Many-to-Many Disjoint Path Covers in Hypercube-Like Interconnection Networks with Faulty Elements*

Jung-Heum Park
The Catholic University of Korea, Korea
Email: j.h.park@catholic.ac.kr

Hee-Chul Kim
Hankuk University of Foreign Studies, Korea
Email: hckim@hufs.ac.kr

Hyeong-Seok Lim
Chonnam National University, Korea
Email: hslim@chonnam.ac.kr

Abstract

A *many-to-many k-disjoint path cover (k-DPC)* of a graph G is a set of k disjoint paths joining k distinct source-sink pairs in which each vertex of G is covered by a path. We deal with the graph $G_0 \oplus G_1$ obtained from connecting two graphs G_0 and G_1 with n vertices each by n pairwise nonadjacent edges joining vertices in G_0 and vertices in G_1. Many interconnection networks such as hypercube-like interconnection networks can be represented in the form of $G_0 \oplus G_1$ connecting two lower dimensional networks G_0 and G_1. In the presence of faulty vertices and/or edges, we investigate many-to-many disjoint path coverability of $G_0 \oplus G_1$ and $(G_0 \oplus G_1) \oplus (G_2 \oplus G_3)$, provided some conditions on the hamiltonicity and disjoint path coverability of each graph G_i are satisfied, $0 \le i \le 3$. We apply our main results to recursive circulant $G(2^m, 4)$ and a subclass of hypercube-like interconnection networks, called *restricted HL-graphs.* The subclass includes twisted cubes, crossed cubes, multiply twisted cubes, Möbius cubes, Mcubes, and generalized twisted cubes. We show that all these networks of degree m with f or less faulty elements have a many-to-many k-DPC joining any k distinct source-sink pairs for any $k \ge 1$ and $f \ge 0$ such that $f + 2k \le m - 1$.

Index Terms: Fault tolerance, network topology, graph theory, fault-hamiltonicity, embedding, strong hamiltonicity, recursive circulants, restricted HL-graphs.

*This work was supported by grant No. R01-2003-000-11676-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

1 Introduction

One of the central issues in various interconnection networks is finding node-disjoint paths concerned with the routing among nodes and the embedding of linear arrays. Node-disjoint paths can be used as parallel paths for an efficient data routing among nodes. Also, each path in node-disjoint paths can be utilized in its own pipeline computation. An interconnection network is often modelled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. In this paper, node(vertex)-disjoint paths are abbreviated to disjoint paths. In the rest of this paper, we will use standard terminology in graphs (see ref. [2]).

Disjoint paths can be categorized as three types: one-to-one, one-to-many, and many-to-many. One-to-one type deals with the disjoint paths joining a single source s and a single sink t. One-to-many type considers the disjoint paths joining a single source s and k distinct sinks $t_1, t_2, \ldots, t_k$. Most of the works done on disjoint paths deal with the one-to-one or one-to-many type. One-to-one and one-to-many disjoint paths were constructed for a variety of networks such as hypercubes[3, 11], star graphs[5], etc. Many-to-many type deals with the disjoint paths joining k distinct sources $s_1, s_2, \ldots, s_k$ and k distinct sinks $t_1, t_2, \ldots, t_k$. In many-to-many type, several problems can be defined depending on whether specific sources should be joined to specific sinks or a source can be freely matched to a sink. The works on many-to-many type have a relative paucity because of its difficulty and some results can be found in [13, 18].

All of three types of disjoint paths in a graph G can be accommodated with the covering of vertices in G. A *disjoint path cover* in a graph G is a set of disjoint paths containing all the vertices in G. A disjoint path cover problem that originated from an interconnection network is concerned with the application where the full utilization of nodes is important. For an embedding of linear arrays in a network, the cover implies every node can be participated in a pipeline computation. As the disjoint path covers which have been studied for interconnection networks, there are one-to-one disjoint path covers in recursive circulants without faulty elements[20, 30] and hypercubes with faulty edges[3], and one-to-many disjoint path covers with faulty elements in some hypercube-like interconnection networks[21]. As the authors know, no results on many-to-many disjoint path covers appeared in the literature with an exception of the earlier version of this paper in [24]. A one-to-one disjoint path cover consisting of k disjoint paths is also known as a k^*-*container*[3, 30].

Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in a graph

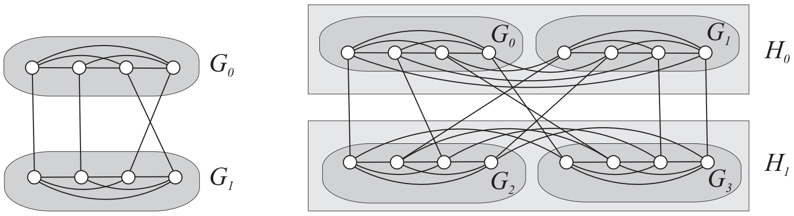
G such that $S \cap T = \emptyset$, we are concerned with many-to-many disjoint paths $P_1, P_2, \ldots, P_k$ in G, P_i joining s_i and t_i, $1 \le i \le k$, that *cover* all the vertices in the graph, that is, $\bigcup_{1 \le i \le k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \emptyset$ for all $i \ne j$. Here $V(P_i)$ and $V(G)$ denote the vertex sets of P_i and G, respectively. We call such a set of k disjoint paths a *many-to-many k-disjoint path cover* (in short, *many-to-many k-DPC*) of G. We call a source or a sink a *terminal*.

The disjoint path cover problems are closely related to well-known hamiltonian problems in graph theory. Since the HAMILTONIAN PATH BETWEEN TWO VERTICES problem is NP-complete[12], so is the MANY-TO-MANY k-DPC problem for all fixed $k \ge 1$. Notice that a graph G has a hamiltonian path between two vertices s and t if and only if the graph G' has a many-to-many k-DPC, where $V(G') = V(G) \cup \{s_i, t_i | 2 \le i \le k\}$, $E(G') = E(G) \cup \{(s, s_i), (s_i, t_i), (t_i, t) | 2 \le i \le k\}$, $s_1 = s$, and $t_1 = t$. The ONE-TO-MANY k-DPC and ONE-TO-ONE k-DPC problems are also NP-complete for all fixed $k \ge 1$. They can be reduced straightforwardly from HAMILTONIAN PATH BETWEEN TWO VERTICES.

On the other hand, embedding of linear arrays and rings into a faulty interconnection network is one of the important problems in parallel processing. The embedding problem can be modelled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges. Fault-hamiltonicity of various interconnection networks was investigated in the literature; for example, arrangement graphs[15, 17], recursive circulants[29], hypercubes[26, 28], star graphs[16, 23, 31], and hypercube-like interconnection networks[25]. A graph G is called *f-fault hamiltonian* (resp. *f-fault hamiltonian-connected*) if there exists a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G \backslash F$ for any set F of faulty elements with $|F| \le f$. For a graph G to be f-fault hamiltonian (resp. f-fault hamiltonian-connected), it is necessary that $f \le \delta(G) - 2$ (resp. $f \le \delta(G) - 3$), where $\delta(G)$ is the minimum degree of G.

To a graph G with a set of faulty elements F, the definition of a many-to-many disjoint path cover can be extended. Given a set of k sources $S = \{s_1, s_2, \ldots, s_k\}$ and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ in $G \backslash F$ such that $S \cap T = \emptyset$, a many-to-many k-disjoint path cover joining S and T is a set of k disjoint paths P_i joining s_i and t_i, $1 \le i \le k$, such that (a) $\bigcup_{1 \le i \le k} V(P_i) = V(G) \backslash F$, (b) $V(P_i) \cap V(P_j) = \emptyset$ for all $i \ne j$, and (c) every edge on each path P_i is fault-free. Such a many-to-many k-DPC is denoted by k-DPC$[\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\} | G, F]$.

Definition 1 *A graph G is called* f-fault many-to-many k-disjoint path coverable *if* $f + 2k \le |V(G)|$



(a) $G_0 \oplus G_1$ (b) $H_0 \oplus H_1$

Figure 1: Examples of $G_0 \oplus G_1$ and $H_0 \oplus H_1$

and for any set F of faulty elements with $|F| \leq f$, G has k-DPC$[\{(s_1,t_1),(s_2,t_2),\ldots,(s_k,t_k)\}|G,F]$ for any set of k sources $S=\{s_1,s_2,\ldots,s_k\}$ and any set of k sinks $T=\{t_1,t_2,\ldots,t_k\}$ in $G\backslash F$ such that $S\cap T=\emptyset$.

For a graph G to be f-fault many-to-many k-disjoint path coverable, it is necessary that $f+2k \leq \delta(G)+1$ (shown in Subsection 2.3). An f-fault many-to-many k-disjoint path coverable graph G can be shown to possess an interesting strong hamiltonicity, say the existence of a hamiltonian path containing a specified set of edges in a given order. Precisely speaking, for any fault set F with $|F| \leq f$ and for any sequence of pairwise nonadjacent $k-1$ edges $((x_1,y_1),(x_2,y_2),\ldots,(x_{k-1},y_{k-1}))$ in $G\backslash F$, there exists a hamiltonian path joining an arbitrary pair of vertices s and t in $G\backslash F$ with $\{s,t\}\cap\{x_i,y_i\}=\emptyset$ for all $1 \leq i \leq k-1$ that passes through the edges in the order given. That is, there exists a hamiltonian path of the form of $(s,\ldots,x_1,y_1,\ldots,x_{k-1},y_{k-1},\ldots,t)$.

We are given two graphs G_0 and G_1 with n vertices each. We denote by V_i and E_i the vertex set and edge set of G_i, $i=0,1$, respectively. We let $V_0=\{v_1,v_2,\ldots,v_n\}$ and $V_1=\{w_1,w_2,\ldots,w_n\}$. With respect to a permutation $M=(i_1,i_2,\ldots,i_n)$ of $\{1,2,\ldots,n\}$, we can "merge" the two graphs into a graph $G_0 \oplus_M G_1$ with $2n$ vertices in such a way that the vertex set $V=V_0\cup V_1$ and the edge set $E=E_0\cup E_1\cup E_2$, where $E_2=\{(v_j,w_{i_j})|1\leq j\leq n\}$. We denote by $G_0\oplus G_1$ a graph obtained by merging G_0 and G_1 w.r.t. an arbitrary permutation M. Here, G_0 and G_1 are called *components* of $G_0\oplus G_1$. When we are given four graphs G_i, $0\leq i\leq 3$, with the same number of vertices, we can also merge them into a graph $H_0\oplus H_1$, where $H_0=G_0\oplus G_1$ and $H_1=G_2\oplus G_3$. Figure 1 shows examples of $G_0\oplus G_1$ and $H_0\oplus H_1$.

Definition 2 *A graph G is called* many-to-many (f,k)-disjoint path coverable *if for any f' and k' such that $k'\leq k$ and $f'+2k'\leq f+2k$, G is f'-fault many-to-many k'-disjoint path coverable.*

That is, a many-to-many (f, k)-disjoint path coverable graph G satisfies all of the following k conditions:

$$\begin{array}{ll} C(k): & G \text{ is } f\text{-fault many-to-many } k\text{-disjoint path coverable;} \\ C(k-1): & G \text{ is } f+2\text{-fault many-to-many } k-1\text{-disjoint path coverable;} \\ \vdots & \\ C(1): & G \text{ is } f+2k-2\text{-fault many-to-many } 1\text{-disjoint path coverable.} \end{array}$$

For a graph G to be many-to-many (f, k)-disjoint path coverable, it is necessary that $f+2k \leq \delta(G)-1$.

In this paper, provided G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian for all $0 \leq i \leq 3$, we will show that $G_0 \oplus G_1$ is many-to-many $(f + 1, k)$-disjoint path coverable and $f + 2k$-fault hamiltonian for any $k \geq 2$ and $f \geq 0$ or for any $k = 1$ and $f \geq 2$, and that $H_0 \oplus H_1$ is many-to-many $(f, k + 1)$-disjoint path coverable and $f + 2k + 1$-fault hamiltonian for any $k \geq 1$ and $f \geq 0$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Note that $\delta(G_0 \oplus G_1) = \min\{\delta(G_0), \delta(G_1)\}+1$ and $\delta(H_0 \oplus H_1) = \min_{0\leq i\leq 3} \delta(G_i)+2$. The fact that a graph G is f-fault many-to-many 1-disjoint path coverable is, by definition, equivalent to that G is f-fault hamiltonian-connected. To prove the main results, we utilize fault-hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$ studied in [25] that provided G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian for each i, $G_0 \oplus G_1$ is $f + 1$-fault hamiltonian-connected and $f+2$-fault hamiltonian for any $f \geq 2$ and $H_0 \oplus H_1$ is $f+2$-fault hamiltonian-connected and $f + 3$-fault hamiltonian for any $f \geq 0$.

We apply our main results to recursive circulant $G(2^m, 4)$ and a subclass of hypercube-like interconnection networks, called *restricted HL-graphs*. The subclass includes twisted cubes[14], crossed cubes[9], multiply twisted cubes[8], Möbius cubes[7], Mcubes[27], and generalized twisted cubes[4]. We will show that all these networks of degree m are f-fault many-to-many k-disjoint path coverable for every $k \geq 1$ and $f \geq 0$ such that $f + 2k \leq m - 1$. Also, we will discuss that "near" bipartite graphs which belong to hypercube-like interconnection networks can have only limited disjoint path coverability by the illustration of twisted m-cubes.

The organization of this paper is as follows. In the next section, we will consider some interesting properties on f-fault many-to-many k-disjoint path coverable graphs, including relationships among the three types of disjoint path covers, sufficiency for some strong-hamiltonicity, and some necessary conditions. In Section 3, we will construct many-to-many disjoint path covers in $G_0 \oplus G_1$ and $H_0 \oplus H_1$, provided each G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian. Many-to-many disjoint path coverability of hypercube-like interconnection networks will be considered in Section 4. Finally in Section 5, concluding remarks of this paper and algorithmic aspects of our construction schemes will be discussed.

Remark 1 Even when there are $p(< k)$ source-sink pairs such that each source is identical with its corresponding sink, that is, when $s_i = t_i$ for all $1 \leq i \leq p$ and $S' \cap T' = \emptyset$, where $S' = \{s_{p+1}, \ldots, s_k\}$ and $T' = \{t_{p+1}, \ldots, t_k\}$, we can construct f-fault many-to-many k-DPC as follows: (a) we first let $P_i = (s_i)$, $1 \leq i \leq p$, a path with one vertex, and then (b) regarding them as virtual faulty vertices, find $f+p$-fault many-to-many $k-p$-DPC. Note that if G is many-to-many (f, k)-disjoint path coverable, then G is $f+p$-fault many-to-many $k-p$-disjoint path coverable for any $p < k$, as well as G is f-fault many-to-many k-disjoint path coverable.

2 Many-to-Many Disjoint Path Coverable Graphs

2.1 Disjoint path covers

First, we are going to discuss about some interesting properties on disjoint path covers of the three types: many-to-many, one-to-many, and one-to-one. Let G be a graph with a set F of faulty elements. Throughout this paper, a path in a graph is represented as a sequence of vertices. A v-w path refers to a path from a vertex v to a vertex w.

Proposition 1 *(a) G is f-fault many-to-many 1-disjoint path coverable if and only if G is f-fault hamiltonian-connected.*
(b) If G is f-fault many-to-many $k(\geq 2)$-disjoint path coverable, then G is f-fault many-to-many $k-1$-disjoint path coverable.

Proof By definition, (a) holds true. To prove (b), we construct an f-fault many-to-many $k-1$-DPC using f-fault many-to-many k-DPC as follows. Given a set of $k-1$ source-sink pairs $\{(s_1, t_1), \ldots, (s_{k-1}, t_{k-1})\}$, letting (x, y) be an edge in $G \backslash F$ such that both x and y are not terminals, we find a k-DPC for $\{(s_1, x), (y, t_1), (s_2, t_2), \ldots, (s_{k-1}, t_{k-1})\}$, and then the s_1-x path and the y-t_1 path are merged with the edge (x, y) into an s_1-t_1 path, which results in an f-fault many-to-many $k-1$-DPC for the given pairs. $\square$

Given a source s and a set of k sinks $T = \{t_1, t_2, \ldots, t_k\}$ with $s \notin T$ in $G \backslash F$, a one-to-many k-disjoint path cover joining s and T is a set of k disjoint paths P_i joining s and t_i, $1 \leq i \leq k$, such that $\bigcup_{1\le i\le k} V(P_i) = V(G)\backslash F$, $V(P_i)\cap V(P_j) = \{s\}$ for all $i \neq j$, and every edge on each path P_i is fault-free. A graph G is called *f-fault one-to-many k-disjoint path coverable* if $f + k + 1 \le |V(G)|$ and for any set F of faulty elements with $|F| \le f$, G has a one-to-many k-disjoint path cover joining every source s and every set T of k distinct sinks in $G\backslash F$ such that $s \notin T$. For a graph G to be f-fault one-to-many k-disjoint path coverable, it is necessary that $f + k \le \delta(G)$. In a similar way, we can also define *f-fault one-to-one k-disjoint path coverable graphs.*

Proposition 2 *The following statements are equivalent (by definition).*
(a) G is f-fault many-to-many 1-disjoint path coverable.
(b) G is f-fault one-to-many 1-disjoint path coverable.
(c) G is f-fault one-to-one 1-disjoint path coverable.

Proposition 3 *(a) If G is f-fault many-to-many k-disjoint path coverable, then G is f-fault one-to-many k-disjoint path coverable.*
(b) If G is f-fault one-to-many k-disjoint path coverable, then G is f-fault one-to-one k-disjoint path coverable[21].

Proof When we are given a single source s and a set of sinks $\{t_1, \ldots, t_k\}$, letting $\{x_2, \ldots, x_k\}$ be the set of $k-1$ vertices in $G\backslash F$ which are adjacent to s via fault-free edges, we find an f-fault many-to-many k-DPC for a set of pairs $\{(s, t_1), (x_2, t_2), \ldots, (x_k, t_k)\}$, and then the edge (s, x_j) and the x_j-t_j path are merged into an s-t_j path for each j, $2 \le j \le k$, which results in an f-fault one-to-many k-DPC for the given source and sinks. □

2.2 Strong hamiltonicity

In an f-fault many-to-many k-disjoint path coverable graph G, if G has no faulty elements, then we can always construct a many-to-many k'-disjoint path cover for any $k' \le k$ in G which does not pass through a specified set of vertices and edges (by regarding them virtual faulty elements) when the number of such vertices and edges are at most f. On one occasion of $k' = 1$, the disjoint path cover is interpreted as a hamiltonian path.

On the contrary, we can think of a hamiltonian path containing a given set of edges. In this subsection, we are concerned with a hamiltonian path (and cycle) which passes through a specified set of edges in a given order. Given two distinct vertices s and t, a sequence of l edges $((x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l))$

in a graph is called *s-t path extendable* if for any two vertices v and w in a multiset $\{s,t\} \cup \{x_j, y_j | 1 \leq j \leq l\}$, either $v \neq w$ or $v = w$ and the unordered pair (v,w) is one of (s,x_1), (y_l,t), or (y_j,x_{j+1}) for some $1 \leq j < l$. The sequence of edges is necessarily s-t path extendable for the graph to have an s-t path $(s,\ldots,x_1,y_1,\ldots,x_l,y_l,\ldots,t)$ passing through the edges in the order given. In [33], the existence of a hamiltonian path containing a given set of directed edges in an orientation of a complete multipartite graph was considered.

Theorem 1 (Strong-hamiltonicity) *(a) If G is f-fault many-to-many $k(\geq 2)$-disjoint path coverable, then for any fault set F with $|F| \leq f$ and for any vertices s, t and any sequence of $k-1$ edges $((x_1,y_1),(x_2,y_2),\ldots,(x_{k-1},y_{k-1}))$ in $G \backslash F$ such that $v \neq w$ for any pair of vertices v and w in a multiset $\{s,t\} \cup \{x_j,y_j | 1 \leq j \leq k-1\}$ (thus, s-t path extendable), there exists an s-t hamiltonian path in $G \backslash F$ that passes through the edges in the order given. That is, there exists a hamiltonian path of the form of $(s,\ldots,x_1,y_1,\ldots,x_{k-1},y_{k-1},\ldots,t)$.*

(b) If the sequence of edges is s-t path extendable and the number of pairs in $\{(s,x_1)|s = x_1\} \cup \{(y_{k-1},t)|y_{k-1} = t\} \cup \{(y_j,x_{j+1})|y_j = x_{j+1}, 1 \leq j < k-1\}$ is $p(< k)$, then every $f+p$-fault many-to-many $k-p$-disjoint path coverable graph has such a hamiltonian path.

Proof To prove (a), we first find an f-fault many-to-many k-DPC for a set of k source-sink pairs $\{(s,x_1),(y_1,x_2),\ldots,(y_{k-2},x_{k-1}),(y_{k-1},t)\}$, and then the k disjoint paths are merged with the $k-1$ edges (x_j,y_j), $1 \leq j \leq k-1$, into a hamiltonian path joining s and t, which is a desired one. For (b), regarding the p pairs (that is, p vertices) as virtual faulty vertices, in a similar way to (a), we find an $f+p$-fault many-to-many $k-p$-DPC for $k-p$ pairs, and then the $k-p$ disjoint paths and p paths with one vertex are merged with the $k-1$ edges into an s-t hamiltonian path. □

A sequence of l edges $((x_1,y_1),(x_2,y_2),\ldots,(x_l,y_l))$ in a graph is called *cycle extendable* if for any two vertices v and w in a multiset $\{x_j,y_j | 1 \leq j \leq l\}$, either $v \neq w$ or $v = w$ and the unordered pair (v,w) is one of (y_l,x_1) or (y_j,x_{j+1}) for some $1 \leq j < l$. Obviously, an edge sequence $((x_1,y_1),(x_2,y_2),\ldots,(x_l,y_l))$ is cycle extendable if and only if the subsequence $((x_1,y_1),(x_2,y_2),\ldots,(x_{l-1},y_{l-1}))$ is y_l-x_l path extendable.

Corollary 1 *(a) If G is f-fault many-to-many $k(\geq 1)$-disjoint path coverable, then for any fault set F with $|F| \leq f$ and for any sequence of k edges $((x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k))$ in $G \backslash F$ such that $v \neq w$ for any pair of vertices v and w in a multiset $\{x_j,y_j | 1 \leq j \leq k\}$ (thus, cycle extendable), there exists a hamiltonian cycle in $G \backslash F$ that passes through the edges in the order given. That is, there exists a* *hamiltonian cycle of the form of* $(y_k, \ldots, x_1, y_1, \ldots, x_{k-1}, y_{k-1}, \ldots, x_k)$.

(b) If the sequence of edges is cycle extendable and the number of pairs in $\{(y_l, x_1)|y_l = x_1\} \cup \{(y_j, x_{j+1})|y_j = x_{j+1}, 1 \leq j < k-1\}$ *is* $p(< k)$*, then every* $f+p$*-fault many-to-many* $k-p$*-disjoint path coverable graph has such a hamiltonian cycle.*

2.3 Necessary conditions

We let G be a graph with n vertices. If G is f-fault many-to-many k-disjoint path coverable, then by definition, $f + 2k \leq n$. Let S and T be the sets of sources and sinks in G, respectively. One might expect that a necessary condition in terms of connectivity can be derived as follows.

Lemma 1 (Connectivity) *If* G *is* f*-fault many-to-many* k*-disjoint path coverable, then* $\kappa(G) \geq f + 2k - 1$*, where* $\kappa(G)$ *is connectivity of* G*.*

Proof Suppose that $\kappa(G) \leq f+2k-2$. If G is a complete graph K_n, then $\kappa(G) = n-1 \leq f+2k-2$, which is a contradiction to that $f+2k \leq n$. Otherwise, there is an s-t vertex cut C with $|C| \leq f+2k-2$. We let $s_1 = s$ and $t_1 = t$, and let S, T, and F be sets of vertices in G so that $C \subseteq S \backslash s_1 \cup T \backslash t_1 \cup F$. Then, there can not exist an s_1-t_1 path which does not pass through any vertex in $S \backslash s_1 \cup T \backslash t_1 \cup F$, and thus G is not f-fault many-to-many k-disjoint path coverable. This completes the proof. □

Corollary 2 (Degree) *For a graph* G *to be* f*-fault many-to-many* k*-disjoint path coverable, it is necessary that* $f + 2k \leq \delta(G) + 1$.

Every bipartite graph is not hamiltonian-connected, and thus for any $f \geq 0$ and $k \geq 1$, it is not f-fault many-to-many k-disjoint path coverable. We are to derive necessary conditions which say that a "near" bipartite graph is not f-fault many-to-many k-disjoint path coverable for large f and k. A set X of vertices and edges in G is called *bipartization set* if $G \backslash X$ is bipartite. A *bipartization number* $b(G)$ of G is the minimum cardinality among all the bipartization sets of G. In connection with bipartization number, minimum vertex/edge deletion bipartite subgraph problems were studied in the literature[1, 34], with applications in computer-aided design of VLSI systems, specifically via minimization problem[6].

Lemma 2 (Bipartization) *If* G *is* f*-fault many-to-many* k*-disjoint path coverable, then* $b(G) > f$*.*

Proof Suppose otherwise, we let the minimum bipartization set be the faulty set. Then, $G \backslash F$ is bipartite, and thus G is not f-fault many-to-many k-disjoint path coverable. □

Lemma 3 (Bicoloring) *Let G be an f-fault many-to-many k-disjoint path coverable graph. Then for any fault set F with $|F| \leq f$ and for any two-coloring vertices in $G\backslash F$ black and white (not necessarily proper coloring, that is, two vertices with the same color may be adjacent), if $n_w \geq 2k$ then $c_b \leq n_w - k$, where n_w is the number of white vertices in $G\backslash F$ and c_b is the number of connected components in the induced subgraph $G\langle B\rangle$ of $G\backslash F$ by the set B of black vertices.*

Proof Suppose for some fault set F with $|F| \leq f$ and for some two-coloring vertices in $G\backslash F$, it holds true that $n_w \geq 2k$ and $c_b > n_w - k$. Letting all the sources and sinks be white, we find a k-DPC in $G\backslash F$. If an s_j-t_j path in the k-DPC passes through r white vertices as intermediate vertices, then it covers at most $r + 1$ connected components in $G\langle B\rangle$. The k disjoint paths pass through $n_w - 2k$ white vertices in total as intermediate vertices, and thus they cover at most $k + (n_w - 2k) = n_w - k$ components in $G\langle B\rangle$. This leads to a contradiction. □

3 Construction of Many-to-Many Disjoint Path Covers

In this section, we are to discuss about constructions of many-to-many disjoint path covers in $G_0 \oplus G_1$ and $H_0 \oplus H_1$ with some faulty elements, provided G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian for all $0 \leq i \leq 3$. Here, $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Precisely speaking, we will prove the following two main theorems.

Theorem 2 *For any $k \geq 2$ and $f \geq 0$ or for any $k = 1$ and $f \geq 2$, if G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian for each $i = 0, 1$, then $G_0 \oplus G_1$ is many-to-many $(f + 1, k)$-disjoint path coverable and $f + 2k$-fault hamiltonian.*

Theorem 3 *For any $k \geq 1$ and $f \geq 0$, if G_i is many-to-many (f, k)-disjoint path coverable and $f + 2k - 1$-fault hamiltonian for each $0 \leq i \leq 3$, then $H_0 \oplus H_1$ is many-to-many $(f, k+1)$-disjoint path coverable and $f + 2k + 1$-fault hamiltonian, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$.*

For a vertex v in $G_0 \oplus G_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained. We let F be the set of faulty elements, and let S and T be the sets of sources and sinks, respectively. We denote by $H[v, w|G, F]$ a hamiltonian path in $G\backslash F$ joining a pair of fault-free vertices v and w in a graph G with a set F of faulty elements. When we find a hamiltonian path/cycle (or a many-to-many DPC), sometimes we regard some fault-free vertices and/or edges as faulty elements. They are called *virtual* faults.

Definition 3 *A vertex v in $G_0 \oplus G_1$ is called* free *if v is fault-free and not a terminal, that is, $v \notin F$ and $v \notin S \cup T$. An edge (v, w) is called* free *if v and w are free and $(v, w) \notin F$.*

Definition 4 *A* free bridge *of a fault-free vertex v in $G_0 \oplus G_1$ is the path $(v, \bar{v})$ of length one if $\bar{v}$ is free and $(v, \bar{v}) \notin F$; otherwise, it is a path $(v, w, \bar{w})$ of length two such that $w \neq \bar{v}$, $(v, w) \notin F$, and $(w, \bar{w})$ is a free edge.*

First of all, we will review results on fault-hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$ studied in [25]. And then, we will consider the existence of pairwise disjoint free bridges for some terminals and will develop five basic procedures for constructing many-to-many disjoint path covers. They play a significant role in proving the main theorems.

3.1 Fault-hamiltonicity

We employ works on fault-hamiltonicity of $G_0 \oplus G_1$ and $H_0 \oplus H_1$ in [25]. They will be utilized later when we need to construct a many-to-many 1-DPC. The problems we are primarily concerned with are, provided G_i is f-fault hamiltonian-connected and $f+1$-fault hamiltonian for each i, whether $G_0 \oplus G_1$ is $f+1$-fault hamiltonian-connected and $f+2$-fault hamiltonian and whether $H_0 \oplus H_1$ is $f+2$-fault hamiltonian-connected and $f+3$-fault hamiltonian. The following two lemmas are concerned with fault-hamiltonicity of $G_0 \oplus G_1$.

Lemma 4 *[25] Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*

(a) for any $f \geq 2$, $G_0 \oplus G_1$ is $f+1$-fault hamiltonian-connected,

(b) for $f = 1$, $G_0 \oplus G_1$ with $2(= f+1)$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and $\bar{s}$ and $\bar{t}$ are the faulty elements(vertices), and

(c) for $f = 0$, $G_0 \oplus G_1$ with $1(= f+1)$ faulty elements has a hamiltonian path joining s and t unless s and t are contained in the same component and the faulty element is contained in the other component.

Lemma 5 *[25] Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*

(a) for any $f \geq 1$, $G_0 \oplus G_1$ is $f+2$-fault hamiltonian, and

(b) for $f = 0$, $G_0 \oplus G_1$ with $2(= f+2)$ faulty elements has a hamiltonian cycle unless one faulty element is contained in G_0 and the other faulty element is contained in G_1.

Contrary to the previous two lemmas, we can obtain fault-hamiltonicity of $G_0 \oplus G_1$ which holds true for any $f \geq 0$, if we reduce the bound on the number of faulty elements in $G_0 \oplus G_1$ by one. The last lemma is concerned with fault-hamiltonicity of $H_0 \oplus H_1$, which also holds true for any $f \geq 0$.

Lemma 6 *[25] Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1$. Then,*

(a) for any $f \geq 0$, $G_0 \oplus G_1$ is f-fault hamiltonian-connected, and

(b) for any $f \geq 0$, $G_0 \oplus G_1$ is $f+1$-fault hamiltonian.

Lemma 7 *[25] Let a graph G_i be f-fault hamiltonian-connected and $f+1$-fault hamiltonian, $i = 0, 1, 2, 3$, and let $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$. Then,*

(a) for any $f \geq 0$, $H_0 \oplus H_1$ is $f+2$-fault hamiltonian-connected, and

(b) for any $f \geq 0$, $H_0 \oplus H_1$ is $f+3$-fault hamiltonian.

3.2 Free bridges

Lemma 8 *Let $G_0 \oplus G_1$ have k source-sink pairs and at most f faulty elements such that $f+2k \leq \Delta - 1$, where Δ is the minimum degree of $G_0 \oplus G_1$.*

(a) For any terminal w in $G_0 \oplus G_1$, there exists a free bridge of w.

(b) For any set of terminals $W_l = \{w_1, w_2, \ldots, w_l\}$ in G_0 such that $l \leq 2k$, there exist l pairwise disjoint free bridges of w_i's, $1 \leq i \leq l$.

(c) For a single terminal w_1 in G_1 and a set of terminals $W_l \backslash w_1 = \{w_2, w_3, \ldots, w_l\}$ in G_0 such that $l \leq 2k$, there exist l pairwise disjoint free bridges of w_i's, $1 \leq i \leq l$.

Proof There are at least Δ candidates for a free bridge of w, and at most $f + 2k - 1$ elements (f faulty elements and $2k-1$ terminals other than w) can "block" the candidates. Since each element blocks at most one candidate, there are at least $\Delta - (f + 2k - 1) \geq 2$ nonblocked candidates, and thus (a) is proved. We prove (b) by induction on l. Before going on, we need some definitions. We call vertices v and $\bar{v}$ and an edge joining them collectively a *column* of v. When $(v, \bar{v})$ (resp. $(v, w, \bar{w})$) is the free bridge of v, we say that the free bridge *occupies* a column of v (resp. two columns of v and w). We are to construct free bridges for W_l satisfying a condition that the number of occupied columns $c(l)$ is less than or equal to $f(l) + t(l)$, where $f(l)$ and $t(l)$ are the numbers of faulty elements and terminals contained in the $c(l)$ occupied columns, respectively. When $l = 1$, there exists a free bridge which satisfies the condition. Assume that there exist pairwise disjoint free bridges for $W_{l-1} = W \backslash w_l$ satisfying the condition. If $(w_l, \bar{w_l})$ is the free bridge of w_l, we are done. Suppose otherwise. There are Δ candidates for a free bridge, and the number of blocking elements is at most $c(l-1)$ plus the number of terminals and faulty elements which are not contained in the $c(l-1)$ occupied columns. Thus, the number of blocking elements is at most $f + 2k - 1$, which implies the existence of pairwise disjoint free bridges for W_l. Obviously, $c(l) = c(l-1) + 2$ and $f(l) + t(l) \geq f(l-1) + t(l-1) + 2$, and thus it satisfies the condition.

Now, let us prove (c). If $(w_1, \bar{w_1})$ is the free bridge of w_1, it occupies one column. If $(w_1, x, \bar{x})$ is the free bridge of w_1 and $\bar{w_1} \notin W_l$, it occupies two columns. For these cases, in the same way as the proof of (b), we can construct pairwise disjoint free bridges satisfying the above condition. When $(w_1, x, \bar{x})$ is the free bridge of w_1 and $\bar{w_1} \in W_l$, letting $w_2 = \bar{w_1}$ without loss of generality, we first find pairwise disjoint free bridges of w_1 and w_2. They occupy three columns, that is, $c(2) = 3$. We proceed to construct free bridges with a relaxed condition that $c(l) \leq f(l) + t(l) + 1$. This relaxation does not cause any problem since the number of blocking elements is at most $f + 2k$, still less than the number of candidates for a free bridge, Δ. $\square$

Remark 2 According to the proof of Lemma 8 (a) and (b), we have at least two choices when we find free bridges of terminals contained in one component.

Remark 3 If G_i satisfies the conditions of Theorem 2 or 3, then $f + 2k \leq \delta - 1$, where $\delta = \min_i \delta(G_i)$. Concerned with Theorem 2, free bridges of type Lemma 8 (b) and (c) exist in $G_0 \oplus G_1$ since $(f+1)+2k \leq \delta(G_0 \oplus G_1) - 1$. Concerned with Theorem 3, free bridges of the two types also exist in $H_0 \oplus H_1$ since $f + 2(k+1) \leq \delta(H_0 \oplus H_1) - 1$.

3.3 Five basic procedures

In a graph $C_0 \oplus C_1$ with two components C_0 and C_1, we are to define some notation. When we are concerned with Theorem 2, C_0 and C_1 correspond to G_0 and G_1, respectively. When we are concerned with Theorem 3, C_0 and C_1 correspond to H_0 and H_1, respectively. We denote by V_0 and V_1 the sets of vertices in C_0 and C_1, respectively. We let F_0 and F_1 be the sets of faulty elements in C_0 and C_1, respectively, and let F_2 be the set of faulty edges joining vertices in C_0 and vertices in C_1. Let $f_i = |F_i|$ for $i = 0, 1, 2$.

We denote by R the set of source-sink pairs in $C_0 \oplus C_1$. It is assumed that $|R| \geq 2$ in this subsection. We also denote by k_i the number of source-sink pairs in C_i, $i = 0, 1$, and by k_2 the number of source-sink pairs between C_0 and C_1. Without loss of generality, we assume that $k_0 \geq k_1$. We let $I_0 = \{1, 2, \ldots, k_0\}$, $I_2 = \{k_0 + 1, k_0 + 2, \ldots, k_0 + k_2\}$, and $I_1 = \{k_0 + k_2 + 1, k_0 + k_2 + 2, \ldots, k_0 + k_2 + k_1\}$. We assume that $\{s_j, t_j | j \in I_0\} \cup \{s_j | j \in I_2\} \subseteq V_0$ and $\{s_j, t_j | j \in I_1\} \cup \{t_j | j \in I_2\} \subseteq V_1$. Among the k_2 sources s_j's, $j \in I_2$, we assume that the free bridges of k_2' sources are of length one and the free bridges of $k_2''(= k_2 - k_2')$ sources are of length two.

First three procedures DPC-A, DPC-B, and DPC-C are applicable when $k_0 \geq 1$, and the last two procedures DPC-D and DPC-E are applicable when $k_2 = |R|$ (equivalently, $k_0 = k_1 = 0$). We denote by B_v the free bridge of a vertex v.

Procedure DPC-A($C_0 \oplus C_1, R, F$)

UNDER the condition of $1 \leq k_0 < |R|$. See Figure 2(a).

1. Find pairwise disjoint free bridges B_{s_j} of s_j for all $j \in I_2$, and let $B_{s_j} = (s_j, \ldots, s_j')$.
2. Find k_0-DPC$[\{(s_j, t_j) | j \in I_0\} | C_0, F_0 \cup F']$, where $F' = V_0 \cap \bigcup_{j \in I_2} V(B_{s_j})$.
3. Find $k_1 + k_2$-DPC$[\{(s_j', t_j) | j \in I_2\} \cup \{(s_j, t_j) | j \in I_1\} | C_1, F_1]$.
4. Merge the two DPC's with the free bridges.

Procedure DPC-B($C_0 \oplus C_1, R, F$)

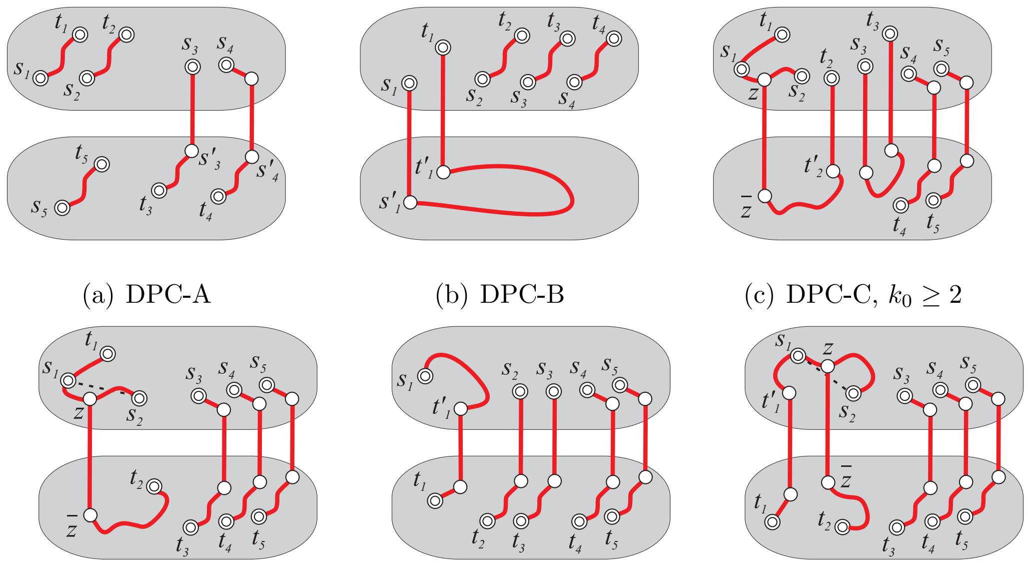
UNDER the condition of $k_0 = |R|$. See Figure 2(b).

1. Let s_1 and t_1 be a pair such that $|X_1| \leq |X_j|$ for all $j \in I_0$, where $X_j = V_0 \cap \{V(B_{s_j}) \cup V(B_{t_j})\}$. Find pairwise disjoint free bridges B_{s_1} and B_{t_1}, and let $B_{s_1} = (s_1, \ldots, s_1')$ and $B_{t_1} = (t_1, \ldots, t_1')$.
2. Find $k_0 - 1$-DPC$[\{(s_j, t_j) | j \in I_0 \backslash 1\} | C_0, F_0 \cup X_1]$.
3. Find $H[s_1', t_1' | C_1, F_1]$.
4. Merge the $k_0 - 1$-DPC and the hamiltonian path with the free bridges.

Keep in mind that under the condition of procedure DPC-C below, for every s_j, $j \in I_2$, $\bar{s}_j = t_{j'}$ for some $j' \in I_2$, and thus for every other fault-free vertex v in C_0, $(v, \bar{v})$ is the free bridge of v.

Procedure DPC-C($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_0 \geq 1$, $k_1 = 0$, $k_2' = 0$, and all the faulty elements are contained in C_0. See Figure 2 (c) and (d).



(a) DPC-A (b) DPC-B (c) DPC-C, $k_0 \geq 2$

(d) DPC-C, $k_0 = 1$ (e) DPC-D (f) DPC-E

Figure 2: Illustration of the five basic procedures.

1. When $k_0 \geq 2$, find pairwise disjoint free bridges B_{t_2}, B_{s_j} and B_{t_j} for all $j \in I_0 \backslash \{1, 2\}$, and B_{s_i} for all $i \in I_2$, and let $B_{t_2} = (t_2, t_2')$, $B_{s_j} = (s_j, s_j')$, $B_{t_j} = (t_j, t_j')$, and $B_{s_i} = (s_i, \ldots, s_i')$. When $k_0 = 1$, find pairwise disjoint free bridges B_{s_j} for all $j \in I_2 \backslash 2$, and let $B_{s_j} = (s_j, \ldots, s_j')$.
2. Find $H[s_2, t_1 | C_0, F_0 \cup F']$, where $F' = V_0 \cap [V(B_{t_2}) \cup \bigcup_{j \in I_0 \backslash \{1,2\}} (V(B_{s_j}) \cup V(B_{t_j})) \cup \bigcup_{j \in I_2} V(B_{s_j})]$ if $k_0 \geq 2$; otherwise, $F' = \{(s_2, s_1)\} \cup (V_0 \cap \bigcup_{j \in I_2 \backslash 2} V(B_{s_j}))$. Let the hamiltonian path be $(s_2, \ldots, z, s_1, \ldots, t_1)$.
3. Let $u = t_2'$ if $k_0 \geq 2$; otherwise, $u = t_2$. Find $k_0 + k_2 - 1$-DPC$[\{(\bar{z}, u)\} \cup \{(s_j', t_j') | j \in I_0 \backslash \{1, 2\}\} \cup \{(s_j', t_j) | j \in I_2 \backslash 2\} | C_1, \emptyset]$.
4. Merge the hamiltonian path and the $k_0 + k_2 - 1$-DPC with the free bridges and the edge $(z, \bar{z})$. Discard the edge (z, s_1).

Procedures DPC-D and DPC-E are concerned with the case of $k_2 = |R|$. Without loss of generality, we assume $f_0 \geq f_1$. This does not conflict with the assumption of $k_0 \geq k_1$.

Procedure DPC-D($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_2 = |R|$ ($k_0 = k_1 = 0$). See Figure 2(e).

1. If $k_2'' \geq 1$, we assume that $(s_1, \bar{s_1})$ is not the free bridge of s_1. Find pairwise disjoint free bridges B_{t_1} and B_{s_j} for all $j \in I_2 \backslash 1$, and let $B_{t_1} = (t_1, \ldots, t_1')$ and $B_{s_j} = (s_j, \ldots, s_j')$.
2. Find $H[s_1, t_1' | C_0, F_0 \cup F']$, where $F' = V_0 \cap \bigcup_{j \in I_2 \backslash 1} V(B_{s_j})$.
3. Find $k_2 - 1$-DPC$[\{(s_j', t_j) | j \in I_2 \backslash 1\} | C_1, F_1 \cup F'']$, where $F'' = V_1 \cap V(B_{t_1})$.

4. Merge the hamiltonian path and the k_2-1-DPC with the free bridges.

Observe that under the condition of procedure DPC-E below, for every source s_j in C_0, $\bar{s}_j = t_{j'}$ for some $j' \in I_2$, and thus for any free vertex v in C_0, $(v, \bar{v})$ is a free edge.

Procedure DPC-E($C_0 \oplus C_1, R, F$)

UNDER the condition that $k_2 = |R|$, $k_2' = 0$, and all the faulty elements are contained in C_0. See Figure 2(f).

1. Find pairwise disjoint free bridges B_{t_1} and B_{s_j} for all $j \in I_2 \backslash \{1,2\}$, and let $B_{t_1} = (t_1, \ldots, t_1')$ and $B_{s_j} = (s_j, \ldots, s_j')$.
2. Find $H[s_2, t_1'|C_0, F_0 \cup F']$, where $F' = \{(s_1, s_2)\} \cup (V_0 \cap \bigcup_{j \in I_2 \backslash \{1,2\}} V(B_{s_j}))$. Let the hamiltonian path be $(s_2, \ldots, z, s_1, \ldots, t_1')$.
3. Find $k_2 - 1$-DPC$[\{(\bar{z}, t_2)\} \cup \{(s_j', t_j) | j \in I_2 \backslash \{1,2\}\} | C_1, F'']$, where $F'' = V_1 \cap V(B_{t_1})$.
4. Merge the hamiltonian path and the $k_2 - 1$-DPC with the free bridges. Discard the edge (s_1, z).

3.4 Proof of Theorem 2

Since G_i is $f+2k-2$-fault hamiltonian-connected and $f+2k-1$-fault hamiltonian, by Lemma 5(a), $G_0 \oplus G_1$ is $f+2k$-fault hamiltonian. To show that $G_0 \oplus G_1$ is many-to-many $(f+1, k)$-disjoint path coverable, it suffices to prove that $G_0 \oplus G_1$ is $f+1$-fault many-to-many k-disjoint path coverable. The fact that $G_0 \oplus G_1$ is $f+3$-fault many-to-many $k-1$-disjoint path coverable can be derived from that each G_i is many-to-many $(f+2, k-1)$-disjoint path coverable and $f+2k-1$-fault hamiltonian, and so forth. For $k=1$ and $f \geq 2$, the theorem is exactly the same as Lemma 4(a). Thus, we assume that

$$k \geq 2,\ f_0 + f_1 + f_2 \leq f + 1,\ \text{and}\ k_0 + k_1 + k_2 = k.$$

Lemmas 9, 10, and 11 are concerned with $k_0 \geq 1$, and Lemmas 12 and 13 are concerned with $k_2 = k$.

Lemma 9 *When $1 \leq k_0 < k$, Procedure DPC-A($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC unless $f_0 = f+1$, $k_1 = 0$, and $k_2' = 0$.*

Proof The existence of pairwise disjoint free bridges in step 1 is due to Lemma 8(b). Unless $f_0 = f+1$, $k_1 = 0$, and $k_2' = 0$, G_0 is $f_0 + k_2' + 2k_2''$-fault k_0-disjoint path coverable since $2k_0 + f_0 + k_2' + 2k_2'' \leq 2k + f$, and thus there exists a k_0-DPC in step 2. Similarly, G_1 is f_1-fault $k_1 + k_2$-disjoint path coverable since $2k_1 + 2k_2 + f_1 \leq 2k + f$. This completes the proof of the lemma. □

Lemma 10 *When $k_0 = k$, Procedure DPC-B($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC unless $f_0 = f+1$ ($k_1 = 0$ and $k_2' = 0$).*

Proof To prove the existence of a $k-1$-DPC in step 2, we will show that $f_0 + |X_1| \leq f+2$. When $|X_1| = 2$, the inequality holds true unless $f_0 = f+1$. When $|X_1| = 3$, the number $f_1 + f_2$ of faulty elements in G_1 or between G_0 and G_1 is at least $k(\geq 2)$, and thus $f_0 + 3 \leq f_0 + f_1 + f_2 + 1 \leq f+2$. When $|X_1| = 4$, analogously to the previous case, $f_0 + 4 \leq f_0 + f_1 + f_2 < f+2$ since $f_1 + f_2 \geq 2k$. The existence of a hamiltonian path joining s_1' and t_1' is due to the fact that $f_1 \leq f + 2k - 2$. □

Lemma 11 *When $k_0 \geq 1$, $f_0 = f+1$, $k_1 = 0$, and $k_2' = 0$, Procedure DPC-C($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC.*

Proof Whether $k_0 \geq 2$ or not, it holds true that $f_0 + |F'| \leq f + 1 + 2(k-2) + 1 = f + 2k - 2$, which implies the existence of a hamiltonian path in step 2. By the construction, $(z, \bar{z})$ is the free bridge of z. Note that $z \neq s_2$ when $k_0 = 1$. The existence of a $k-1$-DPC in step 3 is straightforward. □

Lemma 12 *When $k_2 = k$, Procedure DPC-D($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC unless $f_0 = f+1$ and $k_2' = 0$.*

Proof The existence of pairwise disjoint free bridges is due to Lemma 8(c). To prove the existence of the hamiltonian path, we will show that $f_0 + |F'| \leq f + 2k - 2$. When $k_2'' \geq 1$, $f_0 + |F'| = f_0 + 2(k_2'' - 1) + k_2' \leq f + 2k - 2$ unless $f_0 = f+1$ and $k_2' = 0$. When $k_2'' = 0$, $f_0 + |F'| = f_0 + k_2' - 1 \leq f + 2k - 2$. The existence of $k_2 - 1$-DPC in step 3 is due to that $f_1 + |F''| \leq f+2$. Note that the assumption of $f_0 \geq f_1$ implies that $f_1 < f+1$. □

Lemma 13 *When $k_2 = k$, $f_0 = f+1$, and $k_2' = 0$, Procedure DPC-E($G_0 \oplus G_1, R, F$) constructs an $f+1$-fault k-DPC.*

Proof The existence of the hamiltonian path is due to the fact that $f_0 + |F'| = f_0 + 2(k_2 - 2) + 1 \leq f + 2k - 2$. Note that z is different from s_1 and s_2, and thus $(z, \bar{z})$ is a free edge. The existence of the $k_2 - 1$-DPC is straightforward. □

3.5 Proof of Theorem 3 for $k \geq 2$ and $f \geq 0$ or for $k = 1$ and $f \geq 2$

By Lemma 7, $H_0 \oplus H_1$ is $f + 2k$-fault hamiltonian-connected and $f + 2k + 1$-fault hamiltonian since each G_i is $f + 2k - 2$-fault hamiltonian-connected and $f + 2k - 1$-fault hamiltonian. To show that

$H_0 \oplus H_1$ is many-to-many $(f, k+1)$-disjoint path coverable, it is sufficient to prove that $H_0 \oplus H_1$ is f-fault many-to-many $k+1$-disjoint path coverable. From that each G_i is many-to-many $(f+2, k-1)$-disjoint path coverable and $f+2k-1$-fault hamiltonian, we can conclude that $H_0 \oplus H_1$ is $f+2$-fault many-to-many k-disjoint path coverable, and so forth.

For any $k \geq 2$ and $f \geq 0$ or for any $k = 1$ and $f \geq 2$, H_i, $i = 0, 1$, is many-to-many $(f+1, k)$-disjoint path coverable and $f + 2k$-fault hamiltonian by Theorem 2. In this subsection, by utilizing mainly these properties of H_i, we are to prove Theorem 3 for any $k \geq 1$ and $f \geq 0$ except only when $k = 1$ and $f = 0, 1$. We assume that

$$f_0 + f_1 + f_2 \leq f \text{ and } k_0 + k_1 + k_2 = k + 1.$$

Similar to the proof of Theorem 2, Lemmas 14, 15, and 16 are concerned with $k_0 \geq 1$, and Lemmas 17 and 19 are concerned with $k_2 = k + 1$.

Lemma 14 *When $1 \leq k_0 < k+1$, Procedure DPC-A$(H_0 \oplus H_1, R, F)$ constructs an f-fault $k+1$-DPC unless $f_0 = f$, $k_1 = 0$, and $k_2' = 0$.*

Proof Unless $f_0 = f$, $k_1 = 0$, and $k_2' = 0$, H_0 is $f_0 + k_2' + 2k_2''$-fault k_0-disjoint path coverable since $2k_0 + f_0 + k_2' + 2k_2'' \leq 2k + f + 1$, and thus there exists a k_0-DPC in step 2. Similarly, H_1 is f_1-fault $k_1 + k_2$-disjoint path coverable since $2k_1 + 2k_2 + f_1 \leq 2k + f + 1$. □

Lemma 15 *When $k_0 = k + 1$, Procedure DPC-B$(H_0 \oplus H_1, R, F)$ constructs an f-fault $k+1$-DPC unless $f_0 = f$ $(k_1 = 0$ and $k_2' = 0)$.*

Proof To prove the existence of a k-DPC in step 2, we will show that $f_0 + |X_1| \leq f + 1$. When $|X_1| = 2$, the inequality holds true unless $f_0 = f$. When $|X_1| = 3$, it holds true that $f_1 + f_2 \geq k + 1$, and thus $f_0 + 3 \leq f_0 + f_1 + f_2 + 1 \leq f + 1$. When $|X_1| = 4$, $f_0 + 4 \leq f_0 + f_1 + f_2 < f + 1$ since $f_1 + f_2 \geq 2(k + 1)$. Obviously, there exists a hamiltonian path in H_1 joining s_1' and t_1'. □

Lemma 16 *When $k_0 \geq 1$, $f_0 = f$, $k_1 = 0$, and $k_2' = 0$, Procedure DPC-C$(H_0 \oplus H_1, R, F)$ constructs an f-fault $k+1$-DPC.*

Proof There exists a hamiltonian path in H_0 joining s_2 and t_1 since $f_0 + |F'| \leq f + 2(k-1) + 1 = f + 2k - 1$. The existence of a k-DPC in step 3 is straightforward. □

Hereafter in this subsection, we have $k_2 = k + 1$ $(k_0 = k_1 = 0)$. Due to Lemma 8(a) and Remark 2, we assume that F'' defined in step 3 of Procedures DPC-D and DPC-E is a subset of $V(G_2)$ or $V(G_3)$. That is, $F'' \cap V(G_2) \neq \emptyset$ if and only if $F'' \cap V(G_3) = \emptyset$.

Lemma 17 *When $k_2 = k+1$, Procedure DPC-D($H_0 \oplus H_1, R, F$) constructs an f-fault $k+1$-DPC unless $f_0 = f$ and $k_2' = 0$.*

Proof To prove the existence of a hamiltonian path in H_0, we will show that $f_0 + |F'| \le f + 2k - 1$. When $k_2'' \ge 1$, $f_0 + |F'| = f_0 + 2(k_2'' - 1) + k_2' \le f + 2k - 1$ unless $f_0 = f$ and $k_2' = 0$. When $k_2'' = 0$, $f_0 + |F'| = f_0 + k_2' - 1 \le f + 2k - 1$. Now, let us consider the existence of a $k_2 - 1$-DPC in step 3. When $f \ge 1$ or $|F''| = 1$, there exists a $k_2 - 1$-DPC in H_1 since $f_1 + |F''| \le f + 1$. Note that from the assumption of $f_0 \ge f_1$, if $f \ge 1$, then $f_1 < f$. When $f = 0$ and $|F''| = 2$ ($k \ge 2$ by the assumption of $k \ge 2$ and $f \ge 0$ or $k = 1$ and $f \ge 2$), the existence of a $k_2 - 1$-DPC is due to the following Lemma 18. □

Lemma 18 *For any $k \ge 2$, if G_i is many-to-many $(0, k)$-disjoint path coverable and $2k - 1$-fault hamiltonian for each $i = 0, 1$, then $G_0 \oplus G_1$ with two faulty vertices in G_0 and no other faulty elements is many-to-many k-disjoint path coverable.*

The proof of Lemma 18 is omitted due to space limit. Of course, Lemma 18 does not say that $G_0 \oplus G_1$ is 2-fault many-to-many k-disjoint path coverable. Note that the total number $2 + 2k$ of faulty elements and terminals is not always less than or equal to $\delta(G_0 \oplus G_1) - 1$. However, we can prove the lemma by utilizing the fact that the fault distribution is restrictive, that is, the two faulty elements are vertices contained in G_0. Furthermore, Remark 2 is useful when we are to find pairwise disjoint free bridges of some terminals in $G_0 \oplus G_1$.

Lemma 19 *When $k_2 = k+1$, $f_0 = f$, and $k_2' = 0$, Procedure DPC-E($H_0 \oplus H_1, R, F$) constructs an f-fault $k+1$-DPC.*

Proof There exists a hamiltonian path in H_0 joining s_2 and t_1' since $f_0 + |F'| = f_0 + 2(k_2 - 2) + 1 = f + 2k - 1$. When $f \ge 1$, there exists a $k_2 - 1$-DPC in H_1 since $|F''| = 2 \le f + 1$. When $f = 0$ (and $|F''| = 2$), the existence of a $k_2 - 1$-DPC is due to Lemma 18. □

3.6 Proof of Theorem 3 for $k = 1$ and $f = 0, 1$

In $H_0 \oplus H_1$, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, H_0 and H_1 are called components and G_i's, $0 \le i \le 3$, are called *subcomponents*. Throughout this paper, when we are concerned with $H_0 \oplus H_1$, we denote by V_0 and V_1 the sets of vertices in H_0 and in H_1, respectively, and E_2 the set of edges joining vertices in H_0 and vertices in H_1. $V(G_i)$ denotes the set of vertices in G_i, and $E_{i,j}$ denotes the set of edges joining vertices in G_i and vertices in G_j, $i \neq j$. We denote by F_0 and F_1 the sets of faulty elements in H_0 and in H_1, respectively, F_2 the set of faulty edges in E_2, and let $f_i = |F_i|$, $i = 0, 1, 2$. We let $l_{i,j} = |E_{i,j}|$ and $n = |V(G_i)|$. Observe that $l_{0,1} = l_{2,3} = n$, $l_{0,2} + l_{0,3} = l_{1,2} + l_{1,3} = n$, $l_{0,2} = l_{1,3}$, and $l_{0,3} = l_{1,2}$. For a vertex v in $H_0 \oplus H_1$, we denote by $\bar{v}$ the vertex adjacent to v which is in a component different from the component in which v is contained, and denote by $\hat{v}$ the vertex which is adjacent to v and contained in the same component with v and in a different subcomponent from v.

Contrary to the proofs given in Subsection 3.5, we can not employ Theorem 2. Instead, Lemma 4 (b) and (c) and Lemma 6 are utilized repeatedly in this subsection. By Lemma 7, it remains to show that $H_0 \oplus H_1$ is f-fault many-to-many $k+1$-disjoint path coverable. We assume that

$$f_0 + f_1 + f_2 \leq f \leq 1 \text{ and } k_0 + k_1 + k_2 = k + 1 = 2.$$

It is also assumed that $k_0 \geq k_1$. Lemmas 20, 21, and 22 are concerned with $k_0 \geq 1$, and Lemmas 23 and 24 are concerned with $k_2 = 2$ ($k_0 = k_1 = 0$).

Lemma 20 *When $k_0 = 1$, an f-fault 2-DPC can be constructed unless $f_0 = f$, $k_1 = 0$, and $k'_2 = 0$.*

Proof We are going to utilize Procedure DPC-A($H_0 \oplus H_1, R, F$). Note that $f_0 + |F'| \leq f + 1$ unless $f_0 = f$, $k_1 = 0$, and $k'_2 = 0$. When there exists $P_0 = H[s_1, t_1|H_0, F_0 \cup F']$, we are done since due to Lemma 6, there exists $H[s_2, t_2|H_1, F_1]$ or $H[s'_2, t_2|H_1, F_1]$ depending on whether $k_1 = 1$ or not. If $k_1 = 1$, then $F' = \emptyset$ and P_0 always exists by Lemma 6. Thus, we assume that $k_1 = 0$ and $k_2 = 1$. Suppose P_0 does not exist. By Lemma 4 (b) and (c), both s_1 and t_1 are contained in the same subcomponent, say G_0, and $F_0 \cup F'$ is contained in G_1. Let $(x, \bar{x})$ be a free edge such that x is in G_1. Three hamiltonian paths $H[s_1, t_1|G_0, F_0]$, $H[s_2, x|G_1, F_0]$, $H[\bar{x}, t_2|H_1, F_1]$, and the edge $(x, \bar{x})$ constitute a 2-DPC. □

Lemma 21 *When $k_0 = 2$, an f-fault 2-DPC can be constructed unless $f_0 = f$ ($k_1 = 0$, $k'_2 = 0$).*

Proof We utilize Procedure DPC-B($H_0 \oplus H_1, R, F$). Observe $|X_1| = 2$. Unless $f_0 = f$, it holds true that $f_0 + |X_1| \leq f + 1$. When there exists $P_0 = H[s_2, t_2|H_0, F_0 \cup X_1]$, we are done. Suppose P_0 does not exist. By Lemma 4 (b) and (c), we assume that s_1 and t_1 are contained in G_0, and that s_2 and t_2 are contained in G_1. A hamiltonian path $H[s_2, t_2|G_1, F_0]$ forms an s_2-t_2 path. It remains to construct an s_1-t_1 path. Find a hamiltonian cycle C_0 in $G_0 \backslash F_0 \cup \{s_1\}$ and let $C_0 = (t_1, x, \ldots, y)$. Assuming $(x, \bar{x})$ is a free edge, an s_1-t_1 path is $(s_1, H[\bar{s_1}, \bar{x}|H_1, F_1], C_0 \backslash (t_1, x))$. Note that by the choice of s_1 and t_1, $(s_1, \bar{s_1})$ is the free bridge of s. □

Lemma 22 *When $k_0 \geq 1$, $f_0 = f$, $k_1 = 0$, and $k'_2 = 0$, an f-fault 2-DPC can be constructed.*

Proof We utilize DPC-C($H_0 \oplus H_1, R, F$). If there exists $P_0 = H[s_2, t_1|H_0, F_0 \cup F']$, the proof is completed. Note that when $k_0 = 1$, P_0 always exists since it is impossible that s_2 and t_1 are contained in one subcomponent and the virtual faulty edge (s_2, s_1) is contained in the other subcomponent. Suppose P_0 does not exist. Then, $k_0 = 2$. We assume that s_2 and t_1 are contained in G_0 and that t_2 is contained in G_1. If we reutilize Procedure DPC-C with the roles of s_2 and t_2 interchanged, we can construct an f-fault 2-DPC. To be specific, we find a hamiltonian path $P'_0 = H[t_2, t_1|H_0, F_0 \cup \{s_2\}]$, letting $P'_0 = (t_2, \ldots, z, s_1, \ldots, t_1)$, and then find $H[\bar{s_2}, \bar{z}|H_1, \emptyset]$ and merge them. Since t_2 and t_1 are contained in different subcomponents, P'_0 always exists. □

Now, let us consider the case when $k_2 = 2$ ($k_0 = k_1 = 0$). Remember that $n \geq f + 4$ since each G_i is f-fault hamiltonian-connected and $f + 1$-fault hamiltonian. We denote by P^R the reverse of a path P, that is, $P^R = (v_l, v_{l-1}, \ldots, v_1)$ for $P = (v_1, v_2, \ldots, v_l)$. Hereafter in this subsection, we denote by P_j an s_j-t_j path, $j = 1, 2$.

Lemma 23 *When $k_2 = 2$, an f-fault 2-DPC can be constructed unless $f_0 = f$ and $k'_2 = 0$.*

Proof We assume $f_0 \geq f_1$. It follows that $f_1 = 0$. We will utilize DPC-D($H_0 \oplus H_1, R, F$). Unless $f_0 = f$ and $k'_2 = 0$, it holds true that $f_0 + |F'| \leq f + 1$. Except for the case that $f = 0$ and $|F''| = 2$, it also holds true that $f_1 + |F''| = |F''| \leq f + 1$. The exceptional case is equivalent to that $f = 0$, (s_1, t_1) is an edge, and $k'_2 = 1$ since (i) we need not consider the case that $f = 0$ and $k'_2 = 0$ in this proof and (ii) Procedure DPC-D chooses s_1 such that $(s_1, \bar{s_1})$ is not the free bridge of s_1, if possible. For the exceptional case, employing Lemma 7(a), we have $P_1 = (s_1, t_1)$ and $P_2 = H[s_2, t_2|H_0 \oplus H_1, F^*]$, where $F^* = \{s_1, t_1\}$.

Now, we have $f_0 + |F'| \leq f + 1$ and $f_1 + |F''| \leq f + 1$. Keep in mind that $F' = V_0 \cap V(B_{s_2})$ and $F'' = V_1 \cap V(B_{t_1})$. From now on, the assumption of $f_0 \geq f_1$ will never be applied to obtain symmetry. If both $H[s_1, t'_1|H_0, F_0 \cup F']$ and $H[s'_2, t_2|H_1, F_1 \cup F'']$ exist, the proof is completed. Suppose otherwise, by Lemma 6 and Lemma 4 (b) and (c), at least one of the following two conditions is satisfied:

A1: $f_0 + |F'| = f + 1$, s_1 and t'_1 are in one subcomponent, and $F_0 \cup F'$ is in the other subcomponent;

A2: $f_1 + |F''| = f + 1$, t_2 and s'_2 are in one subcomponent, and $F_1 \cup F''$ is in the other subcomponent.

First, we assume that A1 is satisfied and that s_1 and t'_1 are in G_0 and s_2 is in G_1. If t_1 and t_2 are contained in the same subcomponent of H_1, then $P_1 = (H[s_1, t'_1|G_0, F_0], B^R_{t_1})$ and for some free edge $(x, \bar{x})$ with

$x \in V(G_1)$, $P_2 = (H[s_2, x|G_1, F_0], H[\bar{x}, t_2|H_1, F_1 \cup F''])$. Now, we assume that t_1 is contained in one subcomponent, say G_2, and t_2 is in the other subcomponent G_3. If there is a free edge $(y, \bar{y}) \in E_{1,2}$ with $y \in V(G_1)$, then $P_1 = (H[s_1, t_1'|G_0, F_0], B^R_{t_1})$ and $P_2 = (H[s_2, y|G_1, F_0], H[\bar{y}, t_2|H_1, F_1 \cup F''])$. Otherwise, there exist a pair of free edges $(a, \bar{a}) \in E_{0,2}$ and $(b, \bar{b}) \in E_{1,3}$ with $a \in V(G_0)$ and $b \in V(G_1)$. Then, $P_1 = (H[s_1, a|G_0, F_0], H[\bar{a}, t_1|G_2, F_1])$ and $P_2 = (H[s_2, b|G_1, F_0], H[\bar{b}, t_2|G_3, F_1])$. When A2 is satisfied, symmetrically to that A1 is satisfied, we can also construct an f-fault 2-DPC. Thus, we have the lemma. □

Lemma 24 *When $k_2 = 2$, $f_0 = f$, and $k_2' = 0$, an f-fault 2-DPC can be constructed.*

Proof It follows that $k_2'' = 2$ and $f_1 = f_2 = 0$, and thus we have $\{\bar{s_1}, \bar{s_2}\} = \{t_1, t_2\}$. When $\bar{s_1} = t_1$, we have $P_1 = (s_1, t_1)$ and $P_2 = H[s_2, t_2|H_0 \oplus H_1, F \cup F^*]$, where $F^* = \{s_1, t_1\}$. Thus, we assume that $\bar{s_1} = t_2$ and $\bar{s_2} = t_1$. To obtain utmost symmetry, we will never apply the assumption of $f_0 = f$. Instead, we assume $f_2 = 0$. Therefore, we have three cases.

Case 1: $s_1, s_2 \in V(G_0)$, $t_1 \in V(G_2)$, and $t_2 \in V(G_3)$.

When $l_{0,3} \geq f+2$, there exists a free edge $(x, \bar{x}) \in E_{0,3}$ with $x \in V(G_0)$. Letting $F^* = \{(s_2, v) | v \in V(G_0)$ and either $\hat{v}$ or $(v, \hat{v})$ is faulty$\}$, we find $H[s_1, x|G_0, F_0 \cup F^*]$. The existence of the hamiltonian path in G_0 is due to that the number of faulty elements in G_0 including the virtual faults is at most f. Let the hamiltonian path be $(s_1, Q_1, z, s_2, Q_2, x)$. Observe that both $\hat{z}$ and $(z, \hat{z})$ are fault-free. Then, let $P_2 = (s_2, Q_2, x, H[\bar{x}, t_2|G_3, F_1])$. We are to construct P_1. There exists a free edge $(y, \bar{y}) \in E_{1,2}$ with $y \in V(G_1)$ such that $y \neq \hat{z}$ since $l_{1,2} = l_{0,3} \geq f + 2$. Then, we have $P_1 = (s_1, Q_1, z, H[\hat{z}, y|G_1, F_0], H[\bar{y}, t_1|G_2, F_1])$. When $l_{0,3} \leq f + 1$, we have $l_{0,2} \geq f + 2$. In a symmetric way, we can also construct a 2-DPC.

Case 2: $s_1 \in V(G_0)$, $s_2 \in V(G_1)$, $t_1 \in V(G_2)$, and $t_2 \in V(G_3)$.

We assume w.l.o.g. $f_0 \geq f_1$ ($f_1 = 0$). If there exist a pair of free edges $(x, \bar{x}) \in E_{0,2}$ and $(y, \bar{y}) \in E_{1,3}$ with $x \in V(G_0)$ and $y \in V(G_1)$, then we have $P_1 = (H[s_1, x|G_0, F_0], H[\bar{x}, t_1|G_2, \emptyset])$ and $P_2 = (H[s_2, y|G_1, F_0], H[\bar{y}, t_2|G_3, \emptyset])$. Suppose otherwise, it follows that $l_{0,2} \leq 1$. When $l_{0,2} = 1$, letting $(p, \bar{p}) \in E_{0,2}$ and $(q, \bar{q}) \in E_{1,3}$ with $p, q \in V_0$, we assume w.l.o.g. that p is faulty. There exist a pair of free edges $(a, \hat{a})$ and $(b, \hat{b})$ with $a, b \in V(G_0)$. Two hamiltonian paths $H[a, b|G_0, F_0]$ and $H[\hat{a}, \hat{b}|G_1, F_0]$ are merged with $(a, \hat{a})$ and $(b, \hat{b})$ into a hamiltonian cycle C_0 in $H_0 \backslash F_0$. Let $C_0 = (s_1, z_1, Q, u_2, s_2, u_1, Q', z_2)$. By the construction, we have $z_1, z_2 \in V(G_0)$ and $u_1, u_2 \in V(G_1)$. Observe that both $(z_1, \bar{z_1})$ and $(z_2, \bar{z_2})$ are free edges in $E_{0,3}$, and that at least one of $(u_1, \bar{u_1})$ and $(u_2, \bar{u_2})$ is a free edge in $E_{1,2}$. Assuming w.l.o.g. that $(u_2, \bar{u_2})$ is a free edge in $E_{1,2}$, we have $P_1 = (s_1, z_1, Q, u_2, H[\bar{u_2}, t_1, |G_2, \emptyset])$ and

$P_2 = (s_2, u_1, Q', z_2, H[\bar{z_2}, t_2|G_3, \emptyset])$.

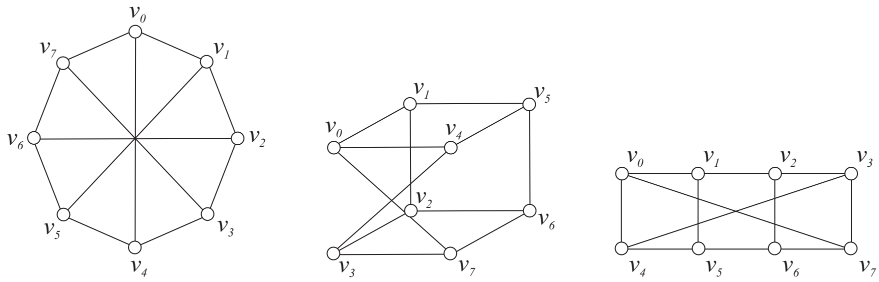
Case 3: $s_1, s_2 \in V(G_0)$ and $t_1, t_2 \in V(G_2)$.

We assume w.l.o.g. $f_0 \geq f_1$ ($f_1 = 0$). When there exists a free edge $(x, \bar{x}) \in E_{0,2}$ with $x \in V(G_0)$, letting $F^* = \{(s_1, v)|v \in V(G_0)$ and either $\hat{v}$ or $(v, \hat{v})$ is faulty$\}$, we find $H[s_2, x|G_0, F_0 \cup F^*]$ and let the hamiltonian path be $(s_2, Q_1, z, s_1, Q_2, x)$. There exists a free edge $(y, \bar{y}) \in E_{1,3}$ with $y \in V(G_1)$ such that $y \neq \hat{z}$ since $l_{0,2} = l_{1,3} \geq 3$. Letting $F^+ = \{(t_1, w)|w \in V(G_2)$ and $\hat{w} = \bar{y}\}$, we find $H[\bar{x}, t_2|G_2, F^+]$ and let the hamiltonian path be $(\bar{x}, Q_3, t_1, u, Q_4, t_2)$. By the construction, $\hat{u} \neq \bar{y}$. Then, $P_1 = (s_1, Q_2, x, \bar{x}, Q_3, t_1)$ and $P_2 = (s_2, Q_1, z, H[\hat{z}, y|G_1, F_0], H[\bar{y}, \hat{u}|G_3, \emptyset], u, Q_4, t_2)$. Suppose there does not exist such a free edge $(x, \bar{x})$. There exists s_i, $i = 0, 1$, such that $\hat{s_i}$ and $(s_i, \hat{s_i})$ are fault-free. We assume s_1 is such a source. We first find a hamiltonian cycle C_0 in $G_0 \backslash F_0 \cup \{s_1\}$. Let $C_0 = (s_2, a, \ldots, b)$ and assume that $\bar{b} \neq \hat{t_2}$. Of course, we have $\bar{a}, \bar{b} \in V(G_3)$. Then, we have $P_2 = (C_0 \backslash (s_2, b), H[\bar{b}, \hat{t_2}|G_3, \emptyset], t_2)$. P_1 is constructed in different ways depending on whether $f = 0$ or not. When $f = 0$, we find a hamiltonian cycle $C_2 = (t_1, c, \ldots, d)$ in $G_2 \backslash t_2$, assuming $\bar{d} \neq \hat{s_1}$, we have $P_1 = (s_1, H[\hat{s_1}, \bar{d}|G_1, F_0], C_2 \backslash (t_1, d))$. When $f = 1$, we choose a free edge $(w, \bar{w}) \in E_{1,2}$ with $w \in V(G_1)$ such that $w \neq \hat{s_1}$. The existence of $(w, \bar{w})$ is due to that $l_{1,2} \geq 3$ or $l_{1,2} = 2$ and G_1 is fault-free. Then, we have $P_1 = (s_1, H[\hat{s_1}, w|G_1, F_0], H[\bar{w}, t_1|G_2, F^*])$, where $F^* = \{t_2\}$. This completes the proof. $\square$

4 Hypercube-Like Interconnection Networks

Vaidya *et al.*[32] introduced a class of hypercube-like interconnection networks, called *HL-graphs*, which can be defined by applying the $\oplus$ operation repeatedly as follows: $HL_0 = \{K_1\}$; for $m \geq 1$, $HL_m = \{G_0 \oplus G_1 | G_0, G_1 \in HL_{m-1}\}$. Then, $HL_1 = \{K_2\}$, $HL_2 = \{C_4\}$, and $HL_3 = \{Q_3,\ G(8,4)\}$. Here, C_4 is a cycle graph with 4 vertices, Q_3 is a 3-dimensional hypercube, and $G(8,4)$ is a recursive circulant which is isomorphic to twisted cube TQ_3 and Möbius ladder as shown in Figure 3. An arbitrary graph which belongs to HL_m is called an *m-dimensional HL-graph.* Recently, it was shown by Park and Chwa in [19] that every nonbipartite HL-graph is hamiltonian-connected and every bipartite HL-graph is hamiltonian-laceable.

Obviously, some m-dimensional HL-graphs such as an m-dimensional hypercube are bipartite. They are not f-faulty many-to-many k-disjoint path coverable for any $f \geq 0$ and $k \geq 1$. Thus, we are to define a subclass of HL-graphs which seems "highly" nonbipartite, and then consider their many-to-



(a) $G(8,4)$ (b) TQ_3 (c) Möbius ladder

Figure 3: Isomorphic graphs.

many disjoint path coverability.

Definition 5 *A subclass of nonbipartite HL-graphs, called* restricted HL-graphs*, is defined recursively as follows:*

- $RHL_m = HL_m$ *for* $0 \leq m \leq 2$*;*
- $RHL_3 = HL_3 \backslash Q_3 = \{G(8,4)\}$*;*
- $RHL_m = \{G_0 \oplus G_1 | G_0, G_1 \in RHL_{m-1}\}$ *for* $m \geq 4$*.*

A graph which belongs to RHL_m *is called an* m-dimensional restricted HL-graph.

It was shown in [25] that many of the nonbipartite hypercube-like interconnection networks such as twisted cube[14], crossed cube[9], multiply twisted cube[8], Möbius cube[7], Mcube[27], generalized twisted cube[4], etc. proposed in the literature are restricted HL-graphs, with the exception of recursive circulants $G(2^m, 4)$[22] and "near" bipartite interconnection networks such as twisted m-cube[10]. Fault-hamiltonicity of restricted HL-graphs was studied in [25] as follows.

Lemma 25 *[25] Every* m*-dimensional restricted HL-graph,* $m \geq 3$*, is* $m-3$*-fault hamiltonian-connected and* $m-2$*-fault hamiltonian.*

In this section, we consider many-to-many disjoint path coverability of restricted HL-graphs, recursive circulants $G(2^m, 4)$, and twisted m-cube. To show that all of them except the twisted m-cube have very good properties in disjoint path coverability, we need a definition.

Definition 6 *A graph* G *is called* fully many-to-many disjoint path coverable *if for any* $k \geq 1$ *and* $f \geq 0$ *such that* $f + 2k \leq \delta(G) - 1$*,* G *is* f*-fault many-to-many* k*-disjoint path coverable.*

4.1 Restricted HL-graphs

We now consider many-to-many disjoint path coverability of restricted HL-graphs. We rely on Theorem 3. From Lemma 25, we know that the 3-dimensional HL-graph $G(8,4)$ is (0-fault) hamiltonian-connected and 1-fault hamiltonian and every 4-dimensional restricted HL-graph $G(8,4) \oplus G(8,4)$ is 1-fault hamiltonian-connected and 2-fault hamiltonian. An arbitrary m-dimensional restricted HL-graph, $m \geq 5$, is isomorphic to $[G_0 \oplus_{M_1} G_1] \oplus_M [G_2 \oplus_{M_2} G_3]$ for some permutations M_1, M_2, and M, where G_0, G_1, G_2, and G_3 are $m-2$-dimensional restricted HL-graphs. Therefore, by an inductive argument utilizing Theorem 3, we can get a theorem as follows.

Theorem 4 *Every m-dimensional restricted HL-graph, $m \geq 3$, is fully many-to-many disjoint path coverable and $m-2$-fault hamiltonian.*

Corollary 3 *Let G be an arbitrary m-dimensional restricted HL-graph, $m \geq 3$. There exists an s-t hamiltonian path in $G \backslash F$ that passes through the edges in the order given for any fault set F with $|F| \leq f$ and for any pair of vertices s and t and any s-t path extendable sequence of $k-1$ edges $((x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}))$ in $G \backslash F$ such that $f + 2k \leq m - 1$ and $v \neq w$ for at least one pair (v, w) in $\{(s, x_1), (y_{k-1}, t), (y_j, x_{j+1}) | 1 \leq j < k-1\}$.*

4.2 Recursive circulants $G(2^m, 4)$

The recursive circulant $G(N, d)$[22] is defined as follows: the vertex set $V = \{v_0, v_1, \ldots, v_{N-1}\}$, and the edge set $E = \{(v_a, v_b) \mid \text{there exists } i,\ 0 \leq i \leq \lceil \log_d N \rceil - 1, \text{ such that } a + d^i \equiv b \pmod{N}\}$. $G(N, d)$ can also be defined as a circulant graph with N vertices and jumps of powers of d, $d^0, d^1, \cdots, d^{\lceil \log_d N \rceil - 1}$. When $N = 2^m$ and $d = 4$, recursive circulant $G(2^m, 4)$ is an m-regular graph with 2^m vertices. According to their recursive structure[22], we can observe that $G(2^m, 4)$ is isomorphic to $[G(2^{m-2}, 4) \times K_2] \oplus_M [G(2^{m-2}, 4) \times K_2]$ for some M. Obviously, every $G(2^m, 4)$ is an HL-graph. Furthermore, every $G(2^m, 4)$ with odd m is a restricted HL-graph. However, not every $G(2^m, 4)$ is a restricted HL-graph. One can check without difficulty that $G(16, 4)$ is not isomorphic to $G(8, 4) \oplus_M G(8, 4)$ for any M, and even $G(16, 4)$ does not have $G(8, 4)$ as a subgraph. Fault-hamiltonicity of recursive circulants $G(2^m, 4)$ was considered in [25, 29].

Lemma 26 *[25, 29] $G(2^m, 4)$, $m \geq 3$, is $m-3$-fault hamiltonian-connected and $m-2$-fault hamiltonian.*

By an inductive argument employing Theorem 3, we can conclude many-to-many disjoint path coverability of $G(2^m, 4)$ as follows.

Theorem 5 *$G(2^m, 4)$, $m \geq 3$, is fully many-to-many disjoint path coverable and $m-2$-fault hamiltonian.*

Corollary 4 *There exists an s-t hamiltonian path in $G(2^m, 4)\backslash F$, $m \geq 3$, that passes through the edges in the order given for any fault set F with $|F| \leq f$ and for any pair of vertices s and t and any s-t path extendable sequence of $k-1$ edges $((x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}))$ in $G(2^m, 4)\backslash F$ such that $f + 2k \leq m - 1$ and $v \neq w$ for at least one pair (v, w) in $\{(s, x_1), (y_{k-1}, t), (y_j, x_{j+1}) | 1 \leq j < k-1\}$.*

4.3 Twisted m-cube

Let (v_0, v_1) and (v_2, v_3) be two nonadjacent edges in an arbitrary cycle (v_0, v_1, v_2, v_3) of length four in hypercube Q_m. The twisted m-cube[10] is constructed as follows. Delete edges (v_0, v_1) and (v_2, v_3) from Q_m. Then connect, via an edge, v_0 to v_2 and v_1 to v_3. Obviously, twisted m-cube is an HL-graph. Due to [19] and [25], we have the following lemma.

Lemma 27 *(a) Twisted m-cube, $m \geq 3$, is hamiltonian-connected, or equivalently, it is 0-fault many-to-many 1-disjoint path coverable[19].*
(b) Twisted m-cube, $m \geq 3$, is 1-fault hamiltonian and not f-fault hamiltonian for any $f \geq 2$[25].

It is hard to expect that twisted m-cube is good in disjoint path coverability since its bipartization number is only two. If we adopt the coloring induced by proper bicoloring (without two adjacent vertices having the same color) of hypercube Q_m, there are exactly two edges joining vertices with the same color in twisted m-cube: one joining two black vertices and the other joining two white vertices.

Theorem 6 *Twisted m-cube, $m \geq 3$, is not f-fault many-to-many k-disjoint path coverable for any $f \geq 0$ and $k \geq 1$ except only for $f = 0$ and $k = 1$.*

Proof The bipartization number of twisted m-cube is two. By Lemma 2, twisted m-cube is not f-fault k-disjoint path coverable for any $f \geq 2$ and $k \geq 1$. Thus, we assume $f = 0, 1$. We rely on Lemma 3. Let us consider the induced coloring of twisted m-cube such that $n_w = n_b = 2^{m-1}$ and $c_w = c_b = 2^{m-1} - 1$. When $f = 0$, $c_b > n_w - k$ for every $k \geq 2$. When $f = 1$, assuming the edge joining two black vertices is faulty, we have that $c_b = 2^{m-1}$ and $c_b > n_w - k$ for every $k \geq 1$. Thus, the proof is completed. □

5 Concluding Remarks

We studied some interesting properties on f-fault many-to-many k-disjoint path coverable graphs such as relationships among the three types of disjoint path covers, sufficiency for some strong-hamiltonicity, and some necessary conditions. And then, we presented construction schemes for many-to-many disjoint path covers in the graphs $G_0 \oplus G_1$ and $H_0 \oplus H_1$ with faulty elements, where $H_0 = G_0 \oplus G_1$ and $H_1 = G_2 \oplus G_3$, provided some conditions on G_i are satisfied for all $0 \leq i \leq 3$. The conditions are constituted with the hamiltonicity and disjoint path coverability in the presence of faulty elements. One of the main results is that in most cases with the exception of $k = 1$ and $f = 0, 1$, the bound on the number of faulty elements in $G_0 \oplus G_1$ is one larger than the minimum bound of G_0 and G_1 to preserve disjoint path coverability. Also, without exception, the bound on the number of disjoint paths in $H_0 \oplus H_1$ is one larger than the minimum bound over all G_i's and the bound on the number of faulty elements in $H_0 \oplus H_1$ is two larger than the minimum bound over all G_i's. By applying the main results to restricted HL-graphs and recursive circulant $G(2^m, 4)$, we concluded that all these networks of degree $m(\geq 3)$ are fully many-to-many disjoint path coverable.

According to the constructions presented in this paper, we can design efficient algorithms for finding many-to-many disjoint path covers in $G_0 \oplus G_1$ and $H_0 \oplus H_1$. We need a bit careful implementation. It is assumed that (i) $G_0 \oplus G_1$ is represented as an adjacency list structure, (ii) $V(G_0) = \{v_1, \ldots, v_n\}$ and $V(G_1) = \{v_{n+1}, \ldots, v_{2n}\}$, (iii) the first node in the linked list of v_i representing vertices adjacent to v_i is $\bar{v}_i$ for every i, (iv) whether a given vertex (or an edge) once located is faulty or not can be determined in a constant time (a status field in each node representing a vertex (or an edge) is sufficient), (v) whether a given vertex is a source s_i or a sink t_j or none can be determined in a constant time, and (vi) the number of faulty edges incident to each vertex is known in advance. We also have such assumptions in $H_0 \oplus H_1$. The first and second nodes in the linked list of v_i are assumed to be $\bar{v}_i$ and $\hat{v}_i$, respectively. We let $T_1(G)$ be the time complexity of an algorithm for finding a hamiltonian cycle or a hamiltonian path between an arbitrary pair of fault-free vertices in a graph G. It can be derived from [25] that

(a) $T_1(G_0 \oplus G_1) = T_1(G_0) + T_1(G_1) + O(n),$ where $n = |V(G_i)|$,

$$\text{(b) } T_1(H_0 \oplus H_1) = \max \begin{cases} T_1(H_0) + T_1(H_1) + O(n), \\ T_1(H_0) + T_1(G_2) + T_1(G_3) + O(n), \\ T_1(G_0) + T_1(G_1) + T_1(H_1) + O(n), \\ T_1(G_0) + T_1(G_1) + T_1(G_2) + T_1(G_3) + O(n). \end{cases}$$

We let $T_2(G)$ be the maximum of $T_1(G)$ and the time complexity for finding an f-fault many-to-many k-disjoint path cover in G for any pair of f and k satisfying $f+2k \le \delta(G)-1$. Then, all the constructions given in this paper (containing the omitted one of the proof of Lemma 18) can be accomplished in time

(c) $T_2(G_0 \oplus G_1) = T_2(G_0) + T_2(G_1) + O(n+\delta^2)$, where $\delta = \min_i \delta(G_i)$,

(d) $$T_2(H_0 \oplus H_1) = \max \begin{cases} T_2(H_0) + T_2(H_1) + O(n+\delta^2), \\ T_2(H_0) + T_2(G_2) + T_2(G_3) + O(n+\delta^2), \\ T_2(G_0) + T_2(G_1) + T_2(H_1) + O(n+\delta^2), \\ T_2(G_0) + T_2(G_1) + T_2(G_2) + T_2(G_3) + O(n+\delta^2). \end{cases}$$

The construction of pairwise disjoint free bridges given in the proof of Lemma 8 leads to a simple greedy algorithm running in time $O(k\delta)$, which is at most $O(\delta^2)$. Therefore, we can conclude that in an arbitrary m-dimensional restricted HL-graph or in recursive circulant $G(2^m, 4)$, an f-fault many-to-many k-disjoint path covers with $f+2k \le m-1$ can be found in time $O(m2^m)$, which is linear to the size of the graphs.

References

[1] A. Abdullah, "On Graph Bipartization," in *Proc. of the IEEE International Symposium on Circuits and Systems ISCAS'92*, San Diego, USA, pp. 1847-1850, 1992.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, 5th printing, American Elsevier Publishing Co., Inc., 1976.

[3] C.-H. Chang, C.-K. Lin, H.-M. Huang, and L.-H. Hsu, "The super laceability of the hypercubes," *Inform. Proc. Lett.* **92(1)**, pp. 15-21, 2004.

[4] F.B. Chedid, "On the generalized twisted cube," *Inform. Proc. Lett.* **55**, pp. 49-52, 1995.

[5] C.C. Chen and J. Chen, "Nearly optimal one-to-many parallel routing in star networks," *IEEE Trans. on Parallel and Distributed Systems* **8(12)**, pp. 1196-1202, 1997.

[6] H.-A. Choi, K. Nakajima, and C.S. Rim, "Graph bipartization and via minimization," *SIAM J. Discrete Math.* **2(1)**, pp. 38-47, 1989.

[7] P. Cull and S. Larson, "The Möbius cubes," in *Proc. of the 6th IEEE Distributed Memory Computing Conf.*, pp. 699-702, 1991.

[8] K. Efe, "A variation on the hypercube with lower diameter," *IEEE Trans. on Computers* **40(11)**, pp. 1312-1316, 1991.

[9] K. Efe, "The crossed cube architecture for parallel computation," *IEEE Trans. on Parallel and Distributed Systems* **3(5)**, pp. 513-524, 1992.

[10] A.-H. Esfahanian, L.M. Ni, and B.E. Sagan, "The twisted n-cube with application to multiprocessing," *IEEE Trnas. Computers* **40(1)**, pp. 88-93, 1991.

[11] S. Gao, B. Novick and K. Qiu, "From hall's matching theorem to optimal routing on hypercubes," *Journal of Combinatorial Theory, Series B.* **74**, pp. 291-301, 1998.

[12] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman & Co., San Francisco, 1979.

[13] Q.P. Gu and S. Peng, "Cluster fault-tolerant routing in star graphs," *Networks* **35(1)**, pp. 83-90, 2000.

[14] P.A.J. Hilbers, M.R.J. Koopman, J.L.A. van de Snepscheut, "The Twisted Cube," in J. Bakker, A. Nijman, P. Treleaven, eds., *PARLE: Parallel Architectures and Languages Europe, Vol. I: Parallel Architectures*, Springer, pp. 152-159, 1987.

[15] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Fault-free hamiltonian cycles in faulty arrangement graphs," *IEEE Trans. Parallel and Distributed Systems* **10(3)**, pp. 223-237, Mar. 1999.

[16] S.-Y. Hsieh, G.-H. Chen, and C.-W. Ho, "Longest fault-free paths in star graphs with edge faults," *IEEE Trans. Computers* **50(9)**, pp. 960-971, Sep. 2001.

[17] H.-C. Hsu, T.-K. Li, J.J.M. Tan, and L.-H. Hsu, "Fault hamiltonicity and fault hamiltonian connectivity of the arrangement graphs," *IEEE Trans. Computers* **53(1)**, pp. 39-53, Jan. 2004.

[18] S. Madhavapeddy and I.H. Sudborough, "A topological property of hypercubes: node disjoint paths," in *Proc. of the 2th IEEE Symposium on Parallel and Distributed Processing SPDP 1990*, pp. 532-539, 1990.

[19] C.-D. Park and K.Y. Chwa, "Hamiltonian properties on the class of hypercube-like networks," *Inform. Proc. Lett.* **91**, pp. 11-17, 2004.

[20] J.-H. Park, "One-to-one disjoint path covers in recursive circulants," *Journal of KISS* **30(12)**, pp. 691-698, 2003 (in Korean).

[21] J.-H. Park, "One-to-many disjoint path covers in a graph with faulty elements," in *Proc. of the International Computing and Combinatorics Conference COCOON 2004*, pp. 392-401, Aug. 2004.

[22] J.-H. Park and K.Y. Chwa, “Recursive circulants and their embeddings among hypercubes,” *Theoretical Computer Science* **244**, pp. 35-62, 2000.

[23] J.-H. Park and H.-C. Kim, “Longest paths and cycles in faulty star graphs,” *J. Parallel Distrib. Comput.* **64(11)**, pp. 1286-1296, 2004.

[24] J.-H. Park, H.-C. Kim, and H.-S. Lim, “Many-to-many disjoint path covers in a graph with faulty elements,” in *Proc. of the International Symposium on Algorithms and Computation ISAAC 2004*, Hong Kong, pp. 742-753, Dec. 2004.

[25] J.-H. Park, H.-C. Kim, and H.-S. Lim, “Fault-hamiltonicity of hypercube-like interconnection networks,” in *Proc. of the IEEE International Parallel and Distributed Processing Symposium IPDPS 2005*, Denver, Apr. 2005.

[26] A. Sengupta, “On ring embedding in hypercubes with faulty nodes and links”, *Inform. Proc. Lett.* **68**, pp. 207-214, 1998.

[27] N.K. Singhvi and K. Ghose, “The Mcube: a symmetrical cube based network with twisted links,” in *Proc. of the 9th IEEE Int. Parallel Processing Symposium IPPS 1995*, pp. 11-16, 1995.

[28] C.-H. Tsai, “Linear array and ring embeddings in conditional faulty hypercubes,” *Theoretical Computer Science* **314**, pp. 431-443, 2004.

[29] C.-H. Tsai, J.J.M. Tan, Y.-C. Chuang, and L.-H. Hsu, “Hamiltonian properties of faulty recursive circulant graphs,” *Journal of Interconnection Networks* **3**, pp. 273-289, 2002.

[30] C.-H. Tsai, J.J.M. Tan, and L.-H. Hsu, “The super-connected property of recursive circulant graphs,” *Inform. Proc. Lett.* **91(6)**, pp. 293-298, 2004.

[31] Y.-C. Tseng, S.-H. Chang, and J.-P. Sheu, “Fault-tolerant ring embedding in a star graph with both link and node failures,” *IEEE Trans. Parallel and Distributed Systems* **8(12)**, pp. 1185-1195, 1997.

[32] A.S. Vaidya, P.S.N. Rao, S.R. Shankar, “A class of hypercube-like networks,” in *Proc. of the 5th IEEE Symposium on Parallel and Distributed Processing SPDP 1993*, pp. 800-803, Dec. 1993.

[33] L. Volkmann and A. Yeo, “Hamiltonian paths, containing a given path or collection of arcs, in close to regular multipartite tournaments,” *Discrete Mathematics* **281(1-3)**, pp. 267-276, 2004.

[34] M. Yannakakis, “Node- and edge-deletion NP-complete problems,” in *Proc. of the 10th ACM Symposium on Theory of Computing STOC 1978*, San Diego, California, pp. 253-264, 1978.