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MANIFOLDS OF NONPOSITIVE CURVATURE AND THEIR BUILDINGS

by KEITH BURNS (*) *and* RALF SPATZIER (**)

Abstract

Let M be a complete Riemannian manifold of bounded nonpositive sectional curvature and finite volume. We construct a topological Tits building $\Delta(\widetilde{M})$ associated to the universal cover of M. If M is irreducible and rank (M) $\geqslant 2$, we show that $\Delta(\widetilde{M})$ is a building canonically associated with a Lie group and hence that M is locally symmetric.

INTRODUCTION

Let M be a complete connected Riemannian manifold of bounded nonpositive sectional curvature and finite volume. For any geodesic γ, let rank γ be the dimension of the space of parallel Jacobi fields along γ. Let rank M be the minimum of the ranks of all geodesics. This definition and the basic structure of such manifolds M with rank M $\geqslant 2$ were discussed in [BBE] and [BBS] (cf. also [E1] and [S]). W. Ballmann in [B] and independently ourselves, though somewhat later in the generality presented here, found the following

Main Theorem. — Let M *be a complete connected Riemannian manifold of finite volume and bounded nonpositive sectional curvature. Then the universal cover* $\widetilde{M}$ *of* M *is a flat Euclidean*

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space, a symmetric space of noncompact type, a space of rank 1 *or a product of spaces of the above types.*

Corollary. — *Suppose in addition that* $\widetilde{M}$ *does not have a Euclidean factor. Then* M *has a finite cover that splits as a Riemannian product of spaces of rank* 1 *and a locally symmetric space.*

This follows from the Main Theorem by [E2, Proposition 4.5]. Using Proposition 4.1 of [E2], the Main Theorem follows from

5.1. *Theorem.* — *If* M *is as in the Main Theorem,* $\widetilde{M}$ *is irreducible and rank* $M \geq 2$, *then* M *is locally symmetric.*

Therefore we may assume that $\widetilde{M}$ is irreducible and has rank at least 2. In particular, we will always assume that $\widetilde{M}$ has no Euclidean factor.

Ballmann's proof relies on Berger's characterization of symmetric spaces by their holonomy [Be, Si, B]. Our approach generalises Mostow's proof of the Mostow-Margulis Rigidity Theorem [M, Ma] and the arguments of Gromov's Rigidity Theorem [BGS]. It is also closely related to Gromov's notion of the Tits distance on the ideal boundary of a manifold with nonpositive curvature [BGS, §4].

Let us give a brief outline of the paper.

Section 1 discusses preliminaries.

In Section 2 we refine the notion of Weyl simplices introduced in [BBS]. Recall that they are subsets of the unit tangent spheres to k-flats F at points $p \in F$ where $k = \text{rank } M$. Weyl simplices are very rigid. In fact, we show that they are all isometric. In Section 3 we define Weyl simplices at infinity. We show that they fit together to form a *spherical Tits building* $\Delta = \Delta(\widetilde{M})$ covering $\widetilde{M}(\infty)$. This is a simplicial complex together with a family $\{ \Sigma \}$ of finite subcomplexes called *apartments* satisfying the axioms

(B1) Δ is *thick* i.e. every codimension 1 simplex in a top dimensional simplex is contained in at least 3 top dimensional simplices;
(B2) every apartment is a Coxeter complex;
(B3) any two elements of Δ belong to an apartment;
(B4) if Σ and Σ' are two apartments containing A and $A' \in \Delta$, then there is an isomorphism of Σ onto Σ' which leaves A, A' and all their faces invariant.

Our version of Axiom B2 is stronger than needed (cf. [T, 3.1]).

The building $\Delta(\widetilde{M})$ is set up to formalise the intersection pattern of the regular k-flats at ∞. For example, any Weyl simplex C in $\widetilde{M}(\infty)$ arises as the intersection $F_1(\infty) \cap F_2(\infty)$ for two regular k-flats F_1 and F_2; see Figure 1.

Buildings are very rigid objects. Quite generally, they arise as the buildings of parabolic subgroups of an algebraic group over some field [T]. Our first aim is to prove that $\Delta(\widetilde{M})$ is the building attached to a real algebraic group. This calls for topology.

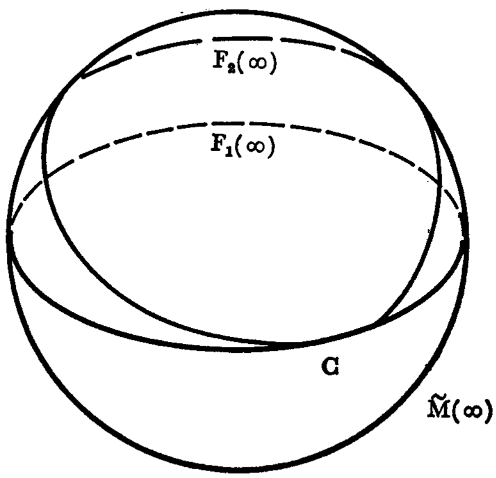

FIG. 1

In [BS] we developed the notion of topological Tits buildings and classified some of them with

Theorem [BS, Main Theorem]. — *Let* Δ *be an infinite, irreducible, locally connected, compact, metric, topologically Moufang building of rank at least* 2. *Then* Δ *is the building of parabolic subgroups of a real simple Lie group* G.

The group G is the group of all automorphisms of Δ which are also homeomorphisms of Δ. Topologically Moufang means that there are plenty of topological automorphisms of Δ [BS, 3.1]. We finish Section 3 by showing that $\Delta(\tilde{M})$ with the topology induced from $\tilde{M}(\infty)$ satisfies all the topological hypotheses of the last theorem.

In Section 4 we show that $\Delta(\tilde{M})$ is irreducible if and only if $\tilde{M}$ is irreducible.

In Section 5 we finally show that M is symmetric. By the above, $\Delta(\tilde{M})$ is the building of a real simple Lie group G. The symmetric space G/K (K a maximal compact subgroup of G) provides us with a model space, as in Gromov's Rigidity Theorem. Adapting Gromov's arguments [BGS, Chapter 4], we show that after a change of scale $\tilde{M}$ is isometric to G/K. Actually our proof is considerably simpler, since $\tilde{M}(\infty)$ already carries a building structure.

We are indebted to V. Schroeder for explaining Gromov's Rigidity Theorem and showing how its proof should be adapted. Before, we could prove the Main Theorem only for compact M. We would also like to thank H. Garland, S. Hurder and A. Katok for their help and encouragement.

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1. Notation and Preliminaries

The results of [BBE] and [BBS] are fundamental to our work. We will use the notation and concepts introduced there. In particular we refer the reader to [BBE, §1] for a survey of basic information about manifolds with nonpositive curvature.

By M we will always denote a complete, connected Riemannian manifold with bounded nonpositive sectional curvature and finite volume. Also we assume that the Riemannian universal cover $\widetilde{M}$ has no Euclidean factor. We denote by k the rank of M [BBE, §2]. Unless otherwise specified, geodesics will have unit speed.

As in [BBE] and [BBS], $\widetilde{M}(\infty)$ denotes the sphere of points at infinity for $\widetilde{M}$ and $\overline{M} = \widetilde{M} \cup \widetilde{M}(\infty)$. If $v \in S\widetilde{M}$ or SM, then γ_v is the unique geodesic with $\dot{\gamma}_v(0) = v$. If $p \in \widetilde{M}$ and $x \in \overline{M}\backslash\{p\}$, then $V(p, x)$ is the unique vector in $S_p \widetilde{M}$ with $\gamma_{V(p,x)}(t) = x$ for some $t \in (0, \infty]$. The geodesic symmetry about a point $p \in \widetilde{M}$ is denoted by σ_p. If F is a flat in $\widetilde{M}$, then $F(\infty)$ is the set of points at infinity for F, i.e.

$$F(\infty) = \{ \gamma_v(\infty) : v \in S\widetilde{M} \text{ is tangent to } F \}.$$

The horosphere $H(v)$ of a unit vector v is defined in [BBE, §1]. If $v \in S\widetilde{M}$ and $p \in \widetilde{M}$, then $v(p)$ is the unique vector of $S_p \widetilde{M}$ asymptotic to v.

The reader might like to review the definitions of regular and p-regular vectors ([BBE, §2] and [BBS, 2.1]). We denote by $\mathscr{R}$ the set of all regular unit vectors. If v is regular or p-regular, $F(v)$ is the unique k-flat to which v is tangent (cf. [BBE, §2] and [BBS, §2]). The strong stable and unstable manifolds and horospheres $W^s(v)$, $W^u(v)$, $H^s(v)$ and $H^u(v)$ of a regular vector v are defined in [BBE, §3].

We call a geodesic γ of $\widetilde{M}$ *periodic* if it is a lift of a closed geodesic in M, and we call $v \in SM$ *periodic* if γ_v is periodic. An isometry φ of $\widetilde{M}$ is an *axial isometry* of a periodic geodesic γ if there is a constant $\tau > 0$ such that $\varphi \circ \gamma(t) = \gamma(t + \tau)$ for all t. We call τ the *period* of φ. Axial isometries of γ arise from the covering transformations of M corresponding to the closed geodesic covered by γ.

1.1. *Lemma. — Let γ be a periodic regular geodesic tangent to the k-flat* F. *Suppose φ is an axial isometry for γ and $x \in \widetilde{M}(\infty)$. If $n \geqslant 0$*

$$\measuredangle_{\gamma(0)}(\varphi^n x, \gamma(\infty)) \leqslant \measuredangle_{\gamma(0)}(x, \gamma(\infty)).$$

If n is large enough, equality holds if and only if $x \in F(\infty)$. Any limit point of $\{ \varphi^n x : n \geqslant 0 \}$ lies in $F(\infty)$.

Proof. — Let $\tau > 0$ be the period of φ. Consider the ideal triangle T with vertices $\gamma(0)$, $\gamma(n\tau)$ and $\varphi^n x$. The sum of its angles at $\gamma(0)$ and $\gamma(n\tau)$ is at most π. Hence $\measuredangle_{\gamma(0)}(\varphi^n x, \gamma(\infty)) \leqslant \measuredangle_{\gamma(n\tau)}(\varphi^n x, \gamma(\infty)) = \measuredangle_{\gamma(0)}(x, \gamma(\infty))$ as φ fixes $\gamma(\infty)$. It is clear that equality holds if $x \in F(\infty)$.

If n is large enough, any parallel Jacobi field along the regular geodesic γ between $\gamma(0)$ and $\gamma(n\tau)$ must be tangent to F. If $\sphericalangle_{\gamma(0)}(\varphi^n x, \gamma(\infty)) = \sphericalangle_{\gamma(0)}(x, \gamma(\infty))$, the ideal triangle T is flat. Hence $\varphi^n x \in F(\infty)$. Since F is invariant under φ, $x \in F(\infty)$.

Finally suppose y is a limit point of $\{ \varphi^n x : n \geqslant 0 \}$. Then

$$\sphericalangle_{\gamma(0)}(\varphi^k y, \gamma(\infty)) = \sphericalangle_{\gamma(0)}(y, \gamma(\infty)) \quad \text{for all } k \geqslant 0.$$

Hence $y \in F(\infty)$. ■

2. Isometry of Weyl Simplices

We define Weyl simplices for so-called ℓ-regular vectors. This extends the definition of the set $\widetilde{\mathscr{C}}(v)$ for p-regular vectors v in [BBS]. Since the set of ℓ-regular vectors is a union of asymptote classes this allows us to define Weyl simplices at infinity from which we then construct a Tits building (cf. Section 3). Our main goal in this section is to show that all Weyl simplices are isometric.

Call $v \in S\widetilde{M}$ *ℓ-regular* if v is asymptotic to a regular vector. Let $\mathscr{L}$ be the set of all ℓ-regular vectors. Since the set $\mathscr{R}$ of all regular vectors is open and dense, so is $\mathscr{L}$.

2.1. *Examples.* — (i) If M has rank 1 then every unit vector v is ℓ-regular. In fact, if γ is a periodic regular geodesic with $\gamma(\infty) \neq \gamma_v(\infty)$ then $\gamma_v(\infty)$ is joined to $\gamma(\infty)$ by a geodesic γ' (cf. Lemma 3.6). Clearly γ' is regular.

(ii) Suppose $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ and $\pi_i : \widetilde{M} \to \widetilde{M}_i$ is the projection onto the i-th factor. If $v \in S\widetilde{M}$, set $v_i = d\pi_i v$ for $i = 1, 2$. Then v is ℓ-regular if and only if $v_i \neq 0$ and $\| v_i \|^{-1} v_i$ is ℓ-regular for $i = 1, 2$.

(iii) Unlike p-regular vectors, ℓ-regular vectors can be tangent to more than one k-flat, where k is the rank of M. Suppose that in Example (ii) both $\widetilde{M}_1$ and $\widetilde{M}_2$ have rank 1. Then $v \in S\widetilde{M}$ is ℓ-regular if $v_1 \neq 0 \neq v_2$. But if either v_i is tangent to a 2-flat in $\widetilde{M}_i$, then v is tangent to a flat of dimension at least 3 (cf. Proposition 2.22).

If $v \in \mathscr{L}$, set

$$A(v) = \{ q \in \widetilde{M} : v(q) \text{ is } p\text{-regular} \}.$$

2.2. *Definition.* — If $v \in \mathscr{L}$, the *Weyl simplex* of v is

$$\widetilde{\mathscr{C}}(v) = \{ w \in S_{\pi v} \widetilde{M} : w(q) \text{ is tangent to } F(v(q)) \text{ for all } q \in A(v) \}.$$

We will see later (cf. Theorem 3.8) that this set actually is a spherical simplex.

It is easy to check that this agrees with Definition 2.4 of [BBS] when v is p-regular. Clearly $\widetilde{\mathscr{C}}(v)$ is closed for any $v \in \mathscr{L}$ and, if φ is an isometry of $\widetilde{M}$, $\widetilde{\mathscr{C}}(d\varphi(v)) = d\varphi(\widetilde{\mathscr{C}}(v))$. When v is p-regular, $\widetilde{\mathscr{C}}(v)$ is a convex subset of the $k - 1$ dimensional unit sphere $S_{\pi v} F(v)$ by Lemma 2.5 of [BBS]. If $v, w \in \mathscr{L}$ are asymptotic, there is a bijection $\widetilde{\mathscr{C}}(v) \to \widetilde{\mathscr{C}}(w)$ defined by $u \to u(\pi w)$. It follows from the Convexity Lemma [BBE, 1.5] that this map is an isometry if v and w are both p-regular.

2.3. *Remark.* — *If v is uniformly recurrent and regular,* $A(v) = \tilde{M}$ *by* [BBS, 2.2]. *Hence the map* $u \to u(q)$ *defines an isometry* $\tilde{\mathscr{C}}(v) \to \tilde{\mathscr{C}}(v(q))$ *for every* $q \in \tilde{M}$.

We now define the interior of a Weyl simplex. If v is p-regular, $\operatorname{Int} \tilde{\mathscr{C}}(v)$ will be the (topological) interior of $\tilde{\mathscr{C}}(v)$ in $S_{\pi v} F(v)$. For $v \in \mathscr{L}$, set

$$\operatorname{Int} \tilde{\mathscr{C}}(v) = \{ w \in \tilde{\mathscr{C}}(v) : w(q) \in \operatorname{Int} \tilde{\mathscr{C}}(v(q)) \quad \text{for all } q \in A(v) \}.$$

Set $\partial\tilde{\mathscr{C}}(v) = \tilde{\mathscr{C}}(v) \backslash \operatorname{Int} \tilde{\mathscr{C}}(v)$.

We begin our proof that all Weyl simplices are isometric by studying Weyl simplices where this is true locally. The first major step will be to show in Proposition 2.12 that all of these Weyl simplices are isometric.

2.4. *Definition.* — A vector $v \in \mathscr{L}$ is *rigid* if v has an open neighborhood $U \subset \mathscr{L}$ such that for every $u \in U$ we have :

(R1) $u \in \operatorname{Int} \tilde{\mathscr{C}}(u)$;
(R2) $\operatorname{Int} \tilde{\mathscr{C}}(u) \subseteq U$;
(R3) $\tilde{\mathscr{C}}(u') = \tilde{\mathscr{C}}(u)$ for every $u' \in \operatorname{Int} \tilde{\mathscr{C}}(u)$;
(R4) $\tilde{\mathscr{C}}(u) \overset{\text{iso}}{=} \tilde{\mathscr{C}}(v)$.

A *rigid Weyl simplex* is the Weyl simplex of a rigid vector.

Note that $\tilde{\mathscr{C}}(v)$ is a $k - 1$ dimensional convex set when v is rigid. This follows from [BBS, 2.7], since U must contain a regular uniformly recurrent vector. The next lemma shows that the set of rigid vectors is dense; it is clearly open. Also it is invariant under the action of isometries of $\tilde{M}$, and Lemma 2.8 shows that it is a union of asymptote classes.

2.5. *Lemma.* — *If $v \in S\tilde{M}$ is regular and uniformly recurrent in both the positive and negative directions, then v is rigid.*

Proof. — Let $U \subseteq \mathscr{R}$ be the neighbourhood of v defined in the Rigidity Lemma [BBS, 2.10]. Recall that [BBS] defined the Weyl chamber $\mathscr{C}(w)$ of a p-regular vector w, and $\mathring{\mathscr{C}}(w)$ as the (topological) interior of $\mathscr{C}(w)$ in $S_{\pi w} F(w)$. It is easy to prove the following:

a) If w is p-regular and $w' \in \mathscr{C}(w)$, then $\tilde{\mathscr{C}}(w') = \tilde{\mathscr{C}}(w)$.

b) If w and w' are p-regular and asymptotic, then the map $u \to u(\pi w')$ on $S_{\pi w} M$ defines isometries from $\tilde{\mathscr{C}}(w)$, $\operatorname{Int} \tilde{\mathscr{C}}(w)$, $\mathscr{C}(w)$ and $\mathring{\mathscr{C}}(w)$ to $\tilde{\mathscr{C}}(w')$, $\operatorname{Int} \tilde{\mathscr{C}}(w')$, $\mathscr{C}(w')$ and $\mathring{\mathscr{C}}(w')$ respectively.

The construction of h_v in the proof of the Rigidity Lemma shows that any $u \in U$ is asymptotic to a vector $u' \in \mathring{\mathscr{C}}(v')$ for some $v' \in W^u(v)$. Moreover it is clear that U can be constructed so that $v' \in \mathscr{R}$ for each $u \in U$. Hence all the vectors in the following argument are p-regular.

We note some obvious consequences of part 1) of the Rigidity Lemma and its proof:

c) $\operatorname{Int}\widetilde{\mathscr{C}}(v') = \mathring{\mathscr{C}}(v')$ and so $\operatorname{Int}\widetilde{\mathscr{C}}(u) = \mathring{\mathscr{C}}(u)$.

d) $\widetilde{\mathscr{C}}(v') \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v)$.

e) $v' \in \mathring{\mathscr{C}}(v')$.

(R1), (R3) and (R4) follow from *a)* — *e)*; (R2) follows from *c)*, since it is clear from 2) of the Rigidity Lemma and its proof that $\mathring{\mathscr{C}}(u) \subseteq \mathrm{U}$ if $u \in \mathrm{U}$. ∎

2.6. *Lemma. — Let a sequence* $\{ v_n \} \subseteq \mathscr{L}$ *converge to* $v \in \mathscr{L}$.

(i) *Then* $\overline{\lim}\, \widetilde{\mathscr{C}}(v_n) \subseteq \widetilde{\mathscr{C}}(v)$.

(ii) *If in addition* v *is rigid,* $\widetilde{\mathscr{C}}(v_n) \to \widetilde{\mathscr{C}}(v)$ *and* $\partial\widetilde{\mathscr{C}}(v_n) \to \partial\widetilde{\mathscr{C}}(v)$ *in the Hausdorff metric on compact subsets of* $\mathrm{S}\widetilde{\mathrm{M}}$.

Proof. — (i) is proved in almost the same way as Lemma 2.8 of [BBS]. (ii) follows easily from (i), since $\widetilde{\mathscr{C}}(v)$ is convex and $\widetilde{\mathscr{C}}(v_n) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v)$ for all large enough n. ∎

2.7. *Lemma. — If* v *is rigid and* $q \in \widetilde{\mathrm{M}}$, *the map* $\widetilde{\mathscr{C}}(v) \to \widetilde{\mathscr{C}}(v(q))$ *given by* $u \to u(q)$ *is an isometry.*

Proof. — Define $\alpha : \mathrm{S}\widetilde{\mathrm{M}} \to \mathrm{S}_q \widetilde{\mathrm{M}}$ by $\alpha(u) = u(q)$. Then α is continuous and for any $w \in \mathscr{L}$, $\alpha \mid \widetilde{\mathscr{C}}(w)$ is a bijection onto $\widetilde{\mathscr{C}}(w(q))$. Choose a sequence $\{ v_n \}$ of uniformly recurrent regular vectors converging to v. By Remark 2.3, $\alpha \mid \widetilde{\mathscr{C}}(v_n)$ is an isometry for each n. Since $\widetilde{\mathscr{C}}(v_n) \to \widetilde{\mathscr{C}}(v)$ by Lemma 2.6, it follows that $\alpha \mid \widetilde{\mathscr{C}}(v)$ is an isometry. ∎

2.8. *Lemma. — If* v *is rigid and* v' *is asymptotic to* v, *then* v' *is rigid.*

Proof. — Let U be an open neighborhood of v satisfying (R1), (R2), (R3) and (R4). Then $\mathrm{U}' = \{ u(q) : u \in \mathrm{U}, q \in \widetilde{\mathrm{M}} \}$ is an open neighborhood of v'. If u' is asymptotic to u, then $\widetilde{\mathscr{C}}(u') = \{ w(\pi u') : w \in \widetilde{\mathscr{C}}(u) \}$ and $\operatorname{Int}\widetilde{\mathscr{C}}(u') = \{ w(\pi u') : w \in \operatorname{Int}\widetilde{\mathscr{C}}(u) \}$. We see easily that U′ satisfies (R1), (R2) and (R3); (R4) follows from the previous lemma. ∎

2.9. *Lemma. — Suppose* $v_0 \in \mathrm{S}\widetilde{\mathrm{M}}$ *is rigid and* $w \in \mathrm{S}\widetilde{\mathrm{M}}$ *has* $\measuredangle(v_0, w(\pi v_0)) < \measuredangle(v_0, \partial\widetilde{\mathscr{C}}(v_0))$. *Then* w *is rigid and* $\widetilde{\mathscr{C}}(w) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v_0)$.

Proof. — If v is close enough to v_0, then v is rigid, $\widetilde{\mathscr{C}}(v) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v_0)$ and it is clear from Lemma 2.6 (ii) that $\measuredangle(v, w(\pi v)) < \measuredangle(v, \partial\widetilde{\mathscr{C}}(v))$. By the Closing Lemma [BBS, 4.5], there is a periodic regular vector v with all the above properties. Let φ be an axial isometry of γ_v. Let $w_n = (d\varphi^n w)(\pi v)$. By Lemma 1.1, $\{ w_n \}$ has a limit vector $w' \in \mathrm{S}_{\pi v}\mathrm{F}(v)$ with $\measuredangle(v, w') \leqslant \measuredangle(v, \partial\widetilde{\mathscr{C}}(v))$. Since $\operatorname{Int}\widetilde{\mathscr{C}}(v)$ is open in $\mathrm{S}_{\pi v}\mathrm{F}(v)$ by [BBS, 2.7], $w' \in \operatorname{Int}\widetilde{\mathscr{C}}(v)$. Hence w' is rigid and $\widetilde{\mathscr{C}}(w') = \widetilde{\mathscr{C}}(v) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v_0)$. For some large m, w_m is rigid and $\widetilde{\mathscr{C}}(w_m) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(w)$. Since w_m is asymptotic to $d\varphi^m w$, it follows from Lemma 2.8 that $d\varphi^m w$ is rigid, and hence that w is rigid. By Lemma 2.7, $\widetilde{\mathscr{C}}(w) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(d\varphi^m w) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(w_m)$ and hence $\widetilde{\mathscr{C}}(w) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v_0)$. ∎

2.10. *Corollary.* — *Suppose* $\{ v_n \}$ *is a sequence of rigid vectors such that* $\sphericalangle(v_n, \partial\widetilde{\mathscr{C}}(v_n))$ *is bounded away from* 0. *If* v_n *converges to* v, *then* v *is rigid.* ∎

Before we show that all rigid Weyl simplices are isometric we introduce:

2.11. *Definition.* — The *center* $c(\widetilde{\mathscr{C}}(v))$ of a Weyl simplex $\widetilde{\mathscr{C}}(v)$ is the unit vector in the same direction as

$$\int_{\widetilde{\mathscr{C}}(v)} \mathrm{I} \, d\mu_{\mathrm{S}}$$

where S is the (unique) great subsphere of smallest dimension which contains $\widetilde{\mathscr{C}}(v)$, μ_{S} is Lebesgue measure on S, and $\mathrm{I} : \mathrm{S}_{\pi v} \widetilde{\mathrm{M}} \to \mathrm{T}_{\pi v} \widetilde{\mathrm{M}}$ is the inclusion.

We list some obvious properties of c which will be used in the following.

(i) If v is rigid, $c(\widetilde{\mathscr{C}}(v)) \in \widetilde{\mathscr{C}}(v)$.

(ii) For any isometry φ of $\widetilde{\mathrm{M}}$,

$$c[\widetilde{\mathscr{C}}(d\varphi(v))] = d\varphi[c(\widetilde{\mathscr{C}}(v))].$$

(iii) If v and v' are rigid and asymptotic, $c(\widetilde{\mathscr{C}}(v))$ and $c(\widetilde{\mathscr{C}}(v'))$ are asymptotic.

(iv) If v is rigid and $v_n \to v$, $c(\widetilde{\mathscr{C}}(v_n)) \to c(\widetilde{\mathscr{C}}(v))$.

2.12. *Proposition.* — *All rigid Weyl simplices are isometric.*

Proof. — By [BBS, 4.5] there is a periodic regular vector w such that γ_w has an axial isometry φ which is a pure translation of $\mathrm{F}(w)$. Lemma 2.5 tells us that w is rigid. By Lemma 1.1, we can assume, after replacing φ by a power φ^m, that if $x \in \widetilde{\mathrm{M}}(\infty)$, then $\sphericalangle_{\pi w}(\varphi x, \gamma_w(\infty)) < \sphericalangle_{\pi w}(x, \gamma_w(\infty))$ unless $x \in \mathrm{F}(w)(\infty)$. Let $\widetilde{\mathscr{D}}$ be a rigid Weyl simplex. We shall show that $\widetilde{\mathscr{D}} \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(w)$. Let D be the set of all rigid vectors in $\mathrm{S}\widetilde{\mathrm{M}}$ that are centers of Weyl simplices isometric to $\widetilde{\mathscr{D}}$. It follows from the properties of the center and Corollary 2.10 that D is closed. Let $\mathrm{D}_0 = \mathrm{D} \cap \mathrm{S}_{\pi w} \widetilde{\mathrm{M}}$.

We shall show below that if $v \in \mathrm{D}_0$ and $v \notin \widetilde{\mathscr{C}}(w)$, there is a vector $v_1 \in \mathrm{D}_0$ with $\sphericalangle(v_1, w) < \sphericalangle(v, w)$. Since D_0 is compact, it follows that there is $v_0 \in \mathrm{D}_0 \cap \widetilde{\mathscr{C}}(w)$. Since v_0 and w are both rigid, we see that if $w_0 \in \mathrm{Int}\, \widetilde{\mathscr{C}}(w)$ is close enough to v_0, then $\widetilde{\mathscr{C}}(w) = \widetilde{\mathscr{C}}(w_0) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v_0) \overset{\text{iso}}{=} \widetilde{\mathscr{D}}$.

Now we construct v_1 from v. Either v belongs to $\mathrm{S}_{\pi w} \mathrm{F}(w)$ or it does not. In the latter case, we take $v_1 = (d\varphi v)(\pi v)$. It is clear that $v_1 \in \mathrm{D}_0$ and $\sphericalangle(v_1, w) < \sphericalangle(v, w)$ by our choice of w and φ, since $\gamma_v(\infty) \notin \mathrm{F}(w)(\infty)$.

In the former case we use the next two lemmas.

2.13. *Lemma. — If* $v \in S_{\pi w} F(w) \backslash \widetilde{\mathscr{C}}(w)$, *there is a vector* $w' \in W^s(w)$ *such that* $v(\pi w')$ *is not tangent to* $F(w')$.

Proof. — Since w is periodic and regular, $A(w) = \widetilde{M}$ which is the union of all $F(w'')$ for $w'' \in W^s(w)$, by [BBE, 2.12]. Since $v \notin \widetilde{\mathscr{C}}(w)$, there is $p \in \widetilde{M}$ with $v(p)$ not tangent to $F(w(p))$. Choose $w' \in W^s(w)$ so that $p \in F(w')$. If $v(\pi w')$ were tangent to $F(w')$, we would have $v(q)$ tangent to $F(w')$ for every $q \in F(w')$. ■

2.14. *Lemma. — Suppose* $w' \in W^s(w)$. *Let* ψ *be an axial isometry of* γ_w. *Then for any* $v \in S_{\pi w} F(w)$, *there is a vector* $v' \in S_{\pi w'} F(w')$ *such that* $\gamma_v(\infty)$ *is a limit point of* $\{\psi^{-n}(\gamma_{v'}(\infty))\}$.

Proof. — Let τ be the period of ψ, so $d\psi^n(w) = g^{n\tau}(w)$, where g^t is the geodesic flow. For $n \geqslant 1$, let $w'_n = g^{n\tau}(d\psi^{-n} w')$. Clearly $w'_n \in H^s(w)$ for each n. Moreover $d_{S\widetilde{M}}(w'_n, w) = d_{S\widetilde{M}}(g^{n\tau} w', g^{n\tau} w) \to 0$ by [BBE, 3.10]. Define $\psi_n : S_{\pi w'} F(w') \to S_{\pi w'_n} F(w'_n)$ by $\psi_n(u) = (d\psi^{-n} u)(\pi w'_n)$. Clearly $\psi_n = P_n \circ d\psi^{-n}$, where P_n is the parallel translation in $F(w'_n)$ from $\psi^{-n}(\pi w')$ to $\pi w'_n$. Hence each ψ_n is an isometry. Since w is regular, $F(w'_n)$ converges to $F(w)$, so $\{\psi_n\}$ has a subsequence that converges to an isometry $\psi : S_{\pi w'} F(w') \to S_{\pi w} F(w)$. Choose $v' = \widetilde{\psi}^{-1}(v)$. Since $\psi_n(v')$ is asymptotic to $d\psi^{-n} v'$, we see that $\gamma_v(\infty)$ is a limit point of $\{\psi^{-n}(\gamma_{v'}(\infty))\}$. ■

Apply Lemma 2.14 with $\psi = \varphi$ and take $v_1 = v'(\pi w)$. Using Lemma 1.1 we see that

$$\begin{aligned} \measuredangle(v_1, w) &= \measuredangle_{\pi w}(\gamma_w(\infty), \gamma_{v'}(\infty)) \\ &\leqslant \lim_{n \to \infty} \measuredangle_{\pi w}(\gamma_w(\infty), \varphi^{-n} \circ \gamma_{v'}(\infty)) = \measuredangle(v, w), \end{aligned}$$

with equality only if $\gamma_{v'}(\infty) \in F(w)(\infty)$. Note that $\gamma_{v'}(\infty) \neq \gamma_v(\infty)$ by our choice of w' in Lemma 2.13, and recall from the beginning of the proof that φ was chosen to fix every point in $F(w)(\infty)$ and so that $\measuredangle_{\pi w}(\varphi x, \gamma_w(\infty)) < \measuredangle_{\pi w}(x, \gamma_w(\infty))$ if $x \in \widetilde{M}(\infty) \backslash F(w)(\infty)$. We see that $\measuredangle(v_1, w) < \measuredangle(v, w)$. Now we show that $v_1 \in D_0$. Let $v_n = (d\varphi^{-n} v')(\pi w)$, so v is a limit vector of $\{v'_n\}$. Since $v \in D$ there is an m such that v'_m is rigid and $\widetilde{\mathscr{C}}(v'_m) \overset{\text{iso}}{=} \widetilde{\mathscr{C}}(v) \overset{\text{iso}}{=} \widetilde{\mathscr{D}}$. It follows that v' is rigid and $\widetilde{\mathscr{C}}(v') \overset{\text{iso}}{=} \widetilde{\mathscr{D}}$. Clearly $c(\widetilde{\mathscr{C}}(v)) = v$ is a limit vector of $\{c(\widetilde{\mathscr{C}}(v'_n))\}$. Since $\measuredangle(v'_n, c(\widetilde{\mathscr{C}}(v'_n))) = \measuredangle(v', c(\widetilde{\mathscr{C}}(v)))$ for all n, we see that $v' = c(\widetilde{\mathscr{C}}(v'))$. Hence $v' \in D$ and thus $v_1 \in D_0$. ■

Before we extend this result to all Weyl simplices we study a further class of vectors—the *r*-periodic vectors.

We will call a vector v *p-rigid* if it is *p*-regular and rigid. We call v *r-rigid* if it is *p*-rigid and *p*-rigid vectors are dense in $S_{\pi v} F(v)$. Note that if v and w are *p*-regular and $w \in S_{\pi v} F(v)$ then $F(v) = F(w)$, since $F(w)$ is the unique *k*-flat tangent to w. If w is also rigid, we see that $\operatorname{Int} \widetilde{\mathscr{C}}(w)$ is open in $S_{\pi v} F(v)$. Moreover every vector in $\operatorname{Int} \widetilde{\mathscr{C}}(w)$ is *p*-rigid. Since all rigid Weyl simplices are isometric and distinct rigid Weyl simplices have disjoint interiors, it follows that there is a number d such that a *p*-rigid vector v is *r*-rigid if and only if $S_{\pi v} F(v)$ contains d *p*-rigid vectors with distinct Weyl simplices.

Note that these d Weyl simplices cover $S_{\pi v} F(v)$ and are the only rigid Weyl simplices in $S_{\pi v} F(v)$.

2.15. *Definition.* — A vector is *r-periodic* if it is periodic, regular and r-rigid.

2.16. *Lemma.* — *If w is r-periodic and $u \in \partial\widetilde{\mathscr{C}}(w)$, then u is not p-regular.*

Remarks. — We will see in Proposition 2.20 that u is not even ℓ-regular. In the case that M is compact, it was already known [BBS, 4.8] that the boundary of the Weyl simplex of a p-regular vector contains no p-regular vectors.

Proof. — It is obvious from the discussion above that there is a p-rigid vector $v \in S_{\pi w} F(w) \setminus \widetilde{\mathscr{C}}(w)$ such that $u \in \widetilde{\mathscr{C}}(v) \cap \widetilde{\mathscr{C}}(w)$. Suppose now that u is p-regular. Applying Lemma 2.6 to sequences in Int $\widetilde{\mathscr{C}}(v)$ and Int $\widetilde{\mathscr{C}}(w)$ which converge to u shows that $\widetilde{\mathscr{C}}(u) \supseteq \widetilde{\mathscr{C}}(v) \cap \widetilde{\mathscr{C}}(w)$. Hence $v(p)$ and $w(p)$ are tangent to $F(u(p))$ for all $p \in A(u)$. As $w(p)$ is always p-regular (by Remark 2.3), we see that $F(w(p)) = F(u(p))$ and $v(p)$ is tangent to $F(w(p))$ for all $p \in A(u)$. Since the set of p-regular vectors is open, $A(u)$ is a neighborhood of $\pi u = \pi w$.

By Lemma 2.13, there is $w' \in W^s(w)$ with $v(\pi w')$ not tangent to $F(w')$. Let ψ be an axial isometry of γ_w and define $w'_n \in W^s(w)$ as in Lemma 2.14. The proofs of Lemmas 2.13 and 2.14 show that $v(\pi w'_n)$ is not tangent to $F(w'_n)$ and $w'_n \to w$. This contradicts the previous paragraph. ∎

2.17. *Lemma.* — *The r-periodic vectors are dense in* $S\widetilde{M}$.

Proof. — Periodic regular vectors are dense, so it will suffice to prove that the set V of all r-rigid vectors is open and dense. Since the set of p-rigid vectors is clearly open and dense, density of V follows from 1) of [BBE, 2.7]. If v' is close to a p-rigid vector v, v' is p-rigid and $S_{\pi v'} F(v')$ is close to $S_{\pi v} F(v)$. If $v \in V$, $S_{\pi v} F(v)$ contains d p-rigid vectors with distinct Weyl simplices. We see using Lemma 2.6 (ii) that if v' is close enough to v, $S_{\pi v'} F(v')$ also has this property, and so $v' \in V$. Thus V is open. ∎

2.18. *Theorem.* — *All ℓ-regular vectors are rigid. All Weyl simplices are isometric. If $v \in \mathscr{L}$, $\widetilde{\mathscr{C}}(v)$ is a $k - 1$ dimensional convex subset of* $S_{\pi v} \widetilde{M}$.

Proof. — First suppose v is a p-regular vector. By Lemma 2.17, v is the limit of a sequence $\{ v_n \}$ of r-periodic vectors. Observe that $\measuredangle(v_n, \partial\widetilde{\mathscr{C}}(v_n))$ is bounded away from 0. For otherwise v would be a limit of vectors in $\partial\widetilde{\mathscr{C}}(v_n)$ which is impossible by Lemma 2.16, since the set of p-regular vectors is open. Since each v_n is rigid, Corollary 2.10 shows that v is rigid.

Every ℓ-regular vector is asymptotic to a p-regular vector, so it follows from Lemma 2.8 that all ℓ-regular vectors are rigid. Hence all Weyl simplices are $k - 1$ dimensional convex sets, which are isometric by Proposition 2.12. ∎

Using this theorem we can restate Lemma 2.6 and Corollary 2.10.

2.19. *Corollary.* — *Suppose* $\{v_n\} \subseteq \mathscr{L}$ *converges to* v.

(i) *If* v *is* ℓ*-regular, then* $\widetilde{\mathscr{C}}(v_n) \to \widetilde{\mathscr{C}}(v)$ *in the Hausdorff metric.*

(ii) *If* $\sphericalangle(v_n, \partial\widetilde{\mathscr{C}}(v_n))$ *is bounded away from* 0, *then* v *is* ℓ*-regular.* ∎

Now it is easy to prove some useful properties of Weyl simplices.

2.20. *Proposition.* — *The boundary of a Weyl simplex cannot contain an* ℓ*-regular vector.*

Proof. — Suppose $v \in \mathscr{L}$ and $w \in \widetilde{\mathscr{C}}(w)$ is ℓ-regular. Consider a sequence $\{w_n\} \subseteq \operatorname{Int}\widetilde{\mathscr{C}}(v)$ which converges to w. By Corollary 2.19, $\widetilde{\mathscr{C}}(w) = \overline{\lim}\,\widetilde{\mathscr{C}}(w_n) = \widetilde{\mathscr{C}}(v)$. But now $w \in \partial\widetilde{\mathscr{C}}(w)$, which is impossible, since w is rigid by Theorem 2.18. ∎

2.21. *Proposition.* — *Every vector* $v \in S\widetilde{M}$ *is contained in a Weyl simplex.*

Proof. — Choose a sequence $\{v_n\}$ of ℓ-regular vectors converging to v. By passing to a subsequence we may assume that the sequence $\{w_n\}$ of centers of $\widetilde{\mathscr{C}}(v_n)$ converges to a vector w. It is clear from Corollary 2.19 that

$$\widetilde{\mathscr{C}}(w) = \overline{\lim}\,\widetilde{\mathscr{C}}(w_n) \supseteq \{v\}. \quad \blacksquare$$

2.22. *Proposition.* — *If* $v \in \mathscr{L}$, *there is a unique* k*-flat* F *through* πv *such that* $\widetilde{\mathscr{C}}(v) \subseteq S_{\pi v}F$. *Moreover* $S_{\pi v}F$ *is a union of Weyl simplices. (Compare Example* 2.1 iii.*)*

Proof. — Uniqueness follows from the fact that $\widetilde{\mathscr{C}}(v)$ is a $k - 1$ dimensional convex subset of $S_{\pi v}\widetilde{M}$. To prove existence, choose a sequence of r-periodic vectors $\{v_n\}$ converging to v. For each n, let $v_n^1, \ldots, v_n^d$ be the centers of the Weyl simplices contained in $S_{\pi v_n}F(v_n)$. By passing to a subsequence we can assume that $F(v_n)$ converges to a k-flat F passing through πv and v_n^i converges to an ℓ-regular vector $v^i \in S_{\pi v}F(v)$ for $i = 1, \ldots, d$. It is clear that $\widetilde{\mathscr{C}}(v^1), \ldots, \widetilde{\mathscr{C}}(v^d)$ are all distinct. It follows that $S_{\pi v}F(v) = \bigcup_{i=1}^{d} \widetilde{\mathscr{C}}(v^i)$, and $\widetilde{\mathscr{C}}(v) = \widetilde{\mathscr{C}}(v^i)$ for some i. ∎

2.23. *Proposition.* — *If* $v \in \mathscr{L}$, *then* $-v \in \mathscr{L}$ *and* $\widetilde{\mathscr{C}}(-v) = -\widetilde{\mathscr{C}}(v) = \{-w : w \in \widetilde{\mathscr{C}}(v)\}$.

Proof. — We prove this first when v is p-regular. Observe that a vector u is p-regular if and only if $-u$ is. Suppose $w \in \widetilde{\mathscr{C}}(v)$ and $-w \notin \widetilde{\mathscr{C}}(-v)$. Then the great circle arc joining v to w contains a vector $u \in \operatorname{Int}\widetilde{\mathscr{C}}(v)$ such that $-u \in \partial\widetilde{\mathscr{C}}(-v)$. Then u is p-regular by [BBS, 2.5] and $-u$ is non-ℓ-regular by Theorem 2.18, which is impossible. It follows that $-\widetilde{\mathscr{C}}(v) \subseteq \widetilde{\mathscr{C}}(-v)$, and hence $\widetilde{\mathscr{C}}(-v) = -\widetilde{\mathscr{C}}(v)$.

Now suppose $v \in \mathscr{L}$ and choose a sequence $\{v_n\}$ of p-regular vectors converging to v. Since $\mathscr{L}$ is open, $\sphericalangle(v_n, \partial\widetilde{\mathscr{C}}(v_n))$ is bounded away from 0, for otherwise v would be a limit of non-ℓ-regular vectors by Proposition 2.20. But $\sphericalangle(-v_n, \partial\widetilde{\mathscr{C}}(-v_n)) = \sphericalangle(v_n, \partial\widetilde{\mathscr{C}}(v_n))$ since $\widetilde{\mathscr{C}}(-v_n) = -\widetilde{\mathscr{C}}(v_n)$. Hence $-v \in \mathscr{L}$ by Corollary 2.19 (ii). By Corollary 2.19 (i), $\widetilde{\mathscr{C}}(v_n) \to \widetilde{\mathscr{C}}(v)$ and $-\widetilde{\mathscr{C}}(v_n) = \widetilde{\mathscr{C}}(-v_n) \to \widetilde{\mathscr{C}}(-v)$. Thus $\widetilde{\mathscr{C}}(-v) = -\widetilde{\mathscr{C}}(v)$. ∎

3. The Tits Building of a Manifold of Nonpositive Curvature

We define Weyl simplices at infinity and show how they give rise to a topological Tits building.

Call a point x at infinity *regular* if it has ℓ-regular representative geodesic rays. Otherwise we call x *singular*. Note that the set $\mathscr{R}(\infty)$ of regular points is open and dense in $M(\infty)$. We call $x \in M(\infty)$ *r-periodic* if $x = \gamma_v(\infty)$ for an r-periodic vector v.

3.1. *Lemma. — The r-periodic points are dense in* $\widetilde{M}(\infty)$.

Proof. — This follows from Lemma 2.17. ∎

3.2. *Definition.* — Let $x \in \mathscr{R}(\infty)$ with geodesic representative γ_v. The *Weyl simplex* of x is the set

$$C(x) = \{ \gamma_w(\infty) : w \in \widetilde{\mathscr{C}}(v) \}.$$

The *interior* of $C(x)$ is the set $\mathring{C}(x) = \{ \gamma_w(\infty) : w \in \operatorname{Int} \widetilde{\mathscr{C}}(v) \}$ and the *boundary* of $C(x)$ is $\partial C(x) = C(x) \backslash C(x)$. If w is the center of $\mathscr{C}(v)$, we call $\gamma_w(\infty)$ the *center* of $C(x)$.

Clearly these definitions do not depend on the choice of representative geodesic for x. Since the Weyl simplex of an ℓ-regular vector is tangent to a (unique) k-flat, we see that $C(x) \subseteq F(\infty)$ for some k-flat F. Note that $C(x)$ is closed in $F(\infty)$ and $\mathring{C}(x)$ is its topological interior as a subset of $F(\infty)$. We see from Proposition 2.21 and 2.20 that

(i) Every point of $\widetilde{M}(\infty)$ lies in a Weyl simplex;

(ii) A point of $M(\infty)$ is regular (singular) if and only if it lies in the interior (boundary) of a Weyl simplex.

3.3. *Proposition. — The set of Weyl simplices is compact in the Hausdorff topology.*

Proof. — Let $\{ x_n \} \subseteq \mathscr{R}(\infty)$. Fix a point p in $\widetilde{M}$. Choose $w_n \in S_p \widetilde{M}$ such that $\gamma_{w_n}(\infty)$ is the centre of $C(x_n)$. Passing to a subsequence, we may assume that $\{ w_n \}$ converges to a vector $w \in S_p \widetilde{M}$. Since all Weyl simplices are isometric, $\measuredangle_p(w_n, \partial\widetilde{\mathscr{C}}(w_n))$ is uniformly bounded away from 0. By Corollary 2.19, w is ℓ-regular and $\widetilde{\mathscr{C}}(w_n) \to \widetilde{\mathscr{C}}(w)$ in the Hausdorff metric. Hence $C(x_n) \to C(y)$ where $y = \gamma_w(\infty)$. ∎

3.4. *Lemma. — If* F *is a regular k-flat, then* $F(\infty)$ *is a union of finitely many Weyl simplices.*

Proof. — If F is a k-flat such that $S_p F$ contains a dense set of regular vectors, then $F(\infty)$ is clearly a finite union of Weyl simplices. Since these k-flats are dense in the space of all regular k-flats by Lemma 2.17, the claim follows from Corollary 2.19. ∎

Now we come to the key lemma of this section. We say that a flat F joins two points $x, y \in M(\infty)$ if $x \in F(\infty)$ and $y \in F(\infty)$.

3.5. *Lemma.* — *Let $v \in S\tilde{M}$ be r-periodic (Definition 2.15). Then any point $y \in \tilde{M}(\infty)$ can be joined to $\gamma_v(-\infty)$ by a regular k-flat F. Moreover, if φ is an axial isometry of γ_v, there is a sequence of integers $n_k \to \infty$ such that $\varphi^{n_k} \mid F(\infty)$ converges to a homeomorphism $\Phi : F(\infty) \to F(v)$ that maps Weyl simplices to Weyl simplices and is the identity on $F(\infty) \cap F(v)(\infty)$.*

Remark. — It is possible for two points in $\tilde{M}(\infty)$ not to be joined by a k-flat. In rank 1 for instance, the Heintze examples [BBE, Introduction] contain 2-flats. If $x \neq y$ are two nonopposite points at infinity of such a 2-flat, then x and y cannot be joined by a geodesic.

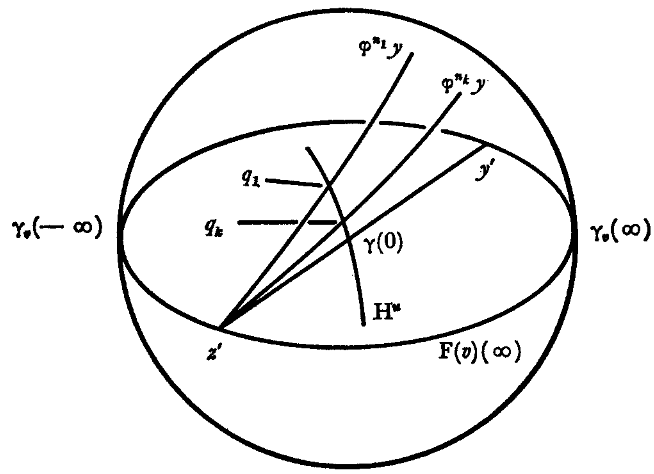

FIG. 2

Proof. — By Corollary 3.4, $F(v)(\infty)$ is the union of finitely many Weyl simplices, which are permuted by φ. After replacing φ by a power of φ if necessary, we may assume that φ fixes each point of $F(v)(\infty)$.

It is clearly sufficient to prove the lemma in the case that y is regular and the center of $C(y)$. By Lemma 1.1 there is an increasing sequence $n_1, n_2, \ldots$ such that $\varphi^{n_k} y \to y' \in F(v)(\infty)$. By Proposition 3.3, y' is regular and is the center of $C(y')$. Let $w = V(\pi v, y')$ and set $z' = \gamma_w(-\infty) \in F(v)(\infty)$. By Proposition 2.23, z' is the center of $C(z')$. Since v is r-periodic and y' is regular, we see from the discussion before Definition 2.15 that w is p-regular. Hence $F(v)$ contains a regular geodesic γ, parallel to γ_w, with $\gamma(-\infty) = z'$ and $\gamma(\infty) = y'$. Let H^u be the strong unstable horosphere of $\dot{\gamma}(0)$.

Consider the continuous injective map $f : H^u \times C(z') \to \tilde{M}(\infty)$ given by $f(p, z'') = \gamma_{V(p, z'')}(-\infty)$. As $H^u \times C(z')$ and $\tilde{M}(\infty)$ have the same dimension, f maps a neighbourhood of $(\gamma(0), z')$ homeomorphically onto a neighbourhood U of y'. Moreover $f(p, z'')$ is the center of its Weyl simplex if and only if $z'' = z'$. Since $\varphi^{n_k} y \to y'$, we can assume that $\varphi^{n_k} y \in U$ for every $k \geqslant 1$. We see that for each k there is a geodesic γ_k joining z' to $\varphi^{n_k} y$ which passes through a point q_k of H^u. As $k \to \infty$, $q_k \to \gamma(0)$ and $\gamma_k \to \gamma$. Since γ is regular, we can assume that every γ_k is regular.

Since φ fixes $F(v)(\infty)$, $\varphi^{-n_k} \circ \gamma_k$ is a regular geodesic joining z' to y for each k. These parallel regular geodesics must all lie in a regular k-flat F. Clearly $y \in F(\infty)$. Also $\{\varphi^{-n_k} q_k\} \subseteq F$. Since $\{q_k\}$ is a bounded sequence in $\widetilde{M}$, φ is an axial isometry for γ_v and $n_k \to \infty$, it is easily shown that $\lim_{k\to\infty} \varphi^{-n_k} w_k = \gamma_v(-\infty)$. Hence $\gamma_v(-\infty) \in F(\infty)$. Finally, since $\varphi^{n_k} F$ contains γ_k and $\{\gamma_k\}$ converges to the regular geodesic γ which lies in $F(v)$, we see that $\varphi^{n_k} F(\infty) \to F(v)(\infty)$. It is clear from Lemma 3.3 that the Weyl simplices of $\varphi^{n_k} F(\infty)$ converge to those of $F(v)(\infty)$. ∎

Before proceeding to construct the Tits building, we extend the argument used in the above proof to join z' to $\varphi^{n_k} y$.

3.6. *Lemma.* — *Let v be a regular vector. There are neighbourhoods* U *of* $\gamma_v(-\infty)$ *and* V *of* $\gamma_v(\infty)$ *and continuous maps* $Q : U \times V \to SM$, $y' : U \times V \to \widetilde{M}(\infty)$ *with* $Q(\gamma_v(-\infty), \gamma_v(\infty)) = v$ *such that for each* $(x, y) \in U \times V$ *the geodesic* $\gamma_{Q(x,y)}$ *joins* x *to* $y'(x, y) \in C(y)$.

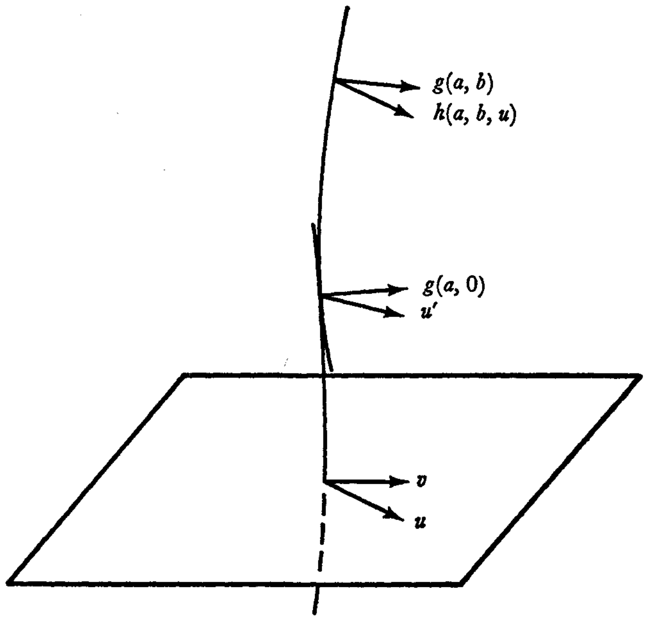

FIG. 3

Proof. — Let B^{n-k} be the unit ball in $\mathbf{R}^{n-k}$. Since the foliations W^s and W^u are transverse near v, there is an injective and continuous map $g : B^{n-k} \times B^{n-k} \to S\widetilde{M}$ such that $g(0, 0) = v$, $g(\cdot, 0)$ is a diffeomorphism onto a neighborhood of v in $W^u(v)$ and $g(a, \cdot)$ is a diffeomorphism onto a neighborhood of $g(a, 0)$ in $W^s(g(a, 0))$ for each $a \in B^{n-k}$. Given $u \in \widetilde{\mathscr{C}}(v)$, we let $h(a, b, u)$ be the vector at $\pi(g(a, b))$ asymptotic to the vector u' at $\pi(g(a, 0))$ that is negatively asymptotic with u. Note that h maps

$\{a\} \times \{b\} \times \widetilde{\mathscr{C}}(v)$ isometrically onto $\widetilde{\mathscr{C}}(g(a, b))$. Also h is continuous and injective. Define

$$\mathrm{H} : \mathrm{B}^{n-k} \times \mathrm{B}^{n-k} \times \widetilde{\mathscr{C}}(v) \times \widetilde{\mathscr{C}}(v) \to \widetilde{\mathrm{M}}(\infty) \times \widetilde{\mathrm{M}}(\infty)$$

by $$\mathrm{H}(a, b, u, w) = (\gamma_{h(a,b,u)}(-\infty), \gamma_{h(a,b,w)}(\infty)).$$

Note that H maps $\{a\} \times \{b\} \times \widetilde{\mathscr{C}}(v) \times \widetilde{\mathscr{C}}(v)$ onto $\mathrm{C}(\gamma_{h(a,b,v)}(-\infty)) \times \mathrm{C}(\gamma_{h(a,b,v)}(\infty))$. Also H is continuous and injective. By invariance of domain, the image of H is a neighbourhood of $(\gamma_v(-\infty), \gamma_v(\infty))$ in $\widetilde{\mathrm{M}}(\infty) \times \widetilde{\mathrm{M}}(\infty)$. Let $\mathrm{U}, \mathrm{V} \subseteq \mathscr{R}(\infty)$ be neighbourhoods of $\gamma_v(-\infty)$ and $\gamma_v(\infty)$ respectively such that $\mathrm{U} \times \mathrm{V} \subseteq \operatorname{im} \mathrm{H}$. Given $(x, y) \in \mathrm{U} \times \mathrm{V}$, let $(a, b, u, w) = \mathrm{H}^{-1}(x, y)$. Then the geodesic $\gamma_{h(a,b,u)}$ joins x to $y' = \gamma_{h(a,b,u)}(\infty)$. Since $y = \gamma_{h(a,b,w)}(\infty)$, we see that $y' \in \mathrm{C}(y)$ because $\mathrm{C}(y) = \mathrm{C}(y') = \mathrm{C}(\gamma_{h(a,b,v)}(\infty))$. It is clear that $\gamma_{h(a,b,u)}$ varies continuously with x and y. We define $\mathrm{Q}(x, y) = h(a, b, u)$. ∎

Consider a regular k-flat F. Let $\Sigma = \Sigma_{\mathrm{F}}$ be the set of Weyl simplices in $\mathrm{F}(\infty)$ and all their intersections. For $\mathrm{A}, \mathrm{B} \in \Sigma$ we say that A is a *face* of B if $\mathrm{A} \subset \mathrm{B}$. Our first goal is to show that Σ with this order relation is a Coxeter complex.

We identify $\mathrm{F}(\infty)$ with $\mathrm{S}_p\mathrm{F}$ for some point $p \in \mathrm{F}$. This gives $\mathrm{F}(\infty)$ the geometric structure of the unit sphere in k-dimensional Euclidean space. Clearly this structure is independent of the choice of p.

Let $\mathrm{C}, \mathrm{C}' \in \Sigma$. Then $\mathrm{C} \cap \mathrm{C}'$ is convex. Hence we can speak of the *codimension* of $\mathrm{C} \cap \mathrm{C}'$, $\operatorname{codim} \mathrm{C} \cap \mathrm{C}'$, in $\mathrm{F}(\infty)$. If $\mathrm{C} \neq \mathrm{C}'$ then $\operatorname{codim} \mathrm{C} \cap \mathrm{C}' \geqslant 1$ since $\mathring{\mathrm{C}} \cap \mathrm{C}' = \emptyset$. If $\operatorname{codim} \mathrm{C} \cap \mathrm{C}' = 1$ then $\mathrm{C} \cap \mathrm{C}'$ lies in the set H of points at infinity of a unique hyperplane in F. We call such a set H a *hypersphere*. Denote by $\mathscr{H}$ the collection of hyperspheres in $\mathrm{F}(\infty)$. Let W be the group generated by the orthogonal reflections in the hyperspheres $\mathrm{H} \in \mathscr{H}$.

3.7. *Lemma.* — (i) *Let* $\mathrm{C}, \mathrm{D} \in \Sigma$. *If* $\operatorname{codim} \mathrm{C} \cap \mathrm{D} = 1$ *and if* w *is the reflection in the hypersphere* H *spanned by* $\mathrm{C} \cap \mathrm{D}$, *then* $w(\mathrm{D}) = \mathrm{C}$.

(ii) *If* $\mathrm{C}, \mathrm{D} \in \Sigma$ *are Weyl simplices, then there is a sequence of Weyl simplices* $\mathrm{C} = \mathrm{C}_0, \mathrm{C}_1, \ldots, \mathrm{C}_m = \mathrm{D}$ *in* Σ *such that* $\operatorname{codim} \mathrm{C}_{i-1} \cap \mathrm{C}_i = 1$ *for* $i = 1, \ldots, m$.

(iii) *If* $w \in \mathrm{W}$ *then* $w(\mathscr{H}) \subset \mathscr{H}$.

Proof. — (i) Because the r-periodic vectors are dense, we may assume by the continuity of Weyl simplices and flats that F contains an r-periodic vector v with $\gamma_v(\infty) \in \mathrm{C}$. Let φ be an axial isometry of γ_v.

By the definition of Weyl simplices, there is a regular k-flat F′ such that $\mathrm{C} \subset \mathrm{F}'(\infty)$ but $\mathring{\mathrm{D}} \cap \mathrm{F}'(\infty) = \emptyset$. By Lemma 3.5 there is a sequence of integers $n_k \to \infty$ such that $\varphi^{n_k} \mid \mathrm{F}'(\infty)$ converges to a homeomorphism $\Phi : \mathrm{F}'(\infty) \to \mathrm{F}(\infty)$ that maps Weyl simplices to Weyl simplices. By Corollary 3.4 there is a Weyl simplex $\mathrm{D}' \subset \mathrm{F}'(\infty)$ such that $\Phi(\mathrm{D}') = \mathrm{D}$. Clearly $\mathrm{D}' \neq \mathrm{C}$ and $\mathrm{D}' \cap \mathrm{C} = \mathrm{D} \cap \mathrm{C}$. As $n \to \infty$, $\varphi^{-n}(\mathrm{D}')$ subconverges to a simplex in $F(\infty)$. Clearly any such limit contains $D \cap C$. Since $\varphi^{n_k} D'$ converges to D it is clear from Lemma 1.1 that $\lim_{n\to\infty} \varphi^{-n} D' = C$. Since all the $\varphi^{-n} D'$ are isometric there is a sequence $m_k \to \infty$ such that $\varphi^{-m_k} D'$ converges to an isometry $\Psi : D' \to C$. Also $\Phi : D' \to D$ is an isometry. Thus the map $\Psi \circ \Phi^{-1} : D \to C$ is an isometry that leaves $C \cap D$ pointwise fixed. Hence $\Psi \circ \Phi^{-1} = w \mid D$. Hence $w(D) = C$.

(ii) Since Σ is finite the set

$$X = \mathrm{U}\{ E \cap E' : E, E' \in \Sigma, \operatorname{codim} E \cap E' \geqslant 2 \}$$

has codimension at least 2 in $F(\infty)$. Since the simplices in Σ cover $F(\infty)$, there is a path $\gamma \subset F(\infty)$ that starts in C, ends in D and does not intersect x. Furthermore we may assume that γ consists of great circle arcs which have only transverse intersections with the hyperspheres $H \in \mathscr{H}$. Clearly the sequence of Weyl simplices that γ intersects satisfies the claim of (ii).

(iii) It suffices to prove this for a reflection w in a hypersphere $H \in \mathscr{H}$. Suppose H is spanned by $C \cap C'$ for some Weyl simplices C, C' in Σ. Let D be a Weyl simplex in $F(\infty)$ and let $C_0 = C, C_1, \ldots, C_m = D$ be a sequence of Weyl simplices as in (ii). By (i) we know that $w(C_0) \in \Sigma$. Suppose that $w(C_i) \in \Sigma$ for $i = 0, \ldots, j-1$. Let w'' be the reflection in the hypersphere H'' spanned by $C_{j-1} \cap C_j$. Clearly $w(H'')$ intersects $\partial w(C_{j-1})$ in a set of codimension 1 in $F(\infty)$. Since $w(C_{j-1}) \in \Sigma$ it is clear that $w(H'') \in \mathscr{H}$. Let $w' \in W$ be the reflection in $w(H'')$. By (i) we have $w''(C_j) = C_{j-1}$ and $w'(w(C_{j-1})) \in \Sigma$. As $w = w' w w''$ we obtain

$$w(C_j) = w' w w''(C_j) = w'(w(C_{j-1})) \in \Sigma.$$

By the obvious induction, we have $w(D) \in \Sigma$. Since any $H \in \mathscr{H}$ is spanned by the intersection of two simplices in Σ, the claim of (iii) follows. ∎

We refer to [T, chapter 2] or [Bou] for the definition and properties of Coxeter complexes.

3.8. *Theorem.* — *The ordered set* $(\Sigma, \subset)$ *is a Coxeter complex. In particular, the faces of a Weyl chamber form a simplex when ordered by inclusion. Moreover,* Σ *is the geometric realization of the Coxeter complex in the* $(k-1)$*-sphere* $F(\infty)$.

Proof. — Since W permutes the Weyl simplices, W is finite. There is no common fixed point of W in $F(\infty)$. In fact suppose x is a common fixed point. Then $x \in \bigcap_{H \in \mathscr{H}} H$ and hence the point opposite to x in $F(\infty)$ is also in $\bigcap_{H \in \mathscr{H}} H$. Then the diameter of any Weyl simplex in Σ is π. This is impossible by [BBS, Lemma 1.6].

Fix a point $p \in F$. For $H \in \mathscr{H}$ let $\hat{H} \subset F$ be the hyperplane passing through p with $\hat{H}(\infty) = H$. By the above, the family of hyperplanes H, $H \in \mathscr{H}$, satisfies the conditions (D1) and (D2) of [Bou, V, §3]. Let $\hat{\Sigma}$ be the set of all cones $\hat{C} \subset F$ based at p with $\hat{C}(\infty) = C$ for some $C \in \Sigma$. Order $\hat{\Sigma}$ by inclusion. Clearly $(\hat{\Sigma}, \subset)$ and $(\Sigma, \subset)$ are isomorphic. By [Bou, V, 3.9, Proposition 7] the cone $\hat{C}$ for $C \in \Sigma$ is a simplicial cone. By [Bou, V, 1.6], $\hat{C}$ with all its faces is a simplex. Hence $(\Sigma, \subset)$ is a complex. By Lemma 3.7 (ii), Σ is a chamber complex. Clearly Σ is thin. Since W maps Weyl simplices to Weyl simplices by Lemma 3.7 (iii), Lemma 3.7 (i) shows that Σ is a Coxeter complex.

That Σ is the geometric realization also follows from this construction. ■

Now we will introduce the Tits building on the sphere at infinity of M. We refer to [T, chapter 3] for the definition and basic properties of buildings.

3.9. *Definition.* — Let Δ be the set consisting of the Weyl simplices at infinity and all their intersections. If $A, B \in \Delta$ we say that A is a *face* of B if $A \subset B$.

By Theorem 3.8, $(\Delta, \subset)$ is a complex.

3.10. *Definition.* — A subcomplex Σ of Δ that is isomorphic to a complex Σ_F for some regular k-flat F is called an *apartment* if the union of the Weyl simplices in Σ is homeomorphic to a $(k-1)$-sphere. The collection of all apartments is denoted by $\mathscr{A}$.

Remark. — Since the set of apartments of a spherical building is unique [T 3.1, 3.26] it is not too crucial exactly how we define an apartment.

3.11. *Theorem.* — *The pair* $(\Delta, \mathscr{A})$ *is a spherical Tits building.*

Proof. — We check the axioms B1-B4 for Tits buildings (cf. Introduction).

(B1) Let $B \in \Delta$ have codimension 1. Then B is a wall of a Weyl simplex C. Let γ_v be a regular geodesic with $\gamma_v(\infty) \in \mathring{C}$ and let $F = F(v)$. Let D_1 be the Weyl simplex in Σ_F that contains B and is adjacent to C. By the definition of Weyl simplices there is a regular geodesic γ_w such that $\gamma_w(\infty) \in \mathring{C}$ and $F(w)(\infty) \cap \mathring{D}_1 = \emptyset$. Let D_2 be the Weyl simplex in $F(w)(\infty)$ adjacent to C and containing B. Now $B \subset C, D_1, D_2$ and hence Δ is thick.

(B2) This axiom follows from Theorem 3.8.

(B3) For $p \in \widetilde{M}$ let σ_p be the geodesic symmetry about p. Since $-\mathscr{C}(v) = \mathscr{C}(-v)$ for any ℓ-regular vector v by Proposition 2.23 it is clear σ_p induces an automorphism of the complex Δ.

Let D_1, D_2 be two Weyl simplices and let γ be a regular geodesic with $\gamma(\infty) \in \mathring{D}_1$. By Lemma 3.6 there is a neighborhood U of $\gamma(-\infty)$ such that any $x \in U$ is joined to a point in $\mathring{D}_1$ by a regular geodesic. By Lemma 3.1 there is an r-periodic point $x \in U$. Let $p \in \widetilde{M}$ be a point on a geodesic joining x to a point in $\mathring{D}_1$. Since $x \in \sigma_p D_1$, Lemma 3.5 shows that there is a regular k-flat F joining $\sigma_p D_1$ to $\sigma_p D_2$. Hence D_1 and D_2 belong to the apartment $\sigma_p \Sigma_F$. Since any two elements of Δ are contained in Weyl simplices, axiom (B3) is proved.

(B4) Let Σ and Σ' be two apartments such that $\Sigma \cap \Sigma'$ contains two elements A and A′ of Δ. We consider first the case where A is a Weyl simplex. After replacing Σ and Σ' by their images under a geodesic symmetry (as in the proof of (B3)), we may assume that the Weyl simplex A contains an r-periodic point x. Let v be an r-periodic vector with $x = \gamma_v(-\infty)$, and φ an axial isometry for γ_v with $x = \lim_{n\to\infty} \varphi^{-n} \pi v$. Choose a sequence of integers $n_k \to \infty$ such that, for any Weyl simplex $C \in \Sigma \cup \Sigma'$, $\varphi^{n_k} C$ converges to a Weyl simplex in $F(v)(\infty)$. This is possible by Lemma 3.5 since $\Sigma \cup \Sigma'$ contains only finitely many simplices. Note that $\varphi^{n_k} \mid \Sigma \cup \Sigma'$ converges to a continuous map $\Phi : \Sigma \cup \Sigma' \to F(v)(\infty)$. Clearly $\mathring{C} \cap \Phi(\Sigma \setminus \mathring{C}) = \emptyset$. Since $\Phi \mid C$ is the identity, $\Phi \mid \Sigma \setminus \mathring{C}$ is a map between $k-1$ dimensional discs that fixes their common boundary. Suppose $p \in F(v)(\infty) \setminus C$ is not in $\Phi(\Sigma \setminus C)$. Let P be the projection along rays emanating from p of $F(v)(\infty) \setminus C$ to ∂C. Then $P \circ \Phi$ is a retract of $\Sigma \setminus C$ onto ∂C. By [Sp, Corollary 4.7.4] this is impossible. Hence $\Phi \mid \Sigma$ and similarly $\Phi \mid \Sigma'$ are surjective. Since the number of Weyl simplices in Σ, Σ' and $\Sigma_{F(v)}$ are the same and Φ is a morphism, $\Phi \mid \Sigma$ and $\Phi \mid \Sigma'$ are isomorphisms. Hence $(\Phi \mid \Sigma')^{-1} \circ (\Phi \mid \Sigma)$ is an isomorphism from Σ to Σ' that fixes $\Sigma \cap \Sigma'$ pointwise.

Now we consider the general case. Choose Weyl simplices C and C' such that $A \subseteq C \in \Sigma$ and $A' \subseteq C' \in \Sigma'$. After replacing Σ and Σ' by their images under a geodesic symmetry, we can assume that $\mathring{C}$ contains an r-periodic point. By Lemma 3.5, there is a regular k-flat F such that $C, C' \subseteq F(\infty)$. We have seen above that there are isomorphisms $\Phi : \Sigma \to \Sigma_F$ and $\Phi' : \Sigma' \to \Sigma_F$ that fix $C \cup A'$ and $A \cup C'$ respectively. Hence $\Phi'^{-1} \circ \Phi : \Sigma \to \Sigma'$ is an isomorphism that fixes A and A' and all their faces. This proves (B4).

Finally notice that Δ is spherical since there are only finitely many Weyl simplices in an apartment. ∎

Note that the chambers of Δ are the Weyl simplices. We will use the two names interchangeably from now on.

Finally we topologize Δ. We refer to [BS, Section 1] for the definition and basic properties of topological buildings. The set Δ_0 of vertices of Δ is a subset of $\widetilde{M}(\infty)$. Give Δ_0 the induced topology. By Proposition 3.3 the space of chambers of Δ, Cham Δ, is closed in Δ_0^k. Hence the set Δ_i of faces of dimension i is closed in Δ_0^i. Therefore Δ is a topological building.

3.12. *Proposition. — The topological building Δ is compact, metric and locally connected.*

Proof. — Clearly Δ is metric. By Proposition 3.3, Δ is compact. To show that Δ is locally connected, let $C \in$ Cham Δ. Choose $v \in \mathscr{L}$ with $\gamma_v(\infty) \in \mathring{C}$. By Theorem 2.18, v is rigid. Let $U \subseteq \mathscr{L}$ be a connected open neighborhood of v satisfying the properties of Definition 2.4. Set $V = \{\gamma_u(\infty) : u \in U\}$. Clearly V is a connected neighborhood of $\mathring{C}$ in $\widetilde{M}(\infty)$ and V is a union of interiors of Weyl simplices. Hence Cham Δ and thus Δ are locally connected. ∎

3.13. *Lemma. — Let γ be a regular geodesic contained in the k-flat* F. *Set $x = \gamma(\infty)$ and $a_t = \sigma_{\gamma(t)} \circ \sigma_{\gamma(0)}$. If $y \in \widetilde{M}(\infty)$ then $\measuredangle_{\gamma(0)}(a_t y, x) \leqslant \measuredangle_{\gamma(0)}(y, x)$ for $t \geqslant 0$. For t suf-*

ficiently big, equality holds if and only if $y \in F(\infty)$. *Moreover any limit point of* $a_t y$ *as* $t \to \infty$ *lies in* $F(\infty)$.

Proof. — Clearly

$$\measuredangle_{\gamma(0)}(a_t y, x) \leqslant \measuredangle_{\gamma(t)}(a_t y, x) = \pi - \measuredangle_{\gamma(t)}(\sigma_{\gamma(0)} y, x)$$
$$\leqslant \pi - \measuredangle_{\gamma(0)}(\sigma_{\gamma(0)} y, x) = \measuredangle_{\gamma(0)}(y, x).$$

If equality holds then $\gamma(t)$, $\gamma(0)$ and $a_t(y)$ span a flat half strip. Suppose t is so large that any Jacobi field along γ that is parallel between $\gamma(0)$ and $\gamma(t)$ is parallel from $\gamma(\infty)$ to $\gamma(-\infty)$. Then $a_t(y)$ and hence y lie in $F(\infty)$. The remaining claim follows as in Lemma 1.1. ∎

Let G be the topological automorphism group of Δ [BS, Section 1]. Clearly the actions on $\tilde{M}(\infty)$ of geodesic symmetries and covering transformations are elements of G. If $C \in \Delta$ let G_C be the stabiliser of C in G. Let Opp C be the set of elements of Δ opposite C. Clearly G_C acts on Opp C. Note that Opp C is open in Cham Δ [BS, 1.9].

3.14. *Lemma.* — *If* $C \in$ Cham Δ *then* G_C *acts transitively on* Opp C.

Proof. — As in the proof of (B3) in Theorem 3.11, we may assume that C contains an r-periodic point x. Let $D \in$ Opp C. We first show that $G_C.D$ contains a neighborhood of D. By Lemma 3.5 there is a regular geodesic γ that joins x to a point $y \in D$. By Lemma 3.6 there is a ball $\mathscr{B}$ about D in Opp C such that: for all $B \in \mathscr{B}$ we can pick a geodesic γ_B depending continuously on B and joining x to a point $y(B)$ in B so that .(D) $= y$. By Proposition 2.23, $\measuredangle_{\gamma_B(0)}(y(B), \partial B) \equiv \measuredangle(x, \partial C) > 0$. We may assume that $\mathscr{B}$ is small enough so that for all $B \in \mathscr{B}$,

$$(*) \qquad \measuredangle_{\gamma_B(0)}(y(B), y) \leqslant \measuredangle_{\gamma_B(0)}(y(B), \partial B).$$

Set $a_t^B = \sigma_{\gamma_B(t)} \circ \sigma_{\gamma_B(0)}$. Then $a_t^B \in G_C$ depends continuously on B and t. It is clear from Lemma 3.13 that we can choose a ball $\mathscr{B}'$ with $D \in \mathscr{B}' \subseteq \mathscr{B}$ such that $a_t^B(D) \in \mathscr{B}$ for all $(B, t) \in \mathscr{B}' \times [0, \infty)$. Let $S = \partial\mathscr{B}'$. By Lemma 3.13 and (*), $a_t^B(D) \to B$ as $t \to \infty y$ Moreover $\measuredangle_{\gamma_B(0)}(a_t^B y, y(B)) \to 0$ monotonically for each B. Hence $a_t^B(D) \to B$ as $t \to \infty$ uniformly for $B \in S$. We get a continuous map $\bar{a} : S \times [0, \infty] \to \mathscr{B}$ such that $\bar{a}(B, \infty) = B$ for all $B \in S$ and $\bar{a}(S \times [0, \infty)) \subseteq G_C.D$. As $a_0^B(D) = D$ for all $B \in S$, $\bar{a}$ gives rise to a continuous map $\bar{\bar{a}} : \mathscr{B}' \to \mathscr{B}$ with $\bar{\bar{a}}(B) = B$ for each $B \in S$ and $\bar{\bar{a}}(\mathring{\mathscr{B}}') \subseteq G_C.D$. Since S is not a retract of $\mathscr{B}'$, we see that $\bar{\bar{a}}(\mathscr{B}') \supseteq \mathscr{B}'$ (cf. the proof of (B4) in Theorem 3.11). It follows that $\mathring{\mathscr{B}}' \subseteq G_C.D$.

Finally we show that G_C is transitive on Opp C. Let v be an r-periodic vector with $\gamma_v(\infty) = x$. Set $E = C(\gamma_v(-\infty))$. Let φ be an axial isometry of γ_v. If $D \in$ Opp C, then $\varphi^{-n} D \to E$ as $n \to \infty$. Since $G_C.E$ contains a neighbourhood of E, we see that $D \in G_C.E$. ∎

3.15. *Proposition.* — *The building* Δ *is topologically Moufang.*

Proof. — Let Σ and Σ' be two apartments in Δ that intersect in a halfapartment A. According to Definition 3.1 of [BS] we have to find $g \in G$ such that $g(\Sigma') = \Sigma$ and g restricts to the identity on A. Let C be a chamber in A. Let D and E be the chambers opposite to C in Σ and Σ' respectively. By Lemma 3.14 there is $g \in G_C$ such that $g(E) = D$. Hence $g(\Sigma') = \Sigma$. Clearly $g|_A = \mathrm{id}_A$. ■

4. Irreducibility

We prove the following criterion for reducibility of the building Δ attached to M in the last section.

4.1. *Theorem.* — *The building* Δ *is reducible if and only if* $\widetilde{M}$ *is reducible.*

Proof. — Since M is simply connected, it is reducible if and only if it is a Riemannian product of two factors of positive dimension. Clearly Δ is reducible if $\widetilde{M}$ is.

Suppose that Δ is reducible. This means that Δ is the join of two Tits buildings Δ_1 and Δ_2. Any vertex of Δ is either a vertex of Δ_1 or of Δ_2. We say that a vertex of Δ is of the *first* or *second kind* if it belongs to Δ_1 or Δ_2 respectively.

4.2. *Lemma.* — *If* $x, y \in \widetilde{M}(\infty)$ *are vertices of* Δ *of different kinds, then* $\measuredangle_q(x,y) = \pi/2$ *for every* $q \in \widetilde{M}$.

Proof. — Let Σ be an apartment containing both x and y. Then Σ is the join of apartments Σ_1 and Σ_2 in Δ_1 and Δ_2 respectively. Since x, y are of different kinds, it is clear that x, y lie in a common chamber C of Δ. By Lemma 2.7 and Theorem 2.18, $\measuredangle_p(x,y)$ is independent of $p \in \widetilde{M}$. Consider a point p in a regular k-flat F such that $F(\infty) \supset C$. Since $F(\infty)$ carries the geometric realization of the Coxeter complex Σ in which Σ_1 and Σ_2 are orthogonal we have that $\measuredangle_p(x,y) = \pi/2$. ■

Now we construct two distributions on $\widetilde{M}$ which will give rise to the desired splitting as a product. For $p \in \widetilde{M}$, $i = 1, 2$, let $V_i(p)$ be the subspace of $T_p\widetilde{M}$ spanned by $D_i(p) = \{v \in S_p\widetilde{M} : v$ points to a vertex of $\Delta(M)$ of the i-th kind$\}$. Then V_1 and V_2 are orthogonal by Lemma 4.2. They span $T\widetilde{M}$ because any vector in $S\widetilde{M}$ lies in a Weyl simplex whose vertices are all in $D_1 \cup D_2$. Clearly $D_i(p)$ varies continuously with p and hence $\dim V_i(p)$ is lower semicontinuous ($i = 1, 2$). Since V_1 and V_2 are complementary, we see that $\dim V_i$ is constant and V_i is a continuous distribution for $i = 1, 2$.

We will say that a C^1 curve $\sigma(s)$ is an *integral curve* of the distribution V_i if $\dot\sigma(s) \in V_i(\sigma(s))$ for all s.

4.3. *Lemma.* — *Let* σ *be an integral curve of* V_1. *Let* $x \in \widetilde{M}(\infty)$ *be a vertex of* $\Delta(M)$ *of the second kind, and let* f *be the Busemann function of a vector pointing toward* x [BBE, p. 179]. *Then* $f \circ \sigma$ *is constant.*

Proof. — For any $q \in \tilde{M}$, $\operatorname{grad}_q f = -V(q, x) \in D_2(q)$ and so is orthogonal to $V_1(q)$. ■

Now consider a point $p \in \tilde{M}$. If $v \in D_2(p)$, then $-v \in D_2(p)$, since $\gamma_{-v}(\infty)$ is a vertex of $\Delta(M)$ by Proposition 2.23 and cannot be of the first kind by Lemma 4.2. It follows from Lemma 4.3 that if σ is an integral curve of V_1 starting at p, then σ lies in

$$S_1(p) = \bigcap_{v \in D_2(p)} H(v) \cap H(-v),$$

where $H(v)$ is the horosphere defined in [BBE, §1]. Conversely, any C^1 curve σ in $S_1(p)$ is an integral curve of V_1. For it is clear that $S_1(q) = S_1(p)$ for any $q \in S_1(p)$. Hence for any s, $\dot\sigma(s)$ is orthogonal to $D_2(\sigma(s))$, and so $\dot\sigma(s) \in V_1(\sigma(s))$.

Thus V_1 is integrable and $S_1(p)$ is its integral submanifold through p. Similarly V_2 is integrable and its integral submanifold through p is

$$S_2(p) = \bigcap_{v \in D_1(p)} H(v) \cap H(-v).$$

4.4. *Lemma.* — *For* $p \in \tilde{M}$, $i = 1, 2$, $S_i(p)$ *is totally geodesic and* $S_i(p) = \exp_p V_i(p)$.

Proof. — It is clear from the Flat Strip Theorem [EO, Proposition 5.1] that $S_i(p) \subseteq \exp_p V_i(p)$.

As $S_i(p)$ is an integral submanifold of V_i, we see that $S_i(p)$ is open in $\exp_p V_i(p)$. It follows that $S_i(p) = \exp_p V_i(p)$, since $S_i(p)$ is obviously closed. Busemann functions are convex, and so it follows that $S_i(p)$ is convex and hence totally geodesic. ■

It follows immediately that each of the distributions V_i is parallel along its own integral curves. Since V_1 and V_2 are orthogonal complements, we also see that each of them is parallel along the integral curves of the other. It follows that V_1 and V_2 are both parallel, and so, by a theorem of de Rham [KN, p. 187], $\tilde{M}$ splits as a Riemannian product. This completes the proof of Theorem 4.1. ■

5. Classification

We adapt the arguments of Gromov's Rigidity Theorem [BGS, Chapter 4] to prove

5.1. *Theorem.* — *If* M *is a complete Riemannian manifold with nonpositive bounded curvature, finite volume and rank at least* 2 *whose universal cover is irreducible, then* M *is locally symmetric.*

The Main Theorem of the Introduction follows using Proposition 4.1 of [E2].

Proof. — Let Δ be the building attached to M as in Section 3. By Propositions 3.12 and 3.15 and Theorem 4.1, Δ is an infinite, irreducible, locally connected, compact, metric, topologically Moufang building of rank at least 2. Let G be the topological automorphism group of Δ and G^0 the connected component of the identity in G. By

[BS, Main Theorem], G^0 is a simple noncompact real Lie group without center. Let $\Delta(G^0)$ be the topological building of parabolic subgroups attached to G^0 [BS, 1.2]. By [BS, Main Theorem], Δ is isomorphic with $\Delta(G^0)$ as a topological building. We will identify Δ with $\Delta(G^0)$. Let $X = G^0/K$ be the symmetric space attached to G^0, where K is a maximal compact subgroup of G^0. As in Section 3, $X(\infty)$ carries the structure of a topological building which we can identify with $\Delta(G^0) = \Delta$. Clearly rank $X =$ rank $\Delta =$ rank M.

Call a k-flat F in $\tilde{M}$ *ℓ-regular* if $F(\infty)$ contains a Weyl simplex C. As in Corollary 3.4 it follows that $F(\infty)$ is the union of finitely many Weyl simplices. Hence $F(\infty)$ determines an apartment Σ_F in Δ.

For the symmetric space X the correspondence $F^* \to \Sigma_{F^*}$ between k-flats F^* in X and apartments in Δ is bijective. Given a regular k-flat F in $\tilde{M}$ we let F^* be the unique k-flat in X with $\Sigma_F = \Sigma_{F^*}$. Next we define a map $\Phi : \tilde{M} \to X$. Let $p \in \tilde{M}$. Then the geodesic symmetry σ_p defines a continuous automorphism of Δ. By [M, 16.2], σ_p determines an involutory isomorphism Θ_p of G^0. As σ_p is continuous, Θ_p is analytic. Thus Θ_p induces an isometry $\theta_p : X \to X$. Since θ_p has order 2 it has a fixed point p^* in X. Suppose q^* is a second fixed point. Then θ_p fixes the geodesic through p^* and q^*. Hence σ_p has fixed points in Δ which is impossible. Therefore θ_p has a unique fixed point p^*. Set $\Phi(p) = p^*$.

5.2. *Lemma.*

(i) *The map* $\Phi : \tilde{M} \to X$ *is continuous.*

(ii) *If* $F \subset \tilde{M}$ *is an ℓ-regular k-flat, then* $\Phi(F) \subset F^*$.

Proof.

(i) Since $\sigma_p \in G$ depends continuously on $p \in \tilde{M}$, it is clear that $\Phi : \tilde{M} \to X$ is continuous.

(ii) Let $p \in F$. Then $\sigma_p F = F$, hence $\sigma_p \Sigma_F = \Sigma_F$. Therefore $\theta_p F^* = F^*$. Since F^* is totally geodesic, F^* contains the fixed point $\Phi(p)$ of θ_p. ■

Call a geodesic γ *maximally singular* if $\gamma(\infty)$ is a vertex of Δ and call a vector v *maximally singular* if $\gamma_v(\infty)$ is a vertex of Δ. Suppose γ is a maximally singular geodesic. Let C_1 and C_2 be two opposite chambers in Star $\gamma(\infty)$. Then $C_1 \cap C_2 = \{\gamma(\infty)\}$. Let F_i be the ℓ-regular k-flat through $\gamma(0)$ and C_i. Then $F_1 \cap F_2 = \gamma$. By Lemma 5.2, $\Phi(\gamma) \subset F_1^* \cap F_2^*$. Since $F_1^*(\infty) \cap F_2^*(\infty) = \{\gamma(\infty), \gamma(-\infty)\}$, $F_1^* \cap F_2^*$ is a maximally singular geodesic in X which we call γ^*. If γ_1 and γ_2 are two parallel maximally singular geodesics, they have the same endpoints and hence γ_1^* is parallel to γ_2^*. Moreover, if δ_1 and δ_2 are any two maximally singular geodesics, the families of geodesics parallel to δ_1^* and δ_2^* make the same angle as do those parallel to δ_1 and δ_2.

5.3. *Lemma.* — *If γ is a maximally singular geodesic then $\Phi \mid \gamma : \gamma \to \gamma^*$ is affine.*

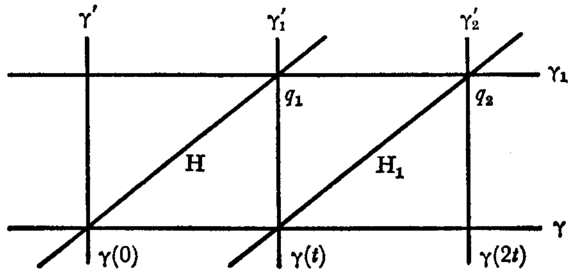

FIG. 4

Proof. — Let F be an ℓ-regular k-flat containing γ. Let $\widetilde{C}$ be a cone in F based at $\gamma(0)$ such that $\widetilde{C}(\infty)$ is a Weyl simplex with $\gamma(\infty)$ as a vertex. Let H be the hyperplane spanned by $\gamma(0)$ and the vertices of $\widetilde{C}(\infty)$ other than $\gamma(\infty)$. Let γ' be the mirror image of γ with respect to H. Then γ' is also maximally singular. Since Δ is irreducible, $\dot\gamma(0)$ and $\dot\gamma'(0)$ are linearly independent. Also γ and γ' are both transversal to H.

Since Φ is continuous, Φ is affine on γ if, for all $n \in \mathbf{N}$ and $t \in \mathbf{R}$,

$$d_X[\Phi(\gamma(nt)), \Phi(\gamma(0))] = n\, d_X[\Phi(\gamma(t)), \Phi(\gamma(0))],$$

where d_X denotes the distance in X. Let H_1 be the hyperplane through $\gamma(t)$ parallel to H. Since the hyperplane H is the span of maximally singular vectors, $\Phi(H)$ is parallel to $\Phi(H_1)$. Let γ_1' and γ_2' be the geodesics parallel to γ' starting at $\gamma(t)$ and $\gamma(2t)$. Set $q_1 = \gamma_1' \cap H$ and $q_2 = \gamma_2' \cap H_1$. Let γ_1 be the geodesic parallel to γ that starts at q_1. Then γ_1 and γ_2' intersect at q_2. As $\Phi(\gamma_1)$ and $\Phi(\gamma_2)$ as well as $\Phi(\gamma')$, $\Phi(\gamma_1')$ and $\Phi(\gamma_2')$ are parallel, we see that

$$d_X[\Phi(\gamma(0)), \Phi(\gamma(t))] = d_X[\Phi(q_1), \Phi(q_2)] = d_X[\Phi(\gamma(t)), \Phi(\gamma(2t))].$$

Hence $$d_X[\Phi(\gamma(2t)), \Phi(\gamma(0))] = 2\, d_X[\Phi(\gamma(t)), \Phi(\gamma(0))].$$

The claim for general n follows similarly. ∎

Now consider an ℓ-regular k-flat F. We will show that $\Phi : F \to F^*$ is affine. Our proof is virtually the same as in [BGS]. Fix a point $p \in F$ and identify F with $\mathbf{R}^k$ so that p is the origin. Let C be a Weyl simplex in $F(\infty)$, and let $\gamma_1, \ldots, \gamma_k$ be the maximally singular geodesics starting at p for which $\gamma_1(\infty), \ldots, \gamma_k(\infty)$ are the vertices of C. Then $\dot\gamma_1(0), \ldots, \dot\gamma_k(0)$ are linearly independent. Every point $q \in F$ can be written uniquely as

$$q = \gamma_1(t_1) + \ldots + \gamma_k(t_k)$$

where $t_i \in \mathbf{R}$. Identify F^* with $\mathbf{R}^k$ using $\Phi(p)$ as the origin. Since the map $\delta \to \delta^*$ on the set of maximally singular geodesics in F preserves parallelism and Φ maps δ into δ^*, we have

$$\Phi(q) = \Phi[\gamma_1(t_1)] + \ldots + \Phi[\gamma_k(t_k)].$$

It follows from Lemma 5.3 that $\Phi \mid F$ is affine.

Since every geodesic of M lies in an ℓ-regular flat, we see that, for each geodesic γ of $\widetilde{M}$, there is a constant $\lambda(\gamma)$ such that

$$d_X[\Phi(q_1), \Phi(q_2)] = \lambda(\gamma)\, d(q_1, q_2)$$

for any points q_1 and q_2 on γ. We will show that $\lambda(\gamma)$ is independent of γ.

5.4. *Lemma.* — *Let p be a point of an ℓ-regular k-flat* F. *Then $\lambda(\gamma)$ is the same for all geodesics γ with $\dot\gamma(0) \in S_p F$.*

Proof. — Since Φ is affine and $S_p F$ is spanned by maximally singular vectors, we can assume that γ is maximally singular. If v and v' are vertices of a Weyl simplex in $S_p F$, then $\measuredangle_p(v, v') \leqslant \pi/2$. Moreover this inequality is strict if v' and v are adjacent in the Coxeter diagram for Δ. Since Δ is irreducible, any maximally singular vectors $w, w' \in S_p F$ can be connected by a finite chain of maximally singular vectors $w_1 = w, w_2, \ldots, w_m = w'$ such that $\measuredangle_p(w_i, w_{i+1}) < \pi/2$ for $i = 1, 2, \ldots, m-1$.

Thus it will suffice to prove that if $v, w \in S_p F$ are maximally singular vectors with $\measuredangle_p(v, w) < \pi/2$, then $\lambda(\gamma_v) = \lambda(\gamma_w)$. Let P be the plane in F spanned by v and w. The circle $S_p P$ is a union of one dimensional faces of Weyl simplices. Since such faces have length at most $\pi/2$ and $\measuredangle_p(v, w) < \pi/2$, we see that there is a maximally singular vector $u \in S_p P$ that is not $\pm v$ or $\pm w$. As $\Phi \mid P$ is affine and preserves the angles between the maximally singular geodesics γ_u, γ_v and γ_w, we see that $\lambda(\gamma_v) = \lambda(\gamma_u) = \lambda(\gamma_w)$. ■

It follows that if F is an ℓ-regular k-flat, then $\Phi \mid F$ is a multiple of an isometry from F to F^* by a scalar λ_F for any ℓ-regular k-flat F. We show that λ_F is constant. Let F and $\widetilde{F}$ be ℓ-regular k-flats through a point $p \in \widetilde{M}$. Let C and $\widetilde{C}$ be Weyl simplices in $F(\infty)$ and $\widetilde{F}(\infty)$. Join C to $\widetilde{C}$ by a gallery $C = C_0, C_1, \ldots, C_m = \widetilde{C}$. Let F_i be the ℓ-regular k-flat through p and C_i. Then $F_i \cap F_{i+1}$ contains a geodesic through p. Hence $\lambda_{F_i} = \lambda_{F_{i+1}}$. Hence $\lambda_F = \lambda_{\widetilde{F}}$ depends only on p. If $q \in \widetilde{M}$ there is an ℓ-regular k-flat through p and q. Hence λ_F is a constant λ.

Since any two points of $\widetilde{M}$ lie in an ℓ-regular k-flat we see that

$$d_X(\Phi(p), \Phi(q)) = \lambda\, d(p, q)$$

for any $p, q \in \widetilde{M}$.

Note that $\lambda \neq 0$. Otherwise Φ maps $\widetilde{M}$ to a point in X. This would mean that $\sigma_p|_\Delta = \sigma_q|_\Delta$ for all $p, q \in \widetilde{M}$. This, however, contradicts Lemma 3.13, for example. Now it is clear that $\widetilde{M}$ is locally symmetric. ■

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