

Manifolds with vanishing Weyl or Bochner curvature tensor

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§1. Introduction.

Let M be a Riemannian manifold of dimension $n > 3$ and denote by g_{ji} , $K_{kji}{}^h$, K_{ji} and K the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

If M is locally conformal to a Euclidean space then M is said to be conformally flat. For a conformally flat M , the Weyl conformal curvature tensor given by

$$(1.1) \quad C_{kji}{}^h = K_{kji}{}^h + \delta_k^h C_{ji} - \delta_j^h C_{ki} + C_k{}^h g_{ji} - C_j{}^h g_{ki}$$

vanishes identically, where

$$(1.2) \quad C_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} K g_{ji}, \quad C_k{}^h = C_{kt} g^{th},$$

g^{th} being contravariant components of the metric tensor. Conversely if $C_{kji}{}^h$ vanishes identically, then M is conformally flat [3], [6].

One of the purposes of the present paper is to prove the following:

THEOREM 1. *In order that a Riemannian manifold of dimension $n > 3$ is conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q on the manifold such that the sectional curvature $K(\sigma)$ with respect to a section σ is the trace of the restriction of Q to σ , i. e. $K(\sigma) = \text{trace } Q/\sigma$, the metric being also restricted to σ .*

Let M be an n -dimensional Kaehlerian manifold and denote by g_{ji} , $F_i{}^h$, $K_{kji}{}^h$, K_{ji} and K the metric tensor, the complex structure tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively. Bochner [1] (see also [4], [9]) introduced a curvature tensor given by

$$(1.3) \quad B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} \\ + F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h),$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}, \quad L_k{}^h = L_{kt} g^{th},$$

$$M_{ji} = -L_{jt} F_i{}^t, \quad M_k{}^h = M_{kt} g^{th}$$

and $F_{ji} = F_j^t g_{ti}$, as a curvature tensor which corresponds to the Weyl conformal curvature tensor in a Riemannian manifold (see also [4], [5], [8], [10]).

Another purpose of the present paper is to prove the following :

THEOREM 2. *In order that the Bochner curvature tensor of a Kaehlerian manifold vanishes, it is necessary and sufficient that there exists a (unique) hybrid quadratic form Q such that the sectional curvature $K(\sigma)$ with respect to a holomorphic section σ is the trace of the restriction of Q to σ , i.e. $K(\sigma) = \text{trace } Q/\sigma$, the metric being also restricted to σ .*

§ 2. Riemannian manifolds with vanishing Weyl conformal curvature tensor.

Suppose that M is a conformally flat Riemannian manifold of dimension $n > 3$, then we have

$$(2.1) \quad C_{kji}{}^h = 0,$$

that is

$$(2.2) \quad K_{kjih} = -g_{kh}C_{ji} + g_{jh}C_{ki} - C_{kh}g_{ji} + C_{jh}g_{ki},$$

where $K_{kjih} = K_{kji}{}^t g_{th}$ and consequently the sectional curvature $K(\sigma)$ with respect to a section σ spanned by vectors X and Y is given by

$$(2.3) \quad K(\sigma) = \frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji} - g_{jh}g_{ki})X^kY^jX^iY^h} \\ = \frac{1}{(X, Y)^2 - (X, X)(Y, Y)} [(Y, Y)C_{ji}X^jX^i \\ - 2(X, Y)C_{ji}X^jY^i + (X, X)C_{ji}Y^jY^i],$$

where (X, Y) denotes the inner product of X and Y . Thus if X and Y are mutually orthogonal unit vectors, then we have

$$(2.4) \quad K(\sigma) = -K_{kjih}X^kY^jX^iY^h \\ = -C_{ji}X^jX^i - C_{ji}Y^jY^i,$$

that is, the sectional curvature $K(\sigma)$ with respect to σ is given by the trace of the restriction of $Q(X, X) = -C(X, X)$ to σ .

Conversely, suppose that the sectional curvature $K(\sigma)$ of a Riemannian manifold M with respect to a section σ spanned by two vectors X and Y is given by

$$(2.5) \quad K(\sigma) = \frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji} - g_{jh}g_{ki})X^kY^jX^iY^h} \\ = -C_{ji}U^jU^i - C_{ji}V^jV^i,$$

where $C_{ji}U^jU^i=C(U, U)$ is a certain quadratic form and U and V are mutually orthogonal unit vectors spanning the section σ .

The expression $-C(U, U)-C(V, V)$ being independent of the choice of mutually orthogonal unit vectors U and V in the section σ , we put

$$(2.6) \quad \begin{aligned} U^h &= \frac{X^h}{\sqrt{(X, X)}}, \\ V^h &= \frac{(X, X)Y^h - (X, Y)X^h}{\sqrt{(X, X)}\sqrt{(X, X)(Y, Y) - (X, Y)^2}}. \end{aligned}$$

Then (2.5) becomes

$$\begin{aligned} & \frac{K_{kjih}X^kY^jX^iY^h}{(g_{kh}g_{ji}-g_{jh}g_{ki})X^kY^jX^iY^h} \\ &= -\frac{1}{(X, X)(Y, Y) - (X, Y)^2} [(Y, Y)C_{ji}X^jX^i \\ & \quad - 2(X, Y)C_{ji}X^jY^i + (X, X)C_{ji}Y^jY^i], \end{aligned}$$

that is,

$$(2.7) \quad \begin{aligned} & K_{kjih}X^kY^jX^iY^h \\ &= g_{jh}Y^jY^hC_{ki}X^kX^i - 2g_{kh}X^kY^hC_{ji}X^iY^j + g_{ki}X^kX^iC_{jh}Y^jY^h. \end{aligned}$$

Since X 's are arbitrary, we have, from (2.7),

$$\begin{aligned} & K_{kjih}Y^jY^h + K_{ijkh}Y^jY^h \\ &= 2g_{jh}Y^jY^hC_{ki} - 2g_{kh}Y^hC_{ji}Y^j - 2g_{ih}Y^hC_{jk}Y^j + 2g_{ki}C_{jh}Y^jY^h, \end{aligned}$$

from which, Y 's being arbitrary,

$$\begin{aligned} & K_{kjih} + K_{khij} + K_{ijkh} + K_{ihkj} \\ &= 4g_{jh}C_{ki} - 2g_{kh}C_{ji} - 2g_{kj}C_{hi} - 2g_{ih}C_{jk} - 2g_{ij}C_{hk} + 4g_{ki}C_{jh}. \end{aligned}$$

Taking the skew-symmetric part with respect to k and j of this equation, we find

$$\begin{aligned} & 2K_{kjih} + K_{khij} - K_{jhik} + K_{ijkh} - K_{ikjh} + 2K_{ihkj} \\ &= 4g_{jh}C_{ki} - 4g_{kh}C_{ji} - 2g_{kh}C_{ji} + 2g_{jh}C_{ki} \\ & \quad - 2g_{ij}C_{hk} + 2g_{ik}C_{hj} + 4g_{ki}C_{jh} - 4g_{ji}C_{kh}, \end{aligned}$$

that is, using the first Bianchi identity,

$$(2.8) \quad K_{kjih} = -g_{kh}C_{ji} + g_{jh}C_{ki} - C_{kh}g_{ji} + C_{jh}g_{ki}.$$

From (2.8), we have, by transvection with g^{kh} ,

$$(2.9) \quad K_{ji} = -(n-2)C_{ji} - g^{kh}C_{kh}g_{ji},$$

from which, by transvection with g^{ji} ,

$$K = -2(n-1)g^{kh}C_{kh},$$

that is,

$$(2.10) \quad g^{kh}C_{kh} = -\frac{1}{2(n-1)} K.$$

Substituting (2.10) into (2.9), we obtain

$$(2.11) \quad C_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} K g_{ji}.$$

Thus (2.8) is equivalent to

$$C_{kji}{}^h = 0,$$

which shows that the Riemannian manifold is conformally flat. Thus Theorem 1 is proved.

REMARK. In [2], Kulkarni also obtained a characterization of a conformally flat space in terms of sectional curvature different from ours.

§ 3. Kaehlerian manifolds with vanishing Bochner curvature tensor.

Suppose that the Bochner curvature tensor of a Kaehlerian manifold vanishes:

$$(3.1) \quad B_{kji}{}^h = 0,$$

then we have

$$(3.2) \quad \begin{aligned} K_{kjih} = & -g_{kh}L_{ji} + g_{jh}L_{ki} - L_{kh}g_{ji} + L_{jh}g_{ki} \\ & - F_{kh}M_{ji} + F_{jh}M_{ki} - M_{kh}F_{ji} + M_{jh}F_{ki} + 2(M_{kj}F_{ih} + F_{kj}M_{ih}), \end{aligned}$$

and consequently the holomorphic sectional curvature $K(\sigma)$ with respect to a holomorphic section σ spanned by X and FX is given by

$$(3.3) \quad \begin{aligned} K(X) = & -\frac{1}{(X, X)^2} K_{klis} X^k F_j{}^t X^j X^i F_h{}^s X^h \\ = & -\frac{8}{(X, X)} L_{ji} X^j X^i, \end{aligned}$$

where we have used

$$M_{ji} = -L_{jt}F_i{}^t, \quad L_{jt} = M_{jt}F_i{}^t$$

and

$$(3.4) \quad L_{qp}F_j{}^q F_i{}^p = L_{ji},$$

that is, L_{ji} are components of a hybrid tensor of type $(0, 2)$, (see Yano [7], Chapter IV).

Thus for the sectional curvature $K(\sigma)$ with respect to a holomorphic

section σ spanned by a unit vector U and its transform FU by F , we have

$$(3.5) \quad K(U) = -8L_{ji}U^jU^i = -4L_{ji}U^jU^i - 4L_{ts}F_j^tU^jF_i^sU^i.$$

Thus the holomorphic sectional curvature $K(\sigma)$ is the trace of the restriction of $Q(X, X) = -4L(X, X)$ to σ .

Conversely suppose that the sectional curvature $K(\sigma)$ of a Kaehlerian manifold with respect to a holomorphic section σ spanned by X and FX is given by

$$(3.6) \quad K(X) = -\frac{K_{ktis}X^kF_j^tX^jX^iF_h^sX^h}{(X, X)^2} \\ = -8L_{ji}U^jU^i,$$

where $L_{ji}U^jU^i$ is a certain quadratic form whose coefficients satisfy (3.4) and

$$(3.7) \quad U^h = \frac{X^h}{\sqrt{(X, X)}}.$$

Then from (3.6) and (3.7), we have

$$(3.8) \quad K_{ktis}F_j^tF_h^sX^kX^jX^iX^h = 8g_{kj}L_{ih}X^kX^jX^iX^h.$$

Since X 's are arbitrary and

$$K_{ktis}F_j^t = K_{jtis}F_k^t, \quad K_{ktis}F_h^s = K_{kths}F_i^s, \\ K_{ktis}F_j^tF_h^s = K_{itks}F_h^tF_j^s,$$

(see Yano [7], Chapter IV), we have, from (3.8),

$$K_{ktis}F_j^tF_h^s + K_{kths}F_i^tF_j^s + K_{ktjs}F_h^tF_i^s \\ = 4[g_{kj}L_{ih} + g_{ki}L_{jh} + g_{kh}L_{ji} + L_{kj}g_{ih} + L_{ki}g_{jh} + L_{kh}g_{ji}],$$

or

$$K_{ktis}F_q^tF_p^s + K_{ktps}F_i^tF_q^s + K_{ktqs}F_p^tF_i^s \\ = 4[g_{kq}L_{ip} + g_{ki}L_{pq} + g_{kp}L_{qi} + L_{kq}g_{ip} + L_{ki}g_{pq} + L_{kp}g_{qi}].$$

Transvecting this with $F_j^qF_h^p$, we find

$$K_{kji h} - K_{ktpj}F_i^tF_h^p - K_{k hqs}F_j^qF_i^s \\ = 4[-F_{jk}M_{ih} + g_{ki}L_{jh} - F_{hk}M_{ij} - M_{kj}F_{hi} + L_{ki}g_{jh} - M_{kh}F_{ji}],$$

where

$$(3.9) \quad M_{ji} = -L_{jt}F_i^t,$$

or

$$K_{kji h} - K_{ktpj}F_i^tF_h^p - K_{khji} \\ = 4[F_{kj}M_{ih} + g_{ki}L_{jh} - F_{kh}M_{ji} + M_{kj}F_{ih} + L_{ki}g_{jh} - M_{kh}F_{ji}],$$

because of $K_{khqs}F_j^qF_i^s = K_{khji}$.

Taking the skew-symmetric part of this equation with respect to k and j and taking account of

$$\begin{aligned} K_{ktpj}F_i^tF_h^p - K_{jtpk}F_i^tF_h^p &= (K_{ktpj} - K_{jtpk})F_i^tF_h^p \\ &= -K_{ktpj}F_i^tF_h^p = -K_{kjih}, \end{aligned}$$

we find

$$\begin{aligned} &2K_{kjih} + K_{kjih} - K_{khji} + K_{jhki} \\ &= 4[2F_{kj}M_{ih} + g_{ki}L_{jh} - g_{ji}L_{kh} - F_{kh}M_{ji} + F_{jh}M_{ki} \\ &\quad + 2M_{kj}F_{ih} + L_{ki}g_{jh} - L_{ji}g_{kh} - M_{kh}F_{ji} + M_{jh}F_{ki}], \end{aligned}$$

or

$$(3.10) \quad \begin{aligned} K_{kjih} &= -[g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} \\ &\quad + F_{kh}M_{ji} - F_{jh}M_{ki} + M_{kh}F_{ji} - M_{jh}F_{ki} - 2(M_{kj}F_{ih} + F_{kj}M_{ih})]. \end{aligned}$$

Transvecting (3.10) with g^{kh} , we find

$$(3.11) \quad K_{ji} = -[(n+4)L_{ji} + g^{kh}L_{kh}g_{ji}],$$

from which, by transvection with g^{ji} ,

$$K = -2(n+2)g^{kh}L_{kh},$$

or

$$(3.12) \quad g^{kh}L_{kh} = -\frac{1}{2(n+2)}K.$$

Substituting (3.12) into (3.11), we obtain

$$K_{ji} = -\left[(n+4)L_{ji} - \frac{1}{2(n+2)}Kg_{ji}\right],$$

that is,

$$(3.13) \quad L_{ji} = -\frac{1}{n+4}K_{ji} + \frac{1}{2(n+2)(n+4)}Kg_{ji}.$$

Thus (3.10) gives

$$(3.14) \quad B_{kji}{}^h = 0,$$

and consequently Theorem 2 is proved.

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