurch, daß ein Ausdruck eine Tautologie, ein sinnvoller Satz oder eine Kontradiktion ist.
>
> 4.464 Die Warheit der Tautologie ist gewiß, des Satzes möglich, der Kontradiktion unmöglich.

Using modern concepts, we can interpret Wittgenstein's position as follows. A bivaluation is a possible world. A possible world can be considered as a description of the external world (i.e. a description of all present, future and past events); propositions can be divided into three classes: tautologies (necessary propositions), antilogies (impossible propositions) and the other ones (possible or contingent propositions).

Following Wittgenstein, the proposition "There will be a sea-battle tomorrow" is possible because it is true in some worlds but not in all of them.

Aristotle and Lukasiewicz argue that if the truth-value (truth or falsity) of such a proposition were fixed, this would entail determinism. Thus, if we reject determinism, this proposition should be neither true nor false.

But if a world is the description of all past, present and future events, then the truth-value of such a proposition must be fixed in all the worlds, in particular in the "real" world. However, this does not entail determinism: in a particular world, for example the real one, given the truth-values of all propositions about an instant of time and about the past events, the truth-value of the propositions about future events in this world are not necessarily determined by this set of truth-values. In fact the only "necessary" propositions at this instant are meaningless propositions, i.e. tautologies. An event which will necessarily happen is of the kind "tomorrow there will be or there will not be a sea-battle".

The proposition "Hiroshima will be destroyed by an atomic bomb" was true in the real world in 1940. But this proposition was false in a possible world in which scientists would not have been able to construct such kinds of bomb. Therefore this proposition was, from the logical point of view, only a possible proposition.

5.2 Partiality.

The third section of Chapter 7 of [22] is devoted to "Partial logics". The idea behind these logics is that there are some *truth-value gaps*, i.e. there are some propositions which have no truth-value; as Malinowski [22, p. 56] writes: "Within these frameworks, sentences which are ascribed neither truth nor falsity are supposed to form a third classification considered a *truth-value gap* (...) Essentially, there are two approaches to truth-value gaps: 'supervaluations' and the matrix approach".

Concerning this issue, we must emphasize that, from the viewpoint of the matrix approach, P. Février had done important work on the subject as early as 1936, trying to give a logical account of Heisenberg's indeterminacy principle.

We will present it briefly and explain how, in fact, in this case it is not necessary to introduce a third value. This method can also be used in the case of the non-truth-functional treatment of gaps.

P. Février's ideas have been perfectly described by J.-L. Destouches in [10, pp. 80–81] (the reader who is interested in technical details may consult [11]):

> Considérons la proposition p:
>
> $p =_d$ "le corpuscule c a la position a"
>
> et la proposition a:
>
> $q =_d$ "le corpuscule c a la vitesse v".
>
> En mécanique quantique on ne peut affirmer simultanément ces deus propositions sans entrainer des contradictions; par suite le produit logique de p et q ne peut suivre les règles ordinaires; ou bien on devra ne pas appliquer l'opération logique à certaines paires de propositions, ou bien on devra en modifier la définition pour qu'appliqué à de telles paires le produit ne soit pas vrai. Il est très mal commode de considérer des opérations qui ne s'appliquent pas à toutes les paires d'éléments de l'ensemble considéré (ici, ensemble de propositions), on a donc avantage à adopter la seconde possibilité. Mais dans les deux cas on voit que l'opération & ne peut s'appliquer à toutes les paires de propositions en donnant le même résultat et en obéissant aux règles classiques. De ce fait, la logique que l'on utilisera dans les théories quantiques ne peut être une logique à règles universelles puisque les règles sur le produit ne sont pas univer selles. Une telle logique ne sera donc pas purement formelle puisque le contenu des propositions interviendra dans les énoncés des règles. Cependant on peut construire une logique adaptée à ce cas comme l'a montré Mlle Février, une telle logique se mettant sous forme symbolique peut être appelé une *logique quasi-formelle.*
>
> Les paires (p, q) de propositions doivent être divisées en deux classes: la *classe des paires composables* qui suivent les règles habituelles ($p\&q$ est vrai si p et q sont toutes les deux vraies) et la *classe des paires incomposables*; pour celles-ci $p\&q$ n'est jamais vrai. De ce fait l'opération & se trouve définie par deux matrices: une pour chaque classe de paires de propositions.
>
> La proposition $p\&q$ pour une paire (p, q) de propositions incomposables n'est jamais vraie, mais on ne peut

> pas la considérer comme fausse au sens habituel (permettant d'affirmer la négation de p ou celle de q); on doit lui attribuer une nouvelle valeur logique $\mathcal{A}$ signifiant: "faux absolu"; ceci conduit à considérer pour les propositions trois valeurs logiques (au moins) qui sont:
>
> $$\mathcal{V} \text{ (vrai)}, \quad \mathcal{F} \text{ (faux)}, \quad \mathcal{A} \text{ (faux absolu)}.$$

It is very illuminating to analyse Destouches' remarks in the light of the conceptual framework established by the Polish School.

The concept of *structurality*, introduced in [19], is a perfect expression of the idea of formal logic. The idea of a *quasi-formal logic* described by Destouches is an example of a non-structural logic. The non-universality mentioned by Destouches is exactly the failure of the rule of substitution.

Destouches delineates two solutions, the first being rejected because of its complexity. However, it seems that nowadays this first solution can be developed without any problems, and choosing it is precisely what permits us to avoid the use of many-valuedness. The idea is simply to exchange the absolute free algebra of propositions with a partial algebra. Although this technical idea has not yet been studied in detail, it seems that its development shall not face insurmountable difficulties, for as D. J. Brown and R. Suszko write in their general study of structural logic, "Although we require A to be a finitary algebra, most of our results can be extended to partial and infinitary algebras" [7, p. 19].

And from the philosophical point of view, it seems that this solution is more satisfactory; at least, it is in accord with Frege's idea of deleting objects without reference (*Bedeutung*).

We can say that a proposition "without reference" in fact does not exist, and sentences "expressing" such kinds of propositions can be viewed as mere optical illusions, similar to Escher's drawings.

The important point is that, nowadays, it is possible to offer a technical account of this kind of problem, introducing gaps not at the semantic level, but at the level of morphology (syntax). And this is a way to avoid the employment of many-valuedness.

In this study we have taken into account only some aspects of Malinowski's [22] book. But our discussion seems to be enough to show that Malinowski's [22] book constitutes a fundamental contribution to the subject.

References

[1] G. Asser. *Einführung in die mathematische Logik, Teil 1: Aus-*

sagenkalkül. B.G. Teubner, Leipzig, 1959.

[2] D. Batens. *Two semantically motivated enrichments of relevant logics*. In: *Essays on Philosophy and Logic (Proceedings of the XXXth Conference on the History of Logic, Dedicated to Roman Suszko, Cracow – 1984)*, J. Perzanowski (ed), Jagiellonian University Press, Cracow, 1987, 65–74.

[3] J.-Y. Béziau. *Logiques construites suivant les méthodes de da Costa*. Logique et Analyse **131–132** 1990, 259–272.

[4] J.-Y. Béziau. *Recherches sur la logique abstraite: les logiques normales*. Acta Universitatis Wratislaviensis, Logika **16** 1994.

[5] J.-Y. Béziau. *Sequents and bivaluations*. 1994, to appear.

[6] J.-Y. Béziau. *La théorie de la valuation*. In: N.C.A. da Costa, *Logiques classiques et non classiques*, Masson, Paris, 1995.

[7] D.J. Brown and R. Suszko. *Abstract logics*. Dissertationes Mathematicae **102** 1973, 5–41.

[8] W.A. Carnielli. *Systematization of finite many-valued logics through the method of tableaux*. Journal of Symbolic Logic **52** 1987, 473–493.

[9] N.C.A. da Costa and J.-Y. Béziau. *La théorie de la valuation en question*. In: *Proceedings of the Ninth Latin American Symposium on Mathematical Logic — Bahia Blanca 1992*, M. Abad (ed), Universidad del Sur, Bahia Blanca, 1994, 95–104.

[10] J.-L. Destouches. *Cours de logique et philosophie générale*. Centre de document universitaire, Fournier & Constans, Paris, 1948.

[11] P. Février. *Les relations d'incertitude d'Heisenberg et la logique*. In: *Travaux du IX^{e} Congres International de Philosophie*, Vol. VI, Hermann, Paris, 1937, 88–94.

[12] J.-Y. Girard. *Three-valued logic and cut-elimination: the actual meaning of Takeuti's conjecture*. Dissertationes Mathematicae **136** 1976.

[13] F. Gonseth. *Les entretiens de Zürich sur les fondements de la méthode des sciences mathématiques — Zürich 1938*. F. Gonseth (ed), Leeman Frères, Zürich, 1941.

[14] N. Grana. *Sulla teoria della valutazioni di N.C.A. da Costa*. Liguori, Naples, 1990.

[15] J. Kotas and N.C.A. da Costa. *Some problems on logical matrices and valorizations.* In: *Proceedings of the Third Brazilian Conference on Mathematical Logic*, A.I. Arruda, N.C.A. da Costa and A. M. Sette (eds), Sociedale Brasileira de Lógica, 1980, 131–145.

[16] A. Loparic. *The method of valuations in modal logic.* In: *Proceedings of the First Brazilian Conference on Mathematical Logic*, A.I. Arruda, N.C.A. da Costa and R. Chuaqui (eds), Marcel Dekker, New York and Basel, 1977, 141–157.

[17] A. Loparic. *A semantial study of some propositional calculi.* Journal of Non-Classical Logic **3** 1986, 73–95.

[18] J. Łos. *O matrycach logicznych.* Travaux de la société des sciences et des lettres de Wrocław, Série B, Nr 19, Wrocław, 1949.

[19] J. Łos and R. Suszko. *Remarks on sentential logics.* Indigationes Mathematicae **20** 1958, 178–183.

[20] J. Łukasiewicz. *Philosophische Bemerkungen zu mehrertigen Systemen des Aussagenkalküls.* Comptes Rendus des Séances de la Société des Lettres de Varsovie, Classe III **23** 1930, 51–77.

[21] G. Malinowski. *Towards the concept of logical many-valuedness.* Folia Philosophica **7** 1990, 97–103.

[22] G. Malinowski. *Many-Valued Logics.* Oxford Logic Guides 25, Clarendon Press, Oxford, 1993.

[23] G. Malinowski. *Inferential many-valuedness.* In: J. Woleński (ed), *Philosophical Logic in Poland*, Kluwer Academic Publishers, Dordrecht, 1994, 75–84.

[24] G. Moisil. *Essais sur les logiques non-chrysipiennes.* Academie de la République Socialiste de Roumanie, Bucarest, 1972.

[25] L.Z. Puga and N.C.A. da Costa. *On the imaginary logic of N.A. Vasil'ev.* Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **34** 1988, 205–211.

[26] R. Suszko. *Remarks on Łukasiewicz's three-valued logic.* Bulletin of the Section of Logic **4** 1975, 87–90.

[27] R. Suszko. *The Fregean Axiom and Polish mathematical logic in the 1920s.* Studia Logica **36** 1977-1978, 377–380.

[28] A. Tarski. *Remarques sur les notions fondamentales de la méthodologie des mathématiques.* Annales de la Société Polonaise de Mathématiques **7** 1929, 270–272.