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Many-Body Problem of Attractive Fermions with Arbitrary Spin in One Dimension

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A set of coupled integral equations which determine the ground state energy of N fermions with arbitrary spin attracting via a delta-function potential are obtained. In the case of $\frac{1}{2}$ spin our equations are equivalent to those of Gaudin.

§ 1. Introduction

Many authors[1)~7)] treated the exact solution of the problem of N particles interacting via a delta-function potential in one dimension. The Hamiltonian is

$$\mathcal{H}=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+4c\sum_{i<j}\delta(x_i-x_j). \qquad (1)$$

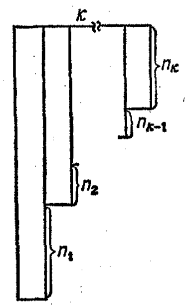

Fig. 1. Young tableau which corresponds to the irreducible representation $[\kappa^{n_\kappa},\cdots,2^{n_2},1^{n_1}]$.

Recently Sutherland[7)] gave the solution of the eigenvalue problem when the wave function transforms like any irreducible representation of S_N. Here we consider the irreducible representation $[\kappa^{n_\kappa},\cdots,2^{n_2},1^{n_1}]$ the Young tableau of which has κ columns (Fig. 1). This corresponds to the problem of fermions with $(\kappa-1)/2$ spin. Eigenvalue problem reduces to a set of coupled transcendental-algebraic equations. For the case $\kappa=3$ Sutherland's equations are

$$\exp(ip_jL)=\prod_{\alpha=1}^{M}\left(\frac{p_j-q_\alpha+ic}{p_j-q_\alpha-ic}\right),\qquad j=1,\cdots,N, \qquad (2a)$$

$$\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\left(\frac{q_\alpha-q_\beta+2ic}{q_\alpha-q_\beta-2ic}\right)=\prod_{j=1}^{N}\left(\frac{q_\alpha-p_j+ic}{q_\alpha-p_j-ic}\right)\prod_{a=1}^{M_1}\left(\frac{q_\alpha-r_a+ic}{q_\alpha-r_a-ic}\right),\qquad \alpha=1,\cdots,M, \qquad (2b)$$

$$\prod_{\substack{b=1\\ b\neq a}}^{M_1}\left(\frac{r_a-r_b+2ic}{r_a-r_b-2ic}\right)=\prod_{\alpha=1}^{M}\left(\frac{r_a-q_\alpha+ic}{r_a-q_\alpha-ic}\right),\qquad a=1,\cdots,M_1, \qquad (2c)$$

$$N=n_1+2n_2+3n_3\,,\qquad M=n_2+2n_3\,,\qquad M_1=n_3\,.$$

Here the p's, q's and r's are quasi-momentums and L is the length of the system. A solution of (2) determines an eigenstate. Its energy eigenvalue is given by

$$E=\sum_{j=1}^{n_1+2n_2+3n_3} p_j^2. \tag{3a}$$

These equations are a generalization of Gaudin's[5] and Yang's[6] equations for $\kappa=2$. Sutherland[7] obtained a set of coupled integral equations which give the ground state energy at the thermodynamic limit in the case of repulsive interaction ($c>0$). He used the fact that the quasi-momentums are all real numbers for the ground state in this case. In the case of attractive interaction ($c<0$) the equations (2) hold, but the quasi-momentums of the ground state are not necessarily real numbers. Therefore Sutherland's integral equations do not hold in the case $c<0$. The integral equations for the case $\kappa=2$, $c<0$ was investigated by Gaudin in detail. We give the integral equations for $\kappa=3$, $c<0$ in § 2 and those for arbitrary value of κ in § 3.

§ 2. Case $\kappa=3$, $c<0$

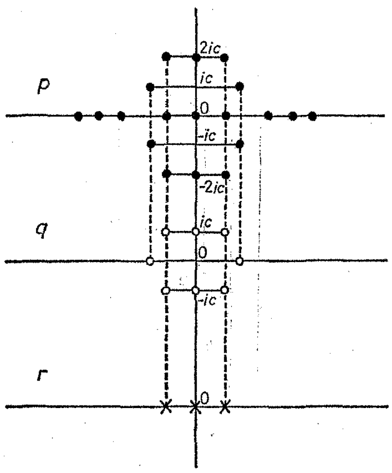

Fig. 2. Distribution of quasi-momentums on the complex plane. In this case $n_1=6$, $n_2=2$, $n_3=3$.

It is expected that the ground state consists of n_1 unpaired electrons, n_2 pairs of two electrons and n_3 pairs of three electrons. Then the quasi-momentums of the ground state are supposed to be

$$p_1, \cdots, p_{n_1}=\text{real};\ q_1, \cdots, q_{n_2}=\text{real};\ r_1, \cdots, r_{n_3}=\text{real},$$

$$p_{n_1+1}, \cdots, p_{n_1+2n_2}=q_1\pm ic, \cdots, q_{n_2}\pm ic, +O(\exp cL),$$

$$q_{n_2+1}, \cdots, q_{n_2+2n_3}=r_1\pm ic, \cdots, r_{n_3}\pm ic, +O(\exp cL),$$

$$p_{n_1+2n_2+1}, \cdots, p_{n_1+2n_2+3n_3}=r_1+\begin{cases}2ic\\0\\-2ic\end{cases}, \cdots, r_{n_3}+\begin{cases}2ic\\0\\-2ic\end{cases},$$

$$+O(\exp cL). \tag{4}$$

The term $O(\exp cL)$ should be zero at the limit $L\to\infty$. Therefore the total energy is given by

$$E=\sum_{i=1}^{n_1} p_i^2+\sum_{j=1}^{n_2}(2q_j^2-2c^2)+\sum_{k=1}^{n_3}(3r_k^2-8c^2). \tag{3b}$$

For $p_j=p_1, \cdots, p_{n_1}$, take the logarithm of (2a)

$$p_jL=2\pi I_j-\sum_{\alpha=1}^{n_2} 2\tan^{-1}\frac{p_j-q_\alpha}{c}-\sum_{\alpha=1}^{n_3} 2\tan^{-1}\frac{p_j-r_\alpha}{2c}. \tag{5a}$$

For $q_\alpha=q_1, \cdots, q_{n_2}$, there must be corresponding p values $p_\alpha, p_{\bar{\alpha}}$. Take the product of (2a):

$$\exp i(p_\alpha+p_{\bar\alpha})L=\exp 2iq_\alpha L=\prod_{\beta=1}^{2n_3+n_2}\left[\left(\frac{p_\alpha-q_\beta+ic}{p_\alpha-q_\beta-ic}\right)\left(\frac{p_{\bar\alpha}-q_\beta+ic}{p_{\bar\alpha}-q_\beta-ic}\right)\right]$$

$$=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n_2}\left(\frac{q_\alpha-q_\beta+2ic}{q_\alpha-q_\beta-2ic}\right)\prod_{a=1}^{n_3}\left[\left(\frac{q_\alpha-r_a+3ic}{q_\alpha-r_a-3ic}\right)\left(\frac{q_\alpha-r_a+ic}{q_\alpha-r_a-ic}\right)\right]\left(\frac{p_\alpha-q_\alpha+ic}{p_\alpha-q_\alpha-ic}\right)\left(\frac{p_{\bar\alpha}-q_\alpha+ic}{p_{\bar\alpha}-q_\alpha-ic}\right).$$

$p_\alpha-q_\alpha-ic$ and $p_{\bar\alpha}-q_\alpha+ic$ are $O(\exp cL)$ and we must be careful in treating them. From (2b) we have

$$\left(\frac{p_\alpha-q_\alpha+ic}{p_\alpha-q_\alpha-ic}\right)\left(\frac{p_{\bar\alpha}-q_\alpha+ic}{p_{\bar\alpha}-q_\alpha-ic}\right)=\prod_{j=1}^{n_1}\left(\frac{q_\alpha-p_j+ic}{q_\alpha-p_j-ic}\right).$$

Then finally we have

$$\exp 2iq_\alpha L=\prod_{j=1}^{n_1}\left(\frac{q_\alpha-p_j+ic}{q_\alpha-p_j-ic}\right)\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n_2}\left(\frac{q_\alpha-q_\beta+2ic}{q_\alpha-q_\beta-2ic}\right)\prod_{a=1}^{n_3}\left[\left(\frac{q_\alpha-r_a+3ic}{q_\alpha-r_a-3ic}\right)\left(\frac{q_\alpha-r_a+ic}{q_\alpha-r_a-ic}\right)\right].$$

Taking the logarithm of this equation we obtain

$$2q_\alpha L=2\pi J_\alpha-\sum_{j=1}^{n_3}2\tan^{-1}\frac{q_\alpha-p_j}{c}-\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n_2}2\tan^{-1}\frac{q_\alpha-q_\beta}{2c}$$

$$-\sum_{a=1}^{n_3}\left(2\tan^{-1}\frac{q_\alpha-r_a}{3c}+2\tan^{-1}\frac{q_\alpha-r_a}{c}\right). \tag{5b}$$

For $r_a=r_1,\cdots,r_{n_3}$, there must be three p-values $p_a, p_{\bar a}, p_{\bar{\bar a}}$ and two q values $q_a, q_{\bar a}$. Take the product of (2a):

$$\exp(3ir_aL)=\prod_{\beta=1}^{n_2}\left(\frac{r_a-q_\beta+3ic}{r_a-q_\beta-3ic}\right)\prod_{\substack{b=1\\ b\neq a}}^{n_3}\left\{\left(\frac{r_a-r_b+4ic}{r_a-r_b-4ic}\right)\left(\frac{r_a-r_b+2ic}{r_a-r_b-2ic}\right)\right\}$$

$$\times\left[\left(\frac{p_a-q_a+ic}{p_a-q_a-ic}\right)\left(\frac{p_{\bar a}-q_a+ic}{p_{\bar a}-q_a-ic}\right)\left(\frac{p_{\bar{\bar a}}-q_a+ic}{p_{\bar{\bar a}}-q_a-ic}\right)\right.$$

$$\left.\times\left(\frac{p_a-q_{\bar a}+ic}{p_a-q_{\bar a}-ic}\right)\left(\frac{p_{\bar a}-q_{\bar a}+ic}{p_{\bar a}-q_{\bar a}-ic}\right)\left(\frac{p_{\bar{\bar a}}-q_{\bar a}+ic}{p_{\bar{\bar a}}-q_{\bar a}-ic}\right)\right].$$

From (2b) and (2c) we can see the last square bracket is replaced by

$$[\cdots]=\prod_{j=1}^{n_1}\left(\frac{r_a-p_j+2ic}{r_a-p_j-2ic}\right)\prod_{\alpha=1}^{n_2}\left(\frac{r_a-q_\alpha+ic}{r_a-q_\alpha-ic}\right).$$

Then we have

$$3r_aL=2\pi K_a-\sum_{j=1}^{n_1}2\tan^{-1}\frac{r_a-p_j}{2c}-\sum_{\alpha=1}^{n_2}\left(2\tan^{-1}\frac{r_a-q_\alpha}{3c}+2\tan^{-1}\frac{r_a-q_\alpha}{c}\right)$$

$$-\sum_{\substack{b\neq a\\ b=1}}^{n_3}\left(2\tan^{-1}\frac{r_a-r_b}{4c}+2\tan^{-1}\frac{r_a-r_b}{2c}\right). \tag{5c}$$

For simplicity we take $n_1, n_2=$even and $n_3=$odd. Going to the limit $c\to 0$ at (5a), (5b) and (5c) we have

$$K_a=\frac{n_3-1}{2},\ \frac{n_3-3}{2},\ \cdots,\ -\frac{n_3-1}{2},$$

$$J_\alpha=\frac{n_2+n_3-1}{2},\ \cdots,\ \frac{n_3+1}{2},\ -\frac{n_3+1}{2},\ \cdots,\ -\frac{n_2+n_3-1}{2},$$

$$I_j=\frac{n_1+n_2+n_3-1}{2},\ \cdots,\ \frac{n_2+n_3+1}{2},\ -\frac{n_2+n_3+1}{2},\ \cdots,\ -\frac{n_1+n_2+n_3-1}{2}.$$

We define functions $\rho_1(p)$, $\rho_2(q)$ and $\rho_3(r)$ as the distributions of $p_1\sim p_{n_1}$, $q_1\sim q_{n_2}$ and $r_1\sim r_{n_3}$ respectively. From (5a), (5b) and (5c) we have

$$2\pi\rho_1(p)=1-\int_{-Q_2}^{Q_2}\frac{2|c|\rho_2(q)dq}{c^2+(p-q)^2}-\int_{-Q_3}^{Q_3}\frac{4|c|\rho_3(r)dr}{4c^2+(p-r)^2}, \tag{6a}$$

$$2\pi\rho_2(q)+\int_{-Q_2}^{Q_2}\frac{4|c|\rho_2(q')dq'}{4c^2+(q-q')^2}=2-\int_{-Q_1}^{Q_1}\frac{2|c|\rho_1(p)dp}{c^2+(p-q)^2}$$
$$-\int_{-Q_3}^{Q_3}\left[\frac{2|c|}{c^2+(q-r)^2}+\frac{6|c|}{9c^2+(q-r)^2}\right]\rho_3(r)dr, \tag{6b}$$

$$2\pi\rho_3(r)+\int_{-Q_3}^{Q_3}\left[\frac{4|c|}{4c^2+(r-r')^2}+\frac{8|c|}{16c^2+(r-r')^2}\right]\rho_3(r')dr'$$
$$=3-\int_{-Q_1}^{Q_1}\frac{4|c|\rho_1(p)dp}{4c^2+(p-r)^2}-\int_{-Q_2}^{Q_2}\left[\frac{2|c|}{c^2+(q-r)^2}+\frac{6|c|}{9c^2+(q-r)^2}\right]\rho_2(q)dq. \tag{6c}$$

From (3b) one obtains the ground state energy

$$E/L=\int_{-Q_1}^{Q_1}p^2\rho_1(p)dp+\int_{-Q_2}^{Q_2}(2q^2-2c^2)\rho_2(q)dq+\int_{-Q_3}^{Q_3}(3r^2-8c^2)\rho_3(r)dr. \tag{6d}$$

Parameters Q_1, Q_2 and Q_3 are determined by

$$n_1/L=\int_{-Q_1}^{Q_1}\rho_1(p)dp,\qquad n_2/L=\int_{-Q_2}^{Q_2}\rho_2(q)dq,\qquad n_3/L=\int_{-Q_3}^{Q_3}\rho_3(r)dr. \tag{6e}$$

For the true ground state, n_1 and n_2 are equal to zero and integral equations for this case are

$$2\pi\rho_3(r)+\int_{-Q_3}^{Q_3}\frac{4|c|\rho_3(r')dr'}{4c^2+(r-r')^2}+\int_{-Q_3}^{Q_3}\frac{8|c|\rho_3(r')dr'}{16c^2+(r-r')^2}=3, \tag{7a}$$

$$E/L=\int_{-Q_3}^{Q_3}(3r^2-8c^2)\rho_3(r)dr, \tag{7b}$$

$$N/L=3\int_{-Q_3}^{Q_3}\rho_3(r)dr. \tag{7c}$$

§ 3. Arbitrary value of κ

Generalization of the integral equations (6) to the arbitrary value of κ is easy.

$$1/2\pi=\sum_{i=1}^{\kappa}[i-1]\rho_i(k),$$

$$2/2\pi=\sum_{i=1}^{\kappa}[i]\rho_i(k)+\sum_{i=2}^{\kappa}[i-2]\rho_i(k),$$

$$3/2\pi=\sum_{i=1}^{\kappa}[i+1]\rho_i(k)+\sum_{i=2}^{\kappa}[i-1]\rho_i(k)+\sum_{i=3}^{\kappa}[i-3]\rho_i(k),$$

$$4/2\pi=\sum_{i=1}^{\kappa}[i+2]\rho_i(k)+\sum_{i=2}^{\kappa}[i]\rho_i(k)+\sum_{i=3}^{\kappa}[i-2]\rho_i(k)+\sum_{i=4}^{\kappa}[i-4]\rho_i(k), \tag{8a}$$

....

$$\kappa/2\pi=\sum_{j=1}^{\kappa}\sum_{i=j}^{\kappa}[i+\kappa-2j]\rho_i(k),$$

where

$$[j]\rho_i(k)=\frac{1}{\pi}\int_{-Q_i}^{Q_i}\frac{j|c|}{(jc)^2+(k-k')^2}\rho_i(k')dk'$$

and $[0]\rho_i(k)=\rho_i(k)$. The ground state energy is given by

$$E/L=\sum_{m=1}^{\kappa}\int_{-Q_m}^{Q_m}\left(mk^2-\frac{m(m^2-1)}{3}c^2\right)\rho_m(k)dk \tag{8b}$$

and Q_i are determined by

$$n_i/L=\int_{-Q_i}^{Q_i}\rho_i(k)dk. \tag{8c}$$

For the true ground state, $Q_1, \cdots, Q_{\kappa-1}$ are zero and integral equation is

$$\kappa=2\pi\rho_\kappa(k)+\sum_{j=1}^{\kappa-1}\int_{-Q_\kappa}^{Q_\kappa}\frac{4j|c|\rho_\kappa(k')dk'}{(2jc)^2+(k-k')^2}, \tag{9a}$$

$$E/L=\int_{-Q_\kappa}^{Q_\kappa}\left\{\kappa k^2-\frac{\kappa(\kappa^2-1)}{3}c^2\right\}\rho_\kappa(k)dk, \tag{9b}$$

$$N/L=\kappa\int_{-Q_\kappa}^{Q_\kappa}\rho_\kappa(k)dk. \tag{9c}$$

In the case $\kappa=2$ these are equivalent to Gaudin's result.[5]

§ 4. Discussion

Lieb and Liniger,[1] Yang[6] and Sutherland[7] also gave the integral equations for the boson problem in the case of repulsive interaction. But in the case of attractive interaction thermodynamic limit of the Boson system does not exist. As was shown by Mc'Guire[2] the ground state energy is $-N(N^2-1)c^2/3$ in this case and is not proportional to N for fixed density. Then it is meaningless to seek the integral equations for Boson system with attractive interaction.

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Note added in proof:

After this paper was prepared, the author was informed that Professor C. N. Yang made mention of the same result without any detailed proof in the lectures given at the Karpacz School, Poland, Feb. 1970.