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Mannheim Partner Curves in 3-Space

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1 Introduction.

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are the very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve Γ, it shares the normal lines with another curve Γ_1, called Bertrand mate or Bertrand partner curve of Γ. In this paper, we are concerned with another kind of associated curves, called Mannheim curve and Mannheim mate (partner curve) in history of differential geometry. In this work, we call them simply as Mannheim pair.

Definition 1. *Let $\mathbb{E}^3$ be the 3-dimensional Euclidean space with the standard inner product $\langle\ ,\rangle$. If there exists a corresponding relationship between the space curves Γ and Γ_1 such that, at the corresponding points of the curves, the principal normal lines of Γ coincides with the binormal lines of Γ_1, then Γ is called a Mannheim curve, and Γ_1 a Mannheim partner curve of Γ. The pair $\{\Gamma, \Gamma_1\}$ is said to be a Mannheim pair.*

From the elementary differential geometry we know clearly about the characterizations of Bertrand pair. But there are rather few works on Mannheim pair. It is just known that a space curve in $\mathbb{E}^3$ is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the formula $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant.

In this paper, we study the Mannheim partner curves in three dimensional

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Euclidean space $\mathbb{E}^3$ and three dimensional Minkowski space $\mathbb{E}_1^3$. We will give the necessary and sufficient conditions for the Mannheim partner curves in Euclidean space $\mathbb{E}^3$ and Minkowski space $\mathbb{E}_1^3$, respectively. In [CH], Prof. B. Y. Chen characterizes the curve which satisfies $\frac{\tau}{\kappa} = as + b$, $a \neq 0$. Here, our examples will give the curve which satisfies $\frac{\tau}{\kappa} = \sinh s$.

2 Mannheim partner curves in $\mathbb{E}^3$.

Let $\Gamma : x(s)$ be a Mannheim curve in $\mathbb{E}^3$ parameterized by its arc length s and $\Gamma_1 : x_1(s_1)$ the Mannheim partner curve of Γ with the arc length parameter s_1. Denote by $\{\alpha(s), \beta(s), \gamma(s)\}$ the Frenet frame field along $\Gamma : x(s)$, that is, $\alpha(s)$ is the tangent vector field, $\beta(s)$ the normal vector field and $\gamma(s)$ the binormal vector field of the curve Γ, respectively. The famous Frenet formulas are given by

$$\begin{cases} \dot{\alpha}(s) = \kappa(s)\beta(s) \\ \dot{\beta}(s) = -\kappa(s)\alpha(s) + \tau(s)\gamma(s) \\ \dot{\gamma}(s) = -\tau(s)\beta(s). \end{cases}$$

Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

Theorem 1. *Let $\Gamma : x(s)$ be a Mannheim curve in $\mathbb{E}^3$ with the arc length parameter s. Then $\Gamma_1 : x_1(s_1)$ is the Mannheim partner curve of Γ if and only if the curvature κ_1 and the torsion τ_1 of Γ_1 satisfy the following equation*

$$\dot{\tau_1} = \frac{\mathrm{d}\tau_1}{\mathrm{d}s_1} = \frac{\kappa_1}{\lambda}(1 + \lambda^2\tau_1^2)$$

for some nonzero constant λ.

Proof. Suppose that $\Gamma : x(s)$ is a Mannheim curve. Then by the definition we can assume that

$$x(s_1) = x_1(s_1) + \lambda(s_1)\gamma_1(s_1) \tag{2.1}$$

for some function $\lambda(s_1)$. By taking the derivative of (2.1) with respect to s_1 and applying the Frenet formulas, we have

$$\alpha\frac{\mathrm{d}s}{\mathrm{d}s_1} = \alpha_1 + \dot{\lambda}\gamma_1 - \lambda\tau_1\beta_1. \tag{2.2}$$

Since γ_1 is coincident with β in direction, we get

$$\dot{\lambda}(s_1) = 0.$$

This means that λ is a nonzero constant. Thus we have

$$\alpha\frac{\mathrm{d}s}{\mathrm{d}s_1} = \alpha_1 - \lambda\tau_1\beta_1. \tag{2.3}$$

On the other hand, we have

$$\alpha = \alpha_1 \cos\theta + \beta_1 \sin\theta, \tag{2.4}$$

where θ is the angle between α and α_1 at the corresponding points of Γ and Γ_1. By taking the derivative of this equation with respect to s_1, we obtain

$$\kappa\beta \frac{\mathrm{d}s}{\mathrm{d}s_1} = -(\kappa_1 + \dot\theta)\sin\theta\alpha_1 + (\kappa_1 + \dot\theta)\cos\theta\beta_1 + \tau_1 \sin\theta\gamma_1.$$

From this equation and the fact that the direction of β is coincident with γ_1, we get

$$\begin{cases} (\kappa_1 + \dot\theta)\sin\theta = 0 \\ (\kappa_1 + \dot\theta)\cos\theta = 0. \end{cases}$$

Therefore we have

$$\dot\theta = -\kappa_1. \tag{2.5}$$

From (2.3), (2.4) and notice that α_1 is orthogonal to β_1, we find that

$$\frac{\mathrm{d}s}{\mathrm{d}s_1} = \frac{1}{\cos\theta} = -\frac{\lambda\tau_1}{\sin\theta}.$$

Then we have

$$\lambda\tau_1 = -\tan\theta.$$

By taking the derivative of this equation and applying (2.5), we get

$$\lambda\dot\tau_1 = \kappa_1(1 + \lambda^2\tau_1^2),$$

that is

$$\dot\tau_1 = \frac{\kappa_1}{\lambda}(1 + \lambda^2\tau_1^2).$$

Conversely, if the curvature κ_1 and torsion τ_1 of the curve Γ_1 satisfy

$$\dot\tau_1 = \frac{\kappa_1}{\lambda}(1 + \lambda^2\tau_1^2)$$

for some nonzero constant λ, then define a curve Γ by

$$x(s_1) = x_1(s_1) + \lambda\gamma_1(s_1) \tag{2.6}$$

and we will prove that Γ is a Mannheim curve and Γ_1 is the partner curve of Γ.

By taking the derivative of (2.6) with respect to s_1 twice, we get

$$\alpha\frac{\mathrm{d}s}{\mathrm{d}s_1} = \alpha_1 - \lambda\tau_1\beta_1, \tag{2.7}$$

$$\kappa\beta\left(\frac{\mathrm{d}s}{\mathrm{d}s_1}\right)^2 + \alpha\frac{\mathrm{d}^2 s}{\mathrm{d}s_1^2} = \lambda\kappa_1\tau_1\alpha_1 + (\kappa_1 - \lambda\dot\tau_1)\beta_1 - \lambda\tau_1^2\gamma_1, \tag{2.8}$$

respectively. Taking the cross product of (2.7) with (2.8) and noticing that

$$\kappa_1 - \lambda\dot{\tau}_1 + \lambda^2\kappa_1\tau_1^2 = 0,$$

we have

$$\kappa\gamma\left(\frac{\mathrm{d}s}{\mathrm{d}s_1}\right)^3 = \lambda^2\tau_1^3\alpha_1 + \lambda\tau_1^2\beta_1. \tag{2.9}$$

By taking the cross product of (2.9) with (2.7), we obtain also

$$\kappa\beta\left(\frac{\mathrm{d}s}{\mathrm{d}s_1}\right)^4 = -\lambda\tau_1^2(1+\lambda^2\tau_1^2)\gamma_1.$$

This means that the principal normal direction β of $\Gamma : x(s)$ coincides with the binormal direction γ_1 of $\Gamma_1 : x_1(s_1)$. Hence $\Gamma : x(s)$ is a Mannheim curve and $\Gamma_1 : x_1(s_1)$ is its Mannheim partner curve. □

Remark 1. *By a simple parameter transformation, the condition*

$$\dot{\tau}_1 = \frac{\kappa_1}{\lambda}(1+\lambda^2\tau_1^2)$$

can be written as

$$\tau_1 = \frac{1}{\lambda}\tan(\int \kappa_1 \mathrm{d}s_1 + c_0).$$

Therefore, for each Mannheim curve, there is an unique Mannheim partner curve.

We have the following Examples (Helices as Mannheim partner curves).

Proposition 1. *Let $\Gamma : x(s)$ be a Mannheim curve in $\mathbb{E}^3$ with the arc length parameter s and $\Gamma_1 : x_1(s_1)$ the Mannheim partner curve of Γ with the arc length parameter s_1. If $\Gamma : x(s)$ is a generalized helix, then $\Gamma_1 : x_1(s_1)$ is a straight line.*

Proof. Let α, β, γ be the tangent, principal normal and binormal vector field of the curve $\Gamma : x(s)$, respectively. From the properties of generalized helices and the definition of Mannheim curves, we have

$$\gamma_1 \cdot p = \beta \cdot p = 0$$

for some constant vector p. Then it is easy to obtain that $\tau_1 = \kappa_1 \equiv 0$. □

Proposition 2. *If a generalized helix is the Mannheim partner curve of some curve $\Gamma : x(s)$ in $\mathbb{E}^3$, then the ratio of torsion and curvature of the curve $\Gamma : x(s)$ is*

$$\frac{\tau}{\kappa} = \frac{c_2}{2}e^{c_1 s} - \frac{1}{2c_2}e^{-c_1 s}$$

for some nonzero constant c_1 *and* c_2 *and* s *is the arc length parameter of* Γ. *In particular, if we put* $c_1 = c_2 = 1$, *we have*

$$\frac{\tau}{\kappa} = \frac{e^s - e^{-s}}{2} = \sinh s.$$

Proof. Let α, β, γ be the tangent, principal normal and binormal vector field of the curve $\Gamma : x(s)$, respectively. From the properties of generalized helices and the definition of Mannheim curves, we have

$$\beta \cdot p = \cos\theta_0$$

for some constant vector p and some constant angle θ_0. From Proposition 1 we know that $\cos q_0 \neq 0$ and $\frac{\tau}{\kappa} \neq$ constant. By taking the derivative of this equation with respect to s twice, we get

$$-\kappa\alpha \cdot p + \tau\gamma \cdot p = 0,$$

$$-\dot{\kappa}\alpha \cdot p + \dot{\tau}\gamma \cdot p = (\kappa^2 + \tau^2)\cos\theta_0.$$

By a direct calculation and using $\kappa = \lambda(\kappa^2 + \tau^2)$, we obtain

$$\alpha \cdot p = \frac{\tau}{\lambda\kappa \dfrac{\mathrm{d}(\tau/\kappa)}{\mathrm{d}s}} \cos\theta_0,$$

$$\gamma \cdot p = \frac{1}{\lambda \dfrac{\mathrm{d}(\tau/\kappa)}{\mathrm{d}s}} \cos\theta_0.$$

Taking the derivative, we have

$$\kappa = \frac{1}{\lambda}\left(1 - \frac{\tau \dfrac{\mathrm{d}^2(\tau/\kappa)}{\mathrm{d}s^2}}{\kappa\left(\dfrac{\mathrm{d}(\tau/\kappa)}{\mathrm{d}s}\right)^2}\right),$$

$$\tau = \frac{\dfrac{\mathrm{d}^2(\tau/\kappa)}{\mathrm{d}s^2}}{\lambda\left(\dfrac{\mathrm{d}(\tau/\kappa)}{\mathrm{d}s}\right)^2},$$

respectively. From these equations, we find that

$$\frac{\tau}{\kappa} = \frac{\dfrac{\mathrm{d}^2(\tau/\kappa)}{\mathrm{d}s^2}}{\left(\dfrac{\mathrm{d}(\tau/\kappa)}{\mathrm{d}s}\right)^2 - \dfrac{\tau}{\kappa}\dfrac{\mathrm{d}^2(\tau/\kappa)}{\mathrm{d}s^2}}.$$

Let $\tau/\kappa = y(s)$, then we get the following differential equation

$$(1+y^2)\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} - y\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 0.$$

Solving this equation, we obtain that

$$y(s) = c_0$$

or

$$y(s) = \frac{c_2}{2}e^{c_1 s} - \frac{1}{2c_2}e^{-c_1 s}$$

for some nonzero constants c_0, c_1 and c_2. Thus, the proposition is proved. □

Remark 2. *It is well known that a twisted curve in $\mathbb{E}^3$ is a generalized helix if and only if the ratio τ/κ is a nonzero constant* (*see* [CA]). *It is also known that a twisted curve is congruent to a rectifying curve if and only if the ratio τ/κ is a nonconstant linear function of the arc length parameter* (*see* [CH]). *The proposition 2 provides some characterizations of the curves whose "slope" τ/κ is hyperbolic sine function in arc length s, i.e., $\tau/\kappa = \sinh s$.*

3 Mannheim partner curves in $\mathbb{E}_1^3$.

Let $\mathbb{E}_1^3$ be the 3-dimensional Minkowski space with the indefinite inner product

$$\langle \cdot, \cdot \rangle = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2$$

in terms of natural coordinates (x_1, x_2, x_3). A vector $\alpha \neq 0$ in $\mathbb{E}_1^3$ is called spacelike, timelike or lightlike, if $\langle \alpha, \alpha \rangle > 0$, $\langle \alpha, \alpha \rangle < 0$ or $\langle \alpha, \alpha \rangle = 0$, respectively. In this section, we extend the main result of Mannheim partner curves in $\mathbb{E}^3$ to the Minkowski 3-space $\mathbb{E}_1^3$. By a similar calculation, we obtain the following theorem.

Theorem 2. *Let $\Gamma : x(s)$ be a curve in $\mathbb{E}_1^3$ and α, β, γ be tangent, principal normal and binormal vector field of $\Gamma : x(s)$, respectively. Then*

(i) *in case that α and β are spacelike vectors, γ is timelike vector, we have the following Frenet formulas*

$$\dot{\alpha} = \kappa\beta; \quad \dot{\beta} = -\kappa\alpha + \tau\gamma; \quad \dot{\gamma} = \tau\beta.$$

The necessary and sufficient condition for Mannheim partner curves is

$$\dot{\tau} = -\frac{\kappa}{\lambda}(1 + \lambda^2\tau^2).$$

(ii) *in case that α and γ are spacelike vectors, β is timelike vector, the corresponding Frenet formulas are*

$$\dot{\alpha} = \kappa\beta; \quad \dot{\beta} = \kappa\alpha + \tau\gamma; \quad \dot{\gamma} = \tau\beta.$$

The necessary and sufficient condition for Mannheim partner curves is

$$\dot{\tau} = \frac{\kappa}{\lambda}(\lambda^2\tau^2 - 1).$$

(iii) *in case that α is timelike vector, β and γ are spacelike vectors, the corresponding Frenet formulas are*

$$\dot{\alpha} = \kappa\beta; \quad \dot{\beta} = \kappa\alpha + \tau\gamma; \quad \dot{\gamma} = -\tau\beta.$$

The necessary and sufficient condition for Mannheim partner curves is

$$\dot{\tau} = \frac{\kappa}{\lambda}(1 - \lambda^2\tau^2).$$

Where λ is a nonzero constant.

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