Manin Involutions for Elliptic Pencils and Discrete Integrable Systems

Matteo Petrera[1] · **Yuri B. Suris**[1] (ORCID) · **Kangning Wei**[1] · **René Zander**[1]

Abstract

We contribute to the algebraic-geometric study of discrete integrable systems generated by planar birational maps: (a) we find geometric description of Manin involutions for elliptic pencils consisting of curves of higher degree, birationally equivalent to cubic pencils (Halphen pencils of index 1), and (b) we characterize special geometry of base points ensuring that certain compositions of Manin involutions are integrable maps of low degree (quadratic Cremona maps). In particular, we identify some integrable Kahan discretizations as compositions of Manin involutions for elliptic pencils of higher degree.

Keywords Elliptic curve · Elliptic pencil · Birational map · Integrable map

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✉ Yuri B. Suris
  suris@math.tu-berlin.de

  Matteo Petrera
  petrera.matteo@googlemail.com

  Kangning Wei
  weikangning12@gmail.com

  René Zander
  rene.zander90@gmail.com

[1] Institut für Mathematik, MA 7-1, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany

1 Introduction

Intimate relation of the theory of integrable systems to algebraic geometry is well appreciated in these days. In the present paper, we address this relation for a very basic class of integrable systems, namely for discrete integrable systems generated by birational maps of $\mathbb{CP}^2$ with a rational integral of motion and an invariant measure with a rational density (whereas the emphasis is put on the integral of motion). In such a system, orbits are confined to invariant curves (level sets of the integral), and on each invariant curve the map induces an automorphism.

For general reasons, invariant curves must have genus zero or one, since only in these cases the induced automorphisms on the invariant curves can be of infinite order (non-periodic). Our main object of interest will be rational elliptic surfaces (i.e., surfaces birationally equivalent to a plane, admitting a fibration by elliptic curves). A classification of pencils of elliptic curves in a plane was given by Bertini, a modern proof of this result is due to Dolgachev [8]. It says that any such pencil is birationally equivalent to a Halphen pencil of index $m \in \mathbb{N}$, in which a generic curve is of degree $3m$ and has multiplicity m at each of nine base points.

Planar maps preserving pencils of elliptic curves appeared over and over again in the theory of discrete integrable systems. Probably, the most prominent example is given by QRT maps [9, 22, 24], which preserve pencils of biquadratic curves. Further examples are given by (autonomous versions of) discrete Painlevé equations [11, 23], as well as the so called HKY maps which preserve pencils of curves of higher degrees [4, 15]. Recently, further examples appeared in the context of the so called Kahan discretization [14], [17, 21], [5–7]. A sort of a classification of such maps, based on the Dolgachev's classification of rational elliptic surfaces, was given in [3] and sounds almost tautologically: a birational map preserving an Halphen pencil (of index m) either preserves each fiber or interchanges the fibers in a nontrivial way.

In the present paper, we are occupied with a construction of integrable maps preserving a pencil of elliptic curves, based only on the pencil itself. The basic idea is to compose two (non-commuting) *birational involutions* preserving the pencil. This construction is almost obvious for QRT maps, where one can always use the horizontal and the vertical switches as suitable involutions (the horizontal switch assigns to any point of a biquadratic curve the second intersection point of the curve with the horizontal line through the original point; the definition of the vertical switch is analogous). In [12, 13] this construction was extended to a (projectively equivalent) case of maps preserving a pencil of quartic curves with two double points, and it was observed that the corresponding involutions are closely related to the so called *Manin involutions*, which were introduced in [16] for pencils of cubic curves. Recall that, for a pencil of cubic curves with nine base points $p_1, \ldots, p_9$, the Manin involution centered at p_1 assigns to each point p the third intersection point of the line $(p_1 p)$ with the curve of the pencil passing through p. In [18], it was shown that Kahan discretizations of canonical Hamiltonian systems with cubic Hamiltonian can be characterized as compositions of Manin involutions in the case of a special geometry of the base points of the cubic pencil, see [19] for a related result. We also mention that compositions of Manin involutions for cubic pencils appeared in the theory of discrete integrable systems already in [10, 26].

Birational involutions of the plane are classical and well studied objects in algebraic geometry. Their classification was given by Bertini [2] and says that every non-trivial birational involution of $\mathbb{P}^2$ is birationally conjugate to exactly one of the following: a de Jonquières involution of degree $d > 2$ (which fixes an irreducible curve of degree d with a unique singular point p which is an ordinary multiple point of multiplicity $d - 2$), a Geiser involution, or a Bertini involution. We refrain from giving precise definitions here and refer the reader to the original source. A modern proof of the Bertini classification was given in [1]. The place of Manin involutions in the Bertini classification is as follows: a Manin involution centered at p_1 for a pencil of cubic curves with nine base points $p_1, \ldots, p_9$ is a de Jonquières involution with center p_1 defined by a hyperelliptic curve of degree $d = 5$ and of genus 3, with a triple point p_1 and eight Weierstrass points $p_2, \ldots, p_9$ (we are indebted to I. Dolgachev for explaining this).

In the present paper, our goal is to use Manin involutions for elliptic pencils to construct integrable dynamical systems. We start by finding a geometric formulation of Manin involutions for elliptic pencils consisting of curves of higher degree. This is achieved via a birational conjugation from the canonical construction for cubic pencils. It seems, however, that the geometric construction directly in terms of the original pencils is not available in the literature. The most non-trivial contribution consists in finding the conditions under which the involutions and their compositions are of low degree, thus producing simple and attractive examples of integrable birational maps.

2 Elliptic Pencils and Cubic Pencils

Throughout the paper, we work over the field $\mathbb{C}$. We consider pencils of curves in $\mathbb{P}^2$, i.e., families of curves $\mathcal{P} = \{C_\lambda\}$ parametrized by $\lambda \in \mathbb{P}^1$,

$$C_\lambda = \left\{ [x_0 : x_1 : x_2] \in \mathbb{P}^2 : F(x_0, x_1, x_2) + \lambda G(x_0, x_1, x_2) = 0 \right\}.$$

Here F, G are two homogeneous polynomials of degree d. The points of the set

$$B = \left\{ [x_0 : x_1 : x_2] \in \mathbb{P}^2 : F(x_0, x_1, x_2) = G(x_0, x_1, x_2) = 0 \right\}$$

are called *base points* of the pencil $\mathcal{P}$. As usual, they are counted with multiplicities. We will assume that the multiplicities of each base point on both curves $F = 0$ and $G = 0$ (and then on all curves of the pencil) are the same. The *type* of the pencil is then

$$(d; (n_1)^1 (n_2)^2 (n_3)^3 \ldots)$$

where d is the degree of the curves of the pencil, n_1 the number of simple base points, n_2 the number of double base points, n_3 the number of triple base points and so on. The pencil itself will be denoted by

$$\mathcal{P}\left(d; p_1^{m_1}, p_2^{m_2}, \ldots p_N^{m_N}\right),$$

which refers to the degree d and the list of base points p_i with their respective multiplicities m_i, so that $N = n_1 + n_2 + n_3 + \ldots$. Multiplicities $m_i = 1$ are usually omitted.

Counting the intersection numbers, we get:

$$d^2 = \sum_k n_k k^2. \tag{1}$$

Through any point $[x_0 : x_1 : x_2] \in \mathbb{P}^2 \setminus B$, there passes a unique curve C_λ of the pencil, with $\lambda = -F(x_0, x_1, x_2)/G(x_0, x_1, x_2)$.

Our main interest is in the *elliptic pencils*, for which generic curves of the pencil are of genus $g = 1$. According to the degree-genus formula, the genus of irreducible curves of the pencil is given by:

$$g = \frac{(d-1)(d-2)}{2} - \sum_k n_k \frac{k(k-1)}{2} = 1. \tag{2}$$

We remark that by virtue of (1), the latter equation is equivalent to

$$3d = \sum_k n_k k, \tag{3}$$

where the right-hand side is the total number of base points (counted with multiplicities).

Examples

1. *A pencil of the type* $(3; 9^1)$ of cubic curves with nine simple base points.
2. *A pencil of the type* $(4; 8^1 2^2)$ of curves of degree 4 with eight simple and two double points. By an automorphism of $\mathbb{P}^2$, we can send the double points to infinity (say, to $[0 : 1 : 0]$ and $[0 : 0 : 1]$), then in the affine coordinates $(x_1/x_0, x_2/x_0)$, we get a pencil of biquadratic curves. Such pencils are pretty well studied and have plenty of applications in the theory of discrete integrable systems [9, 22].
3. *A pencil of the type* $(6; 6^1 3^2 2^3)$ of curves of degree 6 with six simple points, three double points and two triple points.
4. *A pencil of the type* $(6; 9^2)$ of curves of degree 6 with nine double points, i.e., a Halphen pencil of index 2.

Remark We do allow *infinitely near base points*, at which the curves of the pencil have to satisfy certain conditions of tangency up to a certain order. In the formulations of our general results about geometry of Manin involutions, we silently assume that the geometry of base points is generic, in particular, that there are no incidental collinearities. However, all our main examples are non-generic with a plenty of incidental collinearities, since it is exactly this feature that allows for a substantial

drop of degree of the resulting birational maps. We hope this will not lead to any confusions.

3 Manin Involutions

For cubic curves, one has a simple geometric interpretation of the addition law. Correspondingly, there is a simple geometric construction of certain birational involutions of $\mathbb{P}^2$ induced by pencils of cubic curves, cf. [16, p. 1376], [25, p. 35]. These were dubbed *Manin involutions* in [9, Sect. 4.2].

Definition 1 (Manin involutions for cubic pencils)
   1) Consider a nonsingular cubic curve C in $\mathbb{P}^2$, and a point $p_0 \in C$. The *Manin involution* on C with respect to p_0 is the map $I_{C,p_0} : C \to C$ defined as follows: for a generic $p \neq p_0$, the image $I_{C,p_0}(p)$ is the unique third intersection point of C with the line $(p_0 p)$; for $p = p_0$, the line $(p_0 p)$ should be interpreted as the tangent line to C at p_0.
   2) Consider a pencil $\mathcal{P} = \{C_\lambda\}$ of cubic curves in $\mathbb{P}^2$ with at least one nonsingular member. Let p_0 be a base point of the pencil. The *Manin involution $I_{\mathcal{P},p_0} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$* is a birational map defined as follows. For any $p \in \mathbb{P}^2$ which is not a base point, $I_{\mathcal{P},p_0}(p) = I_{C_\lambda,p_0}(p)$, where C_λ is the unique curve of the pencil through the point p.

For elliptic pencils of degree higher than 3, the geometric construction of Manin involutions seems to be unknown. The only exception are the vertical and the horizontal switches in biquadratic pencils, of which the famous QRT maps are composed [9, 22]. They can be immediately translated to a construction of *generalized Manin involutions* for quartic pencils with two double points, with respect to the both double points [13]. The definition of the generalized Manin involution I_{C,p_0} for a quartic curve C and a double point $p_0 \in C$, resp. of the generalized Manin involution $I_{\mathcal{P},p_0}$ for a quartic pencil with two double points, one of them being p_0, literally coincides with Definition 1. This is justified by the fact that any line through a double point $p_0 \in C$ still intersects the quartic curve C at two further points.
   The main goal of this paper is to elaborate on the geometric definition of Manin involutions in arbitrary elliptic pencils birationally equivalent to a Halphen pencil of index 1, i.e., to a cubic pencil.
   To find a birational equivalence, one can resolve the multiple base points by means of suitable birational transformations. Often, the simplest way of doing this is by a sequence of *quadratic Cremona transformations*. Recall that a generic quadratic Cremona transformation $\phi : \mathbb{P}^2_1 \dashrightarrow \mathbb{P}^2_2$ has three distinct fundamental points $\mathcal{I}(\phi) = \{p_1, p_2, p_3\}$ which are blown up to three lines $(q_2 q_3)$, $(q_3 q_1)$, $(q_1 q_2)$, respectively. The three lines $(p_2 p_3)$, $(p_3 p_1)$, $(p_1 p_2)$ are blown down to the points q_1, q_2, q_3, respectively, which build the indeterminacy set of the inverse map: $\mathcal{I}(\phi^{-1}) = \{q_1, q_2, q_3\}$. A practical way to construct such a map consists in finding homogeneous polynomials $\phi(x_0, x_1, x_2)$ of degree 2 vanishing at the fundamental points p_1, p_2, p_3. Geometrically, we are speaking about the set of conics in $\mathbb{P}^2_1$ through p_1, p_2, p_3. The

space of solutions of this linear system is two-dimensional: $\alpha\phi_0 + \beta\phi_1 + \gamma\phi_2$, where ϕ_0, ϕ_1, ϕ_2 are homogeneous polynomials of (x_0, x_1, x_2) of degree 2. The map

$$\phi : [x_0 : x_1 : x_2] \mapsto [u_0 : u_1 : u_2] = [\phi_0(x_0, x_1, x_2) : \phi_1(x_0, x_1, x_2) : \phi_2(x_0, x_1, x_2)] \tag{4}$$

is the sought after birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. A different choice of a basis ϕ_0, ϕ_1, ϕ_2 of the net corresponds to a linear projective transformation of the target plane $\mathbb{P}^2_2$.

Note that the pre-image of a generic line $au_0 + bu_1 + cu_2 = 0$ in the target plane $\mathbb{P}^2_2$ is the conic $a\phi_0 + b\phi_1 + c\phi_2 = 0$ (passing through p_1, p_2, p_3) in the source plane $\mathbb{P}^2_1$. It follows that for any regular point p of ϕ, the pencil of lines $\mathcal{P}(1; q)$ through $q = \phi(p)$ in $\mathbb{P}^2_2$ corresponds to the pencil of conics $\mathcal{P}(2; p, p_1, p_2, p_3)$ in $\mathbb{P}^2_1$.

4 Example: A Quartic Pencil with Two Double Base Points

4.1 Geometry of the Base Points

Consider an elliptic pencil in $\mathbb{P}^2$ of the type $(4; 8^1 2^2)$,

$$\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2).$$

Thus, $\mathcal{E}$ consists of quartic curves with 8 simple base points $p_1, \ldots, p_8$ and two double base points p_9, p_{10}. The position of the ten base points is not arbitrary: for a generic configuration of ten points, there exists just one curve of degree 4 through these points, having the prescribed two of them as double points. On the other hand, for a generic configuration of nine points, there is a one-parameter family (a pencil) of curves of degree 4 through these points, having the prescribed two of them as double points (nine incidence conditions plus four second order conditions, altogether 13 linear conditions, while a generic curve of degree 4 has 14 non-homogeneous coefficients). Counting the intersection numbers, we see that all curves of the pencil pass through a further simple point (indeed, seven simple points and two double points contribute $7 \times 1 + 2 \times 4 = 15$ to the intersection number 16).

More information on the configuration of the ten base points is contained in the following statement.

Proposition 1 *In a generic pencil $\mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$, one of the curves is reducible and consists of the line $(p_9 p_{10})$ and a cubic curve passing through all ten base points $p_1, \ldots, p_{10}$.*

Proof Fix any point $p \in (p_9 p_{10})$ different from p_9, p_{10}, and consider the unique curve C of the pencil through p. If the line $(p_9 p_{10})$ would not be a component of this curve, then the intersection number of C with the line $(p_9 p_{10})$ would be at least $2 \times 2 + 1 = 5$, a contradiction. Thus, the curve C is reducible and contains the line $(p_9 p_{10})$ as one of the components. Another component is a cubic curve through $p_1, \ldots, p_{10}$ (with p_9, p_{10} being simple points on the cubic). $\qquad\square$

Remark 1 If the reducible curve C happens to contain $(p_9 p_{10})$ as a double line, then the remaining component is a conic through eight base points $p_1, \ldots, p_8$.

4.2 Birational Reduction to a Cubic Pencil

Consider a pencil $\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$. Let $\phi : \mathbb{P}_1^2 \dashrightarrow \mathbb{P}_2^2$ be a quadratic Cremona map with the fundamental points p_1, p_9, p_{10}. Thus, ϕ blows down the lines $(p_9 p_{10})$, $(p_1 p_{10})$, $(p_1 p_9)$ to points denoted by q_1, q_9, q_{10}, respectively, and blows up the points p_1, p_9, p_{10} to the lines $(q_9 q_{10})$, $(q_1 q_{10})$, $(q_1 q_9)$. All other base points p_i, $i = 2, \ldots, 8$, are regular points of ϕ, their images will be denoted by $q_i = \phi(p_i)$.

Proposition 2 *Under the map ϕ:*

a) *Quartic curves of the original pencil $\mathcal{E}$ in $\mathbb{P}_1^2$ correspond to curves of a cubic pencil*

$$\mathcal{P}(3; q_2, \ldots, q_8, q_9, q_{10})$$

 with nine base points in $\mathbb{P}_2^2$; the point q_1 is not a base point of the latter pencil.
b) *For $i = 2, \ldots, 8$, the pencil of lines $\mathcal{P}(1; q_i)$ in $\mathbb{P}_2^2$ corresponds to the pencil of conics*

$$\mathcal{P}(2; p_i, p_1, p_9, p_{10})$$

 in $\mathbb{P}_1^2$.
c) *The pencils of lines $\mathcal{P}(1; q_9)$, $\mathcal{P}(1; q_{10})$ in $\mathbb{P}_2^2$ correspond to the pencils of lines*

$$\mathcal{P}(1; p_9), \quad \mathcal{P}(1; p_{10})$$

 in $\mathbb{P}_1^2$.

Proof

a) The total image of a quartic curve $C \in \mathcal{E}$ is a curve of degree 8. Since C passes through p_1, its total image contains the line $(q_9 q_{10})$. Since C passes through p_9 and p_{10} with multiplicity 2, its total image contains the lines $(q_1 q_{10})$ and $(q_1 q_9)$ with multiplicity 2. Dividing by the linear defining polynomials of all these lines, we see that the proper image of C is a curve of degree $8 - 5 = 3$. This curve has to pass through all points q_i, $i = 2, \ldots, 8$. The curve C of degree 4 has no other intersections with the line $(p_9 p_{10})$ different from two double points p_9 and p_{10}, therefore its proper image does not pass through q_1. On the other hand, the curve C of degree 4 has one additional intersection point with each of the lines $(p_1 p_9)$ and $(p_1 p_{10})$, different from the simple point p_1 and the double point p_9, resp. p_{10}. Therefore, its proper image passes through q_{10}, resp. q_9, with multiplicity 1.
b) This follows from the fact that p_i, $i = 2, \ldots, 8$, are regular points of ϕ.

c) Consider the total pre-image of a line through q_9. It is a conic through p_1, p_9, p_{10} whose defining polynomial vanishes on the line $(p_1 p_{10})$. Thus, the conic is reducible and contains that line. Dividing by the defining polynomial of this line (of degree 1), we see that the proper pre-image is a line which must pass through p_9. Similarly, the proper pre-image of a line through q_{10} is a line through p_{10}. □

Let S be the elliptic surface obtained from $\mathbb{P}^2$ by blowing up the ten base points p_i, $i = 1, \ldots, 10$. Let E_i be the exceptional divisor classes of the blow-ups. The Picard group of S is $\mathrm{Pic}(S) = \mathbb{Z}D \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_{10}$. The class of a generic curve of the pencil $\mathcal{E}$ is

$$4D - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 2E_9 - 2E_{10}. \tag{5}$$

The quadratic Cremona map of Proposition 2 corresponds to the following change of basis of the Picard group:

$$\begin{cases} D' & = 2D - E_1 - E_9 - E_{10}, \\ E_1' & = D - E_9 - E_{10}, \\ E_9' & = D - E_1 - E_{10}, \\ E_{10}' & = D - E_1 - E_9, \end{cases} \tag{6}$$

and $E_i' = E_i$ for $i = 2, \ldots, 8$.

One can check that E_1' is a redundant class, in the sense that the class (5) of a general curve of the pencil is expressed through $E_2', \ldots, E_{10}'$ only:

$$4D - E_1 - \ldots - E_8 - 2E_9 - 2E_{10} = 3D' - E_2' - E_3' - \ldots - E_{10}'. \tag{7}$$

This corresponds to the fact that q_1 is not a base point of the ϕ-image of the pencil $\mathcal{E}$. Note that $E_1' = D - E_9 - E_{10}$ is the class of (the proper transform of) the line $(p_9 p_{10})$ in $\mathbb{P}^2$. Blowing down E_1' on S, we obtain the surface S' which is a minimal elliptic surface (blow-up of $\mathbb{P}^2$ at nine points), whose anti-canonical divisor class coincides with (7).

Statement b) of Proposition 2 translates to relations $D' - E_i' = 2D - E_1 - E_9 - E_{10} - E_i$ in the Picard group (for $i = 2, \ldots, 8$), while statement c) translates as $D' - E_9' = D - E_9$ and $D' - E_{10}' = D - E_{10}$.

4.3 Manin Involutions

In the new coordinates, where the pencil consists of cubic curves, Manin involutions I_{q_i} with respect to the base points q_i of the pencil are defined as in Definition 1: for a point q which is not a base point, $I_{q_i}(q)$ is the unique third intersection of the the line $(q_i q)$ with the cubic curve of the pencil passing through q. We now pull back this construction to the original pencil in the old coordinates.

Definition 2 (Manin involutions for pencils of the type $(4; 8^1 2^2)$)

Consider a pencil $\mathcal{E} = \mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$. There are two kinds of Manin involutions.

1) Involutions $I_{i,j}^{(2)}$, $i, j \in \{1, \ldots, 8\}$, defined in terms of the pencil of conics

$$\mathcal{C}_{i,j} = \mathcal{P}(2; p_i, p_j, p_9, p_{10}).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique conic of $\mathcal{C}_{i,j}$ passing through p, and a unique quartic curve of $\mathcal{E}$ passing through p. We set $I_{i,j}^{(2)}(p) = p'$, where p' is the unique further intersection point of those two curves. This intersection is unique, since the intersection number of the conic with the quartic is $2 \times 4 = 8$, while the intersections at the points p_i, p_j, p_9, p_{10}, and p count as $1+1+2+2+1 = 7$.

2) Involutions $I_9^{(1)}$, $I_{10}^{(1)}$ defined in terms of the pencils of lines

$$\mathcal{P}(1; p_9), \quad resp. \quad \mathcal{P}(1; p_{10}).$$

For instance, the involution I_9 is defined as follows. Given a point p which is not a base point of $\mathcal{E}$, we set $I_9^{(1)}(p) = p'$, where p' is the unique third intersection of the line $(p_9 p)$ and the quartic curve of $\mathcal{E}$ passing through p. This intersection is unique, since p_9 is a double point of the curve.

Indeed

1) Due to point b) of Proposition 2, for any $i = 2, \ldots, 8$, the Manin involution with respect to q_i is conjugated to the map defined as above in terms of conics through p_1, p_9, p_{10}, and p_i. Remarkably, while in the construction of the conjugating Cremona map the roles of the simple base points p_1 and p_i are asymmetric, in the resulting map $I_{1,i}^{(2)}$ the points p_1 and p_i are on an equal footing. More generally, $I_{i,j}^{(2)} = I_{j,i}^{(2)}$, where the map on the left-hand side should be understood as conjugated to I_{q_j} under the quadratic Cremona map with the fundamental points p_i, p_9, p_{10}, while the right-hand side should be understood as conjugated to I_{q_i} under the quadratic Cremona map with the fundamental points p_j, p_9, p_{10}.

2) Due to point c) of Proposition 2, Manin involutions I_{q_9}, $I_{q_{10}}$ on $\mathbb{P}_2^2$ are conjugated to the maps $I_9^{(1)}$, $I_{10}^{(1)}$ on $\mathbb{P}_1^2$ defined in terms of lines through p_9, p_{10}, respectively. Again, while the construction depends on the choice of a simple base point p_1, the resulting map does not depend on this choice.

The involution $I_{i,j}^{(2)}$ has all base points of the pencil as singularities (indeterminacy points). For instance, it blows up the point p_k to the conic through p_i, p_j, p_k, p_9, p_{10}. However, a composition

$$I_{j,k}^{(2)} \circ I_{i,j}^{(2)}$$

with three distinct simple base points p_i, p_j, p_k is well defined at p_k and maps it to p_i. Moreover, this composition can be characterized as the unique map acting on the elliptic curves of the pencil as the shift mapping p_k to p_i. In particular, this composition does not depend on j.

5 Example: A Sextic Pencil with Three Double Base Points and Two Triple Base Points

5.1 Birational Reduction to a Cubic Pencil

Consider an elliptic pencil in $\mathbb{P}^2$ of the type $(6; 6^1 3^2 2^3)$,

$$\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3),$$

consisting of curves of degree 6 with six simple base points $p_1, \ldots, p_6$, three double base points p_7, p_8, p_9, and two triple base points p_{10}, p_{11}. We reduce it to a cubic pencil in two steps.

Step 1. Apply a quadratic Cremona map ϕ' with the fundamental points p_9, p_{10}, p_{11} (the both triple base points and one of the double base points). Thus, ϕ' blows down the lines $(p_{10}p_{11})$, $(p_9 p_{11})$, $(p_9 p_{10})$ to the points denoted by q_9, q_{10}, q_{11}, respectively, and blows up the points p_9, p_{10}, p_{11} to the lines $(q_{10}q_{11})$, (q_9q_{11}), (q_9q_{10}). All other base points $p_i, i = 1, \ldots, 8$ are regular points of ϕ' and their images are denoted by $q_i = \phi'(p_i)$.

Proposition 3 *The change of variables ϕ' maps a pencil*

$$\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$$

of sextic curves to a pencil

$$\mathcal{P}(4; q_1, \ldots, q_6, q_{10}, q_{11}, q_7^2, q_8^2)$$

of quartic curves with eight simple base points and two double base points. The point q_9 is not a base point of the latter pencil.

Proof The total image of a curve $C \in \mathcal{E}$ is a curve of degree 12. Since C passes through p_9, p_{10}, p_{11} with the multiplicities 2,3,3, its total image contains the lines $(q_{10}q_{11})$, (q_9q_{11}), (q_9q_{10}) with the same multiplicities. Dividing by the linear defining polynomials of all these lines, we see that the proper image of C is a curve of degree $12 - 8 = 4$. This curve passes through all points $q_i, i = 1, \ldots, 8$ (for $i = 7, 8$ with multiplicity 2). The curve C of degree 6 has no other intersections with the line $(p_{10}p_{11})$ different from two triple points p_{10} and p_{11}, therefore its proper image does not pass through q_9. On the other hand, the curve C of degree 6 has one additional intersection point with each of the lines $(p_9 p_{10})$ and $(p_9 p_{11})$, different from the double point p_9 and the triple point p_{10}, respectively p_{11}. Therefore, its proper image passes through q_{11}, resp. q_{10}, with multiplicity 1. □

Step 2. Apply a quadratic Cremona map ϕ'' with the fundamental points q_7, q_8 (the both double base points), and one of the simple base points. As we know from Proposition 2, the image of the pencil $\mathcal{P}(4; q_1, \ldots, q_6, q_{10}, q_{11}, q_7^2, q_8^2)$ under ϕ'' is a pencil of cubic curves with nine base points. The nature of the composition $\phi'' \circ \phi'$ depends on the choice of the simple base point q_i designated as the third fundamental point of ϕ'', and is different in the cases $i = 1, \ldots, 6$ and $i = 10, 11$.

It turns out that the first option contains all the possibilities for the different sorts of Manin involutions, therefore we restrict our attention to this case, taking, for definiteness, $i = 6$. Thus, let ϕ'' have three fundamental points q_6, q_7, q_8. It blows down the lines (q_6q_7), (q_6q_8), (q_7q_8) to points r_8, r_7, r_6, respectively, and blows up the points q_6, q_7, q_8 to the lines (r_7r_8), (r_6r_8), (r_6r_7). All other base points q_i, $i = 1, \ldots, 5, 10, 11$ are regular points of ϕ, their images will be denoted by $r_i = \phi(q_i)$. As follows from Propositions 3, 2, we have:

Proposition 4 *The change of coordinates $\phi = \phi'' \circ \phi' : \mathbb{P}_1^2 \dashrightarrow \mathbb{P}_2^2$ maps a pencil*

$$\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$$

of sextic curves in $\mathbb{P}_1^2$ to a pencil

$$\mathcal{P}(3; r_1, \ldots, r_5, r_7, r_8, r_{10}, r_{11})$$

of cubic curves with nine base points in $\mathbb{P}_2^2$. The points r_6 and r_9 are not base points of this cubic pencil.

Properties of the birational change of coordinates $\phi = \phi'' \circ \phi'$ on $\mathbb{P}^2$ are easily obtained. It is a Cremona map of degree 4 which blows down the lines $(p_9 p_{10})$, $(p_9 p_{11})$, $(p_{10} p_{11})$ to the points r_{11}, r_{10}, r_9, respectively, and blows down the conics $C(p_6, p_7, p_9, p_{10}, p_{11})$, $C(p_6, p_8, p_9, p_{10}, p_{11})$, $C(p_7, p_8, p_9, p_{10}, p_{11})$ to the points r_8, r_7, r_6, respectively. Moreover, ϕ blows up the points p_9, p_{10}, p_{11} to the lines $(r_{10}r_{11})$, (r_9r_{11}), (r_9r_{10}), respectively, and the points p_6, p_7, p_8 to the conics $C(r_7, r_8, r_9, r_{10}, r_{11})$, $C(r_6, r_8, r_9, r_{10}, r_{11})$, $C(r_6, r_7, r_9, r_{10}, r_{11})$, respectively. Points p_i, $i = 1, \ldots, 5$, are regular points of ϕ, their images are $r_i = \phi(p_i)$. The pre-image of a generic line in $\mathbb{P}^2$ is a quartic curve passing through $p_6, \ldots, p_{11}$ (the points p_{10} and p_{11} being of multiplicity 2). In particular, for any regular point p, the pencil of lines $\mathcal{P}(1; r)$ through $r = \phi(p)$ in $\mathbb{P}_2^2$ corresponds to the pencil

$$\mathcal{P}(4; p, p_6, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2)$$

of quartic curves in $\mathbb{P}_1^2$.

Proposition 5 *The change of coordinates $\phi = \phi'' \circ \phi' : \mathbb{P}_1^2 \dashrightarrow \mathbb{P}_2^2$ has the following properties:*

a) *For $i = 1, \ldots, 5$, the pencil of lines $\mathcal{P}(1; r_i)$ in $\mathbb{P}_2^2$ corresponds to the pencil*

$$\mathcal{P}(4; p_i, p_6, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2)$$

   of quartic curves in $\mathbb{P}_1^2$.

b) *For $i = 10, 11$, the proper pre-images of lines of the pencil $\mathcal{P}(1; r_i)$ in $\mathbb{P}_2^2$ are cubics of the respective pencil*

$$\mathcal{P}(3; p_6, p_7, p_8, p_9, p_{10}^2, p_{11}), \quad \mathcal{P}(3; p_6, p_7, p_8, p_9, p_{10}, p_{11}^2)$$

   in $\mathbb{P}_1^2$.

c)   For $i = 7, 8$, the proper pre-images of lines of the pencil $\mathcal{P}(1; r_i)$ in $\mathbb{P}_2^2$ are conics of the respective pencil

$$\mathcal{P}(2; p_7, p_9, p_{10}, p_{11}), \quad \mathcal{P}(2; p_8, p_9, p_{10}, p_{11})$$

in $\mathbb{P}_1^2$.

Proof

a)   This follows from the fact that p_i, $i = 1, \ldots, 5$ are regular points of ϕ.
b)   Consider the total pre-image of a line through r_{10}. It is a quartic curve passing through $p_6, \ldots, p_{11}$, having p_{10}, p_{11} as double points. Its defining polynomial vanishes on the line $(p_9 p_{11})$, which blows down to r_{10}. Thus, the quartic is reducible and contains that line. Dividing by the defining polynomial of the line, we see that the proper pre-image is a cubic passing through $p_6, p_7, p_8, p_{10}, p_{11}$, with p_{10} being a double point.
c)   Consider the total pre-image of a line through r_7. It is a quartic curve passing through $p_6, \ldots, p_{11}$, having p_{10}, p_{11} as double points. Its defining polynomial vanishes on the conic $C(p_6, p_8, p_9, p_{10}, p_{11})$, which blows down to r_7. Thus, the quartic is reducible and contains that conic. Dividing by the defining polynomial of the conic, we see that the proper pre-image is a conic passing through p_7, p_9, p_{10}, p_{11}.

$\square$

Let S be the elliptic surface obtained from $\mathbb{P}^2$ by blowing up the eleven base points p_i, $i = 1, \ldots, 11$. Let E_i be exceptional divisor classes of the blow ups. The Picard group of S is $\mathrm{Pic}(S) = \mathbb{Z}D \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_{11}$. The class of a generic curve of the pencil is

$$6D - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 - 2E_8 - 2E_9 - 3E_{10} - 3E_{11}. \quad (8)$$

The quadratic Cremona map ϕ' corresponds to the following change of basis of $\mathrm{Pic}(S)$:

$$\begin{cases} D' = 2D - E_9 - E_{10} - E_{11}, \\ E_9' = D - E_{10} - E_{11}, \\ E_{10}' = D - E_9 - E_{11}, \\ E_{11}' = D - E_9 - E_{10}, \end{cases} \quad (9)$$

and $E_i' = E_i$ for $i = 1, \ldots, 8$. The Cremona map ϕ'' corresponds to the following change of basis of the Picard group:

$$\begin{cases} D'' = 2D' - E_6' - E_7' - E_8', \\ E_6'' = D' - E_7' - E_8', \\ E_7'' = D' - E_6' - E_8', \\ E_8'' = D' - E_6' - E_7', \end{cases} \quad (10)$$

and $E_i'' = E_i'$ for $i = 1, \ldots, 5$ and $i = 9, 10, 11$. Composing (9), (10), we easily compute:

$$\begin{cases} D'' = 4D - E_6 - E_7 - E_8 - 2E_9 - 2E_{10} - 2E_{11}, \\ E_6'' = 2D - E_7 - E_8 - E_9 - E_{10} - E_{11}, \\ E_7'' = 2D - E_6 - E_8 - E_9 - E_{10} - E_{11}, \\ E_8'' = 2D - E_6 - E_7 - E_9 - E_{10} - E_{11}, \\ E_9'' = D - E_{10} - E_{11}, \\ E_{10}'' = D - E_9 - E_{11}, \\ E_{11}'' = D - E_9 - E_{10}, \end{cases} \qquad (11)$$

and $E_i'' = E_i$ for $i = 1, \ldots, 5$. One can check that the classes

$$E_6'' = 2D - E_7 - E_8 - E_9 - E_{10} - E_{11}, \quad E_9'' = D - E_{10} - E_{11}$$

are redundant, in the sense that the class (8) of a general curve of the pencil $\mathcal{E}$ is expressed through E_i'', $i \neq 6, 9$:

$$6D - E_1 - \ldots - E_6 - 2E_7 - 2E_8 - 2E_9 - 3E_{10} - 3E_{11}$$
$$= 3D'' - E_1'' - \ldots - E_5'' - E_7'' - E_8'' - E_{10}'' - E_{11}''. \qquad (12)$$

This reflects the fact that r_6, r_9 are not base points of the resulting cubic pencil. The redundant classes are the classes (of the proper transforms) of the conic $C(p_7, p_8, p_9, p_{10}, p_{11})$, resp. of the line $(p_{10}p_{11})$ in $\mathbb{P}_1^2$. The surface S' obtained by blowing down E_6'' and E_9'' on S, is a minimal elliptic surface, whose anti-canonical divisor class coincides with (12). Generic fibers of S' are exactly the lifts of generic curves of the initial sextic pencil $\mathcal{E}$.

Note that statements of Proposition 5 translate as the following relations in Pic(S):

$$D'' - E_i'' = 4D - E_i - E_6 - E_7 - E_8 - 2E_9 - 2E_{10} - 2E_{11}, \quad i = 1, \ldots, 5,$$
$$D'' - E_{10}'' = 3D - E_6 - E_7 - E_8 - E_9 - 2E_{10} - E_{11},$$
$$D'' - E_7'' = 2D - E_7 - E_9 - E_{10} - E_{11}.$$

5.2 Manin Involutions

We pull back the standard construction of Manin involutions for the cubic pencil in $\mathbb{P}_2^2$ by means of the map ϕ to the original pencil in $\mathbb{P}_1^2$.

Definition 3 (Manin involutions for pencils of the type $(6; 6^1 3^2 2^3)$)
    Consider a pencil $\mathcal{E} = \mathcal{P}(6; p_1, \ldots, p_6, p_7^2, p_8^2, p_9^2, p_{10}^3, p_{11}^3)$. There are three kinds of Manin involutions.
    1) Involutions $I_{i,j,k}^{(4)}$, $i, j \in \{1, \ldots, 6\}$, $k \in \{7, 8, 9\}$. E.g., $I_{i,j,9}^{(4)}$ is defined in terms of quartic curves of the pencil

$$\mathcal{Q}_{i,j,9} = \mathcal{P}(4; p_i, p_j, p_7, p_8, p_9^2, p_{10}^2, p_{11}^2).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique quartic curve of $\mathcal{Q}_{i,j,9}$ through p, and a unique sextic curve of $\mathcal{E}$ through p. We set $I_{i,j,9}^{(4)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the quartic with the sextic is $4 \times 6 = 24$, while the intersections at the points p_i, p_j, p_7, p_8, p_9, p_{10}, p_{11}, and p count as $1 + 1 + 2 + 2 + 4 + 6 + 6 + 1 = 23$. Involutions $I_{i,j,k}^{(4)}$ with $k = 7, 8$ are defined similarly.

2) Involutions $I_{i,k}^{(3)}$, $i \in \{1, \ldots, 6\}$, $k \in \{10, 11\}$. E.g., $I_{i,10}^{(3)}$ is defined in terms of cubic curves of the pencil

$$\mathcal{K}_{i,10} = \mathcal{P}(3; \, p_i, \, p_7, \, p_8, \, p_9, \, p_{10}^2, \, p_{11}).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique cubic curve of $\mathcal{K}_{i,10}$ through p, and a unique sextic curve of $\mathcal{E}$ through p. We set $I_{i,10}^{(3)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the cubic with the sextic is $3 \times 6 = 18$, while the intersections at the points p_i, p_7, p_8, p_9, p_{10}, p_{11}, and p count as $1 + 2 + 2 + 2 + 6 + 3 + 1 = 17$. Involutions $I_{i,11}^{(3)}$ are defined similarly.

3) Involutions $I_{i,j}^{(2)}$, $i, j \in \{7, 8, 9\}$, defined in terms of conics of the pencil

$$\mathcal{C}_{i,j} = \mathcal{P}(2; \, p_i, \, p_j, \, p_{10}, \, p_{11}).$$

Given a point p which is not a base point of $\mathcal{E}$, there is a unique conic of $\mathcal{C}_{i,j}$ through p, and a unique sextic curve of $\mathcal{E}$ through p. We set $I_{i,j}^{(2)}(p) = p'$, where p' is the unique further intersection point of these two curves. This intersection is unique, since the intersection number of the conic with the sextic is $2 \times 6 = 12$, while the intersections at the points p_i, p_j, p_{10}, p_{11}, and p count as $2 + 2 + 3 + 3 + 1 = 11$.

6 Quadratic Manin Maps for Special Cubic Pencils

In this section, we consider pencils of cubic curves,

$$\mathcal{E} = \mathcal{P}(3; \, p_1, \ldots, p_9).$$

Generically, a Manin involution for a cubic pencil is a birational map of degree 5 for which all base points of the pencil are singularities (indeterminacy points). Indeed, consider $I_{\mathcal{E}, p_j}$. For any base point $p_i \neq p_j$, all curves $\mathcal{C}_\lambda$ of the pencil pass through p_i, p_j, and have one further intersection point with the line $(p_i \, p_j)$. As a result, $I_{\mathcal{E}, p_j}$ blows up any base point p_i ($i \neq j$) to the line $(p_i \, p_j)$. For the same reason $I_{\mathcal{E}, p_i}$ blows down this line to p_j. Thus:

Proposition 6 *For a cubic pencil, and for any two distinct base points p_i and p_j, the Manin transformation $I_{\mathcal{E}, p_i} \circ I_{\mathcal{E}, p_j}$ is regular at p_i and maps it to p_j.*

For a similar reason, some base points become regular points of Manin involutions if there are collinearities among them:
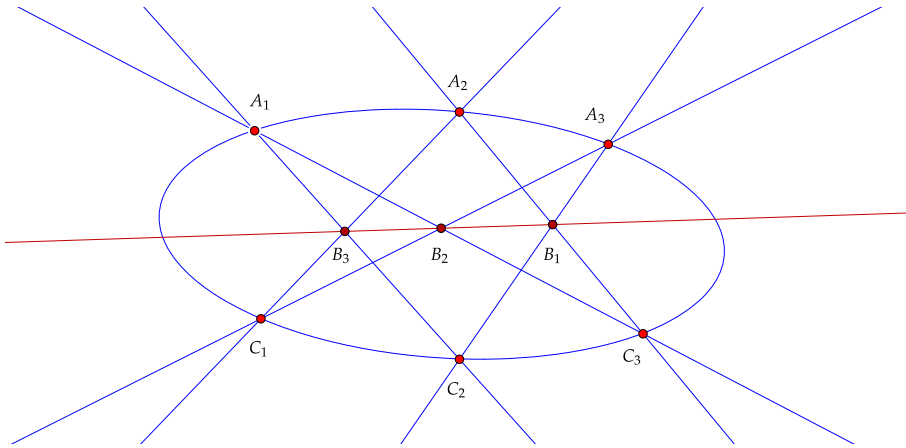
Fig. 1 Pascal configuration of base points of a cubic pencil

Proposition 7 *For a cubic pencil, if three distinct base points p_i, p_j, p_k are collinear, then $I_{\mathcal{E},p_i}$ is regular at p_j and at p_k and interchanges these two points.*

We will say that the nine points A_i, B_i, C_i, $i = 1, 2, 3$, form a *Pascal configuration*, if the six points A_1, A_2A_3, C_1, C_2, C_3 lie on a conic, and

$$B_1 = (A_2C_3) \cap (A_3C_2), \quad B_2 = (A_3C_1) \cap (A_1C_3), \quad B_3 = (A_1C_2) \cap (A_2C_1).$$

By Pascal's theorem, the points B_1, B_2, B_3 are collinear. See Fig. 1

Theorem 1 *Let the points A_i, B_i, C_i, $i = 1, 2, 3$, form a Pascal configuration. Consider the pencil $\mathcal{E}$ of cubic curves with these base points. Then the map*

$$f = I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} \tag{13}$$
$$= I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2} \tag{14}$$
$$= I_{\mathcal{E},A_3} \circ I_{\mathcal{E},B_3} = I_{\mathcal{E},B_3} \circ I_{\mathcal{E},C_3} \tag{15}$$

is a birational map of degree 2, with $\mathcal{I}(f) = \{C_1, C_2, C_3\}$ and $\mathcal{I}(f^{-1}) = \{A_1, A_2, A_3\}$. It has the following singularity confinement patterns:

$$(C_2C_3) \to A_1 \to B_1 \to C_1 \to (A_2A_3), \tag{16}$$
$$(C_3C_1) \to A_2 \to B_2 \to C_2 \to (A_3A_1), \tag{17}$$
$$(C_1C_2) \to A_3 \to B_3 \to C_3 \to (A_1A_2). \tag{18}$$

Proof We start with the following property of the addition law on a nonsingular cubic curve $\mathcal{C}$. Let $P_1, P_2, P_3, P_4 \in \mathcal{C}$, then

$$P_1 - P_3 = P_4 - P_2 \quad \Leftrightarrow \quad P_1 + P_2 = P_3 + P_4 \quad \Leftrightarrow \quad (P_1P_2) \cap (P_3P_4) \in \mathcal{C}.$$

Thus, on any cubic curve $\mathcal{C} \in \mathcal{E}$, we have the following relations:

$$
\begin{aligned}
(A_1B_2) \cap (A_2B_1) = C_3 \in \mathcal{C} &\Rightarrow A_1 - B_1 = A_2 - B_2 &\Rightarrow I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2}, \\
(B_1C_2) \cap (B_2C_1) = A_3 \in \mathcal{C} &\Rightarrow B_1 - C_1 = B_2 - C_2 &\Rightarrow I_{\mathcal{E},B_1} \circ I_{\mathcal{E},C_1} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2}, \\
(A_1C_2) \cap (B_1B_2) = B_3 \in \mathcal{C} &\Rightarrow A_1 - B_1 = B_2 - C_2 &\Rightarrow I_{\mathcal{E},A_1} \circ I_{\mathcal{E},B_1} = I_{\mathcal{E},B_2} \circ I_{\mathcal{E},C_2}.
\end{aligned}
$$

This proves the coincidence of all six representations in (13)–(15). Now it follows from Proposition 6 that f has only three indeterminacy points, $\mathcal{I}(f) = \{C_1, C_2, C_3\}$, and similarly, $\mathcal{I}(f^{-1}) = \{A_1, A_2, A_3\}$. Moreover, Proposition 6 implies the relations in the middle part of the singularity confinement patterns (16)–(18). The blow-up and blow-down relations are shown with the help of Proposition 7 as follows: $f(C_3) = I_{\mathcal{E},A_2} \circ I_{\mathcal{E},B_2}(C_3) = I_{\mathcal{E},A_2}(A_1) = (A_1A_2)$. □

One can achieve a nice canonical form of the map from Theorem 1 by sending points B_1, B_2, B_3 to infinity (cf. [18]).

Theorem 2 *For a pencil of cubic curves with the base points building a Pascal configuration, perform a linear projective transformation of $\mathbb{P}^2$ sending the Pascal line $\ell(B_1, B_2, B_3)$ to infinity. Let (x, y) be the affine coordinates on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$. In these coordinates, the map $f : (x, y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by (13)–(15) is characterized by the following property. There exist constants $a_1, \ldots, a_9 \in \mathbb{C}$ such that f admits a representation through two bilinear equations of motion of the form*

$$
\begin{cases}
\widetilde{x} - x = a_2 x\widetilde{x} + a_3(x\widetilde{y} + \widetilde{x}y) + a_4 y\widetilde{y} + a_6(x + \widetilde{x}) + a_7(y + \widetilde{y}) + a_9, \\
\widetilde{y} - y = -a_1 x\widetilde{x} - a_2(x\widetilde{y} + \widetilde{x}y) - a_3 y\widetilde{y} - a_5(x + \widetilde{x}) - a_6(y + \widetilde{y}) - a_8.
\end{cases}
\tag{19}
$$

These equations serve as the Kahan discretization [14, 17] of the Hamiltonian equations of motion

$$
\begin{cases}
\dot{x} = \partial H/\partial y = a_2 x^2 + 2a_3 xy + a_4 y^2 + 2a_6 x + 2a_7 y + a_9, \\
\dot{y} = -\partial H/\partial x = -a_1 x^2 - 2a_2 xy - a_3 y^2 - 2a_5 x - 2a_6 y - a_8,
\end{cases}
\tag{20}
$$

for the Hamilton function

$$
H(x, y) = \tfrac{1}{3}a_1 x^3 + a_2 x^2 y + a_3 xy^2 + \tfrac{1}{3}a_4 y^3 + a_5 x^2 + 2a_6 xy + a_7 y^2 + a_8 x + a_9 y. \tag{21}
$$

Proof This is a result of a symbolic computation with MAPLE, presented in [18]. □

7 Quadratic Manin Maps for Special Pencils of the Type $(4; 8^1 2^2)$

We describe the geometry of base points of a pencil of the type $(4; 8^1 2^2)$ for which one can find compositions of Manin involutions which are quadratic Cremona maps.

–   Let p_2, p_3, p_6, p_7 be four points of $\mathbb{P}^2$ in general position (no three of them collinear).

–   Consider three intersection points of three pairs of opposite sides of the complete quadrangle with these vertices:

$$A = (p_2 p_6) \cap (p_3 p_7), \quad B = (p_2 p_3) \cap (p_6 p_7), \quad C = (p_2 p_7) \cap (p_3 p_6). \quad (22)$$

Consider the projective involutive automorphism σ of $\mathbb{P}^2$ fixing the point C and the line $\ell = (AB)$ (pointwise). The points of the pairs (p_2, p_7) and (p_3, p_6) correspond under σ.

–   Choose a point $p_9 \in (p_3 p_7)$, and define $p_{10} \in (p_2 p_6)$ so that p_9, p_{10} correspond under σ, or, in other words, so that the line $(p_9 p_{10})$ passes through C.

–   Let $\mathcal{C} \in \mathcal{P}(2; p_2, p_3, p_6, p_7)$ be any conic of the pencil through the specified four points. Define:

$$
\begin{aligned}
p_1 &= \quad \text{the second intersection point of } \mathcal{C} \text{ with } (p_{10} p_7), \\
p_4 &= \quad \text{the second intersection point of } \mathcal{C} \text{ with } (p_9 p_6), \\
p_5 &= \quad \text{the second intersection point of } \mathcal{C} \text{ with } (p_{10} p_3), \\
p_8 &= \quad \text{the second intersection point of } \mathcal{C} \text{ with } (p_9 p_2).
\end{aligned}
$$

Recall that A, B, C are vertices of a self-polar triangle for $\mathcal{C}$. In particular, the projective involution σ leaves $\mathcal{C}$ invariant. The points of the pairs (p_1, p_8) and (p_4, p_5) correspond under σ.

We will call the pencil $\mathcal{E} = \mathcal{P}(p_1, \ldots, p_8, p_9^2, p_{10}^2)$ a *projectively symmetric quartic pencil with two double points*. See Fig. 2.

Theorem 3 *Let $\mathcal{E} = \mathcal{P}(p_1, \ldots, p_8, p_9^2, p_{10}^2)$ be a projectively symmetric quartic pencil with two double points. Then:*

a) The projective involution σ can be represented as

$$\sigma = I_{1,8}^{(2)} = I_{2,7}^{(2)} = I_{3,6}^{(2)} = I_{4,5}^{(2)}. \quad (23)$$
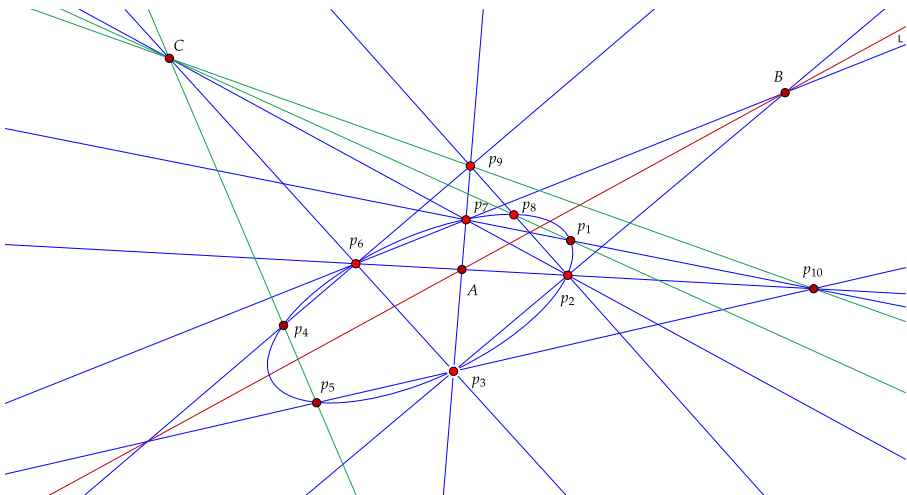

Fig. 2 Geometry of base points of a special quartic pencil $\mathcal{P}(4; p_1, \ldots, p_8, p_9^2, p_{10}^2)$

b) The map

$$f = I_{i,k}^{(2)} \circ I_{j,k}^{(2)} \tag{24}$$

$$= I_9^{(1)} \circ \sigma = \sigma \circ I_{10}^{(1)} \tag{25}$$

with $(i, j) \in \{(1, 2), (2, 3), (3, 4), (5, 6), (6, 7), (7, 8)\}$ and $k \in \{1, \ldots, 8\} \setminus \{i, j\}$, is a birational map of degree 2, with $\mathcal{I}(f) = \{p_4, p_8, p_{10}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_5, p_9\}$. It has the following singularity confinement patterns:

$$(p_8 p_{10}) \to p_1 \to p_2 \to p_3 \to p_4 \to (p_5 p_9), \tag{26}$$

$$(p_4 p_{10}) \to p_5 \to p_6 \to p_7 \to p_8 \to (p_1 p_9), \tag{27}$$

$$(p_4 p_8) \to p_9 \to p_{10} \to (p_1 p_5). \tag{28}$$

c) We have:

$$f^2 = I_9^{(1)} \circ I_{10}^{(1)}. \tag{29}$$

Proof We start with a geometric interpretation of the addition law on a generic curve $\mathcal{C} \in \mathcal{E}$. Recall that the pencil $\mathcal{E}$ can be reduced to a pencil of cubic curves by means of the quadratic Cremona map ϕ based at p_k, p_9, p_{10} for some $k = 1, \ldots, 8$. Lines in the target plane $\mathbb{P}_2^2$, where the cubic pencil is considered, correspond in the source plane $\mathbb{P}_1^2$ of the pencil $\mathcal{E}$ to conics through p_k, p_9, p_{10}. Now let $p, q, r, s \in \mathcal{C}$, then, assuming that neither of the points p_k, p_9, p_{10} is among p, q, r, s, we have:

$$p - r = s - q \quad \Leftrightarrow \quad p + q = r + s \quad \Leftrightarrow \quad \big(\phi(p)\phi(q)\big) \cap \big(\phi(r)\phi(s)\big) \in \phi(\mathcal{C}).$$

The geometry of the pencil $\mathcal{E}$ ensures the existence of a large number of quadruples of base points which, together with p_9, p_{10}, lie on a conic. Namely, the following sextuples are conconical:

$$(p_1, p_2, p_7, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8, \quad p_2 \leftrightarrow p_7 \text{ under } \sigma, \tag{30}$$

$$(p_1, p_3, p_6, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8, \quad p_3 \leftrightarrow p_6 \text{ under } \sigma, \tag{31}$$

$$(p_1, p_4, p_5, p_8, p_9, p_{10}) \text{ because } p_1 \leftrightarrow p_8, \quad p_4 \leftrightarrow p_5 \text{ under } \sigma, \tag{32}$$

$$(p_2, p_3, p_6, p_7, p_9, p_{10}) \text{ because } p_2 \leftrightarrow p_7, \quad p_3 \leftrightarrow p_6 \text{ under } \sigma, \tag{33}$$

$$(p_2, p_4, p_5, p_7, p_9, p_{10}) \text{ because } p_2 \leftrightarrow p_7, \quad p_4 \leftrightarrow p_5 \text{ under } \sigma, \tag{34}$$

$$(p_3, p_4, p_5, p_6, p_9, p_{10}) \text{ because } p_3 \leftrightarrow p_6, \quad p_4 \leftrightarrow p_5 \text{ under } \sigma. \tag{35}$$

The sextuples (30), (33) and (35) lie on reducible conics $\ell(p_1, p_7, p_{10}) \cup \ell(p_2, p_8, p_9)$, $\ell(p_2, p_6, p_{10}) \cup \ell(p_3, p_7, p_9)$ and $\ell(p_3, p_5, p_{10}) \cup \ell(p_4, p_6, p_9)$, respectively. One has, additionally, two more sextuples lying on reducible conics:

$$(p_1, p_4, p_6, p_7, p_9, p_{10}) \quad - \quad \text{on a reducible conic } \ell(p_1, p_7, p_{10}) \cup \ell(p_4, p_6, p_9), \tag{36}$$

$$(p_2, p_3, p_5, p_8, p_9, p_{10}) \quad - \quad \text{on a reducible conic } \ell(p_3, p_5, p_{10}) \cup \ell(p_2, p_8, p_9). \tag{37}$$

- From (31), (37), (36), (34) there follows:

$$\begin{cases} C(p_1, p_6, p_3, p_9, p_{10}) \cap C(p_2, p_5, p_3, p_9, p_{10}) \ni p_8, \\ C(p_1, p_6, p_4, p_9, p_{10}) \cap C(p_2, p_5, p_4, p_9, p_{10}) \ni p_7, \\ C(p_1, p_6, p_7, p_9, p_{10}) \cap C(p_2, p_5, p_7, p_9, p_{10}) \ni p_4, \\ C(p_1, p_6, p_8, p_9, p_{10}) \cap C(p_2, p_5, p_8, p_9, p_{10}) \ni p_3. \end{cases}$$

We explain how these relations are used, taking the first one as example. The intersection $C(p_1, p_6, p_3, p_9, p_{10}) \cap C(p_2, p_5, p_3, p_9, p_{10})$ consists of p_3, p_9, p_{10}, and p_8. Upon the quadratic Cremona map ϕ based at p_3, p_9, p_{10}, this means that the lines $(q_1 q_6)$ and $(q_2 q_5)$ intersect at q_8, where $q_i = \phi(p_i)$ (the blow-ups of other three intersection points do not belong to the proper image of the conics). The point q_8 is one of the base points of the cubic pencil $\phi(\mathcal{E})$. Thus, the four relations above imply

$$I_{1,k}^{(2)} \circ I_{2,k}^{(2)} = I_{5,k}^{(2)} \circ I_{6,k}^{(2)}, \quad k = 3, 4, 7, 8. \tag{38}$$

- From (35), (33), (37) there follows:

$$\begin{cases} C(p_2, p_6, p_1, p_9, p_{10}) \cap C(p_3, p_5, p_1, p_9, p_{10}) \supset (p_1 p_9), \\ C(p_2, p_6, p_4, p_9, p_{10}) \cap C(p_3, p_5, p_4, p_9, p_{10}) \supset (p_4 p_9), \\ C(p_2, p_6, p_7, p_9, p_{10}) \cap C(p_3, p_5, p_7, p_9, p_{10}) \supset (p_7 p_9), \\ C(p_2, p_6, p_8, p_9, p_{10}) \cap C(p_3, p_5, p_8, p_9, p_{10}) \supset (p_8 p_9). \end{cases}$$

Again, we explain how these relations are used, taking the first one as example. The intersection $C(p_2, p_6, p_1, p_9, p_{10}) \cap C(p_3, p_5, p_1, p_9, p_{10})$ consists of the point p_{10} and the line $(p_1 p_9)$. Upon the quadratic Cremona map ϕ based at p_1, p_9, p_{10}, the point p_{10} is blown up to a line which does not belong to the proper image of the conics, while the line $(p_1 p_9)$ is blown down to the point q_{10} through which the proper images of the both conics pass. Thus, the lines $(q_2 q_6)$ and $(q_3 q_5)$ intersect at q_{10}, which is a base point of the pencil $\phi(\mathcal{E})$. Summarizing, the four relations above imply

$$I_{2,k}^{(2)} \circ I_{3,k}^{(2)} = I_{5,k}^{(2)} \circ I_{6,k}^{(2)}, \quad k = 1, 4, 7, 8. \tag{39}$$

- From (31), (32), (33), (34), there follows:

$$\begin{cases} C(p_3, p_6, p_1, p_9, p_{10}) \cap C(p_4, p_5, p_1, p_9, p_{10}) \ni p_8, \\ C(p_3, p_6, p_2, p_9, p_{10}) \cap C(p_4, p_5, p_2, p_9, p_{10}) \ni p_7, \\ C(p_3, p_6, p_7, p_9, p_{10}) \cap C(p_4, p_5, p_7, p_9, p_{10}) \ni p_2, \\ C(p_3, p_6, p_8, p_9, p_{10}) \cap C(p_4, p_5, p_8, p_9, p_{10}) \ni p_1. \end{cases}$$

Exactly as before, these four relations imply

$$I_{3,k}^{(2)} \circ I_{4,k}^{(2)} = I_{5,k}^{(2)} \circ I_{6,k}^{(2)}, \quad k = 1, 2, 7, 8. \tag{40}$$

- In exactly the same way we prove that

$$I_{1,k}^{(2)} \circ I_{2,k}^{(2)} = I_{6,k}^{(2)} \circ I_{7,k}^{(2)}, \quad k = 3, 4, 5, 8. \tag{41}$$

and

$$I_{1,k}^{(2)} \circ I_{2,k}^{(2)} = I_{7,k}^{(2)} \circ I_{8,k}^{(2)}, \quad k = 3, 4, 5, 6. \tag{42}$$

This completes the proof of coincidence of all representations (24), as well as the middle part of the singularity confinement patterns (26), (27).

– One sees immediately that $I_9^{(1)} \circ \sigma$ is a shift with respect to the addition law on the curves of $\mathcal{E}$, sending $p_1 \to p_2 \to p_3 \to p_4$, while $\sigma \circ I_{10}^{(1)}$ is a shift sending $p_5 \to p_6 \to p_7 \to p_8$. Therefore, these shifts must coincide with f. This proves (25) and the middle part of the singularity confinement pattern (28).

– Collecting all the results, we see that $\mathcal{I}(f) = \{p_4, p_8, p_{10}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_5, p_9\}$, so that f must be a quadratic Cremona map.

– It remains to show the blow-up and blow-down relations in the singularity confinement patterns (26)–(28). To see the blow-down relations on the left, we use the representation $f = I_9^{(1)} \circ \sigma$. The involution σ is a projective automorphism and has no singularities, so it suffices to study the blowing down patterns of $I_9^{(1)}$. By definition of the map $I_9^{(1)}$, it is clear that it blows down the line $(p_1 p_9)$ to the point p_1, and blows down the line $(p_5 p_9)$ to the point p_5. Since f is a quadratic Cremona map, the same holds true for the involution $I_9^{(1)}$; there follows that $I_9^{(1)}$ must blow up p_9 to the line $(p_1 p_5)$, which finishes the proof. For the blow-up relations on the right part of (26)–(28), we use $f = \sigma \circ I_{10}^{(1)}$ in a similar manner. □

We now turn to canonical forms of projectively symmetric quartic pencils with two double points, which can be achieved by projective automorphisms of $\mathbb{P}^2$. The most popular one corresponds to the choice $p_9 = [0 : 1 : 0]$, $p_{10} = [1 : 0 : 0]$, so that the quartic curves become *biquadratic* ones. Denote the inhomogeneous coordinates on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$ by (u, v). We can arrange $p_2 = (1, \beta)$, $p_7 = (\beta, 1)$, $p_3 = (\beta, -1)$, $p_6 = (-1, \beta)$, so that $\ell = \{u - v = 0\}$, $C = (p_2 p_7) \cap (p_3 p_6) = [-1 : 1 : 0]$, $A = (p_2 p_6) \cap (p_3 p_7) = (\beta, \beta)$, and σ is the Euclidean reflection at the line ℓ,

$$\sigma(u, v) = (v, u).$$

The pencil $\mathcal{E}$ of biquadratics reads

$$\alpha(\alpha + 1)(u^2 + v^2 - 1) - (\alpha + 1)uv + \beta(u + v) - \beta^2 - \lambda(u^2 - 1)(v^2 - 1) = 0, \tag{43}$$

and is symmetric under σ. Involutions $I_9^{(1)}$ and $I_{10}^{(1)}$ are nothing but the standard vertical and horizontal QRT switches for this pencil, and the map $f = I_9^{(1)} \circ \sigma = \sigma \circ I_{10}^{(1)}$ of Theorem 3 is given by

$$f : (u, v) \mapsto (\tilde{u}, \tilde{v}), \quad \begin{cases} \tilde{u} = v, \\ \tilde{v} = \dfrac{\alpha uv + \beta u - 1}{u - \alpha v - \beta}. \end{cases} \tag{44}$$

It is the "QRT root" of $f^2 = I_9^{(1)} \circ I_{10}^{(1)}$.

To arrive at another canonical form of projectively symmetric quartic pencils with two double points, we perform a linear projective change of variables in $\mathbb{P}^2$, given in the inhomogeneous coordinates by

$$u = \frac{1 + \beta x + y}{x}, \quad v = \frac{1 + \beta x - y}{x}. \tag{45}$$

Upon substitution (45) and some straightforward simplifications, we come to the following system (cf. [20]):

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = (1 - 2\alpha) - 2\alpha\beta(x + \widetilde{x}) + \left(1 - \beta^2(1 + 2\alpha)\right)x\widetilde{x} - (1 + 2\alpha)y\widetilde{y}. \end{cases} \tag{46}$$

In order to give an intrinsic geometric characterization of this canonical form, we will need the following observation.

Proposition 8 *The following five intersection points are collinear:*

$$(p_1 p_8) \cap (p_2 p_7), \quad (p_1 p_5) \cap (p_3 p_7), \quad (p_2 p_5) \cap (p_3 p_8), \quad (p_1 p_6) \cap (p_4 p_7), \quad (p_2 p_6) \cap (p_4 p_8).$$

Proof The triple of intersection points

$$(p_1 p_8) \cap (p_2 p_7), \quad (p_1 p_5) \cap (p_3 p_7), \quad (p_2 p_5) \cap (p_3 p_8)$$

lies on the Pascal line for the hexagon $(p_1, p_5, p_2, p_7, p_3, p_8)$, while the triple of intersection points

$$(p_1 p_8) \cap (p_2 p_7), \quad (p_1 p_6) \cap (p_4 p_7), \quad (p_2 p_6) \cap (p_4 p_8)$$

lies on the Pascal line for the hexagon $(p_1, p_6, p_2, p_7, p_4, p_8)$. These hexagons correspond under σ, therefore this holds true also for their Pascal lines. Moreover, the Pascal lines share the point $(p_1 p_8) \cap (p_2 p_7) = C$, therefore they must coincide. $\square$

We will call the line containing the five intersection points from Proposition 8 the *double Pascal line*.

Theorem 4 *For a projectively symmetric pencil of quartic curves with two double points, perform a linear projective transformation of $\mathbb{P}^2$ sending the double Pascal line to infinity. By a subsequent affine change of coordinates (x, y) on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$, arrange that ℓ coincides with the axis $y = 0$, $p_9 = (0, -1)$, and $p_{10} = (0, 1)$. In these coordinates, the map $f : (x, y) \mapsto (\widetilde{x}, \widetilde{y})$ defined by (24)–(25) is characterized by the following property. There exist $a_0, \ldots, a_3 \in \mathbb{C}$ with $a_0 + a_3 = 2$ such that f admits a representation through two bilinear equations of motion of the form*

$$\begin{cases} \widetilde{x} - x = x\widetilde{y} + \widetilde{x}y, \\ \widetilde{y} - y = a_0 - a_1(x + \widetilde{x}) - a_2 x\widetilde{x} - a_3 y\widetilde{y}. \end{cases} \tag{47}$$

Proof A symbolic computation with MAPLE. $\square$

8 Quadratic Manin Maps for Special Pencils of the Type $(6; 6^1 3^2 2^3)$

In this section, we consider the following example considered in detail in [7, 17, 21, 27]. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map given in the non-homogeneous coordinates $\mathbf{x} = (x, y)$ on the affine part $\mathbb{C}^2 \subset \mathbb{P}^2$ by two bilinear relations between (x, y) and $(\widetilde{x}, \widetilde{y}) = f(x, y)$:

$$
\begin{aligned}
\widetilde{\mathbf{x}} - \mathbf{x} = \ & \gamma_1(\ell_2(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_2(\widetilde{\mathbf{x}})\ell_3(\mathbf{x}))J\nabla\ell_1 \\
& + \gamma_2(\ell_1(\mathbf{x})\ell_3(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_3(\mathbf{x}))J\nabla\ell_2 \\
& + \gamma_3(\ell_1(\mathbf{x})\ell_2(\widetilde{\mathbf{x}}) + \ell_1(\widetilde{\mathbf{x}})\ell_2(\mathbf{x}))J\nabla\ell_3,
\end{aligned}
\tag{48}
$$

where $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$, $\ell_i(\mathbf{x}) = a_i x + b_i y$ are linear forms with $a_i, b_i \in \mathbb{C}$, and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This map is the Kahan discretization of the quadratic flow

$$
\dot{\mathbf{x}} = \gamma_1\ell_2(\mathbf{x})\ell_3(\mathbf{x})J\nabla\ell_1 + \gamma_2\ell_1(\mathbf{x})\ell_3(\mathbf{x})J\nabla\ell_2 + \gamma_3\ell_1(\mathbf{x})\ell_2(\mathbf{x})J\nabla\ell_3,
\tag{49}
$$

which can be put as

$$
\dot{\mathbf{x}} = \frac{1}{(\ell_1(\mathbf{x}))^{\gamma_1-1}(\ell_2(\mathbf{x}))^{\gamma_2-1}(\ell_3(\mathbf{x}))^{\gamma_3-1}}J\nabla H_0(\mathbf{x}),
\tag{50}
$$

where

$$
H_0(\mathbf{x}) = (\ell_1(\mathbf{x}))^{\gamma_1}(\ell_2(\mathbf{x}))^{\gamma_2}(\ell_3(\mathbf{x}))^{\gamma_3}.
\tag{51}
$$

Integrability of the Kahan discretization (48) was demonstrated in [7, 17, 21] for $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1), (1, 1, 2)$, and $(1, 2, 3)$. Sections 6 and 7 deal with generalizations of the cases $(\gamma_1, \gamma_2, \gamma_3) = (1, 1, 1), (1, 1, 2)$, the present one deals with $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$.

As shown in [7, 17, 21], the map f defined by (48) with $(\gamma_1, \gamma_2, \gamma_3) = (1, 2, 3)$ admits an integral of motion:

$$
H(\mathbf{x}) = \frac{H_0(\mathbf{x})}{L_+(\mathbf{x})L_-(\mathbf{x})M_+(\mathbf{x})M_-(\mathbf{x})Q(\mathbf{x})},
\tag{52}
$$

where

$$
\begin{aligned}
L_\pm(\mathbf{x}) &= 1 \pm 3d_{31}\ell_2(\mathbf{x}), \\
M_\pm(\mathbf{x}) &= 1 \pm (3d_{23}\ell_1(\mathbf{x}) - d_{12}\ell_3(\mathbf{x})), \\
Q(\mathbf{x}) &= 1 - \left(9d_{31}^2\ell_2^2(\mathbf{x}) + 16d_{12}^2\ell_3^2(\mathbf{x})\right),
\end{aligned}
$$

with

$$
d_{ij} = a_i b_j - a_j b_i.
$$

Thus, the phase space of f is foliated by the pencil of invariant curves

$$
C_\lambda = \left\{ H_0(\mathbf{x}) - \lambda L_+(\mathbf{x})L_-(\mathbf{x})M_+(\mathbf{x})M_-(\mathbf{x})Q(\mathbf{x}) = 0 \right\}.
\tag{53}
$$

The pencil has $\deg = 6$ and contains two reducible curves: $C_0 = \{H_0(x, y) = 0\}$, consisting of the lines $\{\ell_i(x, y) = 0\}$, $i = 1, 2, 3$, with multiplicities 1,2,3, and C_∞, consisting of the conic $Q(x, y) = 0$ and the four lines $L_\pm(x, y) = 0$, $M_\pm(x, y) = 0$.

All curves C_λ pass through the set of base points which is defined as $C_0 \cap C_\infty$. See Fig. 3. One easily computes the 11 (distinct) base points of the pencil. They are given by:

– six base points of multiplicity 1 on the line $\ell_1 = 0$:

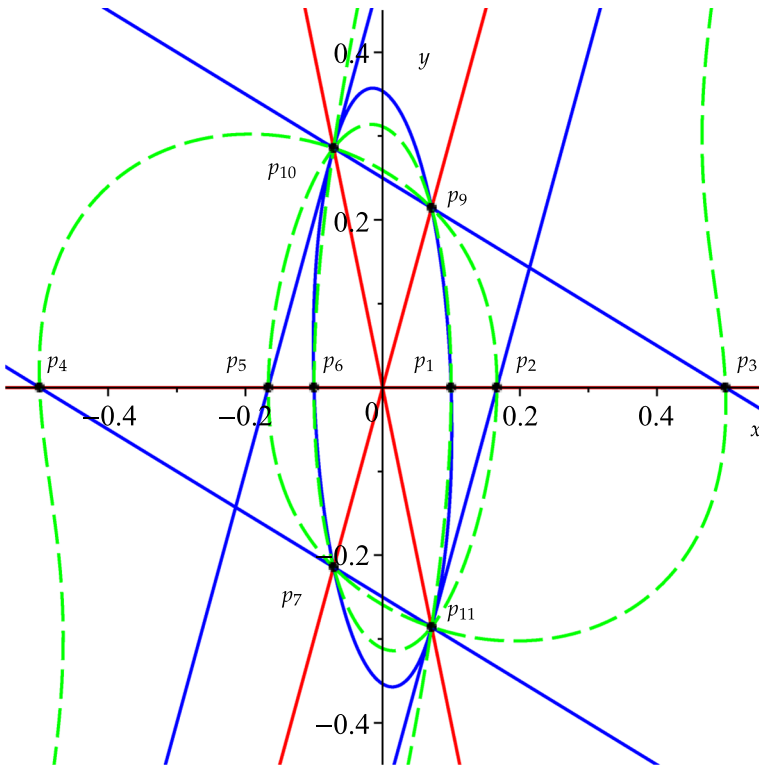$$p_1 = \left(-\frac{b_1}{5d_{12}d_{31}}, \frac{a_1}{5d_{12}d_{31}} \right), \quad p_2 = \left(-\frac{b_1}{3d_{12}d_{31}}, \frac{a_1}{3d_{12}d_{31}} \right), \quad p_3 = \left(-\frac{b_1}{d_{12}d_{31}}, \frac{a_1}{d_{12}d_{31}} \right),$$

$$p_4 = -p_3, \quad p_5 = -p_2, \quad p_6 = -p_1;$$

– three base points of multiplicity 2 on the line $\ell_2 = 0$:

$$p_7 = \left(-\frac{b_2}{4d_{12}d_{23}}, \frac{a_2}{4d_{12}d_{23}} \right), \quad p_8 = [b_2 : -a_2 : 0], \quad p_9 = -p_7;$$

– and two base points of multiplicity 3 on the line $\ell_3 = 0$:

$$p_{10} = \left(-\frac{b_3}{3d_{23}d_{31}}, \frac{a_3}{3d_{23}d_{31}} \right), \quad p_{11} = -p_{10}.$$

Fig. 3 The curves C_0, C_∞, $C_{-0.002}$ of the sextic pencil (in red, blue and green, respectively) for $\ell_1(\mathbf{x}) = y/6$, $\ell_2(\mathbf{x}) = 3x - y$, $\ell_3(\mathbf{x}) = 4x + y$

The map f is a quadratic Cremona map with the indeterminacy points $\mathcal{I}(f) = \{p_6, p_9, p_{11}\}$ and $\mathcal{I}(f^{-1}) = \{p_1, p_7, p_{10}\}$, and has the following singularity confinement patterns:

$$(p_9 p_{11}) \to p_1 \to p_2 \to p_3 \to p_4 \to p_5 \to p_6 \to (p_7 p_{10}),$$
$$(p_6 p_{11}) \to p_7 \to p_8 \to p_9 \to (p_1 p_{10}),$$
$$(p_6 p_9) \to p_{10} \to p_{11} \to (p_1 p_7).$$

Unlike the previous two sections, we will not derive here general geometric conditions for the base points of the pencil which ensure that certain compositions of Manin involutions are quadratic Cremona maps. Rather, we will give here the corresponding statements for the pencil (53).

Theorem 5 *The map f can be represented as compositions of the Manin involutions in the following ways:*

$$f = I^{(4)}_{i,k,m} \circ I^{(4)}_{j,k,m} = I^{(3)}_{i,n} \circ I^{(3)}_{j,n}$$

for any $(i, j) \in \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$, $k \in \{1, \ldots, 6\} \setminus \{i, j\}$, and $m \in \{7, 8, 9\}$, $n \in \{10, 11\}$.

Proof Symbolic computation with MAPLE. $\qquad\qquad\square$

9 Conclusions

The contribution of this paper is two-fold:

– Finding geometric description of Manin involutions for elliptic pencil consisting of curves of higher degree, birationally equivalent to cubic pencils (Halphen pencils of index 1).
– Characterizing special geometry of base points ensuring that certain compositions of Manin involutions are integrable maps of low degree (quadratic Cremona maps). As particular cases, we identify some integrable Kahan discretizations as compositions of Manin involutions.

It should be mentioned that both issues can and should be studied also for Halphen pencils of index $m > 1$. For instance, for a Halphen pencil of index 2, $\mathcal{P}(6; p_1^2, ..., p_9^2)$, one can propose the following construction of involutions I_{p_i}, $i = 1, \ldots, 9$. For any $p \in \mathbb{P}^2$ different from the base points, consider the cubic curve through p and $p_2, \ldots, p_9$. Its intersection number with the Halphen's curve of degree 6 through p is $3 \times 6 = 18$. The intersections with eight base points $p_2, \ldots, p_9$ and with p count as $8 \times 2 + 1 = 17$, so there is exactly one remaining intersection point p'. We declare $p' = I_{p_1}(p)$. One can see that the so defined involutions $I_{p_i} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ are Bertini involutions (of degree 17 in the generic situation). However, one can show that the birational map of degree 3 from [3, eq. (6)] is a composition of two such involutions, due to very special geometry of the base points, involving infintely

near ones. We hope to be able to identify in the future work further low degree integrable birational maps as compositions of fundamental involutions defined by elliptic pencils.

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