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PROBABILISTIC CHARACTERISTICS SUPPORTING THE MANAGEMENT OF A PRODUCTION-SUPPLY SYSTEM

The construction of a model and quantitative measures have been presented aimed at improving the efficiency of a production-supply system described by a three-dimensional stochastic process. For this purpose, the laws governing the functioning of the system have been described, corresponding to the three different states of the stock level in the subsystem *M*. These laws generate the quantitative model of the examined system, which enables the construction of the proposed quantitative measures supporting the managing process of such a system.

Keywords: *production-supply system*, *model of a system*, *system characteristics*

1. The system and theoretical description of its operation

Production systems and supply subsystems have been studied by various authors (e.g., [1–4, 8, 9, 11–17]). This article is a continuation of the research carried out in [4, 8, 9, 11, 15, 16], in particular in [5, 6]. It is dedicated to building a new probabilistic model of the operation of such systems. Based on this model, quantitative measures are defined, with the aim of improving how the system operates.

It is assumed that there are two streams of inputs provided in a continuous manner (e.g., by conveyor belts, pipelines, transmission lines) to the recipient *E* (e.g., a power

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station), whose operation requires the constant supply of raw materials α (e.g. carbon): a stream $y_1(t)$ of production by a subsystem T_1 and a stream $y_2(t)$ of production by a subsystem T_2.

Random changes in the processes $y_1(t)$, $y_2(t)$ and unplanned breaks in operation (failures) of the subsystems T_1 and T_2 are factors reducing the operational efficiency of such a system. This efficiency can be increased – while reducing the potential for interruptions in the supply of a sufficient quantity of the product to the recipient E – by locating a warehouse-reservoir M with specific volume V in the environment of the recipient E. The product streams $y_1(t)$, $y_2(t)$ are collected in the subsystem M if the stock level $z(t)$ in the warehouse M does not exceed the level V and when $y_1(t) > a$ or $y_2(t) > a$. If the present content of the element M is equal to V, and $y_1(t) > a$ or $y_2(t) > a$, then the intensity of the streams $y_1(t)$, $y_2(t)$ is restricted to the level a. When the warehouse M is empty and $y_1(t) < a$ and $y_2(t) < a$, then a situation which is unfavourable to the recipient E arises. Determination of the probability of this event has a practical meaning. The coordination of the subsystems T_1, T_2 , M and E is presented in Fig. 1.

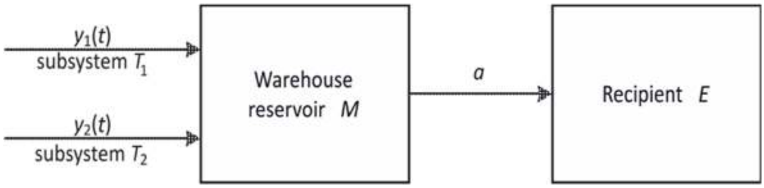

Fig. 1. General description of the operation of the considered production-supply system

The operation of the examined system is described by a three-dimensional process ($y_1(t)$, $y_2(t)$, $z(t)$). We assume that the subsystems T_1 and T_2 operate independently, and the processes $y_1(t)$ and $y_2(t)$, controlling the stock level of the warehouse $z(t)$, are Markov processes with a finite number of states. Let us denote the states of the product stream $y_1(t)$, describing the intensity of supply to the subsystem M, by: $y_{11}, y_{12}, \ldots, y_{1n}$, and the states of the product stream $y_2(t)$ by : $y_{21}, y_{22}, \ldots, y_{2m}$.

The intensities of transitions between states (levels of supply) of the processes $y_1(t)$ and $y_2(t)$ are denoted by $\pi_{jk}^{(1)}$ and $\pi_{sk}^{(2)}$, respectively, and written in the following form:

$$y_{1j} \xrightarrow{\pi_{jk}^{(1)}} y_{1k} \qquad \text{for } j \neq k \tag{1}$$

$$y_{2s} \xrightarrow{\pi_{si}^{(2)}} y_{2i} \qquad \text{for } s \neq i \tag{2}$$

Obtaining the appropriate measures supporting the process of managing the system requires the derivation of the probabilities of the system states. Let $P(y_1(t), y_2(t), z(t))$ be the probability that at a fixed point of time t the supply stream of the product $y_1(t)$ will be in the following state:

$$y_1\colon y_{11}, y_{12}, \ldots, y_{1n} \tag{3}$$

and the supply stream $y_2(t)$ will be in the state

$$y_2\colon y_{21}, y_{22}, \ldots, y_{2m} \tag{4}$$

and, at the same time, the stock level of the warehouse (container) M will be z.

There is no probability mass for any specific z, $0 < z < V$, i.e.

$$P(y_1(t), y_2(t), z(t) = z) = 0$$

because there are uncountably many values of $z \in (0, V)$.

Therefore, probability density functions $f_{x_{2k}}^{x_{1k}}(z, t)$ are introduced, describing probabilities of the forms:

$$P(a_1 \le z(t) \le b_1,\ x_1(t) = x_{1k},\ x_2(t) = x_{2i}) = \int_{a_1}^{b_1} f_{x_{2i}}^{x_{1k}}(z, t)dz \tag{5}$$

where $0 \le a_1 < b_1 \le V$, x_{1k} is the kth state of the auxiliary introduced process

$$x_1(t) = y_1(t) - a(x_{1k} = y_{1k} - a, k = 1, 2, ..., n)$$

while x_{2i} is the ith state of the process

$$x_2(t) = y_2(t) - a(x_{2i} = y_{2i} - a, i = 1, 2, ..., m)$$

Next, let us denote by $Q_{x_{2i}}^{x_{1k}}(\{0\}, t)$, $Q_{x_{2i}}^{x_{1k}}(\{V\}, t)$ probabilities of the form:

$$P(z(t) = 0,\ x_1(t) = x_{1k},\ x_2(t) = x_{2i}),\ P(z(t) = V,\ x_1(t) = x_{1k},\ x_2(t) = x_{2i})$$

We will analyse the operation of the system by considering three different cases: $0 < z(t) < V$, $z(t) = 0$, $z(t) = V$, because these cases correspond to specific operating conditions of the system.

In order to obtain quantitative measures connected with improving the efficiency of system operation (Fig. 1), it is sufficient to determine the probabilities represented by equation (5) and the probabilities $Q_{x_{2i}}^{x_{1k}}(\{0\}, t)$, $Q_{x_{2i}}^{x_{1k}}(\{V\}, t)$.

Equation (5) expresses the probability that at a fixed time t, the inventory level in warehouse M is in the range $(a1, b1)$ and the states of the processes $x_1(t)$, $x_2(t)$ are equal to x_{1k} and x_{2i}, respectively.

Similarly the expressions $Q_{x_{2i}}^{x_{1k}}(\{0\}, t)$, $Q_{x_{2i}}^{x_{1k}}(\{V\}, t)$ are the probabilities of the inventory levels being $z(t) = 0$ and $z(t) = V$, respectively, and the processes $x_1(t)$ and $x_2(t)$ being in the states x_{1k} and x_{2i} at time t.

Analysis of the operation of the system will be carried out for the case when the inventory level fulfils the condition: $0 < z(t) < V$. Hence, at that moment there are no immediate constraints on the rate at which subsystems T_1 and T_2 can supply the warehouse and the recipient's requirements are guaranteed.

Calculation of the probability of the set of states

$$(a_1 < z(t) < b_1, x_1(t) = x_{1k}, x_2(t) = x_{2i})$$

according to equation (5) requires knowledge of the density function $f_{x_{2i}}^{x_{1k}}(z, t)$. Hence, we will derive an equation, which this function fulfils. According to how the system operates in the analysed case, we have [5]:

$$\begin{aligned} f_{x_{2i}}^{x_{1k}}(z, t+\tau) \approx f_{x_{2i}}^{x_{1k}}&[z-(x_{1k}+x_{2i}+a)\tau, t][1-(\pi_k^{(1)}+\pi_i^{(2)})\tau] \\ &+\sum_{k'\neq k} f_{x_{2i}}^{x_{1k'}}[z-(x_{1k'}+x_{2i}+a)\tau, t](1-(\pi_i^{(2)}\tau)\pi_{k'k}^{(1)}\tau \\ &+\sum_{i'\neq i} f_{x_{2i'}}^{x_{1k}}[z-(x_{1k}+x_{2i'}+a)\tau, t](1-(\pi_k^{(1)}\tau)+\pi_{i'i}^{(2)}\tau \\ &+\sum_{\substack{k'\neq k\\ i'\neq i}} f_{x_{2i'}}^{x_{1k'}}[z-(x_{1k'}+x_{2i'}+a)\tau, t](\pi_{k'k}^{(1)}\tau)+\pi_{i'i}^{(2)}\tau \end{aligned} \tag{6}$$

where:

$$\pi_k^{(1)} = \sum_{l\neq k} \pi_{kl}^{(1)} \tag{7}$$

$$\pi_i^{(2)} = \sum_{l\neq i} \pi_{il}^{(2)} \tag{8}$$

The intuition of relationship (6) is as follows: the first element in this equation gives the probability of remaining in the state (x_{1k}, x_{2i}). This probability is 1 minus the sum of the intensities of the outflows from the state (x_{1k}, x_{2i}) (see, e.g., [7]). In our case, the intensity of the outflows from the state x_{1k} is equal to $\pi_k^{(1)}$ (definition (1), formula (7), and from the state x_{2i} is equal to $\pi_i^{(2)}$ (definition (2), formula (8)). Equation (6) is based on the fact that in "simple" processes double changes in the state occur at a rate of order higher than τ. This is taken into account by the asymptotic equality $\approx$, which means that the expression $o(\tau)$, fulfilling the following condition

$$\lim_{\tau\to 0}\frac{o(\tau)}{\tau}=0 \tag{9}$$

is omitted.

Irrespective of changes in the states x_{1k} and x_{2i}, the changes in the stock level of the warehouse z follow according to the initial state. They are controlled by both of the processes $y_1(t)$ and $y_2(t)$. If at time t the state is (x_{1k}, x_{2i}), then in an interval of time of length τ the stock level increases by $(x_{1k} + x_{2i} + a)\tau$. Thus, if at time $t + \tau$ the stock level of the warehouse is z, at time t it had to be $z - (x_{1k} + x_{2i} + a)\tau$. This fact is taken into account in the first component of formula (6). The other components can be explained using analogous arguments. After using Taylor's formula, equation (6) takes the form:

$$\begin{aligned}
f_{x_{2i}}^{x_{1k}}(z,\, t+\tau) \approx & \left\{ f_{x_{2i}}^{x_{1k}}(z,\, t) + \frac{\partial f_{x_{2i}}^{x_{1k}}(z,\, t)}{\partial z}[-(x_{1k}+x_{2i}+a)\tau] + o[-(x_{1k}+x_{2i}+a)\tau] \right\} \\
& \times\left[1-(\pi_k^{(1)}+\pi_i^{(2)})\tau\right] + \sum_{k'\neq k}\left\{ f_{x_{2i}}^{x_{1k'}}(z,\, t) + \frac{\partial f_{x_{2i}}^{x_{1k'}}(z,\, t)}{\partial z} + o[-(x_{1k'}+x_{2i}+a)\tau] \right\} \\
& \times(1-\pi_i^{(2)}\tau)\pi_{k'k}^{(2)}\tau \\
& +\sum_{i'\neq i}\left\{ f_{x_{2i'}}^{x_{1k}}(z,\, t) + \frac{\partial f_{x_{2i'}}^{x_{1k}}(z,\, t)}{\partial z}[-(x_{1k}+x_{2i'}+a)\tau] + o[-(x_{1k}+x_{2i'}+a)\tau] \right\} \\
& \times(1-\pi_k^{(1)}\tau)\pi_{i'i}^{(2)}\tau \\
& +\sum_{\substack{k'\neq k \\ i'\neq i}}\left\{ f_{x_{2i'}}^{x_{1k'}}(z,\, t) + \frac{\partial f_{x_{2i'}}^{x_{1k'}}(z,\, t)}{\partial z}[-(x_{1k'}+x_{2i'}+a)\tau] + o[-(x_{1k'}+x_{2i'}+a)\tau] \right\} \\
& \times\pi_{k'k}^{(1)}\pi_{i'i}^{(2)}\tau^2
\end{aligned} \tag{10}$$

After applying simple transformations to equation (10), the asymptotic form of equation (10) changes into equation (11), which is satisfied by the density functions $f_{x_{2i}}^{x_{1k}}(z,\, t)$ specifying the probabilities expressed by formula (5). This equation has the form:

$$\frac{\partial f_{x_{2i}}^{x_{1k}}(z,\,t)}{\partial z} = -\frac{\partial f_{x_{2i}}^{x_{1k}}(z,\,t)}{\partial z}(x_{1k}+x_{2i}+a) - f_{x_{2i}}^{x_{1k}}(z,\,t)\left[\pi_k^{(1)}+\pi_i^{(2)}\right] + \sum_{k'\neq k} f_{x_{2i}}^{x_{1k'}}(z,\,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'}}^{x_{1k}}(z,\,t)\pi_{i'i}^{(2)} \tag{11}$$

for $0 < z < V$; $k = 1, 2, \ldots, n$; $i = 1, 2, \ldots, m$.

Moving on to the analysis of the system in the other two cases, we present the relationships specifying the probabilities given by $Q_{x_{2i}}^{x_{1k}}(\{0\},\,t)$, $Q_{x_{2i}}^{x_{1k}}(\{V\},\,t)$.

Utilizing the operating conditions of the system in the case $z(t) = 0$, we obtain the following equations, see [6]:

$$\begin{aligned}
Q_{x_{2i}}^{x_{1k}}(\{0\},\,t+\tau) &= P(z(t+\tau)=0,\ x_1(t+\tau)=x_{1k},\ x_2(t+\tau)=x_{2i}) \\
&\approx Q_{x_{2i}}^{x_{1k}}(\{0\},\,t)[1-(\pi_k^{(1)}+\pi_i^{(2)})\tau] + P(0<z(t)<-(x_{1k}+x_{2i}+a)\tau \\
&\quad x_1(t)=x_{1k},\ x_2(t)=x_{2i})[1-(\pi_k^{(1)}+\pi_i^{(2)})\tau] \\
&\quad + \sum_{\substack{k'\neq k \\ -(x_{1k'}+x_{2i}+a)\,\geq\,0}} P(z(t)=0,\ x_1(t)=x_{1k'},\ x_2(t)=x_{2i})\pi_{k'k}^{(1)}\tau(1-\pi_i^{(2)}\tau) \\
&\quad + \sum_{\substack{k'\neq k \\ -(x_{1k'}+x_{2i}+a)\geq 0}} P(0<z(t)<-(x_{1k'}+\,x_{2i}+a)\tau,\ x_1(t)=x_{1k'},\ x_2(t)=x_{2i})\pi_{k'k}^{(1)}\tau(1-\pi_i^{(2)}\tau) \\
&\quad + \sum_{\substack{i'\neq i \\ -(x_{1k}+x_{2i'}+a)\geq 0}} P(0<z(t)<-(x_{1k}+\,x_{2i'}+a)\tau,\ x_1(t)=x_{1k},\ x_2(t)=x_{2i'})\pi_{i'i}^{(2)}\tau(1-\pi_k^{(1)}\tau) \\
&\quad + \sum_{\substack{i'\neq i \\ -(x_{1k}+x_{2i'}+a)\geq 0}} P(z(t)=0,\ x_1(t)=x_{1k},\ x_2(t)=\ x_{2i'})\pi_{i'i}^{(2)}\tau(1-\pi_k^{(1)}\tau) \\
&\quad + \sum_{\substack{i'\neq i \\ k'\neq k \\ -(x_{1k'}+x_{2i'}+a)\geq 0}} P(0<z(t)<-(x_{1k'}+\,x_{2i'}+a)\tau,\ x_1(t)=x_{1k'},\ x_2(t)=x_{2i'})\pi_{i'i}^{(2)}\tau\pi_{k'k}^{(1)}\tau \\
&\quad + \sum_{\substack{i'\neq i \\ k'\neq k \\ -(x_{1k'}+x_{2i'}+a)\geq 0}} P(z(t)=0,\ x_1(t)=x_{1k'},\ x_2(t)=\ x_{2i'})\pi_{i'i}^{(2)}\tau\pi_{k'k}^{(1)}\tau
\end{aligned} \tag{12}$$

where: $\pi_k^{(1)}$ and $\pi_i^{(2)}$ are described by formulas (7) and (8).

The intuition of equation (12) is as follows: the first element of the equation gives the probability of remaining in the state ($z = 0$, $x_1 = x_{1k}$, $x_2 = x_{2i}$). This probability is one minus the sum of the outflow intensities from the state ($x_1 = x_{1k}$, $x_2 = x_{2i}$) (see, e.g., [7]). In our case, according to definitions (1) and (2), the outflow intensity from the state x_{1k} is $\pi_k^{(1)}$ (formula (7)) and the outflow intensity from the state x_{2i} is $\pi_i^{(2)}$ (formula (8)).

However, the level of $z = 0$ at time $t + \tau$ can also be achieved, if at time t the state is (x_{1k}, x_{2i}) and the warehouse is partially filled such that: $0 \le z(t) < -(x_{1k} + x_{2i} + a)\tau$. An initial stock level of $(a - y_{1k} - y_{2i})\tau = (-x_{1k} - x_{2i} - a)\tau \ge 0$ at time t reaches the zero state by time $t + \tau$. This fact takes into account the second term of formula (12). The other components can be explained using analogous arguments, together with the fact that in "simple" processes double changes of state occur at a rate of a higher order than τ. This is taken into account by the asymptotic equation $\approx$, which means that the expression $o(\tau)$, satisfying condition (9), was omitted.

The asymptotic equation (12), after application of formula (5), the mean value theorem for integrals (see, e.g., [10]) and simple transformations, changes into the following form:

$$\begin{aligned}\frac{\partial Q_{x_{2i}}^{x_{1k}}(\{0\}, t)}{\partial t} &= -(\pi_k^{(1)} + \pi_i^{(2)}) Q_{x_{2i}}^{x_{1k}}(\{0\}, t) + f_{x_{2i}}^{x_{1k}}(0, t)(x_{1k} + x_{2i} + a) \\ &+ \sum_{\substack{k' \ne k \\ -(x_{1k'} + x_{2i} + a) \ge 0}} Q_{x_{2i}}^{x_{1k'}}(\{0\}, t)\pi_{k'k}^{(1)} \\ &+ \sum_{\substack{i' \ne i \\ -(x_{1k} + x_{2i'} + a) \ge 0}} Q_{x_{2i'}}^{x_{1k}}(\{0\}, t)\pi_{i'i}^{(2)} \ \text{ for } -(x_{1k} + x_{2i} + a) \ge 0 \end{aligned} \tag{13}$$

Of course, for $-(x_{1k} + x_{2i} + a) < 0$, the equation

$$Q_{x_{2i}}^{x_{1k}}(\{0\}, t) = 0 \tag{14}$$

is satisfied because under the condition $(x_{1k} + x_{2i} + a) > 0$, i.e., when $y_{1k} + y_{2i} > a$, the state $z(t) = 0$ cannot ever be achieved.

Moving on to analysing the operation of the system in the case where it reaches the state $z(t) = V$, we will determine the relationships satisfied by the probabilities $Q_{x_{2i}}^{x_{1k}}(\{V\}, t)$.

The system operates differently than in the two previous cases. Arguing analogously as in the case of the state $z(t) = 0$, we obtain a formula for the probabilities $Q_{x_{2i}}^{x_{1k}}(\{V\}, t + \tau)$ in the following form (see [6]):

$$Q_{x_{2i}}^{x_{1k}}(\{V\},\, t+\tau) = P(z(t+\tau) = V,\ x_1(t+\tau) = x_{1k},\ x_2(t+\tau) = x_{2i})$$

$$\approx Q_{x_{2i}}^{x_{1k}}(\{V\},\, t)[1-(\pi_k^{(1)} + \pi_i^{(2)})\tau]$$

$$+ P(V-(x_{1k}+x_{2i}+a)\tau < z(t) < V,\ x_1(t) = x_{1k},\ x_2(t) = x_{2i})[1-(\pi_k^{(1)} + \pi_i^{(2)})\tau]$$

$$+ \sum_{\substack{k'\neq k \\ (x_{1k'}+x_{2i}+a)\geq 0}} Q_{x_{2i}}^{x_{1k'}}(\{V\},\, t)\pi_{k'k}^{(1)}\tau(1-\pi_i^{(2)}\tau) + \sum_{\substack{i'\neq i \\ (x_{1k}+x_{2i'}+a)\geq 0}} Q_{x_{2i'}}^{x_{1k}}(\{V\},\, t)\pi_{i'i}^{(2)}\tau(1-\pi_k^{(1)}\tau)$$

$$+ \sum_{\substack{k'\neq k, i'\neq i \\ (x_{1k'}+x_{2i'}+a)\geq 0}} Q_{x_{2i'}}^{x_{1k''}}(\{V\},\, t)\pi_{i'i}^{(2)}\tau\pi_{k'k}^{(1)}\tau$$

$$+ \sum_{\substack{k'\neq k \\ (x_{1k'}+x_{2i}+a)\geq 0}} P(V-(x_{1k'}+\ x_{2i}+a)\tau < z(t) < V,\ x_1(t) = x_{1k'},\ x_2(t) = x_{2i})\pi_{k'k}^{(1)}\tau(1-\pi_i^{(2)}\tau)$$

$$+ \sum_{\substack{i'\neq i \\ (x_{1k}+x_{2i'}+a)\geq 0}} P(V-(x_{1k}+\ x_{2i'}+a)\tau < z(t) < V,\ x_1(t) = x_{1k},\ x_2(t) = x_{2i'})\pi_{i'i}^{(2)}\tau(1-\pi_k^{(1)}\tau)$$

$$+ \sum_{\substack{k'\neq k, i'\neq i \\ (x_{1k'}+x_{2i'}+a)>0}} P(V-(x_{1k'}+\ x_{2i'}+a)\tau < z < V,\ x_1(t) = x_{1k'},\ x_2(t) = x_{2i'})\pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau \qquad (15)$$

for $x_{1k} + x_{2i} + a \geq 0$.

Analogously to the previous case, after application of formula (5), the mean value theorem for integrals and simple analytical transformations, relation (15) transforms into an equation of the form:

$$\frac{\partial Q_{x_{2i}}^{x_{1k}}(\{V\},\, t)}{\partial t} = -(\pi_k^{(1)} + \pi_i^{(2)})Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) + \sum_{\substack{k'\neq k \\ (x_{1k'}+x_{2i}+a)\geq 0}} Q_{x_{2i}}^{x_{1k}}(\{V\},\, t)\pi_{k'k}^{(1)}$$

$$+ \sum_{\substack{i'\neq i \\ (x_{1k}+x_{2i'}+a)\geq 0}} Q_{x_{2i'}}^{x_{1k}}(\{V\},\, t)\pi_{i'i}^{(2)} + f_{x_{2i}}^{x_{1k}}(V,\, t)(x_{1k}+x_{2i}+a) \qquad (16)$$

for $x_{1k} + x_{2i} + a \geq 0$.

Whereas for $(x_{1k} + x_{2i} + a) < 0$,

$$Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) = 0 \qquad (17)$$

since under the condition $(x_{1k} + x_{2i} + a) < 0$, i.e. when $y_{1k} + y_{2i} < a$, the state $z(t) = V$ cannot ever be achieved.

2. A quantitative model of the system and the probabilistic characteristics of management support for such a system

Relations (11), (13), (14), (16), (17) create a probabilistic model of the system. Thus it takes the following form:

$$\frac{\partial f_{x_{2i}}^{x_{1k}}(z,\,t)}{\partial z} = -\frac{\partial f_{x_{2i}}^{x_{1k}}(z,\,t)}{\partial z}(x_{1k}+x_{2i}+a) - f_{x_{2i}}^{x_{1k}}(z,\,t)\left[\pi_k^{(1)}+\pi_i^{(2)}\right] + \sum_{k'\neq k} f_{x_{2i}}^{x_{1k'}}(z,\,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'}}^{x_{1k}}(z,\,t)\pi_{i'i}^{(2)} \tag{18}$$

for $0 < z < V$; $k = 1, 2, \ldots, n$; $i = 1, 2, \ldots, m$.

$$\frac{\partial Q_{x_{2i}}^{x_{1k}}(\{0\},\,t)}{\partial t} = -(\pi_k^{(1)}+\pi_i^{(2)})Q_{x_{2i}}^{x_{1k}}(\{0\},\,t) + f_{x_{2i}}^{x_{1k}}(0,\,t)(x_{1k}+x_{2i}+a) + \sum_{\substack{k'\neq k\\ -(x_{1k'}+x_{2i}+a)\geq 0}} Q_{x_{2i}}^{x_{1k'}}(\{0\},\,t)\pi_{k'k}^{(1)} + \sum_{\substack{i'\neq i\\ -(x_{1k}+x_{2i'}+a)\geq 0}} Q_{x_{2i'}}^{x_{1k}}(\{0\},\,t)\pi_{i'i}^{(2)} \tag{19}$$

for $-(x_{1k}+x_{2i}+a)\geq 0$ and for $-(x_{1k}+x_{2i}+a) < 0$, $Q_{x_{2i}}^{x_{1k}}(\{0\},\,t) = 0$.

$$\frac{\partial Q_{x_{2i}}^{x_{1k}}(\{V\},\,t)}{\partial t} = -(\pi_k^{(1)}+\pi_i^{(2)})Q_{x_{2i}}^{x_{1k}}(\{V\},\,t) + f_{x_{2i}}^{x_{1k}}(V,\,t)(x_{1k}+x_{2i}+a) + \sum_{\substack{k'\neq k\\ -(x_{1k'}+x_{2i}+a)\geq 0}} Q_{x_{2i}}^{x_{1k'}}(\{V\},\,t)\pi_{k'k}^{(1)} + \sum_{\substack{i'\neq i\\ -(x_{1k}+x_{2i'}+a)\geq 0}} Q_{x_{2i'}}^{x_{1k}}(\{V\},\,t)\pi_{i'i}^{(2)} \tag{20}$$

for $-(x_{1k}+x_{2i}+a)\geq 0$ and for $x_{1k}+x_{2i}+a < 0$, $Q_{x_{2i}}^{x_{1k}}(\{V\},\,t) = 0$.

The mathematical description of the system expressed by relations (18)–20) will now be applied to present quantitative measures that can be used by a decision-making body in order to increase the effectiveness of system management, as well as in the design phase of such a system.

If at time t a random event occurs, e.g., the rate of supply by the production subsystem represented by the product streams $y_1(t)$, $y_2(t)$ is less than the requirements of a receiver E whose operations are continuous (e.g., production of electricity, water, gas, oil), and the contents of the warehouse M are zero, then the recipient is forced to use other supply sources or to limit its own production. This leads to financial losses in the considered system (the recipient's losses). The probability of such an unfavourable event, namely a shortage in the product supply to the customer at time t, is denoted by $w_1(t)$ and is expressed by the formula

$$w_1(t) = \sum_{(x_{1k}+x_{2i}+a)<0} Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) \tag{21}$$

Another unfavourable state of the system occurs when the quantity of the product supplied by the production subsystem exceeds the requirements of the receiver E and the warehouse M is full. In this case, the production subsystem (sender) is not able to send its product to the recipient E, which also leads to losses in the system (the sender's losses). The probability $w_2(t)$ of such an event can be calculated from the formula

$$w_2(t) = \sum_{(x_{1k}+x_{2i}+a)>0} Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) \tag{22}$$

These two characteristics $w_1(t)$ and $w_2(t)$ are thus measures for assessing the degree of losses in production, for the recipient and the production subsystem, respectively.

As an index for evaluating the degree of utilization of warehouse M at time t in the system, we may use the probability of the following random: the stock level is positive, but lower than V.

The probability $w_3(t)$ of this event is expressed by the formula:

$$w_3(t) = \sum_{x_{1k},\, x_{2i}} Q_{x_{2i}}^{x_{1k}}(V,\, t) \tag{23}$$

where: $Q_{x_{2i}}^{x_{1k}}(V,\, t) = P(0 < z(t) < V, x_1(t) = x_{1k}, x_2(t) = x_{2i})$ (see (5)).

If the value $w_3(t)$ is close to one, then there is low probability that either a deficit occurs (in the product supply to the customer) when the warehouse is empty or sender losses occur, when the warehouse M is full. In the case where the value of the indicator $w_3(t)$ is nearly zero, the phenomenon of sender loss or recipient deficit may occur with a high probability.

The probabilities of the occurrence of the limiting states of the subsystem M at time t are defined by the formulas:

$$w_4(t) = \sum_{(x_{1k}+x_{2i}+a)\le 0} Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) \tag{24}$$

$$w_5(t) = \sum_{(x_{1k}+x_{2i}+a)\ge 0} Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) \tag{25}$$

Formula (24) expresses the probability that the warehouse M is empty at time t, and relation (25) – the probability of the subsystem M being full at moment t.

The probabilities of the warehouse M being partially full, of the process $x_1(t)$ being in state x_{1k}, and of the process $x_2(t)$ being in state x_{2i} at moment t, respectively, can be determined from the following formulas:

$$w_6(t) = P(c_1 < z(t) < c_2) = \sum_{i,k} \int_{c_1}^{c_2} f_{x_{2i}}^{x_{1k}}(z,\, t)dz, 0 \le c_1 < c_2 \le V \tag{26}$$

$$w_7(t) = P(x_1(t) = x_{1k}) = \sum_i \left[Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) + Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) + \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz \right] \tag{27}$$

$$w_8(t) = P(x_2(t) = x_{2i}) = \sum_k \left[Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) + Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) + \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz \right] \tag{28}$$

The chance of the following random events occurring at time t:

$$w_9(t) = P(x_1(t) = x_{1k}, x_2(t) = x_{2i}), \quad w_{10}(t) = P(0 < z(t) < V, x_2(t) = x_{2i})$$

$$w_{11}(t) = P(0 < z(t) = V, x_1(t) = x_{1k}), \quad w_{12}(t) = P(z(t) = 0, x_1(t) = x_{1k})$$

$$w_{13}(t) = P(z(t) = 0, x_2(t) = x_{2i}), \quad w_{14}(t) = P(z(t) = V, x_1(t) = x_{1k})$$

$$w_{15}(t) = P(z(t) = V, x_2(t) = x_{2i}), \quad w_{16}(t) = P(z(t) = 0, x_1(t) = x_{1k}, x_2(t) = x_{2i})$$

$$w_{17}(t) = P(z(t) = V, x_1(t) = x_{1k}, x_2(t) = x_{2i}), \quad w_{18}(t) = P(0 < z(t) < V, x_1(t) = x_{1k}, x_2(t) = x_{2i})$$

can be determined using the quantitative model of the system expressed by the relationships (18)–(20), using the following formulas:

$$w_9(t) = Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) + Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) + \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz \tag{29}$$

$$w_{10}(t) = \sum_k \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz, \quad w_{11}(t) = \sum_i \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz \tag{30}$$

$$w_{12}(t) = \sum_i Q_{x_{2i}}^{x_{1k}}(\{0\},\, t), \quad w_{13}(t) = \sum_k Q_{x_{2i}}^{x_{1k}}(\{0\},\, t) \tag{31}$$

$$w_{14}(t) = \sum_i Q_{x_{2i}}^{x_{1k}}(\{V\},\, t), \quad w_{15}(t) = \sum_k Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) \tag{32}$$

$$w_{16}(t) = Q_{x_{2i}}^{x_{1k}}(\{0\},\, t), \quad w_{17}(t) = Q_{x_{2i}}^{x_{1k}}(\{V\},\, t) \tag{33}$$

$$w_{18}(t) = \int_0^V f_{x_{2i}}^{x_{1k}}(z,\, t)dz \tag{34}$$

3. Summary

The proposed probabilistic description of the operation of a production-supply system represents an aggregated variant of the process of supplying products to the subsystem M, which takes into account both the production subsystem, and the supply subsystem. The resulting model of the system generated by the three highlighted sets of states of the subsystem M allows us to determine measures which support the management process of the system. These measures are given by the relations (21)–(34), which depend on the values of the parameters $\pi_{jk}^{(1)}$, $\pi_{si}^{(2)}$, x_{1k}, x_{2i}, a, V, i.e., they are functions of these quantities. Therefore, we can optimize the values of the listed characteristics by appropriate changes in these parameters, and thus increase the efficiency of the system.

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