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Many-body wave scattering by small bodies and applications

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Abstract

A rigorous reduction of the many-body wave scattering problem to solving a linear algebraic system is given bypassing solving the usual system of integral equation. The limiting case of infinitely many small particles embedded into a medium is considered and the limiting equation for the field in the medium is derived. The impedance boundary conditions are imposed on the boundaries of small bodies. The case of Neumann boundary conditions (acoustically hard particles) is also considered. Applications to creating materials with a desired refraction coefficient are given. It is proved that by embedding suitable number of small particles per unit volume of the original material with suitable boundary impedances one can create a new material with any desired refraction coefficient. The governing equation is a scalar Helmholtz equation, which one obtains by Fourier transforming the wave equation.

1 Introduction

This paper can be considered as a continuation of [15], but it is essentially self-contained. It uses some of the ideas and results from [10], [8], [11], [16], [17]. Applications of our theory to creating materials with desired refraction coefficient are discussed in [12], [13], [14], [18]. Wave scattering by small bodies is a classical branch of science: it was originated by Rayleigh in 1871. In [3] one finds a discussion of wave scattering by a small particle. In [2] there is a review of the low frequency scattering theory and formulas for scattering by small balls and ellipsoids are given. In [10] the theory is developed for small bodies of arbitrary shapes: explicit analytical formulas are given for calculating capacitances of the conductors of arbitrary shapes, electric and magnetic polarizability tensors for homogeneous bodies of arbitrary shapes, and for $S-$matrix for acoustic and electromagnetic (EM) wave scattering by small bodies of arbitrary shapes with any desired accuracy ([19], [10]). In [15] the many-body scattering problem was reduced to solving linear algebraic systems bypassing the usual study of a system of integral equations. In this paper we apply the approach proposed in [15] and study the limiting behavior of the scattering solution when the number of small bodies tends to infinity in such a way that the characteristic size a of the small particles is related to their number M so that $M = O(\frac{1}{a})$ in Theorem 2, and $M = O(\frac{1}{a^3})$ in Theorem 3. Sufficient conditions for convergence of the scattering solution in this limiting process are given. We prove that these conditions are, in some sense, also necessary for convergence. The limit of the scattering solution is a function, which satisfies some differential or integral-differential equations. These equations describe the behavior of the wave field in the new medium, obtained in the limit.

There is a large literature on the calculation of the effective dielectric permittivity and magnetic permeability of the composite materials (Maxwell-Garnett and Bruggeman recipes and their numerous versions, see [22], [5]). In the literature mostly a uniform random distribution of the inclusions is assumed and the resulting homogenized medium is described by effective constant dielectric permittivity and magnetic permeability, which can be tensors. In this work the propagation and scattering of scalar waves are discussed, and the "homogenized" medium is described not by a constant refraction coefficient, but by a refraction coefficient which is a function of spatial variables.

In [4] boundary value problems were studied for positive operators for the Dirichlet boundary conditions in domains which are obtained from some domain by extracting many small bodies from it. We study in this paper the scattering problem in similar domains, but our operator is not positive and we use impedance boundary conditions. The methods in [4] do not seem to be appicable to our problem by the above reasons. Our assumptions lead to new physical phenomena. For example, the new material, created by embedding many small particles according to the recipe, given in Theorem 2, allows one to get a refraction coefficient with any desired real and imaginary parts, so that a desired absorption of energy in this material can be obtained.

Let us formulate the problem. Consider first a bounded domain $D \subset \mathbb{R}^3$ filled with a material with a known refraction coefficient $n_0^2(x)$. The governing equation is:

$$L_0 u_0 := \big(\nabla^2 + k^2 n_0^2(x)\big) u_0 = 0 \quad \text{in } \mathbb{R}^3. \tag{1.1}$$

We assume that $n_0^2(x) = 1$ in $D' = \mathbb{R}^3 \backslash D$, $k = \text{const} > 0$, and $n_0^2 =$

$\max_{x\in D}|n_0^2(x)| < \infty$. The operator L_0 can be written as a Schrödinger operator:

$$L_0 = \nabla^2 + k^2 - q_0(x), \quad q_0(x) := k^2[1 - n_0^2(x)], \tag{1.2}$$

and $q_0 = 0$ in D'. One has

$$n_0^2(x) = 1 - k^{-2}q_0(x),$$

so there is a one-to-one correspondence between $n_0^2(x)$ and $q_0(x)$. If $n_0^2(x)$ is known, then one knows the scattering solution:

$$\begin{aligned} &L_0u_0 = 0 \quad \text{in } \mathbb{R}^3, \\ &u_0(x) = e^{ik\alpha\cdot x} + A_0(\beta,\alpha)\,\frac{e^{ikr}}{r} + o\Big(\frac{1}{r}\Big), \quad r = |x| \to \infty, \beta := \frac{x}{r}. \end{aligned} \tag{1.3}$$

The coefficient $A_0(\beta,\alpha)$ is called the scattering amplitude, the unit vector $\alpha \in S^2$ is given, α is the direction of the incident plane wave $e^{ik\alpha\cdot x}$, S^2 is the unit sphere in $\mathbb{R}^3$, $\beta \in S^2$ is the direction of the scattered wave, $k > 0$ is a wave number, which we assume fixed throughout the paper. By this reason we do not show the k-dependence of A and u_0.

Let $G(x,y)$ be the resolvent kernel of L_0 satisfying the radiation condition (or the limiting absorption principle):

$$L_0G(x,y) = -\delta(x-y) \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

This function $G(x,y)$ is known because $q_0(x)$ is known.

Consider now the scattering problem for many small bodies D_m embedded in D, $1 \le m \le M$:

$$L_0u_M = 0 \quad \text{in } \mathbb{R}^3\setminus \bigcup_{m=1}^{M} D_m, \tag{1.5}$$

$$u_M = u_0 + A_M(\beta,\alpha)\,\frac{e^{ikr}}{r} + o\Big(\frac{1}{r}\Big), \quad r = |x| \to \infty, \quad \frac{x}{r} = \beta, \tag{1.6}$$

$$\frac{\partial u_M}{\partial N} = \zeta_m u_M \quad \text{on } S_m := \partial D_m, \quad 1 \le m \le M, \tag{1.7}$$

where u_0 is the solution ot the scattering problem (1.3). Here N is the normal to S_m pointing out of D_m, ζ_m is a complex number, the boundary impedance, $\operatorname{Im}\zeta_m \le 0$, S_m is uniformly $C^{1,\lambda}$ with respect to m, $1 \le m \le M$. By $C^{1,\lambda}$ surface we mean the surface with local equation $x_3 = f(x_1,x_2)$, where $f \in C^{1,\lambda}$, $\lambda > 0$. We assume throughout this paper that

$$n_0ka \ll 1, \quad d \gg a, \tag{1.8}$$

where

$$a = \frac{1}{2}\max_m \operatorname{diam}D_m, \quad d = \min_{m\neq j} \operatorname{dist}(D_m, D_j). \tag{1.9}$$

By $V_m := |D_m|$ the volume of D_m is denoted, and by $|S_m|$ the surface area of S_m is denoted.

One can prove (see Section 3) that problem (1.5) – (1.7) has at most one solution if $\operatorname{Im}\zeta_m \leq 0$, $1 \leq m \leq M$, and $\operatorname{Im} q_0(x) \leq 0$.

We look for the solution to problem (1.5) – (1.7) of the form

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,s)\sigma_m(s)ds, \tag{1.10}$$

where σ_m should be found from the boundary conditions (1.7). For any σ_m the function (1.10) solves equation (1.5) and satisfies condition (1.6):

$$A_M(\beta,\alpha) = \frac{1}{4\pi}\sum_{m=1}^{M} \int_{S_m} u_0(s,-\beta)\sigma_m(s)ds. \tag{1.11}$$

Formula (1.11) follows from (1.6), (1.10) and the Ramm's lemma ([9], formulas (5.1.31), (5.1.36)):

$$G(x,y) = \frac{e^{ik|x|}}{4\pi|x|} u_0(y,\alpha) + o\Big(\frac{1}{|x|}\Big), \quad |x| \to \infty,\ \frac{x}{|x|} = -\alpha, \tag{1.12}$$

where $u_0(x,\alpha)$ is the scattering solution. A similar formula was proved earlier in [7], p. 46, for the resolvent kernel of the Laplacian in the exterior of a bounded obstacle, (and even earlier, in [6], for some unbounded obstacles). The scattering amplitude for problem (1.5) – (1.7) is

$$A(\beta,\alpha) = A_0(\beta,\alpha) + A_M(\beta,\alpha), \tag{1.13}$$

where A_0 is defined in (1.3) and A_m is defined in (1.6). If ka is sufficiently small, then k^2 is not a Dirichlet eigenvalue of the operator $\nabla^2 - q_0(x)$ in D_m, $1 \leq m \leq M$. If

$$\operatorname{Im}\zeta_m \leq 0, \quad 1 \leq m \leq M; \quad \operatorname{Im} q_0(x) \leq 0, \tag{1.14}$$

then the unique solution to problem (1.5) – (1.7) can be found in the form (1.10).

Theorem 1 *Assume* (1.8) *and* (1.14)*. Then problem* (1.5) – (1.7) *has a solution of the form* (1.10) *and this is the unique solution of the problem* (1.5) –(1.7)*.*

Proof of Theorem 1 is given in Section 3.

Let

$$g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad g_0(x,y) := \frac{1}{4\pi|x-y|}. \tag{1.15}$$

Note that

$$G(x,y) = g(x,y) - \int_D g(x,z)q_0(z)G(z,y)dz. \tag{1.16}$$

We need two lemmas.

Lemma 1 *If*

$$|t-x|\leq a, \quad |x-y|\geq d\gg a, \tag{1.17}$$

then

$$|g(t,y)-g(x,y)|\leq c\Big(\frac{a}{d^2}+\frac{ka}{d}\Big), \tag{1.18}$$

where $c>0$ stands for various positive constants independent of a and d.

Lemma 2 *If* (1.17) *holds, then*

$$|G(t,y)-G(x,y)|\leq c\Big(\frac{a}{d^2}+\frac{ka}{d}\Big). \tag{1.19}$$

These lemmas are proved in Section 3.

We denote by $\tilde{D}$ an arbitrary subdomain of D independent of a, by

$$\mathcal{N}_a(\tilde{D}):=\mathcal{N}(\tilde{D}):=\sum_{D_m\subset\tilde{D}}1$$

the number of particles (small bodies) in $\tilde{D}$, and assume that the small particle D_m shrinks to a point $x\in D$ as $a\to 0$. Since in the limiting process dependence on m disappear, we denote the limiting point x without giving it any subindex. The functions $h(x)$, $\text{Im}h\leq 0$, and $N(x)\geq 0$ in Theorem 2 are arbitrary continuous functions which we can choose as we wish. They do not depend on a. They determine the refraction coefficient of the new material, created by embedding small particles, as $a\to 0$. We assume for simplicity that

$$|S_m|=c_1a^2, \quad J_m=c_2a^3, \quad V_m=c_3a^3, \quad M=O(a^{-1}), \quad d=O(a^{1/3}),$$

where c_j, $j=1,2,3$, are positive constants, independent of a, $|S_m|$ is the surface area of the boundary of $m-$th body, V_m is its volume, and $J_m:=\int_{S_m}\int_{S_m}\frac{dsdt}{|s-t|}$, where ds and dt are surface area elements, and s,t are points on S_m. These assumptions are not repated in the formulation of Theorem 2.

Let us formulate our results under simplifying but physically reasonable assumptions (see [21]).

Theorem 2 *Assume that*

$$\lim_{\substack{a\to 0\\ x_m\in D_m, x_m\to x}}\frac{\zeta_m J_m}{4\pi|S_m|}=h(x), \tag{1.20}$$

and for any subdomain $\tilde{D}\subset D$ the following relation holds:

$$\lim_{a\to 0}a\mathcal{N}_a(\tilde{D})=\int_{\tilde{D}}N(x)dx. \tag{1.21}$$

Under these assumptions there exists the limit:

$$\lim_{M\to\infty} u_M(x) = u(x) := u(x,\alpha). \tag{1.22}$$

This $u(x)$ solves the equations:

$$u(x) = u_0(x) - \int_D G(x,y)\,p(y)\,u(y)dy, \tag{1.23}$$

and

$$Lu := [\nabla^2 + k^2 - q(x)]u = 0 \quad in\ \mathbb{R}^3, \tag{1.24}$$

where the potential q is of the form:

$$q(x) = q_0(x) + p(x), \quad p(x) = \frac{4\pi c_1^2\,N(x)h(x)}{c_2[1+h(x)]}\,, \tag{1.25}$$

and u satisfies the radiation condition:

$$u = e^{ik\alpha\cdot x} + A(\beta,\alpha)\,\frac{e^{ikr}}{r} + o\Big(\frac{1}{r}\Big), \quad r = |x| \to \infty, \tag{1.26}$$

where

$$A(\beta,\alpha) = A_0(\beta,\alpha) - \frac{1}{4\pi}\int_D u_0(y,-\beta)p(y)u(y,\alpha)dy, \tag{1.27}$$

and $u_0(y,-\beta)$ is the scattering solution defined in (1.3)*.*

Theorem 3 *Assume that $\zeta_m = 0$, $1 \le m \le M$, and the following limits exist:*

$$\lim_{a\to 0} \sum_{D_m \subset \tilde{D}} V_m \beta_{pj}^{(m)} = \int_{\tilde{D}} \beta_{pj}(y)\nu(y)dy, \tag{1.28}$$

$$\lim_{a\to 0} \sum_{D_m \subset \tilde{D}} V_m = \int_{\tilde{D}} \nu(y)dy, \tag{1.29}$$

where $\tilde{D} \subset D$ are arbitrary, independent of a, the functions $\nu(y) \ge 0$ and $\beta_{pj}(y)$ are continuous in D, and $\beta_{pj}^{(m)}$ is the magnetic polarizability tensor of the body D_m, defined in (2.38)*-*(2.39)*, see below.*

Then the function $u_M(x)$, defined in (1.10)*, tends to the limit:*

$$\lim_{M\to\infty} u_M(x) = \mathcal{U}(x) = \mathcal{U}(x,\alpha), \tag{1.30}$$

and $\mathcal{U}(x)$ solves the equation:

$$\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\Delta\mathcal{U}(y)\nu(y)dy - \sum_{p,j=1}^{3}\int_D \frac{\partial G(x,y)}{\partial y_p}\,\frac{\partial \mathcal{U}(y)}{\partial y_j}\,\beta_{pj}(y)\nu(y)dy. \tag{1.31}$$

If all the small particles are balls of radius $a > 0$, then

$$V_m = \frac{4\pi a^3}{3}, \quad |S_m| = 4\pi a^2, \quad J_m = 16\pi^2 a^3, \quad \int_{S_m} \frac{dt}{4\pi|s-t|} = a, \ s \in S_m.$$

In this case

$$\int_{S_m} \frac{dt}{|s-t|} = \frac{1}{|S_m|} \int_{S_m} \int_{S_m} \frac{dtds}{4\pi|s-t|},$$

that is, the mean value of the integral $\int_{S_m} \frac{dt}{|s-t|}$ on the surface S_m equals to this integral. If S_m is not a sphere, this mean value is an approximate value of the above integral.

Note that under the assumptions of Theorem 2 one has $M = O(a^{-1})$, while under the assumptions of Theorem 3 one has $M = O(a^{-3})$ (see formula (2.50) below). Therefore, one needs many more particles to deal with the Neumann boundary condition, that is, with acoustically hard particles, than with the impedance boundary condition with large boundary impedance $\zeta = O(a^{-1})$. We discuss at the end of Section 4 in more detail the question concerning the compatibility of the assumption (1.8), namely $d \gg a$, and the existence of the limits (1.28) and (1.29). It will be shown that the assumption $d \gg a$ is compatible with the existence of the limit (1.29) only if $\nu(y)$ is sufficiently small, and in this case the existence of the limit (1.28) is also compatible with the assumption $d \gg a$.

In Section 2 Theorems 2 and 3 are proved. In Section 3 Theorem 1 and Lemmas 1, 2 are proved. In Section 4 some examples are given, the significance of the compatibility of the assumptions $d \gg a$ and (1.21), (1.28) – (1.29) is discussed, and a possible application of our results to creating materials with a desired refraction coefficient is described. In Section 5 some estimates of the effective field are given. These estimates imply the convergence of this field as $a \to 0$. The results of Section 5 are used in the proof of Theorem 2.

2 Proof of Theorem 2

Let us look for the solution to problem (1.5) – (1.7) of the form:

$$u_M = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x,s)\sigma_m(s)ds, \tag{2.1}$$

where $G(x,y)$ is the resolvent kernel of L_0, see (1.4), and σ_m are arbitrary functions at the moment. For any σ_m the function (2.1) solves equation (1.5) and satisfies the radiation condition (1.6). Since problem (1.5) – (1.7) has at most one solution, the function (2.1) is the unique solution to (1.5) – (1.7) provided that σ_m are chosen so that the boundary conditions (1.7) are satisfied. Since diam D_m, $1 \le m \le M$, are small, let us write (2.1) as

$$u_M = u_0(x) + \sum_{m=1}^{M} G(x, x_m)Q_m + \sum_{m=1}^{M} \int_{S_m} [G(x, s) - G(x, x_m)]\sigma_m(s)ds, \quad (2.2)$$

where $x_m \in D_m$ is a point inside D_m and

$$Q_m := \int_{S_m} \sigma_m(s)ds. \quad (2.3)$$

The choice of $x_m \in D_m$ is arbitrary because diam $D_m \le 2a$ is small. We will prove that $Q_m \neq 0$, give an analytic formula for Q_m (formula (2.20) below), and approximate the field u_M in (2.2) by the expression:

$$u_M = u_0(x) + \sum_{m=1}^{M} G(x, x_m)Q_m.$$

The error of this approximate formula is of order $\max(\frac{a}{d}, ka)$, see estimate (2.7) below. Therefore this error tends to zero as $a \to 0$ since $d = O(a^{1/3})$. Let us estimate the term

$$E_m := \int_{S_m} [G(x, s) - G(x, x_m)]\sigma_m(s)ds. \quad (2.4)$$

By the inequality (1.19) one gets

$$|E_m| \le c\Big(\frac{a}{d^2} + \frac{ka}{d}\Big)|Q_m|, \quad |x - x_m| \ge d \gg a. \quad (2.5)$$

We will prove below that $Q_m = O(a)$, see formula (2.20), and, since $|G(x, x_m)| \le cd^{-1}$ if $|x - x_m| \ge d > 0$, one has:

$$\Big|G(x, x_m)Q_m\Big| = O(\frac{a}{d}). \quad (2.6)$$

Let us prove that under our assumptions the term E_m is much smaller than $O(\frac{a}{d})$. Using again inequality (1.19), one gets:

$$|E_m| \le c(ad^{-2} + kad^{-1})O(a).$$

Therefore,

$$|E_m| \le O(\frac{a^2}{d^2} + ka\frac{a}{d}) \ll O(\frac{a}{d}), \quad (2.7)$$

because $ka \ll 1$ and $a \ll d$ by assumption. So, our claim is verified. Moreover,

$$\sum_{m=1}^{M} |E_m| \ll \sum_{m=1}^{M} |G(x, x_m)Q_m|$$

if $|x-x_m| \geq d \gg a$, because $M = O(\frac{1}{a})$.

To find Q_m, we use the boundary condition (1.7). Let us write $u(x) := u_M(x)$ in a neighborhood of S_j as

$$u_M(x) := u_e(x) + \int_{S_j} G(x,s)\sigma_j(s)ds, \quad |x-x_j| \leq 2a, \tag{2.8}$$

where u_e is the effective field acting on the $j-$th small particle from outside:

$$u_e(x) := u_M(x) - \int_{S_j} G(x,s)\sigma_j(s)ds = u_0(x) + \sum_{m\neq j} G(x,x_m)Q_m + O(\frac{a}{d}), \quad |x-x_j| \leq 2a, \tag{2.9}$$

and $u_e(x) := u_M(x), \quad \min_m |x-x_m| \geq d$.

We neglect the error term $O(\frac{a}{d})$ in what follows. From (2.8) and (1.7) one gets:

$$0 = u_{eN}(s) - \zeta_j u_e(s) + \frac{A_j\sigma_j - \sigma_j}{2} - \zeta_j T_j \sigma_j, \quad s \in S_j, \tag{2.10}$$

where $u_{eN}(s)$ is the normal derivative of u_e at the point $s \in S_j$. One can rewrite this equation as:

$$\sigma_j = A_j\sigma_j - 2\zeta_j T_j\sigma_j - 2\zeta_j u_e(s) + 2u_{eN}(s).$$

Here the operators A_j and T_j are defined as follows:

$$T_j\sigma_j := \int_{S_j} G(s,t)\sigma_j(t)dt \simeq \int_{S_j} \frac{\sigma_j(t)dt}{4\pi|s-t|}, \tag{2.11}$$

$$A_j\sigma_j := 2\int_{S_j} \frac{\partial G(s,t)}{\partial N_s}\sigma_j(t)dt \simeq 2\int_{S_j} \frac{\partial}{\partial N_s}\frac{1}{4\pi|s-t|}\sigma_j(t)dt := A\sigma_j, \tag{2.12}$$

and we have used the following approximations:

$$G(x,y) = g_0(x,y)[1 + O(|x-y|)], \quad |x-y| \to 0; \quad g_0(x,y) := \frac{1}{4\pi|x-y|}, \tag{2.13}$$

$$\frac{\partial G(x,y)}{\partial y_p} = \frac{\partial g_0}{\partial y_p}\left[1 + O\big(|x-y|^2 \left|\ln|x-y|\right|\big)\right], \quad |x-y| \to 0. \tag{2.14}$$

Note that (see [10], p. 96, formula (7.21)):

$$\int_{S_j} A_j\sigma_j ds = -\int_{S_j} \sigma_j ds. \tag{2.15}$$

Indeed,

$$\int_{S_j} ds \int_{S_j} \frac{\partial}{\partial N_s}\frac{1}{2\pi|s-t|}\sigma_j(t)dt = \in t_{S_j} dt\sigma_j(t)\int_{S_j} ds\frac{\partial}{\partial N_s}\frac{1}{2\pi|s-t|} = -\int_{S_j}\sigma_j(t)dt.$$

The integral

$$\int_{S_j} \frac{\partial}{\partial N_s} \frac{1}{2\pi|s-t|} ds = -1, \quad t \in S_j,$$

is well known in potential theory for surfaces $S_j \in C^{1,\lambda}$.

Integrating (2.10) over S_j, using formula (2.15), and the divergence theorem, one gets:

$$Q_j = -\zeta_j u_e(x_j)|S_j| - \zeta_j \int_{S_j} ds \int_{S_j} \frac{\sigma_j(t)dt}{4\pi|s-t|} + \int_{D_j} \Delta u_e dy. \tag{2.16}$$

The function $u_e(y)$ is smooth, so

$$\int_{D_j} \Delta u_e(y)dy = V_j \Delta u_e(x_j)[1 + o(1)], \quad a \to 0, \tag{2.17}$$

where $V_j = |D_j|$ is the volume of D_j and we have used the smallness of the diameter of D_j, that is, the smallness of a.

Let us write

$$\begin{aligned}\int_{S_j} ds \int_{S_j} \frac{\sigma_j(t)dt}{4\pi|s-t|} &= \int_{S_j} dt\, \sigma_j(t) \int_{S_j} \frac{ds}{4\pi|s-t|} \\ &= Q_j \frac{1}{S_j} \int_{S_j} dt \int_{S_j} \frac{ds}{4\pi|s-t|} = \frac{Q_j J_j}{4\pi|S_j|}, \quad J_j := \int_{S_j} \int_{S_j} \frac{ds\, dt}{|s-t|}.\end{aligned} \tag{2.18}$$

Here we approximated the continuous on S_j function $\int_{S_j} \frac{ds}{|s-t|}$ by its mean value $\frac{1}{|S_j|} \int_{S_j} dt \int_{S_j} \frac{ds}{|s-t|}$.

If S_j is a sphere of radius a, then

$$\int_{|s|=a} \frac{ds}{|s-t|} = 4\pi a, \quad |t| = a, \tag{2.19}$$

so in this case equation (2.18) is exact.

From (2.16) – (2.18) one finds a formula for Q_j:

$$Q_j = -\frac{\zeta_j |S_j|}{1 + \frac{\zeta_j J_j}{4\pi|S_j|}} u_e(x_j). \tag{2.20}$$

We neglected the term $V_j\, \Delta u_e(x_j) = O(a^3)$ which is much smaller than $|\zeta_j|\,|S_j| = O(a)$ as $a \to 0$, because $|S_j| = O(a^2)$ and $|\zeta_j| = O(\frac{1}{a})$. The quantity $J_j = O(a^3)$. Therefore $\frac{\zeta_j J_j}{4\pi|S_j|} = O(1)$. We choose

$$\zeta_j = \frac{H(x_j)}{a}\,, \tag{2.21}$$

where $H(x)$ is a continuous function in D, which we can choose as we wish subject to the condition $\operatorname{Im} H \leq 0$, because $\operatorname{Im} \zeta_j \leq 0$.

If the small particles are all of the same shape and size, then

$$|S_j| = c_1 a^2, \quad J_j = c_2 a^3,$$

where $c_1, c_2 > 0$ are some constants independent of j, $1 \leq j \leq M$.

Then

$$\frac{\zeta_j J_j}{4\pi |S_j|} = \frac{H(x_j) c_2}{4\pi c_1} := h(x_j), \tag{2.22}$$

and

$$\zeta_j |S_j| = H(x_j) c_1 a. \tag{2.23}$$

Formulas (2.2), (2.20), (2.22) and (2.23) imply:

$$u_M(x) = u_0(x) - \sum_{m=1}^{M} G(x, x_m) \frac{4\pi c_1^2 c_2^{-1} h(x_m) a}{1 + h(x_m)} u_M(x_m), \tag{2.24}$$

where $|x - x_m| \geq d \gg a$, and we replaced $u_e(x_m)$ by $u_M(x_m)$ because their difference (see (2.9)) is of order $O(\frac{a}{d}) \ll 1$. Indeed

$$|u_M(x) - u_e(x)| \leq \int_{S_j} |G(x, s)| \, |\sigma_j(s)| ds \leq \frac{c}{d} |Q_j| \leq \tilde{c} \, \frac{a}{d}, \quad |x - x_j| \geq d \gg a, \tag{2.25}$$

where $c, \tilde{c} > 0$ are some constants independent of a.

If the assumption (1.21) holds, then

$$\lim_{a \to 0} \sum_{m=1}^{M} G(x, x_m) \frac{4\pi c_1^2 c_2^{-1} h(x_m)}{1 + h(x_m)} u_M(x_m) a = \int_D G(x, y) \frac{4\pi c_1^2 c_2^{-1} h(y)}{1 + h(y)} u(y) N(y) dy. \tag{2.26}$$

To pass to the limit in (2.26) one uses lemmas 5 and 6 of Section 5 and the following lemma.

Lemma 3 *Assume that $x_m \in D_m$, $\operatorname{diam} D_m \leq 2a$, f is a continuous function in D with a possible exception of a point y_0 in a neighborhood of which it is absolutely integrable, for example, it admits an estimate $|f(y)| \leq \frac{c}{|y - y_0|^b}$, $b < 3$, and assume that*

$$\lim_{a \to 0} a \sum_{D_m \subset \tilde{D}} 1 = \int_{\tilde{D}} N(x) dx \quad \forall \tilde{D} \subset D \tag{2.27}$$

for any subdomain $\tilde{D} \subset D$, where $N(x)$ is a continuous function. Then there exists the limit

$$\lim_{a \to 0} \sum_{m=1}^{M} f(y_m) a = \int_D f(y) N(y) dy. \tag{2.28}$$

Remark 1 In our case $f(y) = G(x,y)\,\frac{4\pi c_1^2 h(y)}{c_2[1+h(y)]}\,u_M(y)$ and (2.27) is the assumption (1.21).

Proof of Lemma 3 Let $D = \bigcup_{p=1}^{P} \overline{\Delta}_p$, where Δ_p and Δ_q do not intersect each other, $\overline{\Delta}_p$ is the closure of the domain Δ_p, and $\lim_{P\to\infty} \max_p \operatorname{diam} \Delta_p = 0$. Choose any point $y^{(p)} \in \Delta_p$ and note that

$$\sup_{y_m \in D_m,\, D_m \subset \Delta_p} |f(y^{(p)}) - f(y_m)| < \varepsilon_p \to 0 \quad \text{as } \operatorname{diam} \Delta_p \to 0. \tag{2.29}$$

Therefore

$$\begin{aligned}
\lim_{a\to 0} \sum_{m=1}^{M} f(y_m)a &= \lim_{a\to 0} \sum_{p=1}^{P} a \sum_{D_m \subset \Delta_p} f(y_m) = \sum_{p=1}^{P} [f(y^{(p)}) + O(\varepsilon_p)] \cdot \lim_{a\to 0} a \sum_{D_m \subset \Delta_p} 1 \\
&= \sum_{p=1}^{P} [f(y^{(p)}) + O(\varepsilon_p)] \int_{\Delta_p} N(y)dy \\
&= \sum_{p=1}^{P} [f(y^{(p)}) + O(\varepsilon_p)] \cdot [N(y^{(p)}) + O(\varepsilon'_p)]\, |\Delta_p|,
\end{aligned} \tag{2.30}$$

where $\lim_{P\to\infty} \max_p |\varepsilon'_p| = 0$. Let $P \to \infty$ in (2.30). Then

$$\lim_{P\to\infty} \sum_{p=1}^{P} [f(y^{(p)}) + O(\varepsilon_p)]\, [N(y^{(p)}) + O(\varepsilon'_p)]\, |\Delta_p| = \int_D f(y)N(y)dy. \tag{2.31}$$

In the above argument we assumed that f is continuous in D. If f has an integrable singularity at a point x_0, then we choose a ball $B(x_0, \delta_\varepsilon)$ centered at x_0 of radius δ_ε such that $\sup_{0<\delta<\delta_\varepsilon} \int_{B(x_0,\delta)} |f(y)|dy < \varepsilon$, where $\varepsilon > 0$ is an arbitrary small fixed number. Then

$$\sup_{0<\delta<\delta_\varepsilon} \int_{B(x_0,\delta)} |f(y)|\, |N(y)|\, dy < c\varepsilon,$$

where $c = \max_{y\in D} |N(y)| > 0$ is a constant independent of ε. Now we apply the above argument to the region $D \backslash B(x_0, \delta)$, where f is continuous and get:

$$\lim_{a\to 0} \sum_{\substack{m=1 \\ y_m \notin B(x_0,\delta)}}^{M} f(y_m)a = \int_{D\backslash B(x_0,\delta)} f(y)N(y)dy. \tag{2.32}$$

The left side of (2.28) in the case of f having integrable singularity at the point x_0 and continuous in $D \backslash x_0$ is understood as the limit of the expression on the left of (2.32) as $\delta \to 0$. This yields (2.28). Lemma 3 is proved. ∎

Passing to the limit $M \to \infty$, or $a \to 0$, in equation (2.24) and using Lemma 3, one gets

$$\begin{aligned} u(x) &= u_0(x) - \int_D G(x,y)\, p(y)\, u(y) dy, \\ p(y) &= \frac{4\pi c_1^2}{c_2} \frac{h(y)N(y)}{1+h(y)}. \end{aligned}$$

Applying the operator $L_0 = \nabla^2 + k^2 - q_0(x)$ to this equation and using (1.3) and (1.4), one obtains equation (1.24). Formulas (1.26) and (1.27) follow from the above equation and from formula (1.12).

This concludes the proof of Theorem 2. ∎

Remark 2 It is possible (and not difficult) to generalize Theorem 2 to the case of particles with different shapes. Since this does not lead to an essentially new result, we do not go into detail. In [10], [19] and [8] one can find analytical formulas for the $S-$matrix for wave scattering by small bodies of arbitrary shapes.

Proof of Theorem 3.

Now we assume $\zeta_m = 0$, $1 \le m \le M$, which means that all the small particles are acoustically hard. In this case equation (2.10) takes the form

$$\sigma_j = A_j \sigma_j + 2u_{e_N}(s), \quad s \in S_j,\ 1 \le j \le M, \tag{2.33}$$

where

$$u_e(x) := u_0(x) + \sum_{m \ne j} \int_{S_m} G(x,s) \sigma_m(s) ds. \tag{2.34}$$

We cannot use approximation (2.2) because the quantity Q_m now is of the same order of magnitude as the integral $\int_{S_m} [G(x,s) - G(x,x_m)]\, \sigma_m(s) ds$, or even smaller than this integral. This is established below. While under the assumptions of Theorem 2 we had $Q_m = O(a)$, now, under the assumptions of Theorem 3, we have $Q_m = O(k^2 a^3)$, which is a much smaller quantity than $O(a)$ because $ka \ll 1$. To estimate the order of magnitude of Q_m, we integrate (2.33) over S_j and use (2.15). The result is:

$$Q_j = \int_{S_j} u_{e_N} ds = \int_{D_j} \Delta u_e\, dx \simeq \Delta u_e(x_j) V_j, \tag{2.35}$$

where V_j is the volume of D_j, and we have used the assumption $d \gg a$. This assumption allows one to claim that $u_e(x)$ is practically constant in the domain D_j in the absence of j-th particle. Differentiation with respect to x brings a factor k. Since we assume that $k > 0$ is fixed, this factor is not important for our argument, but to make the dimensionality of the term $V_j \Delta u_e$ clear, we may write $V_j \Delta u_e = O(k^2a^3)$. This quantity has dimensionality of length since ka is dimensionless.

We now prove that the term $E_m := \int_{S_m}[G(x,s)-G(x,x_m)]\sigma_m(s)ds$, which was neglected under the assumptions of Theorem 2, because it was much smaller than $|G(x,x_m)Q_m|$, is now, under the assumption $\zeta_m = 0$, $1 \leq m \leq M$, of the same order of magnitude as $|G(x,x_m)Q_m|$, namely $O(k^2a^3d^{-1})$, or even larger. We have

$$\int_{S_m}[G(x,s)-G(x,x_m)]\sigma_m(s)ds \\ = \int_{S_m} \nabla_y G(x,y)\big|_{y=x_m} \cdot (s-x_m)\,\sigma_m(s)ds, \quad |x-x_m| \geq d \gg a, \tag{2.36}$$

where we have used the assumption $|x-x_m| \gg a$ and kept the main term in the Taylor's expansion of the function $G(x,s)-G(x,x_m)$.

Recall, that

$$\int_{S_m}(s-x_m)_p\,\sigma_m(s)ds = -V_m\,\beta_{pj}^{(m)}\,\frac{\partial u_e(y)}{\partial y_j}\Big|_{y=x_m}, \tag{2.37}$$

where one sums up over index $j = 1,2,3$, $\beta_{pj}^{(m)}$ is the magnetic polarizability tensor defined in [10], (p.55, formulas (5.13)-(5.15) and p.62, formula (5.62)), and $(s-x_m)_p$ is the p-th component of the vector $s-x_m$.

Namely, if

$$\sigma = A\sigma - 2N_j, \tag{2.38}$$

then

$$\int_S s_p\,\sigma(s)ds = V\beta_{pj}\,, \tag{2.39}$$

where V is the volume of the body with boundary S, N_j is the j-th component of the exterior unit normal N to S, the role of the point x_m from equation (2.37) is played by the origin, which is located inside S, and the role of S_m is played by S. Equation (2.33) with $j=m$ can be written as

$$\sigma_m = A_m\sigma_m - 2N_j\Big(-\frac{\partial u_e(y)}{\partial y_j}\Big|_{y=x_m}\Big), \tag{2.40}$$

where one sums up over j (but not over m). Compare (2.40) and (2.38) and get (2.37).

Formulas for the tensor $\beta_{pj} = \alpha_{pj}(\gamma)\Big|_{\gamma=-1}$ for bodies of arbitrary shapes were derived in [10], p.55, formula (5.15), so one may consider the tensor β_{pj} known for bodies of arbitrary shapes. In the cited formula $\alpha_{pj}(\gamma)$ is the polarizability tensor of a dielectric body with a constant dielectric permittivity ϵ_i and S is the surface of this body. The parameter $\gamma = \frac{\epsilon_i-\epsilon_e}{\epsilon_i+\epsilon_e}$, where ϵ_e is the dielectric permittivity of the surrounding medium. The case $\gamma = -1$ occurs when $\epsilon_i = 0$. This is the case, for example, in the problem of calculation the magnetic dipole moment of a superconductor placed in a homogeneous magnetic field: in the superconductor the magnetic induction vector $B = 0$, which means that the magnetic permeability μ_i of such body is zero, $\mu_i = 0$, see [3]. That is why the tensor β_{pj} is called magnetic polarizability tensor in [10].

From (2.36) and (2.37) it follows that

$$\int_{S_m} \left[G(x,s) - G(x,x_m)\right] \sigma_m(s)ds = -\frac{\partial G(x,y)}{\partial y_p}\Big|_{y=x_m} \frac{\partial u_e(y)}{\partial y_j}\Big|_{y=x_m} V_m \beta_{pj}^{(m)}, \tag{2.41}$$

where one sums up over the repeated indices p, j, but nor over m. The quantity on the right of (2.41) is of the order $O(k^2a^3d^{-1})$ if $kd \geq 1$, that is, of the same order as $|G(x,x_m)Q_m|$, provided that $|x-x_m| \geq d \gg a$, and it is of the order $O(ka^3d^{-2})$ if $kd < 1$. Indeed, $\beta_{pj}^{(m)} = O(1)$, $V_m = O(a^3)$, and $\big|\nabla_y G(x,y)\big| \leq c \max\big(\frac{k}{d}, \frac{1}{d^2}\big)$.

Let us prove the estimate

$$\big|\nabla_y G(x,y)\big| \leq c \max\big(\frac{k}{d}, \frac{1}{d^2}\big) \qquad \text{for} \qquad |x-y| \geq d \gg a,$$

where $c > 0$ is a constant independent of d.

We have

$$G(x,y) = g(x,y) - \int_D g(x,z)q_0(z)G(z,y)dz := g - \mathcal{T}G,$$

where $\mathcal{T}$ is compact as an operator in $L^p(D), p \geq 1$, under our assumptions, namely, $D \subset R^3$ is a bounded domain, $q_0(x)$ is a bounded piecewise-continuous function. From this equation we get

$$\nabla_y G(x,y) = \nabla_y g(x,y) - \mathcal{T}\nabla_y G.$$

Clearly,

$$\nabla_y g(x,y) = g(x,y)\Big(ik - \frac{1}{|x-y|}\Big)\frac{y-x}{|x-y|},$$

so

$$|\nabla_y g(x,y)| \leq 2\max\big(\frac{k}{4\pi d}, \frac{1}{4\pi d^2}\big) = \frac{1}{2\pi}\max\big(\frac{k}{d}, \frac{1}{d^2}\big), \quad |x-y| \geq d > 0.$$

Thus

$$|\nabla_y G(x,y)| \leq |\nabla_y g(x,y)|\big[1 + c\int_D \frac{1}{|x-z|}|\nabla_y G(z,y)|dz \frac{|x-y|^2}{|ik|x-y|-1|}\big] := |\nabla_y g|(1+cI),$$

where

$$I := \int_D |\nabla_y G(z,x)| \frac{dz}{|x-z|} \frac{|x-y|^2}{\sqrt{1+k^2|x-y|^2}} \leq c\int_D \frac{dz}{|z-y|^2|x-z|} \frac{|x-y|^2}{\sqrt{1+k^2d^2}}.$$

One has

$$\int_D \frac{dz}{|z-y|^2|x-z|} \leq c\Big|\ln|x-y|\Big|, \quad x,y\in D \quad c=c(D),$$

and

$$\sup_{x,y\in D} |\ln|x-y|\big||x-y|^2 \leq c,$$

where $c=c(D)$ is a constant. Therefore

$$I \leq \frac{c}{\sqrt{1+k^2d^2}} \leq c,$$

and

$$|\nabla_y G(x,y)| \leq c\max\big(\frac{k}{d}, \frac{1}{d^2}\big)\frac{1}{\sqrt{1+k^2d^2}} \leq c\max\big(\frac{k}{d}, \frac{1}{d^2}\big),$$

as claimed.

If $\frac{k}{d} \geq \frac{1}{d^2}$, i.e. $kd \geq 1$, then $|\nabla_y G(x,y)| \leq c\frac{k}{d}$, $|x-y| \geq d > 0$.

If $\frac{k}{d} < \frac{1}{d^2}$, i.e. $kd < 1$, then $|\nabla_y G(x,y)| \leq \frac{c}{d^2}$, $|x-y| \geq d > 0$.

Therefore, the right side of (2.41) is $O\big(\frac{k^2a^3}{d}\big)$ if $kd \geq 1$, in which case it is of the same order as the term $G(x,x_m)Q_m$. If $kd<1$, then the right side of (2.41) is $O\big(\frac{ka^3}{d^2}\big)$, in which case it may become larger than the term $G(x,x_m)Q_m$ because the ratio $\frac{ka^3}{d^2}/\frac{k^2a^3}{d} = \frac{1}{kd} > 1$ provided that $kd<1$.

Writing the field (2.1) in the form

$$u_M(x) = u_0(x) + \sum_{m=1}^M G(x,x_m)Q_m + \sum_{m=1}^M \int_{S_m} \big[G(x,s) - G(x,x_m)\big]\,\sigma_m(s)ds \quad (2.42)$$

and using formulas (2.35) and (2.41), one gets:

$$u_M(x) = u_0(x) + \sum_{m=1}^M G(x,x_m)\Delta u_e(x_m)\,V_m - \sum_{m=1}^M \frac{\partial G(x,x_m)}{\partial y_p}\,\frac{\partial u_e(x_m)}{\partial y_j}\,V_m\beta_{pj}^{(m)}(x_m), \quad (2.43)$$

and over the repeated indices p,j one sums up.

Let $a\to 0$, $M\to\infty$. We want to give sufficient conditions for passing to this limit in (2.43).

Lemma 4 *Assume that for any subdomain $\tilde{D}\subset D$ the following limits exist:*

$$\lim_{a\to 0} \sum_{D_m\subset\tilde{D}} V_m\beta_{pj}^{(m)}(x_m) = \int_{\tilde{D}} \beta_{pj}(y)\,\nu(y)dy, \quad (2.44)$$

$$\lim_{a\to 0} \sum_{D_m\subset\tilde{D}} V_m = \int_{\tilde{D}} \nu(y)dy. \quad (2.45)$$

Then the limiting form of equation (2.43) *is:*

$$\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\,\Delta\mathcal{U}(y)\,\nu(y)dy - \int_D \frac{\partial G(x,y)}{\partial y_p}\,\frac{\partial \mathcal{U}(y)}{\partial y_j}\,\beta_{pj}(y)\,\nu(y)dy, \tag{2.46}$$

where one sums up over the repeated indices p, j.

Remark 3 If one assumes that $\nu(y)$ vanishes near the boundary S of D and integrates the last integral in (2.46) by parts, one gets

$$\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\Big\{\Delta\mathcal{U}(y)\nu(y) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial \mathcal{U}(y)}{\partial y_j}\,\beta_{pj}(y)\,\nu(y)\Big)\Big\}dy. \tag{2.47}$$

Applying the operator L_0 to both sides of (2.47) and using (1.4) one gets:

$$L_0\mathcal{U} + \nu(y)\Delta\mathcal{U}(x) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial \mathcal{U}(y)}{\partial y_j}\,\beta_{pj}(y)\,\nu(y)\Big) = 0. \tag{2.47'}$$

Remark 4 If all the small particles are identical, then $V_m = c_3 a^3$, where the positive constant c_3 does not depend on m, and $\beta_{pj}^{(m)} = \beta_{pj}$. Then

$$\lim_{a\to 0} \sum_{D_m\subset\tilde{D}} V_m = \lim_{a\to 0}\Big[c_3 a^3 \sum_{D_m\subset\tilde{D}} 1\Big] = \lim_{a\to 0}[c_3 a^3 \mathcal{N}(\tilde{D})], \tag{2.48}$$

where $\mathcal{N}(\tilde{D})$ is the number of small particles in the domain $\tilde{D}$. For the limit (2.48) to exist it is sufficient that

$$\mathcal{N}(\tilde{D}) = \frac{\int_{\tilde{D}} \nu(y)dy}{c_3 a^3}, \tag{2.49}$$

where $\nu(y) \geq 0$ is a continuous function, and the limit in (2.48) is equal to $\int_{\tilde{D}} \nu(y)dy$.

One can write (2.49), with $|\tilde{D}| = dy,\ y \in \tilde{D}$, as

$$N(y)dy = \frac{\nu(y)}{c_3 a^3}\,dy, \tag{2.50}$$

where $N(y)$ is defined by the above formula. In contrast to Theorem 2, where $M = O(\frac{1}{a})$, we now have $M = O(\frac{1}{a^3})$.

Similarly,

$$\lim_{a\to 0} \sum_{D_m\subset\tilde{D}} V_m \beta_{pj}^{(m)}(x_j) = \lim_{a\to 0}[c_3 a^3\,\beta_{pj}\,\mathcal{N}(\tilde{D})] = \beta_{pj}\int_{\tilde{D}} \nu(y)dy. \tag{2.51}$$

We gave in this Remark some practically realizable sufficient conditions for the existence of the limits (2.44) and (2.45).

Let us verify that if the limits (2.44) – (2.45) exist, then the limit of the right side of equation (2.43) exists, and, denoting this limit by

$$\mathcal{U}(x) = \lim_{a\to 0} u_M(x),$$

one obtains the limiting form of equation (2.43):

$$\begin{aligned} \mathcal{U}(x) &= u_0(x) + \int_D G(x,y)\, \Delta\mathcal{U}(y)\, \nu(y)dy \\ &\quad - \int_D \sum_{p,j=1}^{3} \frac{\partial G(x,y)}{\partial y_p} \frac{\partial \mathcal{U}(y)}{\partial y_j}\, \beta_{pj}(y)\, \nu(y)dy, \end{aligned} \tag{2.52}$$

which is equation (2.46).

We took into account that

$$\lim_{a\to 0} u_e(x) = \mathcal{U}(x).$$

This is so because, as $a \to 0$, the input of a single particle into the field $\mathcal{U}(x)$ tends to zero.

Let us verify the existence of the limit of the right side of equation (2.43). We use, as in the proof of Lemma 3, a representation of D of the form $D = \bigcup_{p=1}^{P} \overline{\Delta}_p$, and assume that

$$\lim_{P\to\infty} \max_{1\le p\le P} \operatorname{diam} \Delta_p = 0. \tag{2.53}$$

Then

$$\begin{aligned} &\lim_{a\to 0} \sum_{m=1}^{M} G(x,y_m)\Delta u_e(x_m) V_m \\ &\quad = \sum_{p=1}^{P} \lim_{a\to 0} \sum_{D_m\subset\Delta_p} G(x,x_m)\Delta u_e(x_m) V_m \\ &\quad = \sum_{p=1}^{P} G(x,y^{(p)})\, \Delta u_e(y^{(p)})(1+\varepsilon_p)\, \lim_{a\to 0} \sum_{D_m\subset\Delta_p} V_m \\ &\quad = \sum_{p=1}^{P} G(x,y^{(p)})\, \Delta u_e(y^{(p)})(1+\varepsilon_p)\, \nu(y^{(p)})(1+\varepsilon'_p)|\Delta_p|, \end{aligned} \tag{2.54}$$

where

$$\lim_{P\to\infty} \max_{p}(|\varepsilon_p| + |\varepsilon'_p|) = 0. \tag{2.55}$$

Let $P \to \infty$ in (2.54) and use (2.55) to get

$$\lim_{P\to\infty} \sum_{p=1}^{P} G(x, y^{(p)})\, \Delta u_e(y^{(p)})\, \nu(y^{(p)})\, |\Delta_p|\, (1 + \varepsilon_p + \varepsilon'_p + \varepsilon_p \varepsilon'_p)$$

$$= \int_D G(x, y)\, \Delta \mathcal{U}(y)\, \nu(y) dy. \tag{2.56}$$

We have replaced $u_e(y^{(p)})$ in the limit $P \to \infty$ by $\mathcal{U}(y)$, because

$$\mathcal{U}(y) - u_e(y) = \int_{S_m} G(y, t)\, \sigma(t) dt = o(1) \quad \text{as } a \to 0, \quad |y - x_m| \geq d \gg a. \tag{2.57}$$

From (2.56) and (2.54) one gets:

$$\lim_{a\to 0} \sum_{m=1}^{M} G(x, x_m)\, \Delta u_e(x_m) V_m = \int_D G(x, y)\, \Delta \mathcal{U}(y)\, \nu(y) dy. \tag{2.58}$$

The singular points $x = y \in D$ of $G(x, y)$ are treated as in the proof of Theorem 2.

The function $|G(x, y)| \leq c|x - y|^{-1}$ as $|x - y| \to 0$, so $|G(x, y)| \in L^1(D)$ as a function of y for any $x \in D$.

Similar arguments, applied to the last sum in (2.43), lead to the formula

$$\lim_{a\to 0} \sum_{m=1}^{M} \frac{\partial G(x, x_m)}{\partial y_p} \frac{\partial u_e(x_m)}{\partial y_j} V_m\, \beta_{pj}^{(m)}(x_m) = \int_D \frac{\partial G(x, y)}{\partial y_p} \frac{\partial \mathcal{U}}{\partial y_j} \beta_{pj}(y)\, \nu(y) dy, \tag{2.59}$$

where one sums up over the repeated indices p, j.

Theorem 3 is proved. ■

In Section 4 we discuss the compatibility of the condition $d \gg a$ and the existence of the limit (1.29).

3 Auxiliary results

In this Section we prove Theorem 1 and Lemmas 1, 2.

Proof of Lemma 1 Let us start with the following observations:

$$\big||t - y| - |x - y|\big| \leq |t - y - (x - y)| = |t - x| \leq a, \tag{3.1}$$

$$\sup_{-a \leq s \leq a} |e^{is} - 1| \leq a, \tag{3.2}$$

$$\left|e^{ik|t-y|} - e^{ik|x-y|}\right| = \left|e^{ik(|t-y|-|x-y|)} - 1\right| \leq ka, \tag{3.3}$$

where the last inequality follows from (3.2).

One has

$$\begin{aligned} \big|g(t,y)-g(x,y)\big| &= \frac{\big||x-y|e^{ik|t-y|}-|t-y|e^{ik|x-y|}\big|}{|x-y||t-y|} \leq \frac{\big||x-y|-|t-y|\big|}{|x-y||t-y|}+ \\ &\quad +\frac{|t-y|\big|e^{ik|t-y|}-e^{ik|x-y|}}{|x-y||t-y|} \qquad (3.4)\\ &\leq \frac{a}{|x-y||t-y|}+\frac{ka}{|x-y|} \leq \frac{a}{d^2(1-\frac{a}{d})}+\frac{ka}{d} \leq O\big(\frac{a}{d^2}\big)+\frac{ka}{d}. \end{aligned}$$

Lemma 1 is proved. ■

Proof of Lemma 2 Let us start with the equation:

$$G(x,y)=g(x,y)-\int_D g(x,z)q_0(z)G(z,y)dz, \qquad (3.5)$$

where q_0 is defined in (1.2). From (3.5) one gets:

$$\begin{aligned} \big|G(t,y)-G(x,y)\big| &\leq \big|g(t,y)-g(x,y)\big|+\big|\int_D [g(t,z)-g(x,z)]q_0(z)G(z,y)dz\big| \\ &\leq O\big(\frac{a}{d^2}\big)+\frac{ka}{d}+c\int_D |g(t,z)-g(x,z)|\frac{dz}{|z-y|}. \qquad (3.6) \end{aligned}$$

Here we have used Lemma 1 and the estimates

$$\sup_{z\in D}|q_0(z)|\leq c_4, \quad |G(z,y)|\leq c_5|z-y|^{-1}, \qquad (3.7)$$

where $c_4, c_5 > 0$ are some constants.

Let us estimate the integral

$$\begin{aligned} I &:= \int_D |g(t,z)-g(x,z)|\frac{dz}{|z-y|} \\ &= \int_{|x-z|\geq\frac{d}{4}, z\in D} \frac{|g(t,z)-g(x,z)|dz}{|z-y|}+\int_{|x-z|\leq\frac{d}{4}, z\in D} \frac{|g(t,z)-g(x,z)|dz}{|z-y|} \\ &:= I_1+I_2. \qquad (3.8) \end{aligned}$$

By Lemma 1, which is applied to I_1 with d replaced by $\frac{d}{4}$, one gets

$$I_1 \leq c\big(\frac{a}{d^2}+\frac{ka}{d}\big)\int_{|x-z|\geq\frac{d}{4}} \frac{dz}{|z-y|} \leq c_1\big(\frac{a}{d^2}+\frac{ka}{d}\big). \qquad (3.9)$$

Here and below we do not write $z \in D$ under the integration sign to simplify the notations.

Let us estimate I_2:

$$I_2 \leq \frac{1}{4\pi}\int_{|x-z|\leq\frac{d}{4}} \frac{\big|e^{ik|t-z|}|x-z| - e^{ik|x-z|}|t-z|\big|}{|z-y||t-z||x-z|}dz. \tag{3.10}$$

One has

$$\begin{aligned}\big|e^{ik|t-z|}|x-z| - |t-z|e^{ik|x-z|}\big| &\leq \big||x-z|-|t-z|\big| + |t-z|\big|e^{ik|t-z|} - e^{ik|x-z|}\big| \\ &\leq |x-t| + |t-z|k\big||t-z|-|x-z|\big| \qquad (3.11) \\ &\leq |x-t| + k|t-z||t-x|.\end{aligned}$$

Thus, with $|x-y|\geq d \gg a$ and $|t-x|\leq a$, one has:

$$\begin{aligned} I_2 &\leq \frac{1}{4\pi}\int_{|x-z|\leq\frac{d}{4}} \frac{(|x-t| + k|t-z||t-x|)}{|z-y||t-z||x-z|}dz \\ &\leq \frac{|t-x|}{4\pi}\Big(\int_{|x-z|\leq\frac{d}{4}} \frac{dz}{|z-y||t-z||x-z|} + k\int_{|x-z|\leq\frac{d}{4}} \frac{dz}{|z-y||x-z|}\Big) \\ &\leq ca\big(\frac{1}{d^2} + \frac{k}{d}\big). \end{aligned} \tag{3.12}$$

From (3.9) and (3.12) the estimate (1.19) follows. Lemma 2 is proved. ■

Proof of Theorem 1 Let us first prove that if conditions (1.14) hold, then problem (1.5) – (1.7) has at most one solution. It is sufficient to prove that the homogeneous problem

$$(\nabla^2 + k^2 - q_0)u = 0 \text{ in } \mathbb{R}^3\setminus\bigcup_{m=1}^M D_m, \tag{3.13}$$

$$\frac{\partial u}{\partial r} - iku = o(\frac{1}{r}), \quad u = O(\frac{1}{r}), \quad r = |x|\to\infty, \tag{3.14}$$

$$u_N = \zeta_m u \text{ on } S_m, \quad 1\leq m\leq M, \tag{3.15}$$

has only the trivial solution if conditions (1.14) hold.

Taking complex conjugate of (3.13) – (3.15) one gets:

$$\big(\nabla^2 + k^2 - \overline{q}_0(x)\big)\overline{u} = 0 \text{ in } \mathbb{R}^3\setminus\bigcup_{m=1}^M D_m, \tag{3.16}$$

$$\frac{\partial \overline{u}}{\partial r} + ik\overline{u} = o\big(\frac{1}{r}\big), \quad \overline{u} = O\big(\frac{1}{r}\big), \ r = |x|\to\infty, \tag{3.17}$$

$$\overline{u}_N = \overline{\zeta}_m\overline{u} \text{ on } S_m, \quad 1\leq m\leq M. \tag{3.18}$$

Multiply (3.13) by $\overline{u}$, (3.16) by u, subtract from the first equation the second one, and integrate over the region $(\mathbb{R}^3\setminus\bigcup_{m=1}^M D_m)\cap B_R := D_R$, where B_R is the ball centered at the origin of radius R. Using Green's formula, one gets:

$$\begin{aligned}
0 &= \int_{D_R} [\overline{u}\nabla^2 u - u\nabla^2\overline{u} - (q_0 - \overline{q}_0)|u|^2]dx \\
&= -2i\int_{D_R} \operatorname{Im} q_0(x)|u|^2 dx + \int_{|x|=R} (\overline{u}\frac{\partial u}{\partial r} - u\frac{\partial \overline{u}}{\partial r})ds \\
&\quad -\sum_{m=1}^{M}\int_{S_m} (\overline{u}\frac{\partial u}{\partial N} - u\frac{\partial \overline{u}}{\partial N})ds. \qquad (3.19)
\end{aligned}$$

Using (3.17) and (3.18) one rewrites (3.19) as follows:

$$0 = -2i\int_{D_R} \operatorname{Im} q_0(x)|u|^2 dx + 2ik\int_{|x|=R} |u|^2 ds + o(1) - 2i\sum_{m=1}^{M}\int_{S_m} \operatorname{Im}\xi_m|u|^2 ds. \qquad (3.20)$$

Letting $R \to \infty$, taking into account that $q_0(x) = 0$ in $D' = \mathbb{R}^3 \setminus D$, and one gets:

$$0 \le \int_{D\setminus\bigcup_{m=1}^M D_m} \operatorname{Im} q_0(x)|u|^2 dx + \sum_{m=1}^{M}\int_{S_m} \operatorname{Im}\zeta_m|u|^2 ds - k\limsup_{R\to\infty}\int_{|x|=R} |u|^2 ds. \qquad (3.21)$$

Since all the terms on the right side of this relation are non-positive by the assumptions (1.14), it follows that

$$\limsup_{R\to\infty}\int_{|x|=R} |u|^2 ds = 0.$$

This implies that $u = 0$ (see, [9], p. 231).

Thus, uniqueness of the solution to problem (1.5) – 1.7 is proved.

Let us prove the existence of the solution to (1.5) – (1.7) of the form (1.10). The existence of the solution of the form (1.10) will be established if one proves the existence of $\sigma_m, 1 \le m \le M$, such that boundary condition (1.7) is satisfied:

$$u_{eN} - \zeta_j u_e + \frac{A_j\sigma_j - \sigma_j}{2} - \zeta_j T_j\sigma_j = 0, \quad 1 \le j \le M. \qquad (3.22)$$

Here u_e, which depends on j, is defined by the formula:

$$u_e := u - \int_{S_j} G(x,s)\sigma_j(s)ds = u_0 + \sum_{m\neq j}\int_{S_m} G(x,s)\sigma_m(s)ds. \qquad (3.23)$$

Under our assumptions $S_m \in C^{1,\lambda}$ uniformly with respect to m. Therefore equation (3.22) is of Fredholm type in the space $L^2(\bigcup_{m=1}^M S_m)$. The corresponding homogeneous equation, i.e., the equation with $u_0 = 0$, cannot have a nontrivial solution because such a solution would generate by formula (1.10) with $u_0 = 0$ a function $u_M(x)=\sum_{m=1}^M\int_{S_m}G(x,s)\sigma_m(s)ds$, which would solve the homogeneous problem (1.5) – (1.7). We have already proved that such a function has to be zero in $\mathbb{R}^3\setminus\bigcup_{m=1}^M D_m$. Thus, $u_M|_{S_m}=0$, $1\leq m\leq M$, and u_M solves the problem:

$$L_0u_M=0 \text{ in } D_m, \quad u_M|_{S_m}=0,\ 1\leq m\leq M. \tag{3.24}$$

If $\operatorname{diam} D_m\leq 2a$ is sufficiently small, then problem (3.24) has only the trivial solution for every $m, 1\leq m\leq M$. Therefore $u_M=0$ in D_m and in $\mathbb{R}^3\setminus\bigcup_{m=1}^M D_m$. Therefore, by the formula for the jumps of the normal derivatives of the single layer potential,

$$\frac{\partial u_M^+}{\partial N}|_{S_m}-\frac{\partial u_M^-}{\partial N}|_{S_m}=\sigma_m,$$

we conclude that $\sigma_m=0$, $1\leq m\leq M$. This implies the existence of the solution to problem (1.5) – (1.7) of the form (1.10).

Theorem 1 is proved. ■

Let us return to the assumptions of Theorem 2, namely,

$$\zeta_m=O(\frac{1}{a}),\quad a\mathcal{N}(\Delta_b(y))=N(y)|\Delta_b(y)|(1+o(1)),$$

where $\Delta_b(y)$ is a cube, centered at the point y with the side $b>0$, and $o(1)$ is related to the limiting process $b\to 0$.

Under these assumptions let us establish an estimate for the function $v_M:=u_M-u_0$, which is uniform with respect to $M\to\infty$, or $a\to 0$. From this estimate it follows that v_M converges, as $a\to 0$, in $L^2(\mathbb{R}^3,(1+|x|)^{-1-\gamma})$, where $\gamma>0$ is an arbitrary fixed constant. The function v_M satisfies the radiation condition at infinity. The function $u_0\in H^2_{loc}(\mathbb{R}^3)$ solves the equation $L_0u_0=0$ in $\mathbb{R}^3$.

Let $D_e:=\mathbb{R}^3\setminus\cup_{m=1}^M D_m$ and $S':=\cup_{m=1}^M S_m$. Let

$$||v||:=\Big(\int_{D_e}|v(x)|^2(1+|x|)^{-1-\gamma}dx\Big)^{1/2},\quad |||v|||=\sum_{m=1}^M\Big(\int_{S_m}(|v_N|^2+|v|^2)ds\Big)^{1/2}.$$

The estimate we wish to prove is:

$$||v_M||\leq c|||u_0|||. \tag{3.25}$$

Here and below $c>0$ stand for various constants independent of a. Inequality similar to (3.25) was used in [20], where a theorem, I have called "Modified Rayleigh Conjecture", is proved.

Let us outline the proof of inequality (3.25).

Step 1. If $M=O(\frac{1}{a})$, then the right side of (3.25) is bounded as $a\to 0$.

Indeed, the number of small particles is $M = O(\frac{1}{a})$ and u_0 is $H^2_{loc}(\mathbb{R}^3)$, so that u_0 and u_{0N} are bounded in $L^2(S_m)$ uniformly with respect to m, $1 \leq m \leq M$. Thus,

$$|||u_0||| \leq \frac{c}{a} \max_{1\leq m\leq M} \Big(\int_{S_m} (|u_{0N}|^2 + |u_0|^2) ds \Big)^{1/2} \leq \frac{c}{a} |S_m|^{1/2} \leq c,$$

where $c > 0$ stand for various constants independent of a.

Step 2. If the inequality (3.25) is false, then there is a sequence $u_0^{(n)}$, $|||u_0^{(n)}||| = 1$, such that $||v_M^{(n)}|| := ||v^{(n)}|| \geq n$.

Define $w^{(n)} := \frac{v^{(n)}}{||v^{(n)}||}$. Then

$$||w^{(n)}|| = 1. \tag{3.26}$$

From the weak compactness of bounded sets in L^2, it follows, that one may select a subsequence, denoted again $w^{(n)}$, such that $w^{(n)}$ converges weakly in $L^2_{loc}(D_e)$ to a function w. The function $w^{(n)}$ solves the problem:

$$\begin{aligned} & L_0 w^{(n)} = 0 \quad in \quad D_e, \\ w_N^{(n)} - \zeta_m w^{(n)} = (\zeta_m u_0^{(n)} - u_{0N}^{(n)})/||v^{(n)}|| \quad & on \quad S_m, 1 \leq m \leq M, \end{aligned} \tag{3.27}$$

and $w^{(n)}$ satisfies the radiation condition.

It follows from (3.27) that $||\nabla^2 w^{(n)}|| < c$, so $||w^{(n)}||_{H^2_{loc}(D_e)} < c$, where $H^2_{loc}(D_e)$ is the Sobolev space (see, for example, book [1], where the theory of these spaces is presented). Thus, one may assume, using the compactness of the embedding from H^2_{loc} into L^2_{loc}, that $w^{(n)}$ converges to w strongly in $L^2_{loc}(D_e)$. This and equation (3.27) imply that $w^{(n)}$ converges to w strongly in $H^2_{loc}(D_e)$, so that w solves equation (3.27), satisfies the radiation condition and the homogeneous boundary condition (3.27), that is, $w_N - \zeta_m w = 0$ on S_m. Therefore, by already proved uniqueness theorem (see the proof of Theorem 1), we conclude that $w = 0$. The terms $u_0^{(n)}/||v^{(n)}||$ and $u_{0N}^{(n)}/||v^{(n)}||$ tend to zero as $n \to \infty$, because $||v^{(n)}|| > n$. Therefore, the limiting function w satisfies the homogeneous boundary condition $w_N = \zeta_m w$ on S_m, $1 \leq m \leq M$.

Let us prove that $|w^{(n)}(x)| < \frac{c}{|x|}$, $|x| > R$, where $R > 0$ is sufficiently large and $c > 0$ does not depend on n.

For $w^{(n)}$ one has a representation by the Green formula in the region $|x| > R$, where $R > 0$ is large enough, so that the ball $B_R := \{x : |x| < R\}$ contains D. Namely

$$w^{(n)}(x) = \int_{|s|=R} (w_r^{(n)} g(x,s) - g_r(x,s) w^{(n)}) ds, \quad |x| > R, \tag{3.28}$$

where the derivatives with respect to r are the derivatives along the normal to the sphere $S_R := \{s : |s| = R\}$, and g is defined in (1.15). It follows from (3.28) that $|w^{(n)}(x)| < \frac{c}{|x|}$ for $|x| > R$, where $c > 0$ is a constant independent of n, because local converegence in H^2 implies that the $L^2(S_R)$-norms of $w^{(n)}$ and of $w_r^{(n)}$ are bounded uniformly with respect to n.

Therefore

$$\lim_{n\to\infty} ||w^{(n)} - w|| = 0, \tag{3.29}$$

because on compact sets $\lim_{n\to\infty} ||w^{(n)} - w||_{H^2_{loc}(D_e)} = 0$, and near infinity the inequality $|w^{(n)}(x)| < \frac{c}{|x|}$ implies that

$$\int_{\{x:|x|>R\}} |w^{(n)}(x)|^2(1+|x|)^{-1-\gamma}dx = O(R^{-\gamma}) \to 0, \;\; R \to \infty,$$

so that (3.29) holds. Because of the uniqueness of the limit, not only a subsequence of $w^{(n)}$ but the sequence itself converges to w as $n \to \infty$.

This leads to a contradiction, because $w = 0$ and (3.26) together with (3.29) imply $||w|| = 1$.

This contradiction proves inequality (3.25).

From inequality (3.25) and Step 1 one concludes that that u_M contains a weakly convergent in $L^2_{loc}(D_e)$ subsequence. By the arguments, similar to the given above, this subsequence converges in $L^2(\mathbb{R}^3, (1+|x|)^{-1-\gamma})$. Its limit solves equation (1.24).

The relation $M = O(\frac{1}{a})$ plays an important role in our proof of Theorem 2 and in Step 1 in the above argument.

4 Application to creating smart materials

Let us ask the following question: can one make a material with a desired refraction coefficient $n^2(x)$ in a bounded domain $D \subset \mathbb{R}^3$, filled by a material with a known refraction coefficient $n_0^2(x)$, for example $n_0^2(x) = n_0^2 = \text{const}$ in D, by embedding into D a number of small particles, each of which is defined by its shape and its boundary impedance?

We give an affirmative answer to this question. Moreover, we give explicit formulas for the number of small particles of characteristic size a to be embedded in the domain D around a point $x \in \Delta$, where $\Delta \subset D$ is a small cube, centered at x, with a side $b \gg a$, and for the boundary impedances ζ_m of these particles (see Theorem 2). Specifically, given the original refraction coefficient $n_0^2(x)$ in D, and the desired refraction coefficient $n^2(x)$, we calculate $q(x) = k^2 - k^2n^2(x)$ and $q_0(x) = k^2 - k^2n_0^2(x)$, and then calculate $p(x) = q(x) - q_0(x) = k^2[n^2(x) - n_0^2(x)]$. Then we find (non-uniquely) three functions: $N(x) \geq 0$, a real-valued function $h_1(x)$, and a non-positive function $h_2(x)$. Define $h(x) := h_1(x) + ih_2(x)$. How does one find these functions is explained below. If $N(x)$ and $h(x) = h_1(x) + ih_2(x)$ are found, then the boundary impedance $\zeta(x)$ is defined by formula (4.2) (see below) and the number of particles in any small cube $\Delta \subset D$ is found by the formula (1.21):

$$\mathcal{N}(\Delta) = \frac{1}{a} \int_{\Delta} N(x) dx,$$

where a is the size of one particle.

Consider first the particles satisfying the assumptions of Theorem 2. More specifically, suppose that all the particles are balls of the same radius a. In this case

$$|S_m| = 4\pi a^2, \quad V = \frac{4\pi}{3} a^3, \quad J_m = \int_{|s|=a} \int_{|t|=a} \frac{ds\, dt}{|s-t|} = 16\pi^2 a^3,$$

so

$$c_1 = 4\pi, \quad c_2 = 16\pi^2, \quad c_3 = \frac{4\pi}{3}, \quad \frac{4\pi c_1^2}{c_2} = 4\pi,$$

and formula (1.25) yields

$$p(x) = \frac{4\pi N(x)\, h(x)}{1 + h(x)}, \tag{4.1}$$

where $h(x)$ is defined by the choice of the boundary impedances by formula (1.20):

$$\zeta(x) = \frac{h(x)}{a}, \tag{4.2}$$

and $N(x)$ is defined by formula (1.21).

If the original refraction coefficient is $n_0^2(x)$, then the corresponding potential is $q_0(x) = k^2[1 - n_0^2(x)]$ by formula (1.2). If the desired refraction coefficient in D is $n^2(x)$, then the corresponding potential is $q(x) = k^2[1 - n^2(x)]$, so

$$p(x) = q(x) - q_0(x) = k^2[n_0^2(x) - n^2(x)]. \tag{4.3}$$

To create a material with the desired refraction coefficient $n^2(x)$ it is sufficient to choose $N(x)$ and $h(x)$ so that (4.1) holds with $p(x)$ defined in (4.3). If the new material with the refraction coefficient $n^2(x)$ has some absorption, that is, $\operatorname{Im} n^2(x) \geq 0$, and $\operatorname{Im} n_0^2 = 0$, then $\operatorname{Im} p(x) \leq 0$. Let us prove that any function $p(x)$ in D with $\operatorname{Im} p \leq 0$, can be obtained (in many ways, non-uniquely) by formula (4.1) with some choices of a nonnegative function $N(x)$ and a function $h(x)$ with $\operatorname{Im} h \leq 0$.

Let $p(x) = p_1(x) + ip_2(x)$, $p_2(x) \leq 0$, and $h(x) = h_1(x) + ih_2(x)$, $h_2(x) \leq 0$. Assume that $p(x)$ is given. Then (4.1) implies

$$p_1 + ip_2 = 4\pi \frac{(h_1 + ih_2)(1 + h_1 - ih_2)}{(1+h_1)^2 + h_2^2} N(x). \tag{4.4}$$

Thus

$$p_1 = 4\pi \frac{h_1 + h_1^2 + h_2^2}{(1+h_1)^2 + h_2^2} N(x), \quad p_2 = 4\pi \frac{h_2}{(1+h_1)^2 + h_2^2} N(x). \tag{4.5}$$

There are many choices of the three functions: $N(x) \geq 0$, $h_2(x) \leq 0$ and a real-valued function $h_1(x)$ such that relations (4.5) hold. For example, if $p_1 > 0$ and $p_2 \neq 0$, then one can choose

$$h_1(x) = 0, \quad h_2(x) = \frac{p_1(x)}{p_2(x)}, \quad N(x) = \frac{p_1^2(x) + p_2^2(x)}{4\pi\, p_1(x)}. \tag{4.6}$$

It is a simple matter to check that relations (4.5) hold with the choice (4.6). Since one has three functions $h_1(x)$, $h_2(x) \leq 0$ and $N(x) \geq 0$ to satisfy two equations (4.5) with $p_2(x) \leq 0$, there are many ways to do this. A particular choice of $h(x) = h_1(x) + ih_2(x)$ and $N(x) \geq 0$ yields the surface impedance $\zeta(x)$ of the particles to be embedded around each point $x \in D$, $\zeta(x) = \frac{h(x)}{a}$ by formula (4.2), and the number of particles per unit volume around the point x, namely, by formula (1.21) this number is $\frac{N(x)}{a}$, so that the number of particles to be embedded in the volume dx around point x is equal to $\frac{N(x)}{a}\, dx$. The smallest distance d between the embedded particles should satisfy the inequality $d \gg a$. One may try to take practically $d > 10a$.

Example 1 Suppose that the elementary subdomain Δ_p, used in the proof of Lemma 3, is a cube with the side $b \gg d$, $x \in \Delta_p$. Let, for example $b = 10^{-2}$cm, $d = 10^{-3}$cm, $a = 10^{-5}$cm. Then there are $\left(\frac{b}{d}\right)^3 = 10^3$ small particles in Δ_p around a point x, the center of Δ_p. The function $N(x)$ in Δ_p in this example is found from the formula $\frac{N(x)}{a}\, b^3 = 10^3$ (use (1.21) with $\tilde{D} = \Delta_p$), so $N(x) = 10^{-5} \cdot 10^3 \cdot 10^6 = 10^4$. The number of small particles, embedded in the cube Δ_p around point x, the center of this cube, is 10^3 in this example. The relative volume of these particles in Δ_p is $10^3 \cdot \frac{4}{3}\pi 10^{-15} \cdot 10^6 = 4.18 \cdot 10^{-6}$, so it is quite small, which is in full agreement with our theory.

The assumption (1.8), specifically, $d \gg a$, is compatible with the requirement (1.21). Indeed, if one denotes by $\mathcal{N}(\tilde{D})$ the left side of (1.21), then $\mathcal{N}(\tilde{D}) = O\left(\frac{1}{a}\right)$ for any $\tilde{D} \subseteq D$.

Let us assume that $\tilde{D}$ is a unit cube, and denote by $\mathcal{N}(\tilde{D})$ the left side of (1.21). The assumption $d \gg a$ implies that the number $\mathcal{N}(\tilde{D})$ of particles in $\tilde{D}$ is $O\left(\frac{1}{d^3}\right)$. These relations are compatible if and only if $O\left(\frac{1}{a}\right) = O\left(\frac{1}{d^3}\right)$, i.e., $d = O(a^{1/3})$. Therefore, it is possible to have $a \to 0$, $\frac{a}{d} \to 0$ and equation (1.21) satisfied.

Let us discuss the new material properties, specifically, anisotropy, when acoustically hard particles are embedded in the domain D, and the assumptions of Theorem 3 are valid. The physical situation is now quite different from the one in Theorem 2. From the physical point of view one can anticipate the drastic difference because the wave scattering by one small acoustically soft particle of the characteristic size a is isotropic and the scattering amplitude is of order a, while the wave scattering by a small acoustically hard particle is anisotropic and the corresponding scattering amplitude is of order k^2a^3, (see [10], chapter 7). We assume that $ka \ll 1$, say $ka < 0.1$, so that the quantity $k^2a^3 = (ka)^2 a$ is 100 times less than a.

Example 2 Let us assume again that the small particles are all balls of the same radius a. Then

$$V_m = \frac{4}{3}\pi a^3, \quad \nu(y)|\Delta_p| = \frac{4}{3}\pi a^3 \mathcal{N}(\Delta_p),$$

where $\mathcal{N}(\Delta_p)$ is the number of small particles in a small cube Δ_p centered at the point y. If b is the size of the edge of the cube Δ_p, then $\nu(y) = 4.18\,\frac{a^3}{b^3}\mathcal{N}(\Delta_p)$, where 4.18 is an approximate value of $\frac{4\pi}{3}$. The magnetic polarizability tensor β_{pj} of a ball of radius a is $\beta_{pj} = -\frac{3}{2}\delta_{pj}$, while the electric polarizability tensor of a perfectly conducting ball is $3\delta_{ij}$, where

$$\delta_{pj} = \begin{cases} 1, & p = j, \\ 0, & p \neq j. \end{cases}$$

These values differ by the factor 4π from the values in [3] because we use the formula $\varphi = \frac{1}{4\pi|x|}$ for the potential of a point charge, while in [3] this potential is $\frac{1}{|x|}$. In our example β_{pj} does not depend on m. Therefore the limit (1.28) exists if the limit (1.29) exists. The limit (1.29) exists if and only if the following limit exists:

$$\frac{4\pi}{3}\lim_{a\to 0} a^3 \sum_{D_m\subset\tilde{D}} 1 = \int_{\tilde{D}} \nu(y)dy, \tag{4.7}$$

where $\nu(y)$ is the function defined in (1.29). Thus, in contrast to Example 1, where $\mathcal{N}(\Delta_p) = O\big(\frac{1}{a}\big)$, we now have $\mathcal{N}(\Delta_p) = O\big(\frac{1}{a^3}\big)$. The relative volume of the small particles in Example 2 is not negligible and does not go to zero as $a \to 0$, in contrast to Example 1.

Let us discuss the compatibility of the condition $d \gg a$ and the existence of the limits (1.28) and (1.29). If the condition $d \gg a$ is compatible with the existence of the limit (1.29), then it is compatible with the existence of the limit (1.28). If the limit (1.29) exists, then $a^3 N(\tilde{D}) = O(1)$, so $N(\tilde{D}) = O(a^{-3})$. On the other hand, $N(\tilde{D}) = O(d^{-3})$. These relations, in general, are not compatible because $d \gg a$. Let us argue more precisely. Let $\tilde{D} = \Delta_p$, where Δ_p is a cube with the edge of size b. Let us assume that the small particles in Δ_p are identical and their characteristic size is a. If (1.29) holds, where $\nu(y)$ is continuous, and if b is small, then the right side of (1.29) equals to $\nu(y)b^3$, $y \in \Delta_p = \tilde{D}$. The left side of (1.29) equals to $c_3 a^3 N(\Delta_p)$, where $V = c_3 a^3$ is the volume of one particle, $c_3 = \frac{4\pi}{3}$ if the particle is a ball of radius a. Thus $N(\Delta_p) = \frac{1}{c_3}\,\nu(y)\,\frac{b^3}{a^3}$. On the other hand, $N(\Delta_p) = \frac{b^3}{d^3}$, provided that one assumes that the centers of the small particles are at the uniform grid, so that there are $\frac{b}{d}$ centers on the segment of length b. If $\frac{1}{c_3}\,\nu(y)\,\frac{b^3}{a^3} = \frac{b^3}{d^3}$, then $\frac{a}{d} = \big(\frac{\nu(y)}{c_3}\big)^{1/3}$. Therefore the condition $d \gg a$

is satisfied only if $\left(\frac{\nu(y)}{c_3}\right)^{1/3} \ll 1$, say $\left(\frac{\nu(y)}{c_3}\right)^{1/3} \leq 0.1$. The number c_3 depends on the shape of the particle. If the particles are balls of radius a, then $c_3 = 4.18$. Therefore $\nu(y) \leq 4.10^{-3}$.

The conclusion is:

The condition $d \gg a$ is compatible with the existence of the limit (1.29) *only if the function $\nu(y)$ in* (1.29) *is sufficiently small.*

In general, equation (1.31) cannot be reduced to a local differential equation for $\mathcal{U}(x)$. However, if $\nu(y)$ is small, one may use perturbation theory to study equation (1.31). However, under an additional assumption, reasonable from the physical point of view, one can reduce integral-differential equation (1.31) to a differential equation. Namely, let us assume that $\nu(y)$ is a continuously differentiable function in D which vanishes near the boundary S.

Under this assumption one can integrate by parts the last integral in (1.31) and get:

$$\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\Big[\Delta\mathcal{U}(y)\nu(y) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial \mathcal{U}(y)}{\partial y_j}\,\beta_{pj}(y)\nu(y)\Big)\Big]. \quad (4.8)$$

Let us apply the operator $L_0 = \nabla^2 + k^2 - q_0(x)$ to (4.8) and use (1.4) to get:

$$[\nabla^2 + k^2 - q_0(y)]\mathcal{U} + \nu(y)\nabla^2\mathcal{U}(x) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial \mathcal{U}(x)}{\partial y}\,\beta_{pj}(y)\,\nu(y)\Big) = 0, \quad (4.9)$$

where $\mathcal{U}(x)$ satisfies the radiation condition of the type (1.6). This is an elliptic equation and the perturbation $\mathcal{P}$ of the operator L_0 is:

$$\mathcal{P}\mathcal{U} := \nu(x)\nabla^2\mathcal{U}(x) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial \mathcal{U}}{\partial y_j}\,\beta_{pj}(y)\,\nu(x)\Big). \quad (4.10)$$

This perturbation is the sum of the terms with positive small coefficient $\nu(y)$ in front of the second derivatives of $\mathcal{U}$ and a term with the first order derivatives of $\mathcal{U}$:

$$\mathcal{P}\mathcal{U} = \nu(x)\Big[\nabla^2\mathcal{U}(x) + \sum_{p,j=1}^{3} \frac{\partial}{\partial x_p}\Big(\frac{\partial \mathcal{U}(x)}{\partial x_j}\,\beta_{pj}(x)\Big)\Big] + \sum_{p,j=1}^{3} \frac{\partial \mathcal{U}(x)}{\partial x_j}\,\beta_{pj}(x)\,\frac{\partial \nu(x)}{\partial x_p}\,. \quad (4.11)$$

If both $\nu(x)$ and $\nabla\nu(x)$ are small, this equation can be studied by perturbation methods. The physical effect on the properties of the new material, created by embedding into D small acoustically hard particles, consists in appearing of anisotropy in the new material: the propagation of waves is described by the integral-differential equation (4.8) or (under the additional assumption on $\nu(y)$, namely: $\nu(y)$ vanishes near the boundary S of D) by the differential equation

(4.9) with variable coefficients in front of the senior (second order) derivatives and the terms with the first order derivatives.

The role of the compatibility of the assumption $d \gg a$ and of the assumption (1.29) is quite important. Although passing to the limit $a \to 0$, justified in the proof of Theorem 3, is based on the assumptions (1.28) and (1.29), but without the assumption $d \gg a$ one cannot expect, in general, that the effective field $u_e(x)$, acting on any single particle, is practically constant on the distances of the order $2a$. This physical assumption is important for our theory.

From the mathematical point of view, if $\nu(x)$ is not sufficiently small, then the existence of the unique solution to equation (4.8) or of the solution to equation (4.9), satisfying the radiation condition, is not guaranteed.

If, on the other hand, the quantity

$$\sup_{x\in\mathbb{R}^3} \big(|\nu(x)| + |\nabla\nu(x)|\big) \ll 1,$$

that is, this quantity is sufficiently small, then one can argue that the norm of the integral operator in (4.8) in $L^2(D)$ is small, so that equation (4.8) has a unique solution in $L^2(D)$. This solution admits a natural extension to the whole space $\mathbb{R}^3$ by the right side of (4.8) because $\nu(y)$ vanishes outside D. Since $G(x,y)$ satisfies the radiation condition, the solution to (4.8) also satisfies this condition. Without the assumption that $|\nu(x)| + |\nabla\nu(x)|$ is sufficiently small, one cannot use the above argument.

With this assumption one may solve equation (4.8) by iterations and find in this way an approximate solution to this equation. The first iteration yields the following approximate solution to equation (4.8):

$$\mathcal{U}(x) = u_0(x) + \int_D G(x,y)\left[\Delta u_0(y)\nu(y) + \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial u_0(y)}{\partial y_j}\,\beta_{pj}(y)\,\nu(y)\Big)\right]dy. \tag{4.12}$$

Formula (4.12) gives the correction to the solution $u_0(x)$ of the unperturbed scattering problem, i.e., the scattering problem in the absence of small bodies. Since one has

$$\Delta u_0 = -k^2 n_0^2(x) u_0,$$

(4.12) can be rewritten as:

$$\begin{aligned}\mathcal{U}(x) = u_0(x) - k^2 \int_D G(x,y) n_0^2(y) u_0(y)\nu(y)dy \\ + \int_D G(x,y) \sum_{p,j=1}^{3} \frac{\partial}{\partial y_p}\Big(\frac{\partial u_0(y)}{\partial y_j}\,\beta_{pj}(y)\nu(y)\Big)dy.\end{aligned} \tag{4.13}$$

In [4], Chapter 3, Section 3, the Neumann problem for the Helmholtz equation with $n_0^2(x) = 1$ was studied in the domain, similar to the one in equation (1.5) and it was proved under the assumptions used in [4], that the main term of the asymptotics of the solution, as the relative volume of the particles tends to zero, is the incident field, while the next term is proportional to this relative volume.

5 Convergence and compactness estimates

Here we prove that the function $u_e(x)$ converges in $C(\mathbb{R}^3)$ to the limit $u(x)$, satisfying the integral equation

$$u(x) = u_0(x) - \int_D G(x,y)\, p(y)\, u(y) dy,$$

where $p(x)$ is defined in (1.25).

Let us outline the steps of our proof.

Step 1. Let Δ_j be the cube with the side $b \gg a^{\frac{1}{6}}$ and center at the point $y^{(j)}$. The union of the cubes Δ_j, $0 \le j \le J$, is a partition of D. Since $a > 0$ is small, and $d = O(a^{1/3})$, one has $b \gg d$.

Lemma 5 *Assume*

$$d = O(a^{1/3}), \quad \sigma = O\big(\frac{1}{a}\big), \quad |S_m| = O(a^2), \quad \mathcal{N}(\Delta_j) = O\big(\frac{b^3}{a}\big). \tag{5.1}$$

Then

$$\lim_{b\to 0} \lim_{a\to 0} \sum_{\substack{x_m \in \Delta_0 \\ x\in\Delta_0,\, |x-x_m|\ge d}} \int_{S_m} G(x,s)\, \sigma_m(s) ds = 0. \tag{5.2}$$

We denote by $\varepsilon(a,b)$ the sum in (5.2)*.*

Step 2. Let us write

$$u_M(x) = u_0(x) + \sum_{\substack{x_m \notin \Delta_0 \\ x\in\Delta_0}} G(x,x_m) Q_m + \varepsilon(a,b) + \eta(a), \tag{5.3}$$

where

$$\lim_{a\to 0} \eta(a) = 0, \quad \lim_{b\to 0} \lim_{a\to 0} \varepsilon(a,b) = 0, \tag{5.4}$$

and by (2.20),

$$Q_m = -f(x_m) a u_e(x_m), \tag{5.5}$$

where $f(x)$ is a continuous function in D, $f(x_m) = \text{const}\, \frac{h(x_m)}{1+h(x_m)}$ in (2.24). We may replace $u_M(x)$ by $u_e(x)$ and neglect the terms $\varepsilon(a,b)$ and $\eta(a)$ in (5.3) because of (5.4). Then (5.3) takes the form

$$u_e(x) = u_0(x) - \sum_{\substack{x_m \notin \Delta_0 \\ x\in\Delta_0}} G(x,x_m)\, f(x_m)\, u_e(x_m) a. \tag{5.6}$$

We prove that, as $a \to 0$, the set $\{u_e\}$ is uniformly bounded and equicontinuous in the space $C(\mathbb{R}^3)$, and uniformly small at infinity. This implies convergence, as $a \to 0$, in $C(\mathbb{R}^3)$ of a subsequence, denoted $\{u_e\}$ again. The limit $u(x)$ of this subsequence satisfies equation (5.14) (see below), which has a unique solution in $C(\mathbb{R}^3)$. Thus, the whole set $\{u_e\}$ converges to $u(x)$ as $a \to 0$.

Let us now give the details of these two steps.

Proof of Lemma 5 Write the sum in (5.2) as

$$\Sigma := \Sigma_1 + \Sigma_2 := \sum_{\substack{x_m \in \Delta_0 \\ |x_m - x| \geq a^{1/6}}} + \sum_{\substack{x_m \in \Delta_0 \\ |x_m - x| < a^{1/6}}}, \tag{5.7}$$

where $\{\Delta_j\}$ is a partition of D into a union of nonintersecting cubes with side b and center $y^{(j)}$, and Δ_0 is the cube, containing point x. In the proof of Lemma 5 we only deal with this cube Δ_0, and we assume the origin at the point x. Note that $a^{1/6} = O(d^{1/2}) \gg d$.

Let us estimate Σ_2, taking $x = 0$ to be the origin:

$$\begin{aligned} \lim_{a\to 0} |\Sigma_2| &\leq \lim_{a \to 0} c \sum_{d \leq |x_m| < a^{1/6}} \int_{S_m} \frac{|\sigma_m(s)| ds}{|s|} \\ &\leq \frac{ca^2}{a} \sum_{d \leq d(i_1^2+i_2^2+i_3^2)^{1/2} \leq a^{1/6}} \frac{1}{d\sqrt{i_1^2+i_2^2+i_3^2}} \\ &\leq \frac{ca}{a^{1/3}} \int_1^{\frac{a^{1/6}}{d}} \frac{r^2 dr}{r} = ca^{2/3}(a^{-1/3} - 1) \leq ca^{1/3} \to 0, \end{aligned} \tag{5.8}$$

where $c > 0$ stands for various constants independent of a, and we have used the assumption $d = O(a^{1/3})$.

Let us estimate Σ_1:

$$\begin{aligned} |\Sigma_1| &\leq ca \sum_{a^{1/6} \leq d(i_1^2+i_2^2+i_2^2)^{1/2} \leq b} \frac{1}{d(i_1^2+i_2^2+i_3^2)^{1/2}} \leq \frac{ca}{a^{1/3}} \int_{a^{-1/6}}^{\frac{b}{a^{1/3}}} r\, dr \\ &= ca^{2/3}\Big(\frac{b^2}{a^{2/3}} - a^{-1/3}\Big) \leq cb^2 \underset{b \to 0}{\longrightarrow} 0, \quad b \gg a. \end{aligned} \tag{5.9}$$

From (5.8) and (5.9) Lemma 5 follows. ■

Lemma 6 *If $\sigma_m = O(\frac{1}{a})$ and $d = O(a^{1/3})$, then*

$$\text{a)} \quad \sup_{0<a<a_0} \sup_{x \in D} |u_e(x)| \leq c, \quad \text{b)} \quad \sup_{0<a<a_0} \sup_{x \in D} |u_e(x+\Delta x) - u_e(x)| \underset{|\Delta x| \to 0}{\longrightarrow} 0, \tag{5.10}$$

Proof. Let us prove a):

$$\Big|\sum_{m=1}^{M}\int_{S_m} G(x,s)\sigma_m\,ds\Big| \le ca\sum_{m=1}^{M}\frac{1}{|x-x_m|} \le ca\sum_{d\le rd\le L}\frac{1}{d\sqrt{i_1^2+i_2^2+i_3^2}}$$
$$\le \frac{ca}{a^{1/3}}\int_1^{L/d} dr\, r = ca^{2/3}\Big(\frac{L^2}{a^{2/3}}-1\Big) \le cL^2, \tag{5.11}$$

$L = \operatorname{diam} D$, $r=\sqrt{i_1^2+i_2^2+i_3^2}$. Thus, a) is proved.

Let us prove b):

$$\Big|\sum_{m=1}^{M}\int_{S_m}\big[G(x+\Delta x)-G(x,s)\big]\sigma_m(s)dy\Big|$$
$$\le ca|\Delta x|\sum_{m=1}^{M}\max\Big(\frac{1}{|x-x_m|},\ \frac{1}{|x-x_m|^2}\Big) \le c|\Delta x|a\sum_{1\le r\le \frac{L}{d}}\frac{1}{d^2(i_1^2+i_2^2+i_3^2)}$$
$$\le c|\Delta x|a^{1/3}\int_1^{\frac{L}{a^{1/3}}} dr = c|\Delta x|\,L. \tag{5.12}$$

Thus b) is proved.

Corollary 1 *If $\sigma_m = O(\frac{1}{a})$ and $d = O(a^{1/3})$, then the set $\{u_e(x)\}_{0<a<a_0}$ contains a convergent in $C(D)$ subsequence:*

$$\lim_{a\to 0}\|u_{e,a}(x)-u(x)\|_{C(D)} = 0. \tag{5.13}$$

We will prove that $u(x) := \lim_{a\to 0} u_e(x)$ satisfies the equation

$$u(x) = u_0(x) - \int_D G(x,y)\,p(y)\,u(y)dy, \quad p(x) := \frac{4\pi h(x)}{1+h(x)}\,N(x). \tag{5.14}$$

Equation (5.14) has a unique solution in $C(D)$. Therefore every subsequence u_e converges to the same limit $u(x)$. Thus, (5.13) holds for the set u_e and not only for its subsequence.

To prove that equation (5.13) has a unique solution, we note that if $u_0 = 0$, then the solution to (5.14) solves the Schrödinger equation

$$\big[\nabla^2 + k^2 - q(x)\big]u = 0 \quad \text{in } \mathbb{R}^3, \quad q := q_0(x) + p(x), \tag{5.15}$$

and satisfies the radiation condition. Since $k^2 > 0$ and $\operatorname{Im} q \le 0$, Theorem 1, proved in Section 3, implies that $u = 0$.

Let us derive equation (5.14) for the limit $u(x)$ of $u_e(x)$ as $a \to 0$. Taking the origin at x, one gets:

$$\begin{aligned}
&\sum_{\substack{x_m \notin \Delta_0 \\ x \in \Delta_0}} G(x, x_m) f(x_m) u_e(x_m) a \\
&\quad = \sum_{\Delta_j \neq \Delta_0} G(x, y^{(j)})\, f(y^{(j)})\, u_e(y^j)(1+\varepsilon_j)\, a \sum_{x_m \in \Delta_j} 1 \\
&\quad = \sum_{\Delta_j \neq \Delta_0} G(x, y^{(j)})\, f(y^{(j)})\, u_e(y^{(j)})\, u_e(y^{(j)})(1+\varepsilon_j)\, a\mathcal{N}(\Delta_j) \\
&\quad = \sum_{\Delta_j \neq \Delta_0} G(x, y^{(j)})\, f(y^{(j)})\, u_e(y^{(j)})(1+\varepsilon_j)\, N(y^{(j)})\, |\Delta_j|\,. \qquad (5.16)
\end{aligned}$$

Here $\lim_{b\to 0} \max_j \varepsilon_j = 0$ if $u_e(x)$ is continuous in D, because $f(y)$, $N(y)$ and $G(x,y)$ are continuous functions in $D \setminus \Delta_0$, G is integral in D, and we have used the relation (1.21). For any fixed a the function $u_e(x)$, defined in (5.6), is continuous in D. Thus, as $\max_j \operatorname{diam} |\Delta_j| \to 0$, the right side of (5.16) tends to the limit

$$\int_D G(x,y)\, f(y)\, N(y)\, u(y) dy,$$

being the Riemann sum for this integral, and $u(x) = \lim_{a\to 0} u_e(x)$, where the limit can be understood as the limit in $C(D)$-norm.

Thus, we have proved that $\zeta_m = \frac{h(x_m)}{a}$, $\sigma_m = O(\frac{1}{a})$, $d = O(a^{1/3})$ and $\mathcal{N}(\Delta) = \frac{1}{a}\int_\Delta N(x)dx\big[1+o(1)\big]$ as $a \to 0$ imply (5.10), the existence of the limit $\lim_{a\to 0} u_e(x) = u(x)$, and equation (5.14).

On the other hand, if (5.10) holds, and

$$\zeta_m = \frac{h(x_m)}{a}, \quad d = O(a^{1/3}), \quad \mathcal{N}(\Delta) = \frac{1}{a}\int_\Delta N(x)dx\big[1+o(1)\big] \quad \text{as } a \to 0,$$

then one can prove that $\sigma_m = O(\frac{1}{a})$.

Let us sketch this proof.

The function $\sigma := \sigma_j$, $j = 1, 2, \ldots.., M$, solves the equation (2.10). Let us study the asymptotics, as $a \to 0$, of the solution to equation (2.10) assuming that $u_{e_N}(s)$ and $u_e(s)$ are continuous functions on S_j, and $\zeta_j = \frac{h_j}{a}$. The choice of ζ_j is in our hands, and we will see that the assumptions about u_e are justified. The equation is

$$\sigma = A\sigma - 2\zeta T\sigma + 2u_{e_N}(s) - 2\zeta u_e(s), \qquad \zeta := \zeta_j, \quad A := A_j, \quad T := T_j. \quad (5.17)$$

Let $\zeta^{-1} := \tau$. We have:

$$T\sigma = \tau\, \frac{(A-I)\sigma}{2} + \tau u_{e_N} - u_e, \quad \tau \to 0. \qquad (5.18)$$

The operator $T : H^0 \to H^1$ is an isomorphism if S_j is a sphere of a sufficiently small radius. Here $H^\ell = H^\ell(S_j)$ are the usual Sobolev spaces.

We wish to prove that the main term of the asymptotics of the solution to (5.18) as $\tau \to 0$ is

$$\sigma_0 = -T^{-1} u_e. \tag{5.19}$$

Equation (5.19) implies $\sigma_0 = O(\frac{1}{a})$. This can be seen from the following argument. Equation

$$\int_{S_j} \frac{\sigma_0(t)\, dt}{4\pi |s-t|} := T\sigma_0 = -u_e \tag{5.20}$$

is the equation for the electrostatic charge density σ_0 on the surface S_j of the perfect conductor D_j, charged to the constant potential $-u_e$. This potential can be considered constant on S_j because $\frac{1}{2} \operatorname{diam} S_j \le a$ is very small. The total charge $Q_0 := \int_{S_j} \sigma_0\, ds = -C u_e$, where $C = O(a)$ is the electric capacitance of the conductor with the surface S_j. Since the surface area is $O(a^2)$ and $\int_{S_j} \sigma_0\, ds = O(a)$, it follows that $\sigma_0 = O(\frac{1}{a})$. If S_j is a sphere of radius a, then $\sigma_0 = \text{const} = -\frac{u_e}{a}$. For an arbitrary smooth surface S_j, such that $\operatorname{diam} S_j \le 2a$, diffeomorphic to a sphere, the estimate $\sigma_0 = O(\frac{1}{a})$ follows from the Hopf lemma (the strong maximum principle), which guarantees that the surface charge density does not vanish on S_j.

Finally, let us check that $\sigma_0 = -T^{-1}u_e$ is the main term of the asymptotics of the solution σ to (5.18) as $\tau \to 0$.

The operator T, $T\sigma = \int_{S_j} \frac{\sigma(t)\, dt}{4\pi|s-t|}$ is selfadjoint and positive in $H^0 = L^2(S_j)$. The quadratic form $(T\sigma, \sigma) := (T\sigma, \sigma)_{L^2(S_j)}$ defines an inner product, and the corresponding norm is equivalent to the norm in $H^{-1/2} := H^{-1/2}(S_j)$ (see its definition in [1]). Let $\sigma_0 := -T^{-1}u_e$. Then equation (5.18) implies

$$\begin{aligned} \big(T(\sigma - \sigma_0),\, \sigma - \sigma_0\big) \;&=\; -\frac{\tau}{2} \|\sigma - \sigma_0\|^2 - \frac{\tau}{2}(\sigma_0, \sigma - \sigma_0) + \tau(u_{e_N}, \sigma - \sigma_0) \\ &\quad + \frac{\tau}{2}\big((A - I)(\sigma - \sigma_0), \sigma - \sigma_0\big) \\ &\quad + \frac{\tau}{2}\big((A - I)\sigma_0,\, \sigma - \sigma_0\big). \end{aligned} \tag{5.21}$$

If S_j is convex, then $\big((A - I)\sigma, \sigma\big) \le 0$. Therefore (5.21) implies

$$\|\sigma - \sigma_0\|^2_{-1/2} \le c\tau \|\sigma - \sigma_0\|_{-1/2}, \tag{5.22}$$

where $c > 0$ is a constant independent of τ, $c = c(\sigma_0)$. As $\tau \to 0$, one gets from (5.22) the inequality

$$\|\sigma - \sigma_0\|_{-1/2} \le c\tau, \quad \tau \to 0. \tag{5.23}$$

Thus, our claim is verified.

One can prove that the norm in (5.23) can be replaced by the norm of $H^{\ell-0.5}(S_j)$ provided that u_e and u_{eN} are sufficiently smooth, but we do not go into detail.

The convexity of S_j is not necessary for our argument, it just simplifies it. Our conclusions hold without this convexity assumption because $A : H^\ell \to H^{\ell+1}$ for smooth S_j.

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