

Many-Brane Extension of the Randall-Sundrum Solution

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Recently, Randall and Sundrum proposed a static solution to the Einstein equations in 5 spacetime dimensions with two 3-branes located at the fixed points of S^1/Z_2 to solve the hierarchy problem. We extend the solution and construct static and also inflationary solutions to the Einstein equations in 5 spacetime dimensions, one of which is compactified on S^1 , with any number of 3-branes whose locations are taken to be arbitrary. We discuss how the hierarchy problem can be explained in our model.

1. *Introduction* Recently, Randall and Sundrum¹⁾ proposed a new interesting mechanism with a single small extra dimension for solving the hierarchy problem between the Planck scale and the weak scale. A key ingredient of this mechanism is that the metric is not factorizable and that the 4-dimensional metric is multiplied by a warp factor which is a rapidly changing function of the extra dimension. They explicitly constructed such a solution to the Einstein equations in 5 spacetime dimensions, one of which is compactified on S^1/Z_2 , with two 3-branes located at the fixed points of S^1/Z_2 . For a solution to exist, it is crucial to take into account the effect of the branes on the bulk gravitational metric.

In the Randall-Sundrum model, the number of 3-branes is 2 and the locations are taken to be the fixed points of S^1/Z_2 . Although this setup is motivated by recent developments in string and M-theory,²⁾ it would be of interest to construct new solutions to the 5-dimensional Einstein equations with many 3-branes. In this paper, we explicitly construct solutions with an arbitrary number of 3-branes which are put at arbitrary positions in the direction of the extra dimension. Our motivation for this is threefold. First, it is known that any number of parallel D-branes can be put at arbitrary positions and that the gauge dynamics depend on the distances of multiple D-branes. Thus, it is physically meaningful to construct solutions corresponding to many 3-branes put at arbitrary positions. Second, many-brane configurations could explain other hierarchy problems, such as the fermion mass hierarchy. The origi-

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nal Randall-Sundrum model gives a solution to the hierarchy problem between the Planck scale and the weak scale, but it does not solve other hierarchy problems. In recent works on large extra dimensions,³⁾ various mechanisms to solve hierarchy problems have been proposed. Some of the authors have pointed out that multiple 3-branes could account for the fermion mass hierarchy.⁴⁾ In this scenario, the mass hierarchy crucially depends on the distances between 3-branes. Thus, again, it is interesting to consider multiple 3-brane configurations with various distances between 3-branes. Third, in any higher-dimensional model, stabilizing extra dimensions is, in general, hard to achieve. In Ref. 5) a mechanism for stabilizing extra dimensions with multiple branes is proposed.

This paper is organized as follows. In 2, we construct static solutions to the Einstein equations in 5 spacetime dimensions with any number of 3-branes put at arbitrary positions. In 3, we discuss the hierarchical structure of our model. In 4, we extend the static solutions found in 2 to inflationary solutions. Conclusions are given in 5.

2. Many-brane configurations In the Randall-Sundrum model, the orbifold fixed points of S^1/Z_2 have been taken as the locations of two 3-branes. Since we would like to consider any number of 3-branes whose locations are taken to be arbitrary, we here take a circle S^1 , rather than S^1/Z_2 , as the compactification of an extra dimension.^{*)} The coordinate ϕ for the extra dimension is taken to extend from 0 to 2π with the identification of $(x^\mu, \phi = 0)$ with $(x^\mu, \phi = 2\pi)$.

Let us consider N parallel 3-brane configurations in 5 spacetime dimensions. The i -th 3-brane may be characterized by the location ϕ_i and the brane tension V_i ($i = 1, 2, \dots, N$). We arrange the locations of the 3-branes such that $0 = \phi_1 < \phi_2 < \dots < \phi_N < 2\pi$. We have here taken the location ϕ_1 of the first 3-brane to be the origin of S^1 for convenience. Since the 5-dimensional spacetime is divided into N domains by N 3-branes, each domain sandwiched between the i -th and the $(i+1)$ -th 3-branes can have a different 5-dimensional cosmological constant Λ_i .^{**)} Thus, the action we start with is given by^{***)}

$$\begin{aligned}
 S &= S_{\text{gravity}} + \sum_{i=1}^N S_i, \\
 S_{\text{gravity}} &= \int d^4x \int_0^{2\pi} d\phi \sqrt{-G} \left\{ 2M^3 R - \sum_{i=1}^N \Lambda_i [\theta(\phi - \phi_i) - \theta(\phi - \phi_{i+1})] \right\}, \\
 S_i &= \int d^4x \sqrt{-g^{(i)}} \{ \mathcal{L}_i - V_i \},
 \end{aligned} \tag{1}$$

where $\phi_{N+1} \equiv 2\pi$ and $\theta(\phi)$ denotes the Heaviside step function defined such that $\theta(\phi) = 1$ for $\phi \geq 0$ and $\theta(\phi) = 0$ for $\phi < 0$. The quantity S_i is the 4-dimensional

^{*)} Solutions for S^1/Z_2 may be obtained from those for S^1 by imposing the Z_2 -symmetry.

^{**)} Solutions to the Einstein equations with even numbers of 3-branes are constructed in Ref. 6), in which the Λ_i have been taken to be identical for all i . Solutions similar to ours, which describe multiple intersecting branes, are given in Ref. 7).

^{***)} For the conventions, see the original paper of Randall-Sundrum.¹⁾

i -th 3-brane action, and the contribution from the Lagrangian \mathcal{L}_i will be ignored in the following analysis.

The 5-dimensional Einstein equations for the above action are

$$\sqrt{-G} \left(R_{MN} - \frac{1}{2} G_{MN} R \right) = -\frac{1}{4M^3} \left[\sum_{i=1}^N A_i [\theta(\phi - \phi_i) - \theta(\phi - \phi_{i+1})] \sqrt{-G} G_{MN} + \sum_{i=1}^N V_i \sqrt{-g^{(i)}} g_{\mu\nu}^{(i)} \delta_M^\mu \delta_N^\nu \delta(\phi - \phi_i) \right]. \tag{2}$$

Here we solve the equations using the metric

$$ds^2 = e^{-2\sigma(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2, \tag{3}$$

taken from Ref. 1). Later we attempt to find solutions describing inflating 3-branes in 5 spacetime dimensions.

With this form of the metric, the Einstein equations (2) reduce to

$$(\sigma'(\phi))^2 = -\frac{r_c^2}{24M^3} \sum_{i=1}^N A_i [\theta(\phi - \phi_i) - \theta(\phi - \phi_{i+1})], \tag{4}$$

$$\sigma''(\phi) = \frac{r_c}{12M^3} \sum_{i=1}^N V_i \delta(\phi - \phi_i). \tag{5}$$

It is not difficult to show that the solution to the above equations has the form

$$\sigma(\phi) = (\lambda_1 - 0)(\phi - \phi_1)\theta(\phi - \phi_1) + (\lambda_2 - \lambda_1)(\phi - \phi_2)\theta(\phi - \phi_2) + \dots + (\lambda_N - \lambda_{N-1})(\phi - \phi_N)\theta(\phi - \phi_N), \tag{6}$$

where the additive integration constant, which is not physically relevant, has been chosen for later convenience. Since $\sigma(\phi)$ is a function on S^1 , it has to be periodic, i.e. $\sigma(2\pi) = \sigma(0)$. This leads to the constraint

$$\sum_{i=1}^N \lambda_i (\phi_{i+1} - \phi_i) = 0. \tag{7}$$

The first equation (4) requires

$$\lambda_i = \sqrt{\frac{-A_i r_c^2}{24M^3}} \quad \text{or} \quad -\sqrt{\frac{-A_i r_c^2}{24M^3}}. \quad (i = 1, 2, \dots, N) \tag{8}$$

We note that every A_i should be negative for the solution to make sense, as pointed out in Ref. 1). This requirement can, however, be relaxed for inflating solutions, as we see below. The second equation (5) requires that the 5-dimensional cosmological constants and the brane tensions be related as

$$\frac{V_i r_c}{12M^3} = \lambda_i - \lambda_{i-1}, \quad (i = 1, 2, \dots, N) \tag{9}$$

with $\lambda_0 \equiv \lambda_N$.*)

Although we have given the exact analytical expression for $\sigma(\phi)$, it may be instructive to determine this geometrically. The solution (6) turns out to be depicted as follows: First, specify distinct N points ϕ_i which correspond to the locations of N 3-branes and choose the values of $\sigma(\phi_i)$ at $\phi = \phi_i$ appropriately. The entire ϕ -dependence of $\sigma(\phi)$ can then be obtained by connecting two adjacent points $\sigma(\phi_i)$ and $\sigma(\phi_{i+1})$ for $i = 1, 2, \dots, N$ by straight lines. In general, each line will be bent at $\phi = \phi_i$. The slope of the line in the region of $\phi_i \leq \phi < \phi_{i+1}$ corresponds to λ_i , which is related to Λ_i through Eq. (8). The difference between the slopes of the adjacent lines at $\phi = \phi_i$ is proportional to the brane tension V_i . Although the solution $\sigma(\phi)$ can completely be specified by $2N$ parameters ϕ_i and $\sigma(\phi_i)$ ($i = 1, 2, \dots, N$), two of them are not physically relevant. An additive constant to $\sigma(\phi)$ can be absorbed into an overall constant rescaling of x^μ , and an overall shift of ϕ_i has no physical consequence.

3. Hierarchical structure Here we discuss the hierarchical structure of the solution derived in 2. To this end, the mass scale of all the parameters (Λ_i, V_i, M and r_c) in the fundamental theory is assumed to be of order the Planck scale.

As mentioned in 2, $\sigma(\phi)$ is specified by ϕ_i and $\sigma(\phi_i)$. Without loss of generality, we can assume that $\sigma(\phi_i) \geq \sigma(\phi_1) = 0$. As was done in Ref. 1), we can derive the 4-dimensional effective theory by performing the ϕ integral. It turns out that the square of the Planck mass on every 3-brane has the same value, i.e.,

$$\begin{aligned} M_{\text{Pl}}^2 &= M^3 r_c \int_0^{2\pi} d\phi e^{-2\sigma(\phi)} \\ &= M^3 r_c \left[\left(\frac{1}{2\lambda_1} - \frac{1}{2\lambda_N} \right) + \sum_{i=2}^N \left(\frac{1}{2\lambda_i} - \frac{1}{2\lambda_{i-1}} \right) e^{-2\sigma(\phi_i)} \right]. \end{aligned} \quad (10)$$

This relation is consistent with the assumption that M and r_c are of order the Planck scale if the λ_i are of order 1 and $\sigma(\phi_i) \gg 1$ for $i = 2, 3, \dots, N$.

Now, the hierarchical structure of our model is obvious. Although the 4-dimensional (effective) Newton constant is observed to be of order M_{Pl}^{-2} for every brane, the physical mass scales for the i -th brane will reduce by the warp factor $e^{-\sigma(\phi_i)}$ from the fundamental parameters not far from the Planck scale. The Randall-Sundrum scenario works well in our model, but we can obtain a variety of the hierarchy between the Planck scale and physical mass scales, which is not seen in the original Randall-Sundrum solution. Thus, in our model the hierarchy problem may be explained as follows: The reason why the hierarchy between the TeV scale and the Planck scale is observed in our world is merely that we happen to live on a 3-brane whose warp factor is of order 10^{-15} .

4. Inflating 3-branes Here, we obtain solutions describing inflating 3-branes in 5 spacetime dimensions. To this end, we use the metric:^{8),9),**)}

$$ds^2 = a(\phi)^2 (-dt^2 + v(t)^2 \delta_{ij} dx^i dx^j) + r_c^2 d\phi^2. \quad (11)$$

*) To obtain the relation for $i = 1$, we need to use the periodicity of $\sigma(\phi)$.

**) Other parametrizations are also possible. For example, see Ref. 10).

With this form of the metric, the Einstein equations (2) reduce to

$$\frac{\ddot{v}(t)}{v(t)} = \left(\frac{\dot{v}(t)}{v(t)}\right)^2, \tag{12}$$

$$\left(\frac{a'(\phi)}{a(\phi)}\right)^2 = \frac{r_c^2}{a(\phi)^2} \left(\frac{\dot{v}(t)}{v(t)}\right)^2 - \frac{r_c^2}{24M^3} \sum_{i=1}^N \Lambda_i [\theta(\phi - \phi_i) - \theta(\phi - \phi_{i+1})], \tag{13}$$

$$\frac{a''(\phi)}{a(\phi)} = -\frac{r_c^2}{24M^3} \left[\sum_{i=1}^N \Lambda_i [\theta(\phi - \phi_i) - \theta(\phi - \phi_{i+1})] + \frac{2}{r_c} \sum_{i=1}^N V_i \delta(\phi - \phi_i) \right], \tag{14}$$

where primes and dots denote derivatives with respect to ϕ and t , respectively. The first equation (12) can easily be solved as

$$v(t) = v(0)e^{Ht}, \tag{15}$$

where H corresponds to the expansion rate along a 3-brane after the coefficient of dt^2 is normalized to unity on the 3-brane.

Although we have not found a simple expression for $a(\phi)$, like Eq. (6), solutions to the above equations turn out to be of the form

$$a(\phi) = \alpha_i e^{\omega_i \phi} + \beta_i e^{-\omega_i \phi} \quad \text{for } \phi_i \leq \phi < \phi_{i+1}. \tag{16}$$

Requiring that $a(\phi)$ be continuous at $\phi = \phi_i$ ($i = 1, 2, \dots, N$) leads to ^{*)}

$$\begin{aligned} \alpha_N e^{2\pi\omega_N} + \beta_N e^{-2\pi\omega_N} &= \alpha_1 + \beta_1, \\ \alpha_{i-1} e^{\omega_{i-1}\phi_i} + \beta_{i-1} e^{-\omega_{i-1}\phi_i} &= \alpha_i e^{\omega_i\phi_i} + \beta_i e^{-\omega_i\phi_i}. \end{aligned} \quad (i = 2, 3, \dots, N) \tag{17}$$

Substituting the expressions (15) and (16) into Eq. (13) leads to

$$(\omega_i)^2 = -\frac{\Lambda_i r_c^2}{24M^3}, \quad (i = 1, 2, \dots, N) \tag{18}$$

$$H^2 = \frac{\Lambda_i \alpha_i \beta_i}{6M^3}. \quad (i = 1, 2, \dots, N) \tag{19}$$

It may be worth noting that the first relation (18) does not necessarily imply that Λ_i is negative. Solutions may exist even when $\Lambda_i > 0$. In this case, ω_i is purely imaginary, and the exponential functions in Eq. (16) should be replaced by trigonometric functions. Furthermore, even if H^2 is negative, we would obtain physically meaningful solutions by analytic continuation.⁹⁾ The last equation (14) can be satisfied provided

$$\begin{aligned} &\left(\omega_i + \frac{V_i r_c}{12M^3}\right) \alpha_i e^{\omega_i \phi_i} - \left(\omega_i - \frac{V_i r_c}{12M^3}\right) \beta_i e^{-\omega_i \phi_i} \\ &= \omega_{i-1} \alpha_{i-1} e^{\omega_{i-1} \phi_i} - \omega_{i-1} \beta_{i-1} e^{-\omega_{i-1} \phi_i}, \end{aligned} \quad (i = 1, 2, \dots, N) \tag{20}$$

where $\omega_0 \equiv \omega_N$, $\alpha_0 \equiv \alpha_N$ and $\beta_0 \equiv \beta_N$.

^{*)} The continuity condition at $\phi = \phi_1 = 0$ implies that $a(2\pi) = a(0)$.

The above solutions for $N = 1$ and 2 (one and two 3-branes) are investigated in Refs. 8) and 9). It, however, seems difficult to analyze the solutions for general N thoroughly. We will not proceed further in this paper. Some of the physical implications of our solutions may be found in Refs. 8) and 9).

5. *Conclusion* In this paper, we have found new solutions to the Einstein equations in 5 spacetime dimensions with many 3-branes. The original Randall-Sundrum solution contains only two 3-branes whose locations are fixed at the orbifold fixed points of S^1/Z_2 , and is static. In our model, any number of parallel 3-branes can be put at arbitrary locations in the direction of the 5th dimension, and the brane tensions and the cosmological constants of the 5-dimensional bulks sandwiched between the 3-branes can, in general, be taken to have different values, though they must satisfy some fine tuning relations for solutions to exist. We have further succeeded in extending the static solutions to the inflationary solutions, though our analysis is far from complete. As in the Randall-Sundrum model, our model can give a solution to the hierarchy problem between the Planck scale and the TeV scale. Although the Randall-Sundrum model does not answer other hierarchy problems such as the fermion mass hierarchy, the existence of multiple 3-branes in our model could offer a mechanism to solve them. A final comment is that although the 5th dimension is compactified on S^1 , our solutions will persist on a non-compact space. This can be seen by simply ignoring the periodicity condition.

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- 1) L. Randall and R. Sundrum, hep-ph/9905221.
 - 2) P. Hořava and E. Witten, Nucl. Phys. **B460** (1996), 506; **B475** (1996), 94.
E. Witten, Nucl. Phys. **B471** (1996), 135.
A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Phys. Rev. **D59** (1999), 086001.
 - 3) N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429** (1998), 263.
I. Antoniadis et al., Phys. Lett. **B436** (1998), 257.
I. Antoniadis, S. Dimopoulos and G. Dvali, Nucl. Phys. **B516** (1998), 70.
K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. **B436** (1998), 55.
I. Antoniadis, S. Dimopoulos, A. Pomarol and M. Quirós, Nucl. Phys. **B544** (1999), 503.
A. Delgado, A. Pomarol and M. Quirós, hep-ph/9812489.
H. Hatanaka, T. Inami and C. S. Lim, Mod. Phys. Lett. **A13** (1998), 2601.
 - 4) N. Arkani-Hamed, S. Dimopoulos, G. Dvali and J. March-Russel, hep-ph/9811448.
N. Arkani-Hamed and S. Dimopoulos, hep-ph/9811353.
 - 5) R. Sundrum, Phys. Rev. **D59** (1999), 085010.
N. Arkani-Hamed, S. Dimopoulos and J. March-Russel, hep-th/9809124.
 - 6) I. Oda, hep-th/9908104.
 - 7) N. Arkani-Hamed, S. Dimopoulos, G. Dvali and N. Kaloper, hep-th/9907209.
C. Csáki and Y. Shirman, hep-th/9908186.
A. E. Nelson, hep-th/9909001.
 - 8) N. Kaloper and A. Linde, Phys. Rev. **D59** (1999), 101303.
T. Nihei, hep-ph/9905487.
 - 9) N. Kaloper, hep-th/9905210.
 - 10) H. B. Kim and H. D. Kim, hep-th/9909053.
H. A. Chamblin and H. S. Reall, hep-th/9903225.