Dept. of Mathematics University of Oslo
Pure Mathematics No 11
ISSN 0806–2439 December 2013

Malliavin differentiability and strong solutions for a class of SDE in Hilbert spaces

F. Flandoli, T. Nilssen, F.N. Proske

Abstract

We consider a class of Hilbert-space valued SDE's where the drift coefficients are non-Lipschitzian in the sense of Hölder-continuity. Using a novel technique based on Malliavin calculus we show in this paper the existence and uniqueness of a mild solution to such equations. We emphasize that our approach does not rely on the Yamada-Watanabe principle. Moreover our method gives the important additional insight that the obtained solution is Malliavin differentiable - a property which was recently shown to play a crucial role in the study of the geometry of certain optimal causal transference plans, [12].

1 Introduction

In a separable Hilbert space H, consider the stochastic differential equation

$$dX_t = AX_t dt + B(t, X_t)dt + \sqrt{Q}dW_t, \ X_0 = x \tag{1}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup e^{tA}, $t \geq 0$; $B : [0,T] \times H \to H$ is continuous, W is a cylindrical Wiener process. If the operator $Q_t = \int_0^t e^{sA} Q e^{sA^*} ds$ is trace class, and suitable linear growth conditions on B are assumed, weak existence is known for equation (1), see [5].

The aim of this paper is to prove Malliavin differentiablility and a direct proof of *strong existence*, under additional assumptions on (A, B) stated in section 1.1. On B we assume Hölder continuity in x uniformly in t. On A we assume certain non-degeneracy condition related to null-controllability. See [2] for the case of Hölder-coefficients. For merely bounded and measurable coefficients, see [9] and the recent work in [3].

1.1 Notations and assumptions

Norm and inner product in H will be denoted $|\cdot|$ and $\langle \cdot, \cdot \rangle$. A complete orthonormal system $\{e_n\}_{n\geq 1}$ in H is assumed to be fixed. If $\varphi : H \to H$, we shall denote its components with respect to $\{e_n\}_{n\geq 1}$ by φ_n: $\varphi_n(x) = \langle \varphi(x), e_n \rangle$.

Given $\alpha, T > 0$, we shall denote by $C([0,T]; C_b^\alpha(H, H))$ the space of all functions $G : [0,T] \times H \to H$ which are continuous and bounded in (t, x), and such that there exists $C > 0$ such that

$$|G(t,x) - G(t,y)| \leq C|x - y|^\alpha, \ x, y \in H, \ t \in [0,T].$$

We denote by $\|G\|_{\alpha,T}$ or simply $\|G\|_\alpha$ the norm

$$\|G\|_\alpha = \sup_{t\in[0,T],\, x\in H} |G(t,x)| + \sup_{t\in[0,T]} \sup_{x\neq y\in H} \frac{|G(t,x) - G(t,y)|}{|x - y|^\alpha}$$

We use the notation $\|G_n\|_\alpha$ also for the similar norm of the components $G_n(t,x) = \langle G(t,x), e_n\rangle$.

We denote by $Lip(H,H)$ the space fo globally Lipschitz continuous maps on H.

Let us now list the assumptions of this paper:

1. The operator A is selfadjoint, with compact resolvent, and $Ae_n = -\alpha_n e_n$, with non-decreasing positive $\{\alpha_n\}_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^{1-\delta}} < \infty \tag{2}$$

2. $B \in C([0,T]; C_b^\alpha(H,H))$ for some $\alpha, T > 0$.

3.

$$e^{tA}(H) \subset Q_t^{1/2}(H) \text{ for all } t > 0 \tag{3}$$

4. The well defined bounded operator $\Lambda_t = Q_t^{-1/2} e^{tA}$ satisfies

$$\int_0^T \|\Lambda_t\|^{1+\theta} dt < \infty \tag{4}$$

for some $\theta \geq \max(\alpha, 1-\alpha)$.

Remark 1 *From assumption (4) we have, in paricular,*

$$\int_0^T \|\Lambda_t\|^{1+\alpha} dt < \infty, \quad \int_0^T \|\Lambda_t\|^{2-\alpha} dt < \infty$$

2 Idea of the method

In this section we do not care about the rigor of the computations. The aim is to explain the idea.

For ever n, consider the following (backward) PDE in H of Kolmogorov type, on some interval $[0,T]$:

$$\begin{aligned} \frac{\partial U_n}{\partial t} + \frac{1}{2} Tr\left(D^2 U_n Q\right) + \langle Ax, DU_n\rangle + \langle B, DU_n\rangle &= B_n \\ U_n(T,x) &= 0 \end{aligned} \tag{5}$$

Notice it is a non-homogeneous equation, opposite to the usual equations of Kolmogorov type; the right-hand-side B_n is the n-component of B. If U_n is a sufficiently regular solution, from Itô‘s formula we get

$$\begin{aligned} dU_n(t,X_t) &= B_n(t,X_t)dt + \langle DU_n(t,X_t), Q^{1/2} dW_t\rangle \\ &= \langle B(t,X_t)dt + DU(t,X_t) Q^{1/2} dW_t, e_n\rangle \end{aligned}$$

namely

$$dU(t,X_t) = B(t,X_t)dt + DU(t,X_t)Q^{1/2}dW_t$$

where $U(t,x) = \sum_n U_n(t,x)e_n$ and where we have used the PDE above. About our vector-valued notations, let us stress that $U(t,\cdot) : H \to H$, hence $DU(t,X_t) \in L(H,H)$. Moreover, for every $v \in H$,

$$\langle DU(t,X_t)v, e_n\rangle = \langle DU_n(t,X_t), v\rangle.$$

Formally speaking, the previous identity gives us a formula for $B(t,X_t)dt$:

$$B(t,X_t)dt = dU(t,X_t) - DU(t,X_t)Q^{1/2}dW_t.$$

We put this formula in equation (1) and get

$$dX_t = AX_t dt + dU(t,X_t) - DU(t,X_t)Q^{1/2}dW_t + Q^{1/2}dW_t.$$

Now we follow the usual variation of constant method and get:

$$\begin{aligned} d_s e^{(t-s)A}X_s =& e^{(t-s)A}dU(s,X_s) \\ &+ e^{(t-s)A}Q^{1/2}dW_s - e^{(t-s)A}DU(s,X_s)Q^{1/2}dW_s \end{aligned}$$

namely

$$\begin{aligned} X_t - e^{tA}x =& \int_0^t e^{(t-s)A}dU(s,X_s) \\ &+ \int_0^t e^{(t-s)A}Q^{1/2}dW_s - \int_0^t e^{(t-s)A}DU(s,X_s)Q^{1/2}dW_s. \end{aligned}$$

Integrating by parts the first integral we finally get the equation

$$\begin{aligned} X_t = e^{tA}(x - U(0,x)) + U(t,X_t) + \int_0^t Ae^{(t-s)A}U(s,X_s)ds \\ + \int_0^t e^{(t-s)A}Q^{1/2}dW_s - I_t(X) \end{aligned} \tag{6}$$

where

$$I_t(X) := \int_0^t e^{(t-s)A}DU(s,X_s)Q^{1/2}dW_s. \tag{7}$$

The non-regular drift B has been removed from the equation, this is the point of the trick. Several new terms appear, which however will be proved to have good Lipschitz properties.

In order to make rigorous this program we need: i) to solve the PDE (5) in a sufficiently regular space to be able to perform the previous computations (one bounded derivative plus an approximation argument is sufficient for this); ii) to prove that all the terms in equation (6) are Lipschitz continuous in the space variable (for this we need a uniform control of first and second derivatives). Morover, we need that the Lipschitz constant of the term $U(t,X_t)$ is small; we get this by taking small T and using the condition $U_n(T,x) = 0$.

3 H-valued Ornstein-Uhlenbeck semigroup

Let R_t be the Ornstein-Uhlenbeck semigroup, defined on $B_b(H)$ as

$$R_t\varphi(x) = E[\varphi(Z_t^x)],\ \varphi \in B_b(H)$$
$$dZ_t^x = AZ_t^x dt + Q^{1/2}dW_t,\ Z_0^x = x.$$

See [6], Chapter 6, for an extensive analysis of it. We introduce the analogous semigroup on H-valued functions:

$$\mathcal{R}_t\Phi(x) = E[\Phi(Z_t^x)],\ \Phi \in B_b(H,H).$$

We have

$$\langle \mathcal{R}_t\Phi(x), h\rangle = R_t\varphi_h(x),\ \varphi_h(x) = \langle \Phi(x), h\rangle,\ h \in H$$

Theorem 2 *Under the assumption (3), we have*

$$\Phi \in UC_b(H,H) \Rightarrow \mathcal{R}_t\Phi \in UC_b^2(H,H)$$

for all $t > 0$. The differential $D\mathcal{R}_t\Phi(x) \in L(H,H)$ at a given point $x \in H$ is the linear opeartor given by

$$D\mathcal{R}_t\Phi(x)g = \int_H \langle \Lambda_t g, Q_t^{-1/2}y\rangle \Phi(e^{At}x + y)N_{Q_t}(dy) \tag{8}$$

or

$$\begin{aligned}\langle D\mathcal{R}_t\Phi(x)g, h\rangle &= \langle DR_t\phi_h(x), g\rangle \\ &= \int_H \langle \Lambda_t g, Q_t^{-1/2}y\rangle \phi_h(e^{At}x + y)N_{Q_t}(dy)\end{aligned}$$

for all $t > 0$, $g, h \in H$. The second derivative $D^2\mathcal{R}_t\Phi(x) \in L(H, L(H,H))$ at a given point $x \in H$, is given by (recall that $D^2\mathcal{R}_t\Phi(x)$ is a linear operator in H, for every $g \in H$)

$$\begin{aligned}&[D^2\mathcal{R}_t\Phi(x)g]k \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle - \langle \Lambda_t g, \Lambda_t k\rangle\right] \Phi(e^{tA}x + y)N_{Q_t}(dy)\end{aligned} \tag{9}$$

or

$$\begin{aligned}&\langle [D^2\mathcal{R}_t\Phi(x)g]k, h\rangle = \langle D^2R_t\phi_h(x)g, k\rangle \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle - \langle \Lambda_t g, \Lambda_t k\rangle\right] \phi_h(e^{tA}x + y)N_{Q_t}(dy)\end{aligned}$$

for all $t > 0$, $g, k, h \in H$. If $\Phi \in UC_b^1(H,H)$ then

$$\langle [D^2\mathcal{R}_t\Phi(x)g]k, h\rangle = \int_H \langle \Lambda_t k, Q_t^{-1/2}y\rangle\langle D\phi_h(e^{tA}x + y), e^{tA}g\rangle N_{Q_t}(dy).$$

Finally,

$$\|D\mathcal{R}_t\Phi(x)\| \le \|\Lambda_t\|\|\Phi\|_0 \tag{10}$$
$$\|D^2\mathcal{R}_t\Phi(x)\| \le \sqrt{2}\|\Lambda_t\|^2\|\Phi\|_0 \tag{11}$$
$$\|D^2\mathcal{R}_t\Phi(x)\| \le \|e^{tA}\|\|\Lambda_t\|\|\Phi\|_1 \tag{12}$$

Proof. **Step 1.** Let us chech that the right hand side of (8), namely the mapping

$$g \mapsto I_{t,x} g := \int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle \Phi(e^{At}x + y) N_{Q_t}(dy)$$

defines a linear bounded operator in H; $x \in H$ and $t > 0$ are given. The integral is a well defined element of H, because

$$\int_H \left| \langle \Lambda_t g, Q_t^{-1/2} y\rangle \Phi(e^{At}x + y)\right|^2 N_{Q_t}(dy)$$

$$\leq \|\Phi\|_0^2 \int_H \left| \langle \Lambda_t g, Q_t^{-1/2} y\rangle \right|^2 N_{Q_t}(dy).$$

Linearity of $I_{t,x}$ is clear; in addition, from this estimate it follows that $I_{t,x}$ is bounded, and $\|I_{t,x}\| \leq \|\Lambda_t\| \|\Phi\|_0$. So inequality (10) well be true when we can say that $I_{t,x} = D\mathcal{R}_t\Phi(x)$.

Step 2. Let us prove that $\mathcal{R}_t\Phi$ is differentiable at x and $I_{t,x}$ is the differential. We have, for $g, h \in H$,

$$\langle \mathcal{R}_t\Phi(x+g) - \mathcal{R}_t\Phi(x) - I_{t,x}g, h\rangle$$

$$= R_t\phi_h(x+g) - R_t\phi_h(x) - \int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle \phi_h(e^{At}x + y) N_{Q_t}(dy).$$

Now, by Theorem 6.2.2 of [6],

$$R_t\phi_h(x+g) - R_t\phi_h(x) = \int_0^1 \langle DR_t\phi_h(x+sg), g\rangle ds$$

$$\int_0^1 \int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle \phi_h(e^{At}(x+sg) + y) N_{Q_t}(dy) ds$$

$$\int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle \left(\int_0^1 \phi_h(e^{At}(x+sg) + y) ds \right) N_{Q_t}(dy).$$

Thus we have

$$\langle \mathcal{R}_t\Phi(x+g) - \mathcal{R}_t\Phi(x) - I_{t,x}g, h\rangle$$

$$\int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle \left[\int_0^1 \phi_h(e^{At}(x+sg) + y) - \phi_h(e^{tA}x + y) ds \right] N_{Q_t}(dy)$$

$$\leq \left(\int_H \langle \Lambda_t g, Q_t^{-1/2} y\rangle^2 N_{Q_t}(dy) \right)^{1/2}$$

$$\left(\int_H \left[\int_0^1 \phi_h(e^{At}(x+sg) + y) - \phi_h(e^{tA}x + y) ds \right]^2 N_{Q_t}(dy) \right)^{1/2}$$

$$\leq |\Lambda g| \omega_t(g) \leq \|\Lambda_t\| |g| \omega_t(g)$$

where

$$\omega_t(g) = \sup_{s \in [0,1], x, y \in H} |\Phi(e^{tA}(x+sg) + y) - \Phi(e^{tA}x + y)|.$$

Since $\Phi \in UC_b(H,H)$,

$$\lim_{g\to 0}\omega_t(g) = 0$$

and thus $\mathcal{R}_t\Phi$ is differentiable at x with differential $I_{t,x}$. One can check that the differential is uniformly continuous in x. Clearly, by (10), it is also bounded. Thus we have proved $\mathcal{R}_t\Phi \in UC_b^1(H,H)$ and all claims about $D\mathcal{R}_t\Phi$.

Step 3. For given t, x let us analyze the right-hand-side of (9). Following [6], Lemma 6.2.7, for every bounded measurable $\varphi : H \to \mathbb{R}$, let us introduce the linear operator $G_\varphi^{t,x}$ in H defined as (we use different notations for the Gaussian measure with respect to the quoted reference)

$$\langle G_\varphi^{t,x}\alpha,\beta\rangle = \int_H \left[\langle \alpha, Q_t^{-1/2}y\rangle\langle \beta, Q_t^{-1/2}y\rangle - \langle\alpha,\beta\rangle\right]\varphi(e^{tA}x+y)N_{Q_t}(dy).$$

It is prove in [6] that $G_\varphi^{t,x}$ is even Hilbert-Schmidt, with Hilbert-Schmidt norm bounded by $2\|\varphi\|_0$. Therefore, in particular, $G_\varphi^{t,x}$ is a bounded linear operator with norm

$$\|G_\varphi^{t,x}\| \le 2\|\varphi\|_0.$$

To understand the right-hand-side of (9), let us introduce the linear mapping in H

$$k \mapsto J_{t,x,g}k := \int_H \left[\langle \Lambda_t g, Q_t^{-1/2}y\rangle\langle \Lambda_t k, Q_t^{-1/2}y\rangle - \langle\Lambda_t g,\Lambda_t k\rangle\right]\Phi(e^{tA}x+y)N_{Q_t}(dy).$$

We have

$$|\langle J_{t,x,g}k,h\rangle\| = |\langle G_{\varphi_h}^{t,x}\Lambda_t g, \lambda_t k\rangle| \le \|G_{\varphi_h}^{t,x}\||\Lambda_t g||\Lambda_t k| \le 2\|\varphi_h\|_0\|\Lambda_t\|^2|g||k|$$

$$\le 2\|\Lambda_t\|^2\|\Phi\|_0|g||k||h|.$$

Thus $J_{t,x,g}$ is bounded and

$$\|J_{t,x,g}\| \le 2\|\Lambda_t\|^2\|\Phi\|_0|g|.$$

Therefor $g \mapsto J_{t,x,g}$ is a bounded linear operator from H to $L(H,H)$, denoted by $J_{t,x}$ in the sequel of the proof (we have $J_{t,x}g = J_{t,x,g}$), and

$$\|J_{t,x}\|_{L(H,L(,H,H))} \le 2\|\Lambda_t\|^2\|\Phi\|_0.$$

If we prove that $J_{t,x}$ is $D^2\mathcal{R}_t\Phi(x)$, we have also proved inequality (11).

The proof of (12) is similar and based on the Hilbert-Schimdt property mentioned above.

Step 4. Given t, x, let us prove that $D\mathcal{R}_t\Phi$ is differentiable at x, and its differential is $J_{t,x}$. Recall that $\langle D\mathcal{R}_t\Phi(x)g,h\rangle$ is equal to $\langle DR_t\phi_h(x),g\rangle$. We have, for $g,h,k \in H$,

$$\langle [D\mathcal{R}_t\Phi(x+g) - D\mathcal{R}_t\Phi(x)]k,h\rangle = \langle DR_t\phi_h(x+g) - DR_t\phi_h(x),k\rangle$$

hence, by Proposition 6.2.2 of [6] this equals,

$$\int_0^1 \langle D^2R_t\varphi_h(x+sg)g,k\rangle ds$$

$$= \int_0^1 \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \phi_h(e^{tA}(xsg) + y) N_{Q_t}(dy) ds$$

$$= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \left(\int_0^1 \phi_h(e^{tA}(xsg) + y) ds \right) N_{Q_t}(dy).$$

Moreover,

$$\langle [J_{t,x} g] k, h \rangle = \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \phi_h(e^{tA} x + y) N_{Q_t}(dy).$$

Therefore

$$\langle [D\mathcal{R}_t \Phi(x+g) - D\mathcal{R}_t \Phi(x) - J_{t,x} g] k, h \rangle$$
$$\int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \psi_{t,x,g,h}(y) N_{Q_t}(dy)$$
$$= \langle G_{\psi_{t,x,g,h}} \Lambda_t g, \Lambda_t k \rangle$$

where

$$\psi_{t,x,g,h}(y) = \int_0^1 [\varphi_h(e^{tA}(x+sg) + y) - \varphi_h(e^{tA} x + y)$$

Hence, by Lemma 6.2.7 of [6],

$$\begin{aligned} |\langle [D\mathcal{R}_t \Phi(x+g) - D\mathcal{R}_t \Phi(x) - J_{t,x} g] k, h \rangle| &= |\langle G_{\psi_{t,x,g,h}} \Lambda_t g, \Lambda_t k \rangle| \\ &\leq 2 \|\psi_{t,x,g,h}\|_0 \|\Lambda_t\|^2 |g||k|. \end{aligned}$$

But

$$|\psi_{t,x,g,h}(y) \leq |h| \int_0^1 [\Phi(e^{tA}(x+sg) + y) - \Phi(e^{tA} x + y)] ds leq |h| \omega_t(g)$$

as in step 2. Therefore $D\mathcal{R}_t \Phi$ is differentiable at x and $D^2 \mathcal{R}_t \Phi$ is $J_{t,x}$. One can check that $D^2 \mathcal{R}_t \Phi$ is uniformly continuous in x. By (11), it is also bounded. We have proved $\mathcal{R}_t \Phi \in UC_b^2(H,H)$ and all claims about $D^2 \mathcal{R}_t \Phi$ when $\Phi \in UC_b(H,H)$. The proof of the claims on $D^2 \mathcal{R}_t \Phi$ when $\Phi \in UC_b^1(H,H)$ is similar and based on Proposition 6.2.9 of [6]. We do not give the details. The proof is complete.

■

4 Non homogenuous Kolmogorov equation

In this section we assume the conditions on A, B, Q stated in the introduction and section 1.1 and we study the sequence on non-homogeneous Kolmogorov equations in H

$$\begin{aligned} \frac{\partial U_n}{\partial t} =& \frac{1}{2} Tr \left(D^2 U_n Q \right) + \langle Ax, DU_n \rangle + \langle B, DU_n \rangle + G_n \\ U_n(0,x) =& 0 \end{aligned} \tag{13}$$

where G_n are the components of a function $G \in C([0,T]; C_b^\alpha(H,H))$. In this section we use forward notations for the PDE, for the sake of simplicity. The final result will apply to the backward PDE (5) (in particular, the assumption $B \in C([0,T]; C_b^\alpha(H,H))$ is invariant by time reversal).

We also show that the H-valued function $U(t,x) = \sum_n U_n(t,x)e_n$ has a meaning and we analyze its properties.

We interpret the PDE (13) as the integral equation

$$U_n(t,x) = \int_0^t R_{t-s}(\langle B(s), DU_n(s)\rangle + G_n(s))(x)ds. \tag{14}$$

Here we write $B(s)$ for $B(s,\cdot)$ and so on. Let us introduce alse the H-valued equation

$$U(t,x) = \int_0^t \mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s))(x)ds. \tag{15}$$

where we have denoted $\sum_n e_n\langle B(s), DU_n(s)\rangle$ by $\langle B(s), D\rangle U(s)$.

We can state the main result of this section. The regularity we prove for U is not optimal, and the theorem is restricted for simplicity of exposition to small T's.

Theorem 3 *Under the assumptions of section 1.1, given*

$$B, G \in C([0,T]; UC_b(H,H)),$$

for T small enough there exists a unique solution U of equation (15) in $C([0,T]; UC_b^1(H,H))$. If we put $K_T := \|DU\|_0$, then

$$\lim_{T\to 0} K_T = 0.$$

Moreover, $DU \in C([0,T]; C_b^\theta(H,H))$, θ such that assumption (4) hold.

If in addition $B, G \in C([0,T]; C_b^\alpha(H,H))$ for some $\alpha > 0$, then $U \in C([0,T]; UC_b^2(H,H))$. Finally, there is a constant $C_T > 0$ such that

$$\|D^2 U_n\|_0 \le C_T \|G_n\|_\alpha \tag{16}$$

for every $n \in \mathbb{N}$.

Proof. **Step 1.** Consider the map $\mathcal{L}$ defined as

$$\mathcal{L}U(t,x) = \int_0^t \mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s))(x)ds.$$

It is defined on functions $U \in C([0,T]; UC_b^1(H,H))$. It is easy to check that $\mathcal{L}U \in C([0,T]; UC_b(H,H))$. But we also have the bound

$$\int_0^t \|D\mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s))(x)\| ds$$

$$\le \int_0^t \|\Lambda_{t-s}\| \|\langle B(s), D\rangle U(s) + G(s)\|_0 ds$$

$$\leq (\|B\|_0 \|DU\|_0 + \|G\|_0) \int_0^t \|\Lambda_s\| ds$$

which implies $\mathcal{L}U \in C([0,T]; UC_b^1(H,H))$ and

$$D\mathcal{L}U(t,x) = \int_0^t D\mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s))(x) ds.$$

Since $\lim_{T\to 0} \int_0^T \|\Lambda\| ds = 0$, and the map $\mathcal{L}$ is linear, it is a contraction in $C([0,T]; UC_b^1(H,H))$ for sufficiently small T (one has to use also an estimate on U in the norm of $C([0,T]; UC_b(H,H))$). Moreover, if U is a solution, then

$$DU(t,x) = \int_0^t D\mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s))(x) ds \tag{17}$$

hence

$$\|DU\|_0 \leq (\|B\|_0 \|DU\|_0 + \|G\|_0) \int_0^t \|\Lambda_s\| ds$$

hence, for T such that $\|B\|_0 \int_0^T \|\Lambda\| ds \leq 1/2$ we have

$$\frac{1}{2} \|DU\|_0 \leq \|G\|_0 \int_0^t \|\Lambda_s\| ds$$

which proves $\lim_{T\to 0} K_T = 0$. We have proved the first claims of the theorem.

Step 2. Let us recall a result from interpolation theory developed in [6], Chapter 2. From Theorem 2.3.3 and the remarks at the beginning of section 2.3.3, for every $\theta\imath(0,1)$ there is a constant $C_\theta > 0$ such that

$$\|\varphi\|_\theta \leq C_\theta \|\varphi\|_0^{1-\theta} \|\varphi\|_1^\theta$$

for every $\varphi \in UC_b^1(H,\mathbb{R})$. The same result is true for $\Phi \in UC_b^1(H,H)$. Indeed, for every $h \in H$ the function $\varphi_h = \langle \Phi(\cdot), h\rangle$ belongs to $UC_b^1(H,\mathbb{R})$, hence

$$|\langle \Phi(x) - \Phi(y), h\rangle \leq C_\theta \|\varphi_h\|_0^{1-\theta} \|\varphi_h\|_1^\theta |h| |x-y|^\theta.$$

But $\|\varphi_h\|_0 \leq \|\Phi\|_0 |h|$ and $\|\varphi_h\|_1 \leq \|\Phi\|_1 |h|$. Hence

$$|\langle \Phi(x) - \Phi(y), h\rangle \leq C_\theta \|\Phi\|_0^{1-\theta} \|\Phi\|_1^\theta |h| |x-y|^\theta$$

which implies

$$\|\Phi\|_\theta \leq C_\theta \|\Phi\|_0^{1-\theta} \|\Phi\|_1^\theta.$$

We also have

$$\|\Phi\|_\theta \leq C_\theta \|\Phi\|_0^{1-\theta} \|D\Phi\|_0^\theta + C_\theta \|\Phi\|_0.$$

Similary, if $\Phi \in UC_b^2(H,H)$, we have

$$\|D\Phi\|_\theta \leq C_\theta \|D\Phi\|_0^{1-\theta} \|D^2\Phi\|_0^\theta + C_\theta \|D\Phi\|_0.$$

Step 3. Let us apply the previous interpolation inequality to $\mathcal{R}_t\Phi$, $\Phi \in UC_b(H,H)$, $t \geq 0$, with $\theta \in (0,1)$:

$$\begin{aligned}
\|D\mathcal{R}_t\Phi\|_\theta &\leq C_\theta\|D\mathcal{R}_t\Phi\|_0^{1-\theta}\|D^2\mathcal{R}_t\Phi\|_0^\theta + C_\theta\|D\mathcal{R}_t\Phi\|_0 \\
&\leq C_\theta(\|\Lambda_t\|\|\Phi\|_0)^{1-\theta}(\sqrt{2}\|\Lambda_t\|^2\|\Phi\|_0)^\theta + C_\theta\|\Lambda_t\|\|\Phi\|_0 \\
&\leq C'_\theta(\|\Lambda_t\|^{1+\theta}+1)\|\Phi\|_0
\end{aligned}$$

for a new constant $C'_\theta > 0$, where we have used inequalities (10) and (11) . Thus from (17) we have

$$\begin{aligned}
\|DU(t)\|_\theta &\leq \int_0^t \|D\mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s)\|_\theta ds \\
&\leq \int_0^t C'_\theta(\|\Lambda_t\|^{1+\theta}+1)\|\langle B(s), D\rangle U(s) + G(s)\|_0 ds \\
&= C'_\theta\|\langle B, D\rangle U + G\|_0 \int_0^t (\|\Lambda_t\|^{1+\theta}+1)ds.
\end{aligned}$$

If θ satisfies the assumption of section 1.1, namely $\int_0^t \|\Lambda_t\|^{1+\theta}ds < \infty$, we deduce that $DU(t) \in C_b^\theta(H,H)$ for each $t \in [0,T]$. Easily one can check that $DU \in C([0,T], C_b^\theta(H,H))$.

Step 4. Assume now $B, G \in C([0,T], C_b^\alpha(H,H))$. Since $\theta \geq \alpha$ (see section 1.1), we know that $\langle B, D\rangle U + G \in C([,T], C_b^\alpha(H,H))$. We use again an interpolation result of [6], see the proof of Lemma 6.4.1: there exists $C''_\alpha > 0$ such that

$$\|D^2R_t\varphi(x)\| \leq C''_\alpha\|\Lambda_t\|^{2-\alpha}\|\varphi\|_\alpha$$

for all $\varphi \in C_b^\alpha(H,\mathbb{R})$. It follows that

$$\|D^2\mathcal{R}_t\Phi(x)\| \leq C''_\alpha\|\Lambda_t\|^{2-\alpha}\|\Phi\|_\alpha$$

for all $\Phi \in C_b^\alpha(H,H)$.

Using these fact, from (17) we have

$$\begin{aligned}
\|D^2U(t)\| &\leq \int_0^t \|D^2\mathcal{R}_{t-s}(\langle B(s), D\rangle U(s) + G(s)\|ds \\
&\leq \int_0^t C''_\theta\|\Lambda_t\|^{2-\alpha}\|\langle B(s), D\rangle U(s) + G(s)\|_\alpha ds \\
&= C'_\theta\|\langle B, D\rangle U + G\|_\alpha \int_0^t \|\Lambda_t\|^{2-\alpha}ds
\end{aligned}$$

.

We have $\int_0^t \|\Lambda_t\|^{2-\alpha}ds < \infty$ (see section 1.1), hence $U \in C([0,T]; UC_b^2(H,H))$.

Step 5. From (17) or directly from equation (14) we have

$$DU_n(t,x) = \int_0^t DR_{t-s}(\langle B(s), DU_n(s)\rangle + G_n(s))(x)ds$$

and thus

$$D^2U_n(t,x) = \int_0^t D^2R_{t-s}(\langle B(s), DU_n(s)\rangle + G_n(s))(x)ds.$$

From the first one of these identities, with the same computations of step 1, we get (on the interval $[0,T]$ found in step 1)

$$\|DU_n\|_0 \leq C_1\|G_n\|_0.$$

As in step 3, we get

$$\|DU_n\|_\alpha \leq C_2\|\langle B, DU_n\rangle + G_n\|_0$$

and thus

$$\|DU_n\|_\alpha \leq (C_1C_2\|B\|_0 + C_2)\|G_n\|_0.$$

Finally, from the equation for $D^2U_n(t,x)$, exactly as in step 4, we prove

$$\|D^2U_n\|_0 \leq C_3\|\langle B, DU_n\rangle + G_n\|_\alpha.$$

Putting together these estimates, we obtain (16). The proof is complete.

■

5 Malliavin Differentiability

5.1 Strong Uniqueness

We fix a filtered probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}_{t\in[0,T]}$ such that W is a $\mathcal{F}_t$-cylindrical Brownian motion on H. A mild solution of equation (1) is a process $X = (X)_{t\in[0,T]}$, which is an $\mathcal{F}_t$-adapted continuous process in H and satisfies

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}B(s, X_s)ds + \int_0^t e^{(t-s)A}\sqrt{Q}dW_s.$$

The stochastic integral is well defined since we have assumed Q_t is of trace class.

The following rewriting is essential to our estimates:

Lemma 4 *Under the assumptions of section 1.1, let U be the solution given by Theorem 3. If $X = (X)_{t\in[0,T]}$ is a mild solution of equation (1), then the equation (6) is satisfied.*

Proof. Having now Theorem 3, the proof is the one given in Section 2. The only point is the application of Itô's formula. In order to use elementary versions of it, one can introduce the approximations

$$dX_t^{j,h} = A_jX_t^{j,h}dt + B(t, X_t^{j,h})dt + P_h\sqrt{Q}dW_t, \; X_0^{j,h} = x$$

where A_j are the Yosida approximations of A, $P_hx = \sum_{i=1}^h x_ie_i$. The computations of Section 2 can be done on these approximations and then one can pass to the limit in the final equation. We omit the details which are classical.

■

Using the previous lemma we proceed to prove pathwise uniqueness for the equation (1).

Theorem 5 *There exists a $T > 0$ such that pathwise uniqueness holds for (1) on $[0,T]$. That is, if X^1 and X^2 are two mild solutions, then we have for leb$\times P$ almost all $(t,x) \in [0,T] \times \Omega$, $X_t^1(\omega) = X_t^2(\omega)$.*

Proof. Assume X^1 and X^2 are two milds solutions, and define $V_t = X_t^1 - X_t^2$. Then, by Lemma 4 we have

$$\begin{aligned}\int_0^T |V_t|^2 dt \leq 3 \int_0^T |U(t,X_t^1) - U(t,X_t^2)|^2 dt \\ + 3 \int_0^T |\int_0^t Ae^{(t-s)A}[U(s,X_s^1) - U(s,X_s^2)]ds|^2 dt \\ + 3 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt.\end{aligned}$$

From Theorem 3 we have

$$|U(t,X_t^1) - U(t,X_t^2)| \leq K_T |X_t^1 - X_t^2|, \; t \in [0,T].$$

To deal with the second term we use the maximal inequality

$$\| \int_0^\cdot Ae^{(\cdot - s)A} f(s) ds \|_{L^2(0,T;H)}^2 \leq C_T \|f\|_{L^2(0,T;H)}^2$$

where C_T is a constant independent of f. Notice, however, that C_T does not converge to 0 as $T \to 0$. We then make the following estimate:

$$\int_0^T |V_t|^2 dt \leq (3 + 3C_T) K_T \int_0^T |V_t|^2 dt + 3 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt.$$

For T small enough we thus have

$$\int_0^T |V_t|^2 dt \leq 6 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt,$$

and in particular

$$\int_0^T E[|V_t|^2] dt \leq 6 \int_0^T E[|I_t(X^1) - I_t(X^2)|^2] dt.$$

The proof will be complete once we find an estimate on the right-and side of the previous inquality. We have

$$\int_0^T E[|I_t(X^1) - I_t(X^2)|^2] dt = \int_0^t \|e^{(t-s)A}(DU(s,X_s^1) - DU(s,X_s^2))\sqrt{Q}\|_{HS}^2 ds.$$

For the kernel, we write

$$
\begin{aligned}
&\|e^{(t-s)A}(DU(s,X_s^1)-DU(s,X_s^2))\sqrt{Q}\|_{HS}^2\\
&=\sum_{n,h\geq 1}\langle e^{(t-s)A}(DU(s,X_s^1)-DU(s,X_s^2))\sqrt{Q}e_h,e_n\rangle^2\\
&=\sum_{n,h\geq 1}e^{-2(t-s)\alpha_n}\langle (DU_n(s,X_s^1)-DU_n(s,X_s^2)),\sqrt{Q}e_h\rangle^2\\
&=\sum_{n\geq 1}e^{-2(t-s)\alpha_n}\sum_{h\geq 1}\langle \sqrt{Q}(DU_n(s,X_s^1)-DU_n(s,X_s^2)),e_h\rangle^2\\
&\leq \|Q\|\sum_{n\geq 1}e^{-2(t-s)\alpha_n}|DU_n(s,X_s^1)-DU_n(s,X_s^2)|^2\\
&\leq \|Q\|\sum_{n\geq 1}e^{-2(t-s)\alpha_n}\|DU_n\|_\infty^2|X_s^1-X_s^2|^2\\
&.
\end{aligned}
$$

From Theorem 3 we have

$$\|DU_n\|_\infty \leq C_T\|B_n\|_\alpha,$$

hence

$$E[|I_t(X^1)-I_t(X^2)|^2]\leq C_T^2\|Q\|\int_0^t\sum_{n\geq 1}e^{-2(t-s)\alpha_n}\|B_n\|_\alpha^2|V_s|^2ds.$$

Therefore

$$
\begin{aligned}
E[\int_0^T|I_t(X^1)-I_t(X^2)|^2dt] &\leq C_T^2\|Q\|\int_0^T\int_0^t\sum_{n\geq 1}e^{-2(t-s)\alpha_n}\|B_n\|_\alpha^2|V_s|^2ds\\
&\leq C_T^2\|Q\|\|B\|_\alpha^2\int_0^T\left(\int_s^T\sum_{n\geq 1}e^{-2(t-s)\alpha_n}dt\right)|V_s|^2ds\\
&\leq C_t^2\|Q\|\|B\|_\alpha^2\left(\int_0^T\sum_{n\geq 1}e^{-2t\alpha_n}dt\right)\int_0^T|V_s|^2ds.
\end{aligned}
$$

By assumption (2) we have $\lim_{T\to 0}\int_0^T\sum_{n\geq 1}e^{-2t\alpha_n}dt=0$, so that for small enough T we have

$$E[\int_0^T|V_s|^2ds]=0$$

which gives the result.

■

Notice that by the Yamada-Watanabe theorem, the previous theorem coupled with weak existence is enough to guarantee strong existence of equation (1). We will not elaborate further on this here.

In this paper we will however use Malliavin calculus to construct the solution. As a by-product of the construction method, we will prove that the solution is Malliavin differentiable.

5.2 Malliavin Differentiability

In the remainder of this section we want to use a compactness criterion for L^2-functionals of W_t based on Malliavin calculus (see Appendix, Theorem 14) to construct Malliavin differentiable mild solutions to (1).

To this end we need some definitions and auxiliary results.

Denote by $L_2(H)$ the space of Hilbert-Schmidt operators from H into itself with norm $\|\cdot\|_{HS}$. In what follows let $M : D(M) \subset H \longrightarrow H$ be a non-negative self-adjoint operator with existing compact inverse M^{-1}. Further consider the space E obtained by completion with respect to the norm $\|\cdot\|_E$ given by

$$\|K\|_E^2 := \sum_{n\geq 1} \|KMe_n\|^2$$

for $K \in L_2(H)$, if defined.

From now on we also assume that $Q^{1/2}M$ has a self-adjoint continuous extension to H such that

$$\left\|e^{(t-u)A}Q^{1/2}\right\|_E^2 \leq C\frac{1}{(t-u)^{1-\delta}} \tag{18}$$

for all $t > u \geq 0$ and

$$\begin{aligned}&\left\|e^{(t-u_1)A}Q^{1/2} - e^{(t-u_2)A}Q^{1/2}\right\|_E^2 \\ \leq\ & C\frac{1}{(t-u_1)^{1-\delta}}\left|u_1 - u_2\right|^{\mu}\end{aligned} \tag{19}$$

for all $t > u_1 > u_2 \geq 0$ and some $\mu > 0$. Further we also assume that

$$\int_H \left\|e^{(u_1-u_2)A}y - y\right\|^2 N_{Q_s}(dy) \leq C\left|u_1 - u_2\right|^{\eta} \tag{20}$$

for all $u_1 > u_2 \geq 0$, $s \geq 0$ and some $\eta > 0$.

Remark 6 *Since $\|K\|_{H.S.} \leq C\|K\|_E$ for all $K \in E$ for a constant C depending on M we also see that*

$$\left\|e^{(t-u)A}Q^{1/2}\right\|_{H.S.}^2 \leq C\frac{1}{(t-u)^{1-\delta}} \tag{21}$$

for all $t > u \geq 0$,

$$\begin{aligned}&\left\|e^{(t-u_1)A}Q^{1/2} - e^{(t-u_2)A}Q^{1/2}\right\|_{H.S.}^2 \\ \leq\ & C\frac{1}{(t-u_1)^{1-\delta}}\left|u_1 - u_2\right|^{\vartheta}\end{aligned} \tag{22}$$

for all $t > u_1 > u_2 \geq 0$.

The next result shows that if B in (1) is "nice" then the $E-$norm of the Malliavin derivative D_uX_t of X_t exists u-a.e., $P-$a.e.

Lemma 7 *Suppose that* $B \in C([0,T]; C_b^1(H,H))$ *in (1) . Further assume the conditions (18) and (19). Then*

$$\begin{aligned} & E\left[\int_0^T \|D_u X_t\|_E^2 \, du\right] \\ \leq \; & C(\frac{1}{\delta}T^\delta + \exp\left(2 \sup_{0 \leq r \leq T} \left\|e^{rA}\right\| \sup_{0 \leq s \leq T} \|DB(s,\cdot)\|_0 \, T\right)) \end{aligned}$$

for all $0 \leq t \leq T$.

Proof. Since

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B(s, X_s) ds + \int_0^t e^{(t-s)A} Q^{1/2} dW_s, 0 \leq t \leq T,$$

we find for all $0 \leq u \leq T$ that

$$D_u X_t = \int_u^t e^{(t-s)A} DB(s, X_s) D_u X_s ds + e^{(t-u)A} Q^{1/2}, u \leq t \leq T.$$

So we obtain by Picard iteration that

$$\begin{aligned} & D_u X_t \\ = \; & e^{(t-u)A} Q^{1/2} + \sum_{n \geq 2} {}_{u \leq s_1 < \ldots < s_{n-1} \leq t} e^{(s_n - s_{n-1})A} DB(s_{n-1}, X_{s_{n-1}}) \ldots \\ & e^{(s_2 - s_1)A} DB(s_1, X_{s_1}) e^{(s_1 - u)A} Q^{1/2} ds_1 \ldots ds_{n-1} \end{aligned} \tag{23}$$

in $L^2(\Omega; L_2(H))$ for all $u \leq t \leq T$.

Hence it follows from (23) $t > u$ that

$$\begin{aligned}
&\|D_u X_t\|_E \\
\leq\ & \left\|e^{(t-u)A}Q^{1/2}\right\|_E + \sum_{n\geq 2} {}_{u\leq s_1<\ldots<s_{n-1}\leq t} \sup_{0\leq r\leq T}\left\|e^{rA}\right\| \|DB(s_{n-1},\cdot)\|_0 \cdots \qquad (24)\\
& \sup_{0\leq r\leq T}\left\|e^{rA}\right\| \|DB(s_1,\cdot)\|_0 \left\|e^{(s_1-u)A}Q^{1/2}\right\|_E ds_1 \ldots ds_{n-1} \\
\leq\ & \left\|e^{(t-u)A}Q^{1/2}\right\|_E + \sum_{n\geq 2} \sup_{0\leq r\leq T}\left\|e^{rA}\right\|^{n-1} \\
& \cdot \frac{1}{(n-1)!}\int_{[u,T]^{n-1}} \|DB(s_{n-1},\cdot)\|_0 \cdot \ldots \cdot \|DB(s_1,\cdot)\|_0 \left\|e^{(s_1-u)A}Q^{1/2}\right\|_E ds_1 \ldots ds_{n-1} \\
\leq\ & \left\|e^{(t-u)A}Q^{1/2}\right\|_E + C\sum_{n\geq 2}\frac{1}{(n-1)!}\left(\sup_{0\leq r\leq T}\left\|e^{rA}\right\| \sup_{0\leq s\leq T}\|DB(s,\cdot)\|_0\right)^{n-1} (T-u)^{n-2} \\
& \cdot \int_u^T \frac{1}{(s_1-u)^{1-\delta}} ds_1 \\
\leq\ & \left\|e^{(t-u)A}Q^{1/2}\right\|_E + C(1+\sum_{n\geq 2}\frac{1}{(n-1)!}\left(\sup_{0\leq r\leq T}\left\|e^{rA}\right\| \sup_{0\leq s\leq T}\|DB(s,\cdot)\|_0 T\right)^{n-1})\frac{1}{\delta}(T-u)^{\delta}\frac{1}{T} \\
\leq\ & \left\|e^{(t-u)A}Q^{1/2}\right\|_E + C_{T,\delta}\exp\left(\sup_{0\leq r\leq T}\left\|e^{rA}\right\| \sup_{0\leq s\leq T}\|DB(s,\cdot)\|_0 T\right). \qquad (25)
\end{aligned}$$

So

$$\begin{aligned}
& \int_0^T \|D_u X_t\|_E^2 du = \int_0^t \|D_u X_t\|_E^2 du \\
\leq\ & C(\int_0^t \frac{1}{(t-u)^{1-\delta}}du + \exp\left(2\sup_{0\leq r\leq T}\left\|e^{rA}\right\| \sup_{0\leq s\leq T}\|DB(s,\cdot)\|_0 T\right)) \\
=\ & C(\frac{1}{\delta}t^{\delta} + \exp\left(2\sup_{0\leq r\leq T}\left\|e^{rA}\right\| \sup_{0\leq s\leq T}\|DB(s,\cdot)\|_0 T\right)) < \infty.
\end{aligned}$$

■

We shall also use the following Lemma

Lemma 8 *Let B in (1) be in $C([0,T];C_b^1(H,H))$. Then $X_t \in D(A^{\gamma/2})$ $P-$a.e. for all $0<\gamma<1$ and*

$$E[\left\|A^{\gamma/2}X_t\right\|^2] \leq C\frac{1}{t^{\gamma}}(1+\|B\|_0^2)$$

for all $0 \leq t \leq T$.

Proof. Since

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}B(s,X_s)ds + \int_0^t e^{(t-s)A}Q^{1/2}dW_s, 0 \leq t \leq T,$$

it is sufficient to prove that

$$E[\|Q_i\|^2] \le C_i \frac{1}{t^\gamma}(1+\|B\|_0^2), i=1,2,3,$$

where

$$\begin{aligned} Q_1 &:= A^{\gamma/2}e^{tA}x, Q_2 := \int_0^t A^{\gamma/2}e^{(t-s)A}B(s,X_s)ds, \\ Q_3 &:= \int_0^t A^{\gamma/2}e^{(t-s)A}dW_s. \end{aligned}$$

Then using the inequality

$$\left\|A^{\gamma/2}e^{(t-s)A}\right\| \le \frac{C_\gamma}{(t-s)^{\gamma/2}}$$

and Itô's isometry the result follows. ■

The next Lemma will be crucial for the application of the compactness criterion Theorem 14 in the Appendix.

Lemma 9 *Assume that $B \in C([0,T];C_b^1(H,H)) \cap C([0,T];C_b^\alpha(H,H))$. Let $X.$ be the mild solution to (1) associated with the coefficient B. Then for all $0 \le \theta \le T$ there exists a $0 < \beta < \frac{1}{2}$ such that*

$$E\left[\int_0^\theta \|D_u X_\theta\|_E^2 \, du\right] \le L_1(\|B\|_\alpha^2) \tag{26}$$

and

$$E\left[\int_0^\theta \int_0^\theta \frac{\|D_{u_1}X_\theta - D_{u_2}X_\theta\|_E^2}{|u_1-u_2|^{1+2\beta}} du_1 du_2\right] \le L_2(\|B\|_\alpha^2) \tag{27}$$

where $L_i, i=1,2$ are non-negative continuous functions on $[0,\frac{1}{V_T}]$ with $V_T \longrightarrow 0$ for $T \longrightarrow 0$.

Proof By applying the chain rule for the Malliavin derivative (see [13]) we know that

$$\begin{aligned} D_u X_t &= DU(t,X_t)D_uX_t + \int_u^t Ae^{(t-s)A}DU(s,X_s)D_uX_s ds \\ &\quad + e^{(t-u)A}Q^{1/2} - e^{(t-u)A}DU(u,X_u)Q^{1/2} \\ &\quad - \int_u^t e^{(t-s)A}D^2U(s,X_s)Q^{1/2}D_uX_s dW_s \\ &= I_1+I_2+I_3+I_4+I_5 \end{aligned} \tag{28}$$

for $u \le t < T \quad P-$a.e., where

$$\begin{aligned} I_1 &:= DU(t,X_t)D_uX_t, \; I_2 := \int_u^t Ae^{(t-s)A}DU(s,X_s)D_uX_s ds, \; I_3 := e^{(t-u)A}Q^{1/2}, \\ I_4 &:= -e^{(t-u)A}DU(u,X_u)Q^{1/2}, \; I_5 := -\int_u^t e^{(t-s)A}D^2U(s,X_s)Q^{1/2}D_uX_s dW_s. \end{aligned}$$

We want to use Gronwall's Lemma to show (26) and (27). To this end we need some estimates of $I_1, ..., I_5$.

1. Estimate for I_1 : By Lemma 7 and the estimates of Theorem 3 we find that

$$\begin{aligned} E[\|I_1\|_E^2] &\leq \|DU\|_0^2 \, E[\|D_u X_t\|_E^2] \\ &\leq K_T \, \|B\|_0^2 \, E[\|D_u X_t\|^2]_E < \infty \end{aligned}$$

for $t > u$, where $\lim_{T \longrightarrow 0} K_T = 0$.

2. Estimate of I_2 : Using the inequalities

$$\left\| A^{\epsilon} e^{At} \right\| \leq \frac{C_{\epsilon}}{t^{\epsilon}}$$

and

$$\|A^{\epsilon} DU(s, X_t)\| \leq C_{\varepsilon,T} \, \|B\|_0$$

for $0 < \epsilon < 1$ (see [8]) we obtain

$$\begin{aligned} & E[\|I_2\|_E^2] \\ \leq \; & (t-u)E[\int_u^t \left\| A e^{(t-s)A} DU(s, X_s) \right\|^2 \|D_u X_s\|_E^2 \, ds \\ \leq \; & (t-u)E[\int_u^t \left\| A^{1-\gamma} e^{(t-s)A} \right\|^2 \|A^{\gamma} DU(s, X_s)\|^2 \, \|D_u X_s\|_E^2 \, ds] \\ \leq \; & (t-u) C_{\varepsilon,T} \, \|B\|_0^2 \, E[\int_u^t \frac{C_{\gamma}}{(t-s)^{2(1-\gamma)}} \, \|D_u X_s\|_E^2 \, ds] \\ = \; & (t-u) C_{\varepsilon,T} C_{\gamma} \, \|B\|_0^2 \int_u^t \frac{1}{(t-s)^{2(1-\gamma)}} E[\|D_u X_s\|_E^2] ds \end{aligned}$$

for $1 > \gamma > 0$ with $2(1-\gamma) < 1$(see (25)).

3. Estimation of I_3: We know from (18) and (19)that

$$\|I_3\|_E^2 \leq C \frac{1}{(t-u)^{1-\delta}}.$$

As for the other two estimates we use the notation of the previous section and let

$$\Phi = \Phi(s) := \langle B(s), D \rangle \, U(s) - B(s).$$

4. Estimate for I_4: Because of our assumptions and Theorem 3 we obtain

$$\begin{aligned}
\|I_4\|_E^2 &= \left\|e^{(t-u)A}DU(u,X_u)Q^{1/2}\right\|_E^2 \\
&= \left\|e^{(t-u)A}DU(u,X_u)Q^{1/2}M\right\|_{H.S.}^2 \\
&= \sum_{n,h\geq 1}\langle e^{(t-u)A}DU(u,X_u)Q^{1/2}Me_h, e_n\rangle^2 \\
&= \sum_{n,h\geq 1} e^{-2\alpha_n(t-u)}\langle DU(u,X_u)Q^{1/2}Me_h, e_n\rangle^2 \\
&= \sum_{n,h\geq 1} e^{-2\alpha_n(t-u)}\langle DU_n(u,X_u), Q^{1/2}Me_h\rangle^2 \\
&= \sum_{n\geq 1} e^{-2\alpha_n(t-u)}\sum_{h\geq 1}\langle Q^{1/2}MDU_n(u,X_u), e_h\rangle^2 \\
&\leq \sum_{n\geq 1} e^{-2\alpha_n(t-u)}\left\|Q^{1/2}M\right\|^2\|DU_n(u,X_u)\|^2 \\
&\leq \left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_s\|\,ds)^2\sum_{n\geq 1}e^{-2\alpha_n(t-u)}\|B_n\|_0^2 \\
&= \left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_s\|\,ds)^2 \cdot \\
&\quad \cdot(\sum_{n\geq 1}(2\alpha_n(t-u))^{(1-\delta)}2e^{-2\alpha_n(t-u)}(2\alpha_n(t-u))^{-(1-\delta)}\|B_n\|_0^2) \\
&\leq C_\delta\left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_s\|\,ds)^2\|B\|_0^2\,2^{-(1-\delta)}\frac{1}{(t-u)^{(1-\delta)}}
\end{aligned}$$

if T is small enough. Hence

$$E[\|I_4\|_E^2]\leq C_\delta\left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_s\|\,ds)^2\|B\|_0^2\,2^{-(1-\delta)}\frac{1}{(t-u)^{(1-\delta)}}.$$

5. Estimate for I_5 : By our assumptions and the estimates of Theorem 2 we get for fixed

$r \in \mathbb{N}$

$$\begin{aligned}
& \sum_{h\geq 1} \left\| e^{(t-s)A} D^2 U(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r] \right\|^2 \\
= & \sum_{n,h\geq 1} \langle e^{(t-s)A} D^2 U(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r], e_n\rangle^2 \\
= & \sum_{n,h\geq 1} e^{-2\alpha_n(t-s)} \langle D^2 U(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r], e_n\rangle^2 \\
= & \sum_{n,h\geq 1} e^{-2\alpha_n(t-s)} (D^2 U_n(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r])^2 \\
= & \sum_{n\geq 1} e^{-2\alpha_n(t-s)} \sum_{h\geq 1} (D^2 U_n(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r])^2 \\
\leq & \sum_{n\geq 1} e^{-2\alpha_n(t-s)} \|Q\| \left\|D^2 U_n\right\|_0^2 \|D_u X_s M e_r\|^2 \\
\leq & C_T \|Q\| \sum_{n\geq 1} e^{-2\alpha_n(t-s)} \|B_n\|_\alpha^2 \|D_u X_s M e_r\|^2 \\
= & C_T \|Q\| \sum_{n\geq 1} (2\alpha_n(t-s))^{(1-\delta)} 2 e^{-2\alpha_n(t-s)} \cdot \\
& \cdot (2\alpha_n(t-s))^{-(1-\delta)} \|B_n\|_\alpha^2 \|D_u X_s M e_r\|^2 \\
\leq & C_\delta C_T \|Q\| \, 2^{-(1-\delta)} \|B\|_\alpha^2 \|D_u X_s M e_r\|^2 \frac{1}{(t-s)^{1-\delta}}
\end{aligned}$$

So it follows from the Itô isometry that

$$\begin{aligned}
& E[\|I_5\|_E^2] \\
= & \sum_{r\geq 1}\sum_{h\geq 1} \int_u^t \left\| e^{(t-s)A} D^2 U(s,X_s)[Q^{1/2}e_h, D_u X_s M e_r] \right\|^2 ds \\
\leq & C_\delta C_T \|Q\| \, 2^{-(1-\delta)} \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{1-\delta}} \|D_u X_s\|_E^2 \, ds.
\end{aligned}$$

So using the above estimates we get

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
\leq & C(K_T \|B\|_\alpha^2 E[\|D_u X_t\|]_E^2 + (t-u) C_{\varepsilon,T} C_\gamma \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{2(1-\gamma)}} E[\|D_u X_s\|_E^2] ds \\
& + C \frac{1}{(t-u)^{1-\delta}} + C_\delta \left\| Q^{1/2} M \right\|^2 C_T (\int_0^T \|\Lambda_s\| \, ds)^2 \|B\|_\alpha^2 \, 2^{-(1-\delta)} \frac{1}{(t-u)^{(1-\delta)}} \\
& + C_\delta C_T \|Q\| \, 2^{-(1-\delta)} \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{1-\delta}} \|D_u X_s\|_E^2 \, ds)
\end{aligned}$$

for $u \leq t \leq T$. Thus

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
\leq\ & \frac{C}{1-CK_T\|B\|_\alpha^2}\cdot \\
& \cdot(C\frac{1}{(t-u)^{1-\delta}}+C_\delta\left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_s\|\,ds)^2\|B\|_\alpha^2 2^{-(1-\delta)}\frac{1}{(t-u)^{(1-\delta)}}) \\
& +\frac{C}{1-CK_T\|B\|_\alpha^2}\cdot \\
& \cdot((t-u)C_{\varepsilon,T}C_\gamma\|B\|_\alpha^2+C_\delta C_T\|Q\|2^{-(1-\delta)}\|B\|_\alpha^2)\int_u^t\frac{1}{(t-s)^{1-\delta}}\|D_u X_s\|_E^2\,ds
\end{aligned}$$

for $u \leq t \leq T$ with T small enough such that $CK_T\|B\|_0 < 1$. Hence by a generalized Lemma of Gronwall for weakly singular kernels (see [1, Theorem 3]) we get

$$E[\|D_u X_t\|_E^2] \leq a(t)+\int_u^t\sum_{n\geq 1}\frac{(g(t)\Gamma(\delta))^n}{\Gamma(n\delta)}(t-s)^{n\delta-1}a(s)ds$$

for $u \leq t \leq T$, where

$$\begin{aligned}
& a(s) \\
:\ =\ & \frac{C}{1-CK_T\|B\|_\alpha^2}\cdot \\
& \cdot(C+C_\delta\left\|Q^{1/2}M\right\|^2 C_T(\int_0^T\|\Lambda_r\|\,dr)^2\|B\|_\alpha^2 2^{-(1-\delta)})\frac{1}{(s-u)^{1-\delta}},
\end{aligned}$$

$$\begin{aligned}
& g(t) \\
:\ =\ & \frac{C}{1-CK_T\|B\|_\alpha^2}((t-u)C_{\varepsilon,T}C_\gamma\|B\|_\alpha^2+C_\delta C_T\|Q\|2^{-(1-\delta)}\|B\|_\alpha^2)
\end{aligned}$$

and where Γ is the Gamma function.

Let us now assume that $n_0\delta < 1$, but $(n_0+1)\delta \geq 1$ for $n_0 \in \mathbb{N}$.

Therefore by using the following relation based on the Beta function

$$\int_u^t(t-s)^{n\delta-1}\frac{1}{(s-u)^{1-\delta}}ds=\frac{\Gamma(n\delta)\Gamma(\delta)}{\Gamma(n\delta+\delta)}(t-u)^{(n+1)\delta-1},$$

where Γ is the Gamma function, we obtain

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
\leq\ & L_1(\|B\|_\alpha)\frac{1}{(t-u)^{1-\delta}}+L_1(\|B\|_\alpha)\sum_{n\geq 1}\frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n}{\Gamma(n\delta)}\frac{\Gamma(n\delta)\Gamma(\delta)}{\Gamma(n\delta+\delta)}(t-u)^{(n+1)\delta-1} \\
\leq\ & L_1(\|B\|_\alpha)\frac{1}{(t-u)^{1-\delta}}+L_1(\|B\|_\alpha)\sum_{n=1}^{n_0-1}\frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n\Gamma(\delta)}{\Gamma((n+1)\delta)}\frac{1}{(t-u)^{1-(n+1)\delta}} \\
& +L_1(\|B\|_\alpha)\sum_{n\geq n_0}\frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n\Gamma(\delta)}{\Gamma((n+1)\delta)}(t-u)^{(n+1)\delta-1}, \qquad (29)
\end{aligned}$$

where L_1 and L_2 are non-negative continuous functions on $[0, \frac{1}{2CK_T}]$, where $K_T \longrightarrow 0$ for $T \longrightarrow 0$.

Altogether we get

$$\begin{aligned} & E[\int_0^T \|D_u X_t\|_E^2 \, du] \\ = \; & E[\int_0^t \|D_u X_t\|_E^2 \, du] \\ \leq \; & L_1(\|B\|_\alpha)\frac{1}{\delta}t^\delta + L_1(\|B\|_\alpha)\sum_{n=1}^{n_0-1}\frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n\Gamma(\delta)}{\Gamma((n+1)\delta)}\frac{t^{(n+1)\delta}}{(n+1)\delta} \\ & + L_1(\|B\|_\alpha)\sum_{n\geq n_0}\frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n T^{(n+1)\delta-1}\Gamma(\delta)}{\Gamma((n+1)\delta)}t. \end{aligned}$$

Let us now show the estimate (27). Assume that $t = \theta \geq u_1 > u_2 \geq 0$. Then it follows from (28)

$$D_{u_2}X_t - D_{u_1}X_t = \sum_{k=1}^{7} J_k,$$

where

$$\begin{aligned} J_1 \; & : \; = DU(t, X_t)(D_{u_2}X_t - D_{u_1}X_t), \\ J_2 \; & : \; = \int_{u_2}^{u_1} Ae^{(t-s)A}DU(s, X_s)D_{u_2}X_s ds, \\ J_3 \; & : \; = \int_{u_1}^{t} Ae^{(t-s)A}DU(s, X_s)(D_{u_2}X_s - D_{u_1}X_s)ds, \\ J_4 \; & : \; = e^{(t-u_2)A}Q^{1/2} - e^{(t-u_1)A}Q^{1/2} \\ J_5 \; & : \; = -(e^{(t-u_2)A}DU(u_2, X_{u_2})Q^{1/2} - e^{(t-u_1)A}DU(u_1, X_{u_1})Q^{1/2}) \\ J_6 \; & : \; = -\int_{u_2}^{u_1} e^{(t-s)A}D^2U(s, X_s)Q^{1/2}D_{u_2}X_s dW_s \\ J_7 \; & : \; = -\int_{u_1}^{t} e^{(t-s)A}D^2U(s, X_s)Q^{1/2}(D_{u_2}X_s - D_{u_1}X_s)dW_s. \end{aligned}$$

Let us first estimate the terms J_4, J_5, J_2 and J_6.

1. Estimation of J_4: By assumption we have

$$\left\| e^{(t-u_2)A}Q^{1/2} - e^{(t-u_1)A}Q^{1/2} \right\|_E^2 \leq C\frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^\mu$$

for all $0 \leq u_2 < u_1 < t$ and some $0 < \mu < 1$.

2. Estimate for J_5 : We can write J_5 as

$$J_5 = T_1 + T_2 + T_3,$$

where

$$\begin{aligned} T_1 &: = -(e^{(t-u_2)A} - e^{(t-u_1)A}) DU(u_2, X_{u_2}) Q^{1/2}, \\ T_2 &: = -e^{(t-u_1)A}((DU(u_2, X_{u_2}) Q^{1/2} - DU(u_2, X_{u_1}) Q^{1/2}) \\ T_3 &: = -e^{(t-u_1)A}(DU(u_2, X_{u_1}) Q^{1/2} - DU(u_1, X_{u_1}) Q^{1/2}) \end{aligned}$$

2.1. T_2 : Because of our assumptions, Theorem 3 and the mean value theorem we obtain

$$\begin{aligned} \|T_2\|_E^2 &= \left\| e^{(t-u_1)A}(DU(u_2, X_{u_2}) Q^{1/2} - DU(u_2, X_{u_1}) Q^{1/2}) \right\|_E^2 \\ &= \left\| e^{(t-u_1)A}(DU(u_2, X_{u_2}) - DU(u_2, X_{u_1})) Q^{1/2} M) \right\|_{H.S.}^2 \\ &= \sum_{n,h\geq 1} \langle e^{(t-u_1)A}(DU(u_2, X_{u_2}) - DU(u_2, X_{u_1})) Q^{1/2} M e_h, e_n \rangle^2 \\ &= \sum_{n,h\geq 1} e^{-2\alpha_n(t-u_1)} \langle (DU(u_2, X_{u_2}) - DU(u_2, X_{u_1})) Q^{1/2} M e_h, e_n \rangle^2 \\ &= \sum_{n,h\geq 1} e^{-2\alpha_n(t-u_1)} \langle (DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1})), Q^{1/2} M e_h \rangle^2 \\ &= \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \sum_{h\geq 1} \langle Q^{1/2} M (DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1})), e_h \rangle^2 \\ &\leq \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2} M \right\|^2 \| DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1}) \|^2 \\ &\leq \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2} M \right\|^2 \int_0^1 \left\| D^2 U_n(u_2, X_{u_2} + s(X_{u_2} - X_{u_1}))(X_{u_2} - X_{u_1}) \right\|^2 ds \\ &\leq \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2} M \right\|^2 \left\| D^2 U_n \right\|_0^2 \| X_{u_2} - X_{u_1} \|^2 \\ &\leq C_T \left\| Q^{1/2} M \right\|^2 \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \| B_n \|_\alpha^2 \| X_{u_2} - X_{u_1} \|^2 \\ &= C_T \left\| Q^{1/2} M \right\|^2 \sum_{n\geq 1} (2\alpha_n(t-u_1))^{(1-\delta)} e^{-2\alpha_n(t-u_1)} (2\alpha_n(t-u_1))^{-(1-\delta)} \| B_n \|_\alpha^2 \| X_{u_2} - X_{u_1} \|^2 \\ &\leq C_T \left\| Q^{1/2} M \right\|^2 \| B \|_\alpha^2 \, 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}} \| X_{u_2} - X_{u_1} \|^2 . \end{aligned}$$

Further we have that

$$\begin{aligned}
& X_{u_1} - X_{u_2} \\
= \ & e^{u_1 A}x - e^{u_2 A}x + \int_0^{u_1} e^{(u_1-s)A}B(s,X_s)ds - \int_0^{u_2} e^{(u_2-s)A}B(s,X_s)ds \\
& + \int_0^{u_1} e^{(u_1-s)A}Q^{1/2}dW_s - \int_0^{u_2} e^{(u_2-s)A}Q^{1/2}dW_s \\
= \ & e^{u_1 A}x - e^{u_2 A}x + \int_0^{u_2} (e^{(u_1-s)A} - e^{(u_2-s)A})B(s,X_s)ds + \int_{u_2}^{u_1} e^{(u_1-s)A}B(s,X_s)ds \\
& \int_0^{u_2} (e^{(u_1-s)A} - e^{(u_2-s)A})Q^{1/2}dW_s + \int_{u_2}^{u_1} e^{(u_1-s)A}Q^{1/2}dW_s.
\end{aligned}$$

On the other hand we find for all $0 < \varepsilon < 1/2$ that

$$\begin{aligned}
& \left\| (e^{(u_1-s)A} - e^{(u_2-s)A})x \right\|^2 \\
= \ & \sum_{k\geq 1} \langle x, e_k\rangle^2 (1 - e^{-(u_1-u_2)\alpha_k})^2 e^{-2(u_2-s)\alpha_k} \\
= \ & \sum_{k\geq 1} \langle x, e_k\rangle^2 ((u_1-u_2)\alpha_k)^{-2\varepsilon} (1 - e^{-(u_1-u_2)\alpha_k})^2 \cdot \\
& \cdot ((u_1-u_2)\alpha_k)^{2\varepsilon} \frac{(u_2-s)^{2\varepsilon}}{(u_2-s)^{2\varepsilon}} e^{-2(u_2-s)\alpha_k} \\
\leq \ & \|x\|^2 (u_1-u_2)^{2\varepsilon} \frac{1}{(u_2-s)^{2\varepsilon}}.
\end{aligned} \tag{30}$$

Hence it follows in connection with (22) and (30) for $T < 1$ that

$$\begin{aligned}
& E[\|X_{u_1} - X_{u_2}\|^2] \\
\leq \ & C(\|x\|^2 \frac{1}{u_2^{2\varepsilon}} (u_1-u_2)^{2\varepsilon} + \|B\|_0^2 \frac{1}{1-2\varepsilon} u_2^{1-2\varepsilon} (u_1-u_2)^{2\varepsilon} + \sup_{0\leq r\leq T} \left\| e^{rA} \right\|^2 \|B\|_0^2 (u_1-u_2) \\
& + \int_0^{u_2} \frac{1}{(u_2-s)^{1-\delta}} (u_1-u_2)^{\vartheta} ds + \int_{u_2}^{u_1} \frac{1}{(u_1-s)^{1-\delta}} ds) \\
= \ & C(\|x\|^2 \frac{1}{u_2^{2\varepsilon}} (u_1-u_2)^{2\varepsilon} + \|B\|_0^2 \frac{1}{1-2\varepsilon} u_2^{1-2\varepsilon} (u_1-u_2)^{2\varepsilon} + \sup_{0\leq r\leq T} \left\| e^{rA} \right\|^2 \|B\|_0^2 (u_1-u_2) \\
& + \frac{1}{\delta} u_2^{\delta} (u_1-u_2)^{\vartheta} + \frac{1}{\delta} (u_1-u_2)^{\delta}) \\
\leq \ & H(\|B\|_0) \frac{1}{u_2^{2\varepsilon}} (u_1-u_2)^{2\varepsilon\wedge\delta\wedge 1},
\end{aligned}$$

where H is a non-negative continuous function on $[0, \infty)$.

Therefore we have

$$\begin{aligned}
& E[\|T_2\|^2] \\
\leq \ & C_T \left\| Q^{1/2} M \right\|^2 \|B\|_\alpha^2 H(\|B\|_0) 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon}} (u_1-u_2)^{2\varepsilon\wedge\delta\wedge 1}.
\end{aligned}$$

for all $u_1 \geq u_2$.

2.2. T_3 : We know that

$$\begin{aligned}
& -e^{(t-u_1)A}(DU(u_2, X_{u_1})Q^{1/2} - DU(u_1, X_{u_1})Q^{1/2})h \\
= & \int_0^{u_1} e^{(t-u_1)A}[D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr - \int_0^{u_2} e^{(t-u_1)A}[D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})Q^{1/2}h]dr \\
= & G_1 + G_2,
\end{aligned}$$

where

$$\begin{aligned}
G_1 \ : \ & = \int_{u_2}^{u_1} e^{(t-u_1)A}[D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr, \\
G_2 \ : \ & = \int_0^{u_2} e^{(t-u_1)A}[D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr - \int_0^{u_2} e^{(t-u_1)A}[D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})Q^{1/2}h]dr.
\end{aligned}$$

We also see that

$$\begin{aligned}
& \left\| e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))drQ^{1/2} \right\|_E^2 \\
= & \left\| e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))drQ^{1/2}M \right\|_{H.S.}^2 \\
= & \sum_{n,h\geq 1} \langle e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))drQ^{1/2}Me_h, e_n\rangle^2 \\
= & \sum_{n,h\geq 1} e^{-2\alpha_n(t-u_1)} \langle \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))drQ^{1/2}Me_h, e_n\rangle^2 \\
= & \sum_{n,h\geq 1} e^{-2\alpha_n(t-u_1)} \langle \int_0^{u_2} (DR_{u_1-r}(\Phi_n)(X_{u_1}) - DR_{u_2-r}(\Phi_n)(X_{u_1}))dr, Q^{1/2}Me_h\rangle^2 \\
= & \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \sum_{h\geq 1} \langle Q^{1/2}M \int_0^{u_2} (DR_{u_1-r}(\Phi_n)(X_{u_1}) - DR_{u_2-r}(\Phi_n)(X_{u_1}))dr, e_h\rangle^2 \\
\leq & \sum_{n\geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2}M \right\|^2 \left\| \int_0^{u_2} (DR_{u_1-r}(\Phi_n)(X_{u_1}) - DR_{u_2-r}(\Phi_n)(X_{u_1}))dr \right\|^2 \\
= & \left\| Q^{1/2}M \right\|^2 \sum_{n\geq 1} (2\alpha_n(t-u_1))^{1-\delta} e^{-2\alpha_n(t-u_1)} (2\alpha_n(t-u_1))^{-(1-\delta)} \cdot \\
& \cdot \left\| \int_0^{u_2} (DR_{u_1-r}(\Phi_n)(X_{u_1}) - DR_{u_2-r}(\Phi_n)(X_{u_1}))dr \right\|^2 \\
\leq & C_\delta \left\| Q^{1/2}M \right\|^2 \left\| \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))dr \right\|^2 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}}.
\end{aligned}$$

On the other hand it follows from the semigroup property of $\mathcal{R}_t$ that

$$D\mathcal{R}_{u_1-r}(\Phi)(x) = D(\mathcal{R}_{u_2-r}(\mathcal{R}_{u_1-u_2}\Phi))(x).$$

This in connection with Theorem 2 gives

$$\int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))hdr$$
$$\int_0^{u_2} \int_H \left\langle \Lambda_{u_2-r}h, Q_{u_2-r}^{-1/2}y \right\rangle ((\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(e^{(u_2-r)A}X_{u_1} + y))N_{Q_{u_2-r}}(dy)dr.$$

Further, we find for $a = e^{(u_2-r)A}X_{u_1}+y$ and arbitrarily small $0 < 2\rho < 1$ by using Burkholder's inequality, Lemma 8 and (21)

$$\begin{aligned}
& \|(\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(a)\|^2 \\
= & \left\|E[\Phi(r, Z^a_{u_1-u_2}) - \Phi(r,a)]\right\|^2 \\
\leq & \|\Phi\|_\alpha^2 \, E[\left\|Z^a_{u_1-u_2} - Z^a_0\right\|^{2\alpha}] \\
= & \|\Phi\|_\alpha^2 \, E[\left\|(e^{(u_1-u_2)A}a - a + \int_0^{u_1-u_2} e^{(u_1-u_2-s)A}Q^{1/2}dW_s\right\|^{2\alpha}] \\
\leq & C\,\|\Phi\|_\alpha^2 \, (\left\|e^{(u_1-r)A}X_{u_1}(\omega_2) - e^{(u_2-r)A}X_{u_1}(\omega_2)\right\|^{2\alpha} \\
& + \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} + E[\left\|\int_0^{u_1-u_2} e^{(u_1-u_2-s)A}Q^{1/2}dW_s\right\|^{2\alpha}]) \\
\leq & C_\alpha \,\|\Phi\|_\alpha^2 \, (\left\|e^{(u_1-r)A}A^{-\rho}(A^\rho X_{u_1}(\omega_2)) - e^{(u_2-r)A}A^{-\rho}(A^\rho X_{u_1}(\omega_2))\right\|^{2\alpha} \\
& + \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} + \int_0^{u_1-u_2} \left\|e^{(u_1-u_2-s)A}Q^{1/2}\right\|^{2\alpha}_{H.S.} ds) \\
\leq & C_\alpha \,\|\Phi\|_\alpha^2 \, (|u_1-u_2|^{2\rho\alpha} \,\|A^\rho X_{u_1}(\omega_2)\|^{2\alpha} + \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} \\
& + \frac{1}{1-(1-\delta)\alpha}(u_1-u_2)^{1-(1-\delta)\alpha}).
\end{aligned}$$

Hence by (20) in connection with Hölder's in inquality we get

$$\begin{aligned}
& \left\|\int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))dr\right\|^2 \\
\leq & (\int_0^{u_2} \|\Lambda_{u_2-r}\| \, (\int_H \left\|((\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(e^{(u_2-r)A}X_{u_1} + y))\right\|^2 N_{Q_{u_2-r}}(dy))^{1/2}dr)^2 \\
\leq & (\int_0^{u_2} \|\Lambda_{u_2-r}\| \, (C_\alpha \,\|\Phi\|_\alpha^2 \, (|u_1-u_2|^{2\rho\alpha} \,\|A^\rho X_{u_1}(\omega_2)\|^{2\alpha} \\
& + \int_H \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} N_{Q_{u_2-r}}(dy)dr + \frac{1}{1-(1-\delta)\alpha}(u_1-u_2)^{1-(1-\delta)\alpha})^{1/2}dr)^2 \\
\leq & C_\alpha (\int_0^{u_2} \|\Lambda_{u_2-r}\| \, dr)^2 (C_\alpha \,\|\Phi\|_\alpha^2 \, (|u_1-u_2|^{2\rho\alpha} \,\|A^\rho X_{u_1}(\omega_2)\|^{2\alpha} \\
& + |u_1-u_2|^{\eta(\alpha/(1-\alpha))} + \frac{1}{1-(1-\delta)\alpha}(u_1-u_2)^{1-(1-\delta)\alpha})
\end{aligned}$$

So for $T<1$ we have

$$
\begin{aligned}
& E[\|G_2\|_E^2] \\
\leq\ & C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 (\int_0^T \|\Lambda_{u_2-r}\| \, dr)^2 (\|\Phi\|_\alpha^2 E[\|A^\rho X_{u_1}\|^{2\alpha}]+1)^2 \\
& \cdot \frac{1}{(t-u_1)^{1-\delta}} |u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha)} \\
\leq\ & C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 (\int_0^T \|\Lambda_{u_2-r}\| \, dr)^2 ((C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2) E[\|A^\rho X_{u_1}\|^{2\alpha}]+1)^2 \\
& \cdot \frac{1}{(t-u_1)^{1-\delta}} |u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha)}
\end{aligned}
$$

On the other hand we know by Lemma 8 in connection with Hölder's inequality that

$$
E[\|A^\rho X_{u_1}\|^{2\alpha}] \leq C(1+\|B\|_\alpha^2)^{\alpha/(1-\alpha)} \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}}
$$

for arbitrarily small $\rho>0$.

$$
\begin{aligned}
& E[\|G_2\|_E^2] \\
\leq\ & C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 (\int_0^T \|\Lambda_{u_2-r}\| \, dr)^2 ((C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2)(1+\|B\|_\alpha^2)^{\alpha/(1-\alpha)}+1)^2 \\
& \cdot \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha)}
\end{aligned}
$$

for arbitrarily small $\rho>0$.

As for the term G_1 we argue just as above and get

$$
\begin{aligned}
& \left\| e^{(t-u_1)A} \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr Q^{1/2} \right\|_E^2 \\
\leq\ & C_\delta \left\|Q^{1/2}M\right\|^2 \left\| \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr \right\|^2 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}}.
\end{aligned}
$$

But

$$
\begin{aligned}
& \left\| \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr \right\|^2 \\
\leq\ & C(\int_{u_2}^{u_1} \|\Lambda_{u_1-r}\| \, dr)^2 \|\Phi\|_0^2 \\
\leq\ & C(\int_0^{u_1-u_2} \|\Lambda_r\| \, dr)^2 C_T (K_1 \|B\|_\alpha^2 + K_2 \|B\|_\alpha)^2 \\
\leq\ & C(\int_0^T \|\Lambda_{u_1-r}\|^{1+\theta} \, dr)^{2/(1+\theta)} C_T (K_1 \|B\|_\alpha^2 + K_2 \|B\|_\alpha)^2 (u_1-u_2)^{\frac{\theta}{1+\theta}}
\end{aligned}
$$

Hence

$$E[\|G_1\|_E^2]$$

$$C_\delta \left\|Q^{1/2}M\right\|^2 2^{-(1-\delta)} C(\int_0^T \|\Lambda_{u_1-r}\|^{1+\theta}\, dr)^{2/(1+\theta)} C_T(K_1\|B\|_\alpha^2 + K_2\|B\|_\alpha)^2 (u_1-u_2)^{\frac{\theta}{1+\theta}} \frac{1}{(t-u_1)^{1-\delta}}.$$

So we obtain

$$\begin{aligned}
& E[\|T_3\|_E^2] \\
\leq\ & C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 (\int_0^T \|\Lambda_{u_2-r}\|\, dr)^2((C_1\|B\|_\alpha^3 + C_2\|B\|_\alpha^2)(1+\|B\|_\alpha^2)^{\alpha/(1-\alpha)}+1)^2 \\
& \cdot \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha)} \\
& +2C_\delta \left\|Q^{1/2}M\right\|^2 2^{-(1-\delta)} (C_1\|B\|_\alpha^3 + C_2\|B\|_\alpha^2)^2 \sup_{0\leq s\leq T} \left\|e^{sA}Q^{1/2}\right\|_{HS}^{2\alpha} (u_1-u_2)^{2\alpha} \frac{1}{(t-u_1)^{1-\delta}}.
\end{aligned}$$

2.3. T_1 : We find that

$$\begin{aligned}
& \left\|(e^{(t-u_2)A} - e^{(t-u_1)A})DU(u_2, X_{u_2})Q^{1/2}\right\|_E^2 \\
=\ & \sum_{h\geq 1} \left\|(e^{(t-u_2)A} - e^{(t-u_1)A})DU(u_2, X_{u_2})Q^{1/2}Me_h\right\|^2 \\
=\ & \sum_{n,h\geq 1} \langle (e^{(t-u_2)A} - e^{(t-u_1)A})DU(u_2, X_{u_2})Q^{1/2}Me_h, e_n\rangle^2 \\
=\ & \sum_{n,h\geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \langle DU_n(u_2, X_{u_2}), Q^{1/2}Me_h\rangle^2 \\
=\ & \sum_{n,h\geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \langle Q^{1/2}MDU_n(u_2, X_{u_2}), e_h\rangle^2 \\
\leq\ & \left\|Q^{1/2}M\right\|^2 \|DU\|_0^2 \sum_{n\geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \\
=\ & \left\|Q^{1/2}M\right\|^2 \|DU\|_0^2 \sum_{n\geq 1} e^{-2(t-u_1)\alpha_n}(1-e^{-(u_1-u_2)\alpha_n})^2 \\
=\ & \left\|Q^{1/2}M\right\|^2 \|DU\|_0^2 \cdot \\
& \cdot \sum_{n\geq 1} (2(t-u_1)\alpha_n)^{1-\delta} e^{-2(t-u_1)\alpha_n} (2(t-u_1)\alpha_n)^{\delta-1} (1-e^{-(u_1-u_2)\alpha_n})^2 \\
=\ & \left\|Q^{1/2}M\right\|^2 \|DU\|_0^2 \cdot \\
& \cdot \sum_{n\geq 1} (2(t-u_1)\alpha_n)^{1-\delta} e^{-2(t-u_1)\alpha_n} 2^{\delta-1}(t-u_1)^{\delta-1}(u_1-u_2)^{1-\delta}((u_1-u_2)\alpha_n)^{\delta-1}(1-e^{-(u_1-u_2)\alpha_n})^2 \\
\leq\ & C_\delta \left\|Q^{1/2}M\right\|^2 \|DU\|_0^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta} \\
=\ & C_T C_\delta \left\|Q^{1/2}M\right\|^2 \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta}.
\end{aligned}$$

Hence

$$
\begin{aligned}
& E[\|T_1\|_E^2] \\
\leq\ & C_T C_\delta \left\|Q^{1/2}M\right\|^2 \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{1-\delta}.
\end{aligned}
$$

Altogether we see for $T<1$ that

$$
\begin{aligned}
& E[\|J_5\|_E^2] \\
\leq\ & C(T,\delta,\alpha,M,Q,\theta,\mu,\vartheta,\varepsilon,A)G(\|B\|_\alpha) \\
& \cdot(\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{1-\delta}+\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon}}(u_1-u_2)^{2\varepsilon\wedge\delta\wedge 1} \\
& +\frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}}\frac{1}{(t-u_1)^{1-\delta}}|u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha\wedge 2\alpha)}),
\end{aligned}
$$

where G is a non-decreasing continuous function on $[0,\infty)$.

3. Estimate for J_2 : The calculation in **2.** for the estimate I_2 shows that

$$
\begin{aligned}
& E[\|J_2\|_E^2] \\
\leq\ & (u_1-u_2)C_{\varepsilon,T}C_\gamma \|B\|_0^2 \int_{u_2}^{u_1}\frac{1}{(u_1-s)^{2(1-\gamma)}}E[\|D_{u_2}X_s\|_E^2]ds
\end{aligned}
$$

Then employing the estimate (29) we have

$$
\begin{aligned}
& E[\|J_2\|_E^2] \\
\leq\ & (u_1-u_2)CC_{\varepsilon,T}C_\gamma\|B\|_0^2 \\
& \cdot(G_1(\|B\|_\alpha)\int_{u_2}^{u_1}\frac{1}{(u_1-s)^{2(1-\gamma)}}\frac{1}{(s-u_2)^{1-\delta}}ds \\
& +G_2(\|B\|_\alpha)\int_{u_2}^{u_1}\frac{1}{(u_1-s)^{2(1-\gamma)}}(s-u_2)^{\delta}ds) \\
=\ & (u_1-u_2)CC_{\varepsilon,T}C_\gamma\|B\|_0^2 \\
& \cdot(G_1(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)-(1-\delta)}\frac{\Gamma(1-2(1-\gamma))\Gamma(1-\delta)}{\Gamma(1-2(1-\gamma)+1-\delta)} \\
& +G_2(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)+\delta}\frac{\Gamma(1-2(1-\gamma))\Gamma(1+\delta)}{\Gamma(1-2(1-\gamma)+1+\delta)}) \\
=\ & C(\varepsilon,T,\gamma,\delta)(G_1(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)+\delta} \\
& +G_2(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)+\delta}),
\end{aligned}
$$

where we used the Beta function and where $0<2(1-\gamma)<1$ and $G_i, i=1,2$ are non-negative functions on an interval $[0,V(T)]$ with $V(T)\longrightarrow\infty$ for $T\longrightarrow 0$.

4. Estimate for J_6 : We argue just as in **5.** for the estimate I_5 and get in connection with the inequality (29) for $T<1$

$$
\begin{aligned}
& \sum_{r\geq 1}\sum_{h\geq 1}\int_{u_2}^{u_1} \left\| e^{(t-s)A} D^2U(s,X_s)[Q^{1/2}e_h, D_{u_2}X_sMe_r]\right\|^2 ds \\
= \; & \sum_{r\geq 1}\sum_{n,h\geq 1}\int_{u_2}^{u_1} \langle e^{(t-s)A} D^2U(s,X_s)[Q^{1/2}e_h, D_{u_2}X_sMe_r], e_n\rangle^2 ds \\
= \; & \sum_{r\geq 1}\sum_{n,h\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \langle D^2U(s,X_s)[Q^{1/2}e_h, D_{u_2}X_sMe_r], e_n\rangle^2 ds \\
= \; & \sum_{r\geq 1}\sum_{n,h\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} (D^2U_n(s,X_s)[Q^{1/2}e_h, D_{u_2}X_sMe_r])^2 ds \\
= \; & \sum_{r\geq 1}\sum_{n\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \sum_{h\geq 1}(D^2U_n(s,X_s)[Q^{1/2}e_h, D_{u_2}X_sMe_r])^2 ds \\
\leq \; & \sum_{n\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \|Q\| \left\|D^2U_n\right\|_0^2 \|D_{u_2}X_s\|_E^2 \, ds \\
\leq \; & C_T \sum_{n\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \|Q\| \, \|B\|_\alpha^2 \, \|D_{u_2}X_s\|_E^2 \, ds \\
\leq \; & C_T F(\|B\|_\alpha) \, \|B\|_\alpha^2 \sum_{n\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \frac{1}{(s-u_2)^{1-\delta}} ds,
\end{aligned}
$$

where F is a non-decreasing continuous function on $[0,\infty)$.

On the other hand we see by integration by parts and Hölder's inequality that

$$
\begin{aligned}
&\sum_{n\geq 1}\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)}\frac{1}{(s-u_2)^{1-\delta}}ds \\
= \quad & \sum_{n\geq 1}(\frac{1}{\delta}e^{-2\alpha_n(t-u_1)}(u_1-u_2)^{\delta}-2\alpha_n\int_{u_2}^{u_1} e^{-2\alpha_n(t-s)}\frac{1}{\delta}(s-u_2)^{\delta}ds) \\
\leq \quad & \sum_{n\geq 1}(\frac{1}{\delta}e^{-2\alpha_n(t-u_1)}(u_1-u_2)^{\delta}+\frac{1}{\delta}2\alpha_n(\int_{u_2}^{u_1} e^{-2\alpha_n(1+\tau)(t-s)}ds)^{1/(1+\tau)}(\int_{u_2}^{u_1}(s-u_2)^{\delta(1+\tau^{-1})}ds)^{1/(1+\tau^{-1})}) \\
= \quad & \sum_{n\geq 1}(\frac{1}{\delta}e^{-2\alpha_n(t-u_1)}(u_1-u_2)^{\delta}+\frac{1}{\delta}2\alpha_n(\frac{1}{2(1+\tau)\alpha_n}(e^{-2\alpha_n(1+\tau)(t-u_1)}-e^{-2\alpha_n(1+\tau)(t-u_2)}))^{1/(1+\tau)} \\
& \cdot(\frac{1}{(1+\tau^{-1})\delta+1})^{1/(1+\tau)}(u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}}) \\
= \quad & \sum_{n\geq 1}(\frac{1}{\delta}(2\alpha_n(t-u_1))^{1-\delta}e^{-2\alpha_n(t-u_1)}\frac{1}{(2\alpha_n(t-u_1))^{1-\delta}}(u_1-u_2)^{\delta}) \\
& +\frac{1}{\delta}(\frac{1}{2(1+\tau)})^{1/(1+\tau)}(\frac{1}{(1+\tau^{-1})\delta+1})^{1/(1+\tau)} \\
& \cdot\sum_{n\geq 1}\frac{1}{\alpha_n^{\frac{1}{1+\tau}-\delta}}(2\alpha_n(t-u_1))^{1-\delta}e^{-2\alpha_n(t-u_1)}\frac{1}{(t-u_1)^{1-\delta}}(1-e^{-2\alpha_n(1+\tau)(u_1-u_2)})^{1/(1+\tau)}(u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}} \\
\leq \quad & C(\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{\delta}+\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}}) \\
\leq \quad & C\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{\delta}
\end{aligned}
$$

for $\tau > 0$ small enough.

So we get

$$
\begin{aligned}
& E[\|J_6\|_E^2] \\
\leq \quad & C_T F(\|B\|_\alpha)\,\|B\|_\alpha^2\,\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{\delta}
\end{aligned}
$$

Let us now consider the terms J_1, J_3 and J_7. But this case just corresponds to the calculations for the estimates of I_1, I_2 and I_5 and we obtain

$$
\begin{aligned}
& E[\|J_1\|_E^2]+E[\|J_3\|_E^2]+E[\|J_7\|_E^2] \\
\leq \quad & K_T\,\|B\|_0^2\,E[\|D_{u_1}X_t-D_{u_2}X_t\|]_E^2 \\
& +(t-u_1)C_{\varepsilon,T}\,\|B\|_0^2\,E[\int_{u_1}^{t}\frac{C_\gamma}{(t-s)^{2(1-\gamma)}}\,\|D_{u_1}X_s-D_{u_2}X_s\|_E^2\,ds] \\
& +C_\delta C_T\,\|Q\|\,2^{-(1-\delta)}\,\|B\|_\alpha^2\int_{u_1}^{t}\frac{1}{(t-s)^{1-\delta}}\,\|D_{u_1}X_s-D_{u_2}X_s\|_E^2\,ds
\end{aligned}
$$

Altogether it follows from the above estimates that

$$
\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq\ & C(\sum_{i=1}^{7} E[\|J_i\|_E^2])) \\
\leq\ & C(K_T \|B\|_\alpha^2 E[\|D_{u_1}X_t - D_{u_2}X_t\|]_E^2 \\
& +C(\varepsilon,T,\gamma,\delta)(G_1(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)+\delta} + G_2(\|B\|_\alpha)(u_1-u_2)^{1-2(1-\gamma)+\delta}) \\
& +C\frac{1}{(t-u_1)^{1-\delta}}|u_1-u_2|^\mu \\
& +C(T,\delta,\alpha,M,Q,\theta,\mu,\vartheta,\varepsilon,A)G(\|B\|_\alpha) \\
& \cdot(\frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^{1-\delta} + \frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon}}(u_1-u_2)^{2\varepsilon\wedge\delta\wedge 1} \\
& +\frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}}\frac{1}{(t-u_1)^{1-\delta}}|u_1-u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha\wedge 2\alpha)}) \\
& +C_T F(\|B\|_\alpha)\|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}}(u_1-u_2)^\delta \\
& +(t-u_1)C_{\varepsilon,T}\|B\|_0^2 E[\int_{u_1}^t \frac{C_\gamma}{(t-s)^{2(1-\gamma)}}\|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds] \\
& +C_\delta C_T \|Q\| 2^{-(1-\delta)}\|B\|_\alpha^2 \int_{u_1}^t \frac{1}{(t-s)^{1-\delta}}\|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds).
\end{aligned}
$$

Since $K_T \longrightarrow 0$ for $T \longrightarrow 0$ and $1 > T > u_1 > u_2$ we get

$$
\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq\ & V(\|B\|_\alpha)\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda \\
& +V(\|B\|_\alpha)\int_{u_1}^t \frac{1}{(t-s)^{1-\delta}}\|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds,
\end{aligned}
$$

where $\lambda = (1-2(1-\gamma)+\delta)\wedge\mu\wedge(1-\delta)\wedge(2\varepsilon\wedge\delta\wedge 1)\wedge((2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha\wedge 2\alpha))\wedge\delta > 0$ and where V is a non-negative continuous function on $[0, \frac{1}{A_T}]$ for $A_T \longrightarrow 0$ for $T \longrightarrow 0$.

So by a Lemma of Gronwall for weakly singular kernels (see [1, Theorem 3]) we get

$$
E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \leq a(t) + \int_{u_1}^t \sum_{n\geq 1}\frac{(g(t)\Gamma(\delta))^n}{\Gamma(n\delta)}(t-s)^{n\delta-1}a(s)ds
$$

for $u_1 \leq t \leq T$, where

$$
a(t) := V(\|B\|_\alpha)\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda
$$

and

$$
g(t) \equiv V(\|B\|_\alpha).
$$

Therefore we have by means of the Beta function

$$\begin{aligned}
&E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq\ & V(\|B\|_\alpha)\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda \\
&+\sum_{n=1}^{n_0-1}\frac{(V(\|B\|_\alpha)\Gamma(\delta))^n\Gamma(\delta)}{\Gamma((n+1)\delta)}\frac{1}{(t-u_1)^{1-(n+1)\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda \\
&+\sum_{n\geq n_0}\frac{(V(\|B\|_\alpha)\Gamma(\delta))^n\Gamma(\delta)}{\Gamma((n+1)\delta)}(t-u_1)^{(n+1)\delta-1}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda \\
\leq\ & E(\|B\|_\alpha)\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}(u_1-u_2)^\lambda,
\end{aligned}$$

where V is a non-negative continuous function on $[0, \frac{1}{A_T}]$ for $A_T \longrightarrow 0$ for $T \longrightarrow 0$ and where $n_0\delta < 1$, but $(n_0+1)\delta \geq 1$ for $n_0 \in \mathbb{N}$.

Thus it follows for all $0 \leq u_1, u_2 < t$ that

$$\begin{aligned}
&E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq\ & E(\|B\|_\alpha)(\frac{1}{(t-u_1)^{1-\delta}}\frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}|u_1-u_2|^\lambda \\
&+\frac{1}{(t-u_2)^{1-\delta}}\frac{1}{u_1^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}}|u_1-u_2|^\lambda)
\end{aligned}$$

Now choose in (27) $\beta > 0$ such that $\kappa := 1 + 2\beta - \lambda < 1$. Then

$$
\begin{aligned}
& \int_0^t \int_0^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1 - u_2|^{\kappa}} du_1 du_2 \\
= & \int_0^t \int_{u_2}^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1 - u_2|^{\kappa}} du_1 du_2 \\
& + \int_0^t \int_0^{u_2} \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1 - u_2|^{\kappa}} du_1 du_2 \\
\leq & \int_0^t \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \int_{u_2}^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{|u_1 - u_2|^{\kappa}} du_1 du_2 \\
& + \int_0^t \frac{1}{(t-u_2)^{1-\delta}} \int_0^{u_2} \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1 - u_2|^{\kappa}} du_1 du_2 \\
= & \frac{\Gamma(\delta)\Gamma(1-\kappa)}{\Gamma(\delta+1-\kappa)} \int_0^t \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_2)^{\kappa-\delta}} du_2 \\
& + \frac{\Gamma((1-2\varepsilon) \vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 1 - \kappa + \delta)} \\
& \cdot \int_0^t \frac{1}{(t-u_2)^{1-\delta}} \frac{1}{u_2^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 2\beta - \lambda - \delta}} du_2 \\
= & \frac{\Gamma(\delta)\Gamma(1-\kappa)}{\Gamma(\delta+1-\kappa)} \frac{\Gamma(1-2\varepsilon \vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 1 - \kappa + \delta))} \\
& \cdot \frac{1}{t^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 2\beta - \lambda - \delta}} \\
& + \frac{\Gamma(1-2\varepsilon \vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 1 - \kappa + \delta)} \\
& \cdot \frac{\Gamma(\delta)\Gamma(1-(2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 2\beta - \lambda - \delta))}{\Gamma(\delta + 1 - (2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 2\beta - \lambda - \delta))} \frac{1}{t^{2\varepsilon \vee 2\rho(\alpha/(1-\alpha)) + 2\beta - \lambda - 2\delta}} \\
< & \infty,
\end{aligned}
$$

since $2\beta - \lambda < 0$ and since $\varepsilon >$ and $\rho > 0$ can be chosen arbitrarily small.

So we get the estimate (27), which completes the proof. ■

Let now $B^n \in C([0,T]; C_b^1(H,H)) \cap C([0,T]; C_b^{\alpha}(H,H)), n \geq 1$ be a sequence of functions and $B \in C([0,T]; C_b^{\alpha}(H,H))$ such that

$$B^n(t,x) \longrightarrow B(t,x) \tag{31}$$

for $n \longrightarrow \infty$ in H for all x and such that

$$\|B^n\|_{\alpha} \leq K \tag{32}$$

for a constant K independent of n. See e.g. [11].

We also need the following Lemma:

Lemma 10 *Suppose that $X_t^n, 0 \leq t \leq T, n \geq 1$ are the unique mild solutions to (1) with respect to the coefficients B^n in (31) and (32). Let $X_t^{n,i} = \langle e_i, X_t^n \rangle$. Then there exists for all i a subsequence $(n_k^i)_{k\geq 1}$ which only depends on (a sufficiently small) T and i such that for all $0 \leq t \leq T$ $X_t^{n_k^i,i}$ converges in $L^2(\Omega)$ for $k \longrightarrow \infty$.*

Proof. We know that

$$X_t^n = e^{tA}x + \int_0^t e^{(t-s)A}B^n(s, X_s)ds + \int_0^t e^{(t-s)A}Q^{1/2}dW_s, 0 \leq t \leq T.$$

So

$$X_t^{n,i} = \langle e_i, e^{tA}x \rangle + \int_0^t \langle e_i, e^{(t-s)A}B^n(s, X_s) \rangle ds + \int_0^t \langle e_i, e^{(t-s)A}Q^{1/2}, dW_s \rangle \ , 0 \leq t \leq T.$$

Hence

$$\begin{aligned}
& X_{t_1}^{n,i} - X_{t_2}^{n,i} \\
= \ & \langle e_i, e^{t_1 A}x - e^{t_2 A}x \rangle + \int_0^{t_2} \langle e_i, (e^{(t_1-s)A} - e^{(t_2-s)A})B^n(s, X_s^n) \rangle ds \\
& + \int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}B^n(s, X_s^n) \rangle ds + \int_0^{t_2} \langle e_i, e^{(t_1-s)A} - e^{(t_2-s)A}Q^{1/2}dW_s \rangle \\
& + \int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}Q^{1/2}dW_s \rangle
\end{aligned} \tag{33}$$

for all $0 \leq t_2 \leq t_1 \leq T$.

Now let f be an element of the Hida test function space $(\mathcal{S}) \subset L^2(\Omega)$. Denote by $(\mathcal{S})^*$ its topological dual (Hida distribution space). See [10] for further information on these spaces. Then $\left\langle (X_{t_1}^{n,i} - X_{t_2}^{n,i}), f \right\rangle_{(\mathcal{S})^*,(\mathcal{S})} = E[(X_{t_1}^{n,i} - X_{t_2}^{n,i})f]$, where $\langle \cdot, \cdot \rangle_{(\mathcal{S})^*,(\mathcal{S})}$ is the dual pairing. So using (33) we get

$$\begin{aligned}
& E[(X_{t_1}^{n,i} - X_{t_2}^{n,i})f] \\
= \ & \langle e_i, e^{t_1 A}x - e^{t_2 A}x \rangle E[f] + \int_0^{t_2} E[\langle e_i, (e^{(t_1-s)A} - e^{(t_2-s)A})B^n(s, X_s^n) \rangle, f]ds \\
& + \int_{t_2}^{t_1} E[\langle e_i, e^{(t_1-s)A}B^n(s, X_s^n) \rangle, f]ds + E[\int_0^{t_2} \langle e_i, e^{(t_1-s)A} - e^{(t_2-s)A}Q^{1/2}dW_s, f] \rangle \\
& + E[\int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}Q^{1/2}dW_s, f]. \rangle
\end{aligned}$$

Thus it follows from (21) and (22)

$$
\begin{aligned}
& \left|E[(X^{n,i}_{t_1} - X^{n,i}_{t_2})f]\right| \\
\leq \; & |\langle e_i, x\rangle| \left|e^{-t_1\alpha_i} - e^{-t_2\alpha_i}\right| |E[f]| + \int_0^{t_2} E[|\langle e_i, B^n(s, X^n_s)\rangle f|] \left|(e^{-(t_1-s)\alpha_i} - e^{(t_2-s)\alpha_i})\right| ds \\
& + \int_{t_2}^{t_1} E[|\langle e_i, B^n(s, X^n_s)\rangle f|] e^{-(t_1-s)\alpha_i} ds + (\int_0^{t_2} \left\|e^{(t_1-s)A} - e^{(t_2-s)A} Q^{1/2}\right\|^2_{H.S.} ds)^{1/2} (E[f^2])^{1/2} \\
& +(\int_{t_2}^{t_1} \left\|e^{(t_1-s)A} Q^{1/2}\right\|^2_{H.S.} ds)^{1/2} (E[f^2])^{1/2} \\
\leq \; & C_{i,T} |t_1 - t_2| \|B^n\| E[|f|] + C(\int_0^{t_2} \frac{1}{(t_2-s)^{1-\delta}} |t_1 - t_2|^{\vartheta} ds)^{1/2} (E[f^2])^{1/2} \\
& +C(\int_{t_2}^{t_1} \frac{1}{(t_1-s)^{1-\delta}} ds)^{1/2} (E[f^2])^{1/2} \\
\leq \; & C(i, T, \vartheta, \delta, K, f) |t_1 - t_2|^{\vartheta \wedge (\delta/2)} .
\end{aligned}
$$

So

$$\sup_{n\geq 1} m^T(\langle (X^{n,i}_{\cdot}, f\rangle_{(\mathcal{S})^*,(\mathcal{S})}, \delta) \longrightarrow 0 \text{ for } \delta \searrow 0,$$

where m^T is the modulus of continuity given by

$$m^T(g, \delta) := \max_{\substack{|t-s|\leq \delta \\ 0\leq t,s\leq T}} |g(t) - g(s)| .$$

So $\left\langle (X^{n,i}_{\cdot}, f \right\rangle_{(\mathcal{S})^*,(\mathcal{S})}$ is relatively compact in $C([0,T])$ for all $f \in (\mathcal{S})$. Since $(\mathcal{S})^*$ is the dual of a countably Hilbertian nuclear space $(\mathcal{S})$, we can apply a result of I. Mitoma [14] and find that there exists for all i a subsequence $(n^i_k)_{k\geq 1}$ which only depends on (a sufficiently small) T and i such that $X^{n^i_k,i}_{\cdot}$ converges in $C([0,T]; (\mathcal{S})^*)$.

On the other hand it follows from Lemma 9 that there exists (for fixed t) a $C < \infty$ and a $0 < \beta < \frac{1}{2}$ such that

$$E\left[\int_0^t \|D_u X^n_t\|^2_E \, du\right] \leq L_1(\|B^n\|^2_\alpha) \leq C < \infty$$

and

$$
\begin{aligned}
& E\left[\int_0^t \int_0^t \frac{\|D_{u_1} X^n_t - D_{u_2} X^n_t\|^2_E}{|u_1 - u_2|^{1+2\beta}} du_1 du_2\right] \\
\leq \; & L_2(\|B^n\|^2_\alpha) \leq C < \infty
\end{aligned}
$$

for all $n \geq 1$, provided that T is sufficiently small.

Then, if we apply Theorem 14 in connection with Remark 15 in the Appendix to the sequence $X^{n^i_k,i}_t$ we see that for all t and i there exists a subsequence $m_l = m^{t,i}_l, l \geq 1$ of $n^i_k, k \geq 1$ and a $\widetilde{X}^i_t \in L^2(\Omega)$ such that

$$X^{n^i_{m_l},i}_t \longrightarrow \widetilde{X}^i_t \text{ for } l \longrightarrow \infty \tag{34}$$

in $L^2(\Omega)$.

We claim that

$$X_t^{n_k^i,i} \longrightarrow \widetilde{X}_t^i \text{ for } k \longrightarrow \infty \text{ in } L^2(\Omega)$$

for all t, i. To see this assume that there exists for some t, i a $\varepsilon > 0$ and a subsequence $\varphi_l, l \geq 1$ such that

$$\left\| X_t^{n_{\varphi_l}^i,i} - \widetilde{X}_t^i \right\|_{L^2(\Omega)} \geq \varepsilon.$$

On the other hand we know by Theorem 14 that there exists a subsequence $\phi_r, r \geq 1$ of such that

$$X_t^{n_{\varphi_{\phi_r}}^i,i} \longrightarrow \widetilde{Y}_t^i \text{ for } r \longrightarrow \infty \text{ in } L^2(\Omega).$$

But since

$$X_t^{n_k^i,i} \longrightarrow \widetilde{X}_t^i \text{ for } k \longrightarrow \infty \text{ in } (\mathcal{S})^*$$

because of (34), we see that

$$\widetilde{Y}_t^i = \widetilde{X}_t^i.$$

But this leads to the contradiction

$$\left\| X_t^{n_{\varphi_{\phi_r}}^i,i} - \widetilde{X}_t^i \right\|_{L^2(\Omega)} \geq \varepsilon.$$

This completes the proof. ■

We are coming to the main result of this section

Theorem 11 *Assume that the functions $B : [0,T] \times H \longrightarrow H$ and $B^n : [0,T] \times H \longrightarrow H, n \geq 1$ satisfy the conditions (31) and (32). Then there exists a Malliavin differentiable unique mild solution $X_t, 0 \leq t \leq T$ to the stochastic differential equation*

$$dX_t = AX_t dt + B(t, X_t)dt + Q^{1/2}dW_t, X_0 = x. \tag{35}$$

Proof. Let $X_t^n, n \geq 1$ be the mild solutions associated with the coefficients B^n and denote by $X_t^{n,i}$ the $i-$th component of X_t^n. Then it follows from Lemma 10 that there exists for all i a subsequence $(n_k^i)_{k\geq 1}$ which only depends on (a sufficiently small) T and i such that for all $0 \leq t \leq T$

$$X_t^{n_k^i,i} \longrightarrow X_t^i \text{ in } L^2(\Omega) \text{ for } k \longrightarrow \infty$$

for some $X_t^i \in L^2(\Omega), 0 \leq t \leq T, i \geq 1$.

Now let us denote by $(\varphi_n)_{n\geq 1}$ the diagonal sequence of the sequences $(n_k^1)_{k\geq 1}, (n_k^2)_{k\geq 1}, (n_k^3)_{k\geq 1}, \ldots$ So

$$X_t^{\varphi_n,i} \longrightarrow X_t^i \text{ for } n \longrightarrow \infty$$

in $L^2(\Omega)$ for all t, i.

We now want to show that

$$X_t^{\varphi_n} \longrightarrow X_t \text{ for } n \longrightarrow \infty \tag{36}$$

in $L^2(\Omega; H)$ for all t, where $X_t = \sum_{k\geq 1} X_t^k e_k$. For this purpose choose a $\epsilon > 0$. By a weak compactness argument we also see from Lemma 8 that

$$E[(\sum_{k\geq 1} \left| X_t^k \right|^2 \alpha_k^{1-\delta})] < \infty.$$

This implies

$$\begin{aligned}
& \sup_{n\geq 1} \sum_{k\geq m} E[\left| X_t^{\varphi_n,k} - X_t^k \right|^2] \\
\leq \; & \sup_{n\geq 1} E[\sum_{k\geq m} \left| X_t^{\varphi_n,k} - X_t^k \right| \frac{1}{\alpha_k^{(1-\delta)/2}} \left| X_t^{\varphi_n,k} - X_t^k \right| \alpha_k^{(1-\delta)/2}] \\
\leq \; & \sup_{n\geq 1} E[(\sum_{k\geq m} \left| X_t^{\varphi_n,k} - X_t^k \right|^2 \frac{1}{\alpha_k^{1-\delta}})^{1/2} (\sum_{k\geq m} \left| X_t^{\varphi_n,k} - X_t^k \right|^2 \alpha_k^{1-\delta})^{1/2}] \\
\leq \; & \sup_{n\geq 1} (E[(\sum_{k\geq m} \left| X_t^{\varphi_n,k} - X_t^k \right|^2 \frac{1}{\alpha_k^{1-\delta}})])^{1/2} (E[(\sum_{k\geq m} \left| X_t^{\varphi_n,k} - X_t^k \right|^2 \alpha_k^{1-\delta})])^{1/2} \\
& C(\sum_{k\geq m} \frac{1}{\alpha_k^{1-\delta}})^{1/2} \\
< \; & \epsilon
\end{aligned}$$

for $m \geq m_0$. Choosing a n_0 such that for all $n \geq n_0$

$$\sum_{k\geq 1}^{m_0 - 1} E[\left| X_t^{\varphi_n,k} - X_t^k \right|^2] < \epsilon$$

we find that

$$E[\left\| X_t^{\varphi_n,k} - X_t \right\|^2] < 2\epsilon$$

for all $n \geq n_0$. So (36) holds.

Finally it follows from dominated convergence that (measurability/continuous modification of $(\omega, t) \longmapsto X_t(\omega)$ can be shown)

$$\begin{aligned}
& E[\left\| \int_0^t e^{(t-s)A} (B^n(s, X_s^{\varphi_n}) - B(s, X_s) ds \right\|^2] \\
\leq \; & 2 \sup_{0\leq s\leq T} \left\| e^{sA} \right\|^2 (E[\int_0^t \left\| B^n(s, X_s^{\varphi_n}) - B^n(s, X_s) \right\|^2 ds] \\
& + E[\int_0^t \left\| B^n(s, X_s) - B(s, X_s) \right\|^2 ds]) \\
\leq \; & 2 \sup_{0\leq s\leq T} \left\| e^{sA} \right\|^2 K^2 (\int_0^t (E[\left\| X_s^n - X_s \right\|^2])^\alpha ds \\
& + E[\int_0^t \left\| B^n(s, X_s) - B(s, X_s) \right\|^2 ds]) \\
\longrightarrow \; & 0 \text{ for } n \longrightarrow \infty.
\end{aligned}$$

From this we see that X_t is a mild solution to (35). Uniqueness was shown in Section 5.1. ■

Remark 12 *Another approach based on the so called $S-$transform to verify X_t as a unique solution to SDE's is discussed in [16], [15].*

Example 13 *Consider the equation*

$$dX(t,\xi) = (\Delta X(t,\xi) + B(t, X(t,\cdot))(\xi))dt + \sigma(-\Delta)^{-\gamma/2} dW(t,\xi)$$

for $t \geq 0$ and $\xi \in [0, 2\pi]$, with periodic boundary conditions. In this case we have $H = L^2(0, 2\pi)$ and $A = \Delta$. We let $Q = (-\Delta)^{-\gamma}$ with $0 < \gamma < \frac{1}{3}, \theta = \frac{1}{2}$ in Section 1.1 and $M = (-\Delta)^{\lambda}$ for a sufficiently small $\lambda > 0$, then the conditions of Section 1.1 and the conditions of Theorem 11.

Proof. Let us show that (19) holds. We have

$$\begin{aligned} & \left\| e^{(t-u_1)A} Q^{1/2} - e^{(t-u_2)A} Q^{1/2} \right\|_E^2 \\ = & \sum_{n=1}^{\infty} \left(e^{-(t-u_1)\alpha_n} \alpha_n^{-\gamma/2} - e^{-(t-u_2)\alpha_n} \alpha_n^{-\gamma/2} \right)^2 \alpha_n^{2\lambda} \\ = & \sum_{n=1}^{\infty} \left(1 - e^{-(u_1-u_2)\alpha_n} \right)^2 e^{-2(t-u_1)\alpha_n} \alpha_n^{2\lambda-\gamma}. \end{aligned}$$

Now, for $\epsilon, \nu \in (0,1)$ we can find positive constants C_ϵ and C_ν such that

$$(1 - e^{-a}) \leq C_\epsilon a^\epsilon \quad \text{and} \quad e^{-2a} \leq C_\nu a^{-\nu}$$

for every $a \geq 0$. Thus, the above is bounded by

$$\begin{aligned} & C_\epsilon |u_1 - u_2|^{2\epsilon} \sum_{n=1}^{\infty} \alpha_n^{2\epsilon+2\lambda-\gamma} e^{-2(t-u_1)\alpha_n}. \\ \leq \quad & C_\epsilon C_\nu |u_1 - u_2|^{2\epsilon} |t - u_1|^{-\nu} \sum_{n=1}^{\infty} \alpha_n^{2\epsilon+2\lambda-\gamma+\nu}. \end{aligned}$$

Rewriting this condition in the Fourier basis on the interval $[0, 2\pi]$, we get the condition

$$\sum_{k \in \mathbb{Z}} k^{4\epsilon+4\lambda-2\gamma+2\nu} < \infty.$$

Let $\nu = 1 - \delta$ and choose ϵ and λ small to get (19). Inequality (18) is proved similary.

To see (20) we write

$$\int_H \|e^{(u_1-u_2)A} y - y\|^2 N_{Q_s}(dy) = E[\|e^{(u_1-u_2)A} \sqrt{Q_s} W_1 - \sqrt{Q_s} W_1\|^2]$$

where $W = (W_t)_{t\geq 0}$ is a cylindrical Brownian motion, $W_t = \sum_{n=1}^{\infty} W_t^n e_n$. We get

$$(e^{(u_1-u_2)A}\sqrt{Q_s} - \sqrt{Q_s})W_1 = \sum_{n=1}^{\infty}(e^{-(u_1-u_2)\alpha_n} - 1)\left(\int_0^s q_n e^{-2u\alpha_n}du\right)^{1/2} W_1^n,$$

so that

$$\begin{aligned} E[\|e^{(u_1-u_2)A}\sqrt{Q_s}W_1 - \sqrt{Q_s}W_1\|^2] &= \sum_{n=1}^{\infty}(e^{-(u_1-u_2)\alpha_n} - 1)^2 \int_0^s q_n e^{-2u\alpha_n}du \\ &\leq C_\epsilon \sum_{n=1}^{\infty}((u_1-u_2)\alpha)^{2\epsilon}\frac{q_n(1-e^{-2s\alpha_n})}{2\alpha_n} \\ &\leq C_\epsilon |u_1-u_2|^{2\epsilon}\sum_{n=1}^{\infty}\alpha_n^{2\epsilon-1-\gamma}. \end{aligned}$$

As before we rewrite this in the Fourier basis we get the condition

$$\sum_{k\in\mathbb{Z}} k^{4\epsilon-2-2\gamma} < \infty$$

which is satisfied for small ϵ.

Finally we show that we have $\int_0^T \|\Lambda_t\|^{1+\theta}dt < \infty$ when $\theta = \frac{1}{2}$. We have

$$\begin{aligned} \|\Lambda_t x\|^2 &= \|A^{(1+\gamma)/2}(I - e^{2tA})^{-1/2}e^{tA}x\|^2 \\ &= t^{-(1+\gamma)}\sum_{n=1}^{\infty}(t\alpha_n)^{1+\gamma}(1-e^{-2t\alpha_n})^{-1}e^{-2t\alpha_n}. \end{aligned}$$

The mapping $s \mapsto \frac{s^{1+\gamma}}{e^{2s}-1}$ is bounded on $(0,\infty)$, so that we get

$$\|\Lambda_t\| \leq Ct^{-(1+\gamma)/2}$$

and thus we get that $t \mapsto \|\Lambda_t\|^{3/2}$ is integrable on any interval $[0,T]$. ■

Appendix

The following result which is based on Malliavin calculus and which is essentially due to [4] provides a compactness criterion for subsets of $L^2(\Omega)$ of square integrable functionals of a cylindrical Wiener process $W_t, 0 \leq t \leq 1$ on the Hilbert space H. See e.g. [17], [13] or [7] for more information about Malliavin calculus.

Theorem 14 *Assume that L is a self-adjoint compact operator on H^* with dense image. Denote by $DX \in L^2(\Omega; L^2([0,1]) \otimes H^*)$ the Malliavin derivative of a square integrable X in* *the domain of D and by $\mathbf{D}_{1,2}$ the space of such functionals. Then for $0 < \beta < 1/2$ and $c > 0$ the set*

$$\begin{aligned}\mathcal{G} \quad = \quad & \Bigg\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \left(\int_0^1 \left\| L^{-1} D_u G \right\|_{L^2(\Omega)} du \right)^{1/2} \\ & + \left(\int_0^1 \int_0^1 \frac{\left\| L^{-1}(D_{u_1} G - D_{u_2} G) \right\|_{L^2(\Omega)}^2}{|u_1 - u_2|^{1+2\beta}} du_1 du_2 \right)^{1/2} \leq c \Bigg\}\end{aligned}$$

is relatively compact in $L^2(\Omega)$.

Remark 15 *Denote by $J : H^* \longrightarrow H$ the standard isometric isomorphism for Hilbert spaces H. Then an example of L which satisfies the conditions of Theorem 14 is given by*

$$La := \langle MJ(a), \cdot \rangle \, , a \in H^*,$$

where $M = A^\tau$ for some sufficiently small $\tau > 0$, where A is the densely defined operator in Section 1.

References

[1] Beesack, P. R.: Comparison theorems and integral inequalities. Proc. Amer. Math. Soc., 20, 61-66 (1969).

[2] Da Prato, G., Flandoli, F.: Pathwise uniqueness for a class of SDE in Hilbert spaces and applications. Journal of Functional Analysis, 259 (2010) 243-267.

[3] Da Prato, G., Flandoli, F., Priola, E., Röckner, M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces with unbounded measurable drift. Preprint: arXiv:1109.0363

[4] Da Prato, G., Malliavin, P., Nualart, D.: Compact families of Wiener functionals. C.R. Acad. Sci. Paris, t. 315, Série I, p. 1287-1291 (1992).

[5] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1992.

[6] Da Prato, G., Zabczyk, J.: Second order partial differential equations in Hilbert spaces, London Mathematical Society Lecture Note Series, 293, Cambridge University Press, Cambridge, 2002.

[7] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance. Universitext, Springer (2009).

[8] Flandoli, F.: Random Perturbation of PDE's and Fluid Dynamic Models. LNM 2015, Springer (2011).

[9] Gyöngy, I., Pardoux, E.: On quasi-linear stochastic partial differential equations. Probability Theory and Related Fields 94. 413-425 (1993).

[10] Hida, T., Kuo, H.-S., Potthoff, J., Streit, L.: White Noise- An Infinite Dimensional Calculus. Kluwer (1993).

[11] Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Graduate Studies in Mathematics. Vol 12. American Mathematical Society (1997).

[12] Lassalle, R.: Causal Transference Plans and Their Monge-Kantorovich Problems. Preprint arXiv:1303.6925

[13] Malliavin , P.: Stochastic Analysis. Grundlehren der Mathematischen Wissenschaften, Vol. 313, Springer-Verlag, Berlin (1997).

[14] Mitoma, I.: Tightness of probabilities on $C([0,1],S')$ and $D([0,1];S')$. Annals of Probability, 11(4), p. 989-999 (1983).

[15] Menoukeu-Pamen, O., Meyer-Brandis, T., Nilssen, T., Proske, F., Zhang, T.: A variational approach to the construction and Malliavin differentiability of strong solutions of SDE's. Mathematische Annalen, Vol. 357 Issue 2, pp 761-799, (2013).

[16] Meyer-Brandis, T., Proske, F.:Construction of strong solutions of SDE's via Malliavin calculus. J. of Funct. Anal., 258, 3922-3953 (2010).

[17] Nualart, D.: The Malliavin Calculus and Related Topics. Springer (1995).