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MANIFOLDS OF ALMOST NONNEGATIVE RICCI CURVATURE

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To investigate relations between Ricci curvature and topology of a Riemannian manifold is one of the main subjects in differential geometry. In this paper, we study the global structure of manifolds of almost nonnegative Ricci curvature by means of the convergence and collapsing phenomena of Riemannian manifolds.

Let M be a compact connected C^∞ Riemannian manifold of dimension n. A classical theorem of Bochner states that if the Ricci curvature $\mathrm{Ric}(M)$ of M is nonnegative, then the first Betti number $b_1(M)$ of M satisfies $b_1(M) \le n$, where the equality takes place if and only if M is isometric to a flat torus. In [14], Gromov extended this result as follows: There is an $\varepsilon > 0$ depending on n and a given constant $D > 0$ such that if the diameter $d(M)$ and the Ricci curvature of M satisfy $d(M) \le D$ and $\mathrm{Ric}(M) > -\varepsilon$, then the first Betti number of M is still bounded by n. Gallot [10] also gave an analytic proof.

Under an additional condition, an upper bound of sectional curvature, we determine the topological and global geometric structure. Let $\mathcal{M}(n, D)$ denote the family of compact Riemannian n-manifolds M with sectional curvature $K(M) \le 1$ and diameter $d(M) \le D$. Our first result is a topological classification by first Betti numbers.

Theorem 1. *There is an $\varepsilon > 0$ depending on n and D such that if $M \in \mathcal{M}(n, D)$ satisfy $\mathrm{Ric}(M) > -\varepsilon$, then M is a fiber bundle over a $b_1(M)$-torus. In particular, if $b_1(M) = n-1$, then M is diffeomorphic to an infranilmanifold, and if $b_1(M) = n$, then M is diffeomorphic to an n-torus.*

A manifold M is called an *infranilmanifold* if a finite covering space of M is a quotient of a simply connected nilpotent Lie group by its lattice. The special case $b_1(M) = n$ of Theorem 1 gives a partial affirmative answer to a conjecture of Gromov [14] (see the end of §2).

Our next goal is to describe the global geometric structure. To do this, we shall collapse Riemannian manifolds in the situation that the greatest lower bound of Ricci curvature tends to zero. In general cases, limit spaces

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may have complicated singularities. In our situation, however, we obtain a stability theorem in the cases of large first Betti number.

Theorem 2. *Let $\{M_i\}_{i\in N}$ be a convergent sequence in $\mathcal{M}(n,D)$ for the Hausdorff distance such that* $\mathrm{Ric}(M_i) > -1/i$, $b_1(M_i) \geq n-1$. *Then the limit space X is isometric to the quotient space of a flat torus by an isometric (not necessarily free) Z_2 action. In the case when each M_i is orientable, X itself is a flat torus.*

As a result of Theorem 2, we obtain from Gromov's compactness theorem [14] that if M_i does not collapse, then the orientable double covering space $\hat{M}_i$ of M_i converges to a flat n-torus for the Lipschitz distance, and from the fiber bundle theorem of Fukaya [9] that if M_i collapses, then there exists an almost Riemannian submersion from $\hat{M}_i$ to a flat torus with an infranilmanifold as a fiber.

The main method in the proof which we employ is the $C^{1,\alpha}$ convergence of Riemannian metrics with respect to harmonic coordinates developed by Greene and Wu [12] and Peters [23]. We shall recall it in §1 as preliminaries. The convergence method, in particular, in noncollapsing situations, has been used by some authors to obtain results about pinching problems ([2], [7], [8], [18], [19] etc.). We have to consider the collapsing situation, however. The key to the proof is to show the convergence of harmonic 1-forms. This is done in §2 with the proof of Theorem 1. The proof of Theorem 2 is deferred to §3.

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1. Preliminaries

In [14], Gromov introduced the notion of *Hausdorff distance* between metric spaces. For subsets A and B in a metric space Z, we denote by $d_H^Z(A,B)$ the infimum of positive ε such that the ε-neighborhoods of A and B include B and A respectively. The Hausdorff distance between two metric spaces X and Y is, by definition

$$d_H(X,Y) = \inf_{Z,f,g} d_H^Z(f(X), g(Y)),$$

where $f\colon X \to Z$ and $g\colon Y \to Z$ are isometric imbeddings.

The *Lipschitz distance* between X and Y is defined as the infimum of numbers L such that

$$e^{-L} \leq d(f(x_1), f(x_2))/d(x_1,x_2) \leq e^L$$

for all $x_1 \neq x_2$ in X, where f runs over the set of all bi-Lipschitz homeomorphisms of X onto Y. The convergence for the Hausdorff distance is characterized in terms of the Lipschitz convergence of discrete subsets: A subset N in X is called an *ε-net* if and only if for any x in X there is x' in N such that $d(x, x') < \varepsilon$.

Proposition [14, 3.5]. (a) *If a sequence $\{X_i\}$ of metric spaces converges to X for the Hausdorff distance, then for every positive ε and $\varepsilon' > \varepsilon$, every ε-net of X is the limit for the Lipschitz distance of a sequence N_i where N_i is an ε'-net of X_i.*

(b) *Conversely, if X and X_i have uniformly bounded diameters, and if for every $\varepsilon > 0$ there exists an ε-net of X which is the limit for the Lipschitz distance of a sequence of ε-nets N_i of X_i, then X_i converges to X for the Hausdorff distance.*

Here we exhibit only a few examples which imply that the assumptions in Theorem 2 are optimal.

Example 1.1 (*Klein bottle*). For each positive ε, consider a Klein bottle M_ε defined by the distance subgroup of isometries of the flat plane R^2 with the generators γ_1, γ_2:

$$\gamma_1(x, y) = (x + 1, y), \qquad \gamma_2(x, y) = (-x, y + \varepsilon).$$

Then M_ε converges for the Hausdorff distance to the interval $S^1/Z_2 = [0, 1/2]$ as $\varepsilon \to 0$.

Example 1.2 (*Three-dimensional nilmanifold*). Let N be the Heisenberg group and Γ the integer lattice:

$$N = \left\{ \begin{pmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; x, y, z \in R \right\}, \qquad \Gamma = \left\{ \begin{pmatrix} 1 & c & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in Z \right\}.$$

Then the quotient space $M = \Gamma \backslash N$ is orientable, and $b_1(M) = 2$. For every $\varepsilon > 0$, define the left invariant metric g_ε on M by

$$\begin{pmatrix} 0 & w & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \in T_e N, \qquad \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\| = \varepsilon^2 u^2 + v^2 + w^2,$$

where e is the identity element of N. The sectional curvature and diameter of g_ε satisfy $|K(g_\varepsilon)| \le 24\varepsilon^2$, $d(g_\varepsilon) \le 2$ and (M, g_ε) converges to a flat torus T^2 with respect to the Hausdorff distance.

Now we briefly sketch the $C^{1,\alpha}$ convergence of Riemannian metrics developed by Greene and Wu [12] and Peters [23] (see also [14] and Katsuda [17]).

A coordinate system $(h^1, \cdots, h^n)$ defined on an open subset of a Riemannian manifold (M, g) is called a *harmonic coordinate system* if and only if $\Delta h^i \equiv 0$, $i = 1, \cdots, n$, where Δ is the Laplacian of (M, g). For positive constants Λ, D, ε, let $\mathcal{C} = \mathcal{C}(n, \Lambda, D, \varepsilon)$ be the family of Riemannian n-manifolds

M such that $|K(M)| \leq \Lambda^2$, $d(M) \leq D$, $i(M) \geq \varepsilon$. Let α, $0 < \alpha < 1$, be a fixed constant. By Jost and Karcher [16], there exists positive constants $R' = R'(n, \Lambda, \varepsilon)$ and $C = C(n, \Lambda, \varepsilon, \alpha)$ such that for each (M, g) in $\mathcal{C}$ and x in M, there is a harmonic coordinate system $\{h^i\}$ on the metric ball $B_{R'}(x, M)$ in M around x of radius R' such that

$$|g_{ij}|_{C^{1,\alpha}}, |g^{ij}|_{C^{1,\alpha}} < C,$$

where $g_{ij} = g(\partial/\partial h^i, \partial/\partial h^j)$ and the $(1, \alpha)$ norms are taken in the h^i coordinates. Furthermore, the eigenvalues of (g_{ij}) are uniformly bounded from below and from above by positive constants. Let $H_r(z, M)$ be a harmonic r-ball around z, namely,

$$H_r(z, M) = \left\{ y \in B_{R'}(x, M); \sum_{i=1}^{n} (h^i(y))^2 \leq r \right\}$$

for harmonic coordinates h^i on $B_{R'}(x, M)$.

Let (M_k, g_k) be a convergent sequence in the family $\mathcal{C}$ for the Hausdorff distance. Then the limit space X carries a $C^{1,\alpha}$ Riemannian metric in the following way: There exist positive numbers $R < R'$ and $Q' \in Z_+$ depending on n, Λ, D, ε such that each (M_k, g_k) in $\mathcal{C}$ is covered by $Q(k)$ harmonic R-balls $\{H_R(z_i, M_i)\}_{i=1,\cdots,Q(k)}$ with $Q(k) \leq Q'$. By passing to a subsequence, we may assume $Q(k) \equiv Q$ for all k. Further, by passing to a subsequence and by using a center of mass technique, we obtain, for a fixed large i and every $k \geq i$, a diffeomorphism $F = F_k \colon M \to M_k$, $M := M_i$, which is $C^{2,\alpha}$ uniformly bounded in harmonic coordinates. Then for each $H_R(z_l, M)$, $1 \leq l \leq Q$, with harmonic coordinates $\{h^p\}$,

$$F^* g_k = g_{(k)ab} (\partial F^a / \partial h^p)(\partial F^b / \partial h^q)\, dh^p\, dh^q,$$

where $F^a = \tilde{h}^a \circ F$, $\{\tilde{h}^a\}$ being harmonic coordinates in $\tilde{M} = M_k$, and $g_{(k)ab}$ is the components of g_k with respect to $\{\tilde{h}^a\}$. Each of $g_{(k)ab}$, $\partial F^a/\partial h^p$, $\partial F^b/\partial h^q$ is uniformly bounded in $C^{1,\alpha}$ norm. Hence by passing to a subsequence Q times, $F^* g_k$ converges to a $C^{1,\alpha}$ metric. We note that $\partial (F^{-1})^p / \partial \tilde{h}^a$ is also uniformly bounded in $C^{1,\alpha}$ norm with respect to $\tilde{h}^a$ coordinates, where $(F^{-1})^p = h^p \circ F^{-1}$.

2. Convergence of harmonic 1-forms

In the nonnegative Ricci curvature case, every harmonic 1-form is parallel [3]. Together with this, the Hodge theory yields Bochner's result in the introduction. We look at the Bochner technique again from the viewpoint of convergence of Riemannian metrics.

Let (M_i, g_i) be a sequence in the family $\mathcal{M}(n, D)$ such that $b_1(M_i) \neq 0$ and $\operatorname{Ric}(M_i) > -\Lambda^2$ for a constant $\Lambda > 0$. Notice that the absolute value of the sectional curvature of M_i is uniformly bounded. Let p_i be a fixed point of M_i and let B and B' be the metric balls in R^n around the origin of radii 1 and 2 respectively. Let $f_i\colon B' \to M_i$ be the composition of a linear isometry of B' onto $B_2(0, T_{p_i}M_i)$ and the exponential mapping of M_i at p_i. By the curvature assumption, f_i has maximal rank, let $\tilde{g}_i$ denote the metric of B' induced from f_i. Since the "injectivity radius" of $(B, \tilde{g}_i)$ in $(B', \tilde{g}_i)$ is uniformly bounded below by a positive constant, we can apply the $C^{1,\alpha}$ convergence as in §1 to the sequence $(B, \tilde{g}_i)$: Cover $(B, \tilde{g}_i)$ by harmonic balls in $(B', \tilde{g}_i)$ whose centers lie in B. By passing to a subsequence, we obtain, for a fixed large i and every $k \geq i$, a diffeomorphism $F = F_k$ of $(B, \tilde{g}_i)$ into $(B', \tilde{g}_k)$ which is $C^{2,\alpha}$ uniformly bounded in harmonic coordinates. Then on each preferred harmonic ball in $(B', \tilde{g}_i)$ with coordinates $\{h^p\}$,

$$\hat{g}_k = \tilde{g}_{(k)ab}(\partial F^a/\partial h^p)(\partial F^b/\partial h^q)\, dh^p\, dh^q,$$

where $\hat{g}_k = F^*\tilde{g}_k$, $F^a = \tilde{h}^a \circ F$, $\{\tilde{h}^a\}$ being harmonic coordinates with respect to $\tilde{g}_k$. Passing to a subsequence, $\hat{g}_k$ converges to a $C^{1,\alpha}$ metric $\hat{g}$.

Notation. For 1-forms ω, η on a Riemannian manifold M, we define its inner product $\langle \omega, \eta \rangle$ by

$$\langle \omega, \eta \rangle = \operatorname{vol}(M)^{-1} \langle \omega, \eta \rangle_{L^2} = \operatorname{vol}(M)^{-1} \int_M (\omega, \eta)\, dv,$$

where dv is the canonical measure of M.

Let ω^k be a harmonic 1-form of (M_k, g_k) such that $\langle \omega, \eta \rangle = 1$, and let $\hat{\omega}^k$ be its lift to B by G_k, where we set $G_k = f_k \circ F_k$.

Proposition 2.1. *A subsequence of $\hat{\omega}^k$ converges to a $C^{1,\alpha}$ 1-form on B.*

Proof. By the Poincare Lemma, there exists a function f^k such that $df^k = \hat{\omega}^k$. Then $\Delta f^k = \delta \hat{\omega}^k = 0$, where δ is the coboundary operator of $(B, \hat{g}^k)$. Namely, on each preferred harmonic ball in $(B', \hat{g}_i)$ with coordinates $\{h^p\}$, f^k satisfies

$$\hat{g}^{pq}_{(k)}(\partial^2 f^k/\partial h^p \partial h^q) + (\partial \hat{g}^{pq}_{(k)}/\partial h^p + \tfrac{1}{2}\hat{g}^{pq}_{(k)}\partial(\log \hat{G}_k)/\partial h^p)\partial f^k/\partial h^q = 0,$$

where $\hat{g}^{pq}_{(k)}$ is the inverse of $\hat{g}_{(k)pq}$, $\hat{g}_{(k)pq} = \hat{g}_k(\partial/\partial h^p, \partial/\partial h^q)$, and $\hat{G}_k = \det(\hat{g}_{(k)pq})$. From $\hat{g}_{(k)pq} = \tilde{g}_{(k)ab}(\partial F^a/\partial h^p)(\partial F^b/\partial h^q)$, we obtain uniform $C^{0,\alpha}$ bounds of $\partial \hat{g}^{pq}_{(k)}/\partial h^p$, $\partial(\log \hat{G}_k)/\partial h^p$ and a uniform positive bound from below for the minimal eigenvalue of $(\hat{g}^{pq}_{(k)})$. It follows from the classical Schauder interior estimate that

$$|f^k|_{C^{2,\alpha}} \leq c_1 |f^k|_{C^0},$$

where and hereafter, we denote by c_i some uniform constants. By adding a constant to f_k, we may assume that $|f^k|_{C^0} \leq c_2|\hat{\omega}^k|_{C^0} \leq c_2|\omega^k|_{C^0}$. According to Li [20, Lemma 8] and Gallot [10, Proposition 3.2], we have

$$|\omega^k|_{C^0} \leq c_3\langle\omega^k, \omega^k\rangle^{1/2}.$$

It follows $|f^k|_{C^{2,\alpha}} \leq c_1c_2c_3 = c_4$, and hence $|\hat{\omega}_p^k|_{C^{1,\alpha}} \leq c_4$ for $\hat{\omega}^k = \hat{\omega}_p^k\, dh^p$. Thus, by Ascoli's theorem, a subsequence of $\hat{\omega}^k$ converges to a $C^{1,\alpha}$ 1-form.

Now we further assume for the sequence (M_i, g_i) that the Ricci curvature of M_i is greater than $-1/i$.

Proposition 2.2. *The limit* 1*-form is parallel with respect to the limit metric.*

Proof. Let $\hat{\omega}$ be the limit $C^{1,\alpha}$ 1-form of $\hat{\omega}^k$. By the Weitzenböck formula,

$$-\tfrac{1}{2}\Delta(|\omega^k|^2) = |D^k\omega^k|^2 + \mathrm{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k),$$

where D^k is the covariant derivative with respect to g_k and $\sharp\omega^k$ is the vector corresponding to ω^k. Integrating this, we have

$$0 = \mathrm{vol}(M_k)^{-1}\int_{M_k}|D^k\omega^k|^2\,dv + \mathrm{vol}(M_k)^{-1}\int_{M_k}\mathrm{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k)\,dv,$$

where

$$\liminf_{k\to\infty}\mathrm{vol}(M_k)^{-1}\int_{M_k}\mathrm{Ric}(M_k)(\sharp\omega^k, \sharp\omega^k)\,dv = 0,$$

from the assumption for Ricci curvature. Hence, by passing to a subsequence, we may assume

$$(*) \qquad \lim_{k\to\infty}\mathrm{vol}(M_k)^{-1}\int_{M_k}|D^k\omega^k|^2\,dv = 0.$$

Now suppose that $\hat{\omega}$ is not parallel with respect to the limit metric $\hat{g}$. We set $\hat{A}_k = |\hat{D}^k\hat{\omega}^k|^2$, $\hat{A} = |\hat{D}\hat{\omega}|^2$, where $\hat{D}^k$ and $\hat{D}$ denote the covariant derivative with respect to $\hat{g}_k$ and $\hat{g}$. Then there is a point m in B such that $\hat{A} > a$ on $B_\delta(m, (B, \hat{g}))$ for positive constants a and δ. It turns out

$$(**) \qquad \hat{A}_k > a/2 \quad \text{on } B_\delta(m, (B, \hat{g}_k))$$

for large k. We set $A_k = |D^k\omega^k|^2$. Together with this, the Bishop and Gromov volume comparison theorem [14] implies

$$1 - \mathrm{vol}\{A_k \leq a/2\}/\mathrm{vol}(M_k) \geq \mathrm{vol}\, B_\delta(q_k, M_k)/\mathrm{vol}(M_k) \geq b(\delta)/b(D),$$

where $q_k = G_k(m)$ and $b(\delta)$ denotes the volume of the δ-ball in the simply connected space of constant curvature -1. Therefore,

$$\mathrm{vol}\{A_k < a/2\}/\mathrm{vol}(M_k) \leq 1 - b(\delta)/b(D)$$

for all large k. However, $(*)$ yields, for every fixed $b > 0$,

$$\lim_{k\to\infty} \operatorname{vol}\{A_k < b\}/\operatorname{vol}(M_k) = 1.$$

This is a contradiction.

Proof of Theorem 1. We suppose the contrary. Then there exists a sequence $\{M_k\}$ in $\mathcal{M}(n, D)$ such that $\operatorname{Ric}(M_k) > -1/k$, and that each M_k does not admit any fibering over a $b_1(M_k)$-torus. We may assume $b_1(M_k) \equiv s \neq 0$ for all k. Let $\{\omega_i^k\}_{1\le i\le s}$ be harmonic 1-forms of M_k giving a basis of the de Rham cohomology group such that $\langle \omega_i^k, \omega_j^k\rangle = \delta_{ij}$, $1 \le i, j \le s$. Cover M_k by metric balls $\{B_{k,\beta}\}_{1\le\beta\le N(k)}$ of radius 1, where we may assume $N(k) \equiv N$ for all k. For each fixed i, $1 \le i \le s$, and β, $1 \le \beta \le N$, let $\hat\omega_{i,\beta}^k$ be the lift of $\omega_i^k | B_{k,\beta}$ to B as in the previous construction. Applying the previous argument to $\{\hat\omega_{i,\beta}^k\}_k$ we obtain a parallel $C^{1,\alpha}$ 1-form $\hat\omega_{i,\beta}$ on B with respect to a $C^{1,\alpha}$ Riemannian metric. It is easily seen that the constant $c_{ij} = (\hat\omega_{i,\beta}, \hat\omega_{j,\beta})$ does not depend on β. It follows from the following calculation that $c_{ij} = \delta_{ij}$:

$$\begin{aligned}|\delta_{ij} - c_{ij}| &\le \lim_{k\to\infty} \operatorname{vol}(M_k)^{-1} \int_{M_k} |(\omega_i^k, \omega_j^k) - c_{i,j}|\, dv \\ &\le \lim_{k\to\infty} \sup_{M_k} |(\omega_i^k, \omega_j^k) - c_{ij}| \\ &= \lim_{k\to\infty} \sup_{B} |(\hat\omega_{i,\beta}^k, \hat\omega_{j,\beta}^k) - c_{ij}| = 0.\end{aligned}$$

It turns out that $\{\omega_i^k\}_{1\le i\le s}$ are pointwise linearly independent for large k. By an argument of Tischler [25], (also compare Lichnerowicz [21]), we can find differentiable functions $\varphi_i\colon M_k \to S^1$, $1 \le i \le s$, such that $\varphi_i^*(dt) = \omega_i^k$. The mapping $\varphi = (\varphi_1, \cdots, \varphi_s)$ is a submersion of M_k onto an s-torus T^s, and M_k is a fiber bundle over T^s.

It remains to show that an S^1-bundle M over T^{n-1} is diffeomorphic to an infranilmanifold. Let h be a flat metric on T^{n-1}. From the bundle structure, in a standard way, we can construct a metric g_ε on M for each $\varepsilon > 0$ so that the projection $(M, g_\varepsilon) \to (T^{n-1}, \varepsilon h)$ is a Riemannian submersion, the fibers are closed geodesic of length ε and that the sectional curvature of g_ε is uniformly bounded as $\varepsilon \to 0$. Since the diameter of g_ε goes to zero as $\varepsilon \to 0$, Gromov's almost flat manifolds theorem [13] implies that M is diffeomorphic to an infranilmanifold.

More generally, an S^1-bundle over an infranilmanifold is an infranilmanifold as the last part of the above proof shows.

In our proof of Theorem 1, of course, the condition for sectional curvature is essential. One might hope to eliminate it by using some methods different from ours, however. In fact, the most acceptable conjecture would be the following one (due to Gromov [14, 5.22] for the case $b_1(M) = n$).

Conjecture. There is an $\varepsilon > 0$ depending on n and D such that if M satisfies $d(M) \leq D$ and $\mathrm{Ric}(M) > -\varepsilon$, then M is a fiber bundle over a $b_1(M)$-torus.

3. Proof of Theorem 2

Let (M_k, g_k) be a sequence in $\mathcal{M}(n, D)$ such that $\mathrm{Ric}(M_k) > -1/k$ and $b_1(M_k) \geq n-1$, and assume that it converges to a metric space X with respect to the Hausdorff distance. For a given x in X, choose p_k in M_k such that $\lim_{k\to\infty} d_H(B_r(p_k, M_k), B_r(x, X)) = 0$ for all r. Let B', B and $G_k \colon B \to M_k$ be as before. The metric $\hat{g}_k = G_k^* g_k$ converges to a $C^{1,\alpha}$ metric $\hat{g}$.

Lemma 3.1. *$(B, \hat{g})$ is a C^∞ flat Riemannian manifold.*

Let $\omega_1^k, \cdots, \omega_{n-1}^k$ be harmonic 1-forms on M_k such that $\langle \omega_i^k, \omega_j^k \rangle = \delta_{ij}$, $1 \leq i, j \leq n-1$. We set $V_i^k = \sharp\omega_i^k$. Let $\hat{V}_i^k$ be the lift of V_i^k to the unit ball B by the map G_k, as in §2. By Proposition 2.2, for a subsequence, $\hat{V}_i^k$ converges to a $C^{1,\alpha}$ parallel vector field $\hat{V}_i$ with respect to $\hat{g}$. Let $\hat{V}_n$ be a unit vector field orthogonal to $\{\hat{V}_1, \cdots, \hat{V}_{n-1}\}$, which is also parallel. Let $\hat{\varphi}_{i,t}^k$ and $\hat{\varphi}_{i,t}$, $1 \leq i \leq n$, be the flow transformations generated by $\hat{V}_i^k$ and $\hat{V}_i$, where a priori $\hat{\varphi}_{i,t}$ is only of class C^1. It is easily seen, however, that $\hat{\varphi}_{i,t}^k$ gets close to an isometry as $i \to \infty$. Hence, $\hat{\varphi}_{i,t}$ is an isometry, so has C^2 differentiability by Calabi and Hartmann [5], which insures by a standard argument the existence of a coordinate system $\{x^i\}$ on B such that $\partial/\partial x^i = \hat{V}_i$. Since $\hat{g}(\partial/\partial x^i, \partial/\partial x^j) = \delta_{ij}$, $\hat{g}$ is a flat metric.

The above argument together with Gromov's compactness theorem shows that if M_k does not collapse, then X is a flat n-manifold with $b_1(X) \geq n-1$.

Now suppose that M_k collapses, namely, the injectivity radius of M_k goes to zero. We follow the argument in [14, 8.33–8.36]. For $r = 1/5$, let $B_0 \subset B$ be the concentric ball of radius r, and let Γ_k be the pseudogroup of isometric imbeddings γ of $(B_0, \hat{g}_k)$ into $(B, \hat{g}_k)$ such that $G_k \circ \gamma = G_k$. Then $B_r(p_k, M_k) = \Gamma_k \backslash B_0$. Let Z be the set of imbeddings φ of B_0 into B such that

$$\tfrac{1}{2} \leq d(\varphi(x), \varphi(y))/d(x, y) \leq 2,$$

where d is the distance induced from $\hat{g}$. By Ascoli's theorem, Z with the compact open topology is compact. By taking a subsequence, we may assume that Γ_i converges to a closed set Γ of Z, which is a pseudogroup of isometric imbeddings of $(B_0, \hat{g})$ into $(B, \hat{g})$. Since $\Gamma_k \backslash B_0$ converges to $\Gamma \backslash B_0$ for the Hausdorff distance, it turns out that $\Gamma \backslash B_0$ is isometric to $B_r(x, X)$.

Now assume that each M_k is orientable. Then we have

Lemma 3.2. *X is a flat manifold.*

Proof. Let V_n^k be the unit vector field orthogonal to $\{V_1^k, \cdots, V_{n-1}^k\}$ so that $\{V_1^k, \cdots, V_n^k\}$ gives the positive orientation, and let $\hat{V}_n^k$ be the lift to B. For every $u = (u^i)$ in R^n, the vector field $\hat{V}^k = \sum_{i=1}^n u^i \hat{V}_i^k$ converges to a parallel field $\hat{V}$. For small t, let $\hat{\varphi}_t^k$ and $\hat{\varphi}_t$ be the flow transformations of $\hat{V}^k$ and $\hat{V}$ respectively. From the Γ_k equivariance of $\hat{\varphi}_t^k$, we obtain the Γ equivariance of $\hat{\varphi}_t$, namely, $\gamma \circ \hat{\varphi}_t = \hat{\varphi}_t \circ \gamma$, $\gamma \in \Gamma$. It follows from the fact that $\hat{\varphi}_t$ is a translation of the direction $\sum u^i \hat{V}_i$, that Γ consists of translations. Let $\hat{\Gamma}$ be the extension of Γ as a closed subgroup of the vector group R^n, which is isomorphic to $Z^k \times R^l$ for some k, l, $k + l \leq n$. Thus there is a natural imbedding of $\Gamma \backslash B_0$ onto a neighborhood of the flat manifold $\hat{\Gamma} \backslash R^n$. Hence X admits the structure of a flat manifold.

To prove X is a flat torus, it suffices to show that X is homogeneous. Let x, y be any points in X with $d(x, y) < 1/10$. We have only to show the existence of an isometry carrying x to y. By the definition of the Hausdorff distance, we may assume that X and M_k are isometrically imbedded in a metric space (Z_k, d_k) so that $d_H Z_k(X, M_k) < \varepsilon_k$, $\lim \varepsilon_k = 0$. Take x_k, y_k in M_k such that $d_k(x_k, x) < \varepsilon_k$, $d_k(y_k, y) < \varepsilon_k$, and let $\hat{x}_k$, $\hat{y}_k$ be points in B_0 such that $G_k(\hat{x}_k) = x_k$, $G_k(\hat{y}_k) = y_k$. By passing to a subsequence, we may assume that $\hat{x}_k$ and $\hat{y}_k$ converge to $\hat{x}$ and $\hat{y}$ respectively. Choose (u^i) in R^n and s so that the flow transformation $\hat{\varphi}_t$ of $\sum u^i \hat{V}_i$ satisfies $\hat{\varphi}_s(\hat{x}) = \hat{y}$. Then we have $d(\hat{\varphi}_s^k(\hat{x}_k), \hat{y}_k) < \delta_k$, $\lim \delta_k = 0$, and hence $d(\varphi_s^k(x_k), y_k) < \delta_k$, where $\hat{\varphi}_t^k$ and φ_t^k are the flow transformations of $\sum u^i \hat{V}_i^k$ and $\sum u^i V_i^k$ respectively. We set $\varphi_k = \varphi_s^k$. It follows from the lifting argument that φ_k is an almost isometry of M_k, that is,

$$|d(\varphi_k(p), \varphi_k(q)) - d(p, q)| < \mu_k$$

for all p, q in M_k, where $\lim \mu_k = 0$.

We next define a (not necessarily continuous) map $\psi_k \colon X \to X$. For any z in X, take z_k in M_k and z' in X such that $d_k(z_k, z) < \varepsilon_k$, $d_k(\varphi_k(z_k), z') < \varepsilon_k$. Then we set $\psi_k(z) = z'$. The map ψ_k just defined satisfies

$$\begin{aligned} &|d(\psi_k(z), \psi_k(w)) - d(z, w)| \\ &\quad < |d(\varphi_k(z_k), \varphi_k(w_k)) - d(z_k, w_k)| + 4\varepsilon_k < \mu_k + 4\varepsilon_k \end{aligned}$$

for every z, w in X, where w_k is a point in M_k with $d_k(w_k, w) < \varepsilon_k$, and

$$\begin{aligned} d(\psi_k(x), y) &\leq d_k(\psi_k(x), \varphi_k(x_k)) + d(\varphi_k(x_k), y_k) + d_k(y_k, y) \\ &< 2\varepsilon_k + \delta_k. \end{aligned}$$

Interpolating ψ_k by, for instance, a center of mass technique, we can construct a differentiable mapping $\Phi_k\colon X \to X$ so as to satisfy the following:

$$|d(\Phi_k(z), \Phi_k(w)) - d(z,w)| < \alpha_k,$$
$$d(\Phi_k(z), \psi_k(z)) < \beta_k, \qquad ||d\Phi_k|| \le c,$$

where $\lim \alpha_k = 0$, $\lim \beta_k = 0$ and c is a constant. By Ascoli's theorem, we may assume Φ_k converges to an isometry Φ of X satisfying $\Phi(x) = y$. Therefore, X is homogeneous, so a flat torus.

We should remark that, in the above proof, the diagonal method is also useful for the construction of Φ.

Finally, we consider the nonorientable case. Suppose that each M_k is nonorientable, and take the orientable double covering space $\hat{M}_k$ of M_k. By applying the previous argument to $\hat{M}_k$, and by passing to a subsequence if necessary, we may assume that $\hat{M}_k$ converges to a flat torus T for the Hausdorff distance. Let γ_k be the deck transformation of $\hat{M}_k$. By the definition of Hausdorff distance, we may assume that $\hat{M}_k$ and T are isometrically imbedded in a metric space (Z_k, d_k) in such a way that $d_H Z_k(\hat{M}_k, T) < \varepsilon_k$, $\lim \varepsilon_k = 0$. By applying the previous argument to γ_k instead of φ_k, we can construct a differentiable mapping $\Phi_k\colon T \to T$ so as to satisfy

(i) $d_k(\Phi_k(x), \gamma_k(x_k)) < \delta_k$ for $x \in T$, $x_k \in M_k$ with $d_k(x, x_k) < \varepsilon_k$,
(ii) $|d(\Phi_k(x), \Phi_k(y)) - d(x,y)| < \mu_k$,
(iii) $d(\Phi_k^2(x), x) < \nu_k$,
(iv) $||d\Phi_k|| \le c$,

where δ_k, μ_k and ν_k tend to zero as $k \to \infty$. By Ascoli's theorem, we may assume that Φ_k converges to an involutive isometry Φ of T. It remains to prove that X is isometric to the quotient space T/Φ. This is done in the following

Assertion. $\lim_{k\to\infty} d_H(M_k, T/\Phi) = 0$.

Let $\pi\colon T \to T/\Phi$ and $\pi_k\colon \hat{M}_k \to M_k$ be the projections, and let ε be a positive number. Let $N_\varepsilon = \{y_j\}$ be a maximal set of points in T/Φ such that any two distinct points have distance greater than ε. Set $\pi^{-1}(y_j) = \{x_j, \Phi(x_j)\}$, and take p_j in $\hat{M}_k$ with $d_k(p_j, x_j) < \varepsilon_k$. Then $N_\varepsilon^k = \{\pi_j(p_j)\}$ is a 2ε-net in M_k for large k, and N_ε^k converges to N_ε for the Lipschitz distance. Hence M_k converges to T/Φ for the Hausdorff distance.

References

[1] M. Berger, *Riemannian manifolds whose Ricci curvature is bounded from below* (note by T. Tsujishita), Osaka University, 1982. (Japanese)

[2] ——, *Sur les variété riemanniennes pincées juste au-dessous de* 1/4, Ann. Inst. Fourier (Grenoble) **33** (1983) 135–150.

[3] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946) 776–797.

[4] P. Buser & H. Karcher, *Gromov's almost flat manifolds*, Asterisque, No. 81, Soc. Math. France, 1981.

[5] E. Calabi & P. Hartmann, *On the smoothness of isometries*, Duke Math. J. **37** (1970) 741–750.

[6] J. Cheeger & D. Gromoll, *The splitting theorem for manifolds of non-negative Ricci curvature*, J. Differential Geometry **6** (1971) 119–129.

[7] O. Durumeric, *A generalization of Berger's almost 1/4-pinched manifolds theorem*. I, Bull. Amer. Math. Soc. (N.S.) **12** (1985) 260–264.

[8] K. Fukaya, *Theory of convergence for Riemannian orbifolds*, Japan. J. Math. **12** (1986) 121–160.

[9] ——, *Collapsing Riemannian manifolds to lower dimensional one*, J. Differential Geometry **25** (1987) 139–156.

[10] S. Gallot, *A Sobolev inequality and some geometric applications*, Spectra of Riemannian Manifolds, Kaigai, Tokyo, 1983, 45–55.

[11] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren 224, Springer, Berlin, 1977.

[12] R. E. Greene & H. Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math., to appear.

[13] M. Gromov, *Almost flat manifolds*, J. Differential Geometry **13** (1978) 231–241.

[14] ——, *Structures métriques pour les variétés riemanniennes*, rédigé par J. Lafontaine et P. Pansu, Cedic-Nathan, Paris, 1980.

[15] K. Grove & H. Karcher, *How to conjugate C^1-close actions*, Math. Z. **132** (1973) 11–20.

[16] J. Jost & H. Karcher, *Geometrische methoden zur Gewinnung von a-priori-Schranken fur harmonische Abbildungen*, Manuscripta Math. **40** (1982) 27–77.

[17] A. Katsuda, *Gromov's convergence theorem and its application*, Nagoya Math. J. **100** (1985) 11–48.

[18] ——, *A pinching problem for locally homogeneous spaces*, Curvature and Topology of Riemannian Manifolds, Lecture Notes in Math., Vol. 1201, Springer, Berlin, 1986, 150–155.

[19] W. Klingenberg & T. Sakai, *Remarks on the injectivity radius estimate for almost 1/4-pinched manifolds*, Curvature and Topology of Riemannian Manifolds, Lecture Notes in Math., Vol. 1201, Springer, Berlin, 1986, 150–155.

[20] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. **13** (1980) 451–468.

[21] A. Lichnerowicz, *Applications harmoniques dans un tore*, C. R. Acad. Sci. Paris **269** (1969) 912–916.

[22] S. Peters, *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*, J. Reine Angew. Math. **349** (1984) 77–82.

[23] ——, *Convergence of Riemannian manifolds*, Compositio Math. **62** (1987) 3–16.

[24] R. Schoen & S. T. Yau, *Complete three dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, Annals of Math. Studies, No. 102, Princeton University Press, Princeton, NJ, 1982, 209–227.

[25] D. Tischler, *On fibering certain foliated manifolds over S^1*, Topology **9** (1970) 153–154.

[26] T. Yamaguchi, *On the number of diffeomorphism classes in a certain class of Riemannian manifolds*, Nagoya Math. J. **97** (1985) 173–192.

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