International Statistical Review (2008), 76, 2, 247–267 doi:10.1111/j.1751-5823.2008.00049.x

Many-valued Logic in Multistate and Vague Stochastic Systems

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Summary

The state of the art in coherent structure theory is driven by two assertions, both of which are limiting: (1) all units of a system can exist in one of two states, failed or functioning; and (2) at any point in time, each unit can exist in only one of the above states. In actuality, units can exist in more than two states, and it is possible that a unit can *simultaneously* exist in more than one state. This latter feature is a consequence of the view that it may not be possible to precisely define the subsets of a set of states; such subsets are called *vague*. The first limitation has been addressed via work labeled 'multistate systems'; however, this work has not capitalized on the mathematics of many-valued propositions in logic. Here, we invoke its truth tables to define the structure function of multistate systems and then harness our results in the context of vagueness. A key contribution of this paper is to argue that many-valued logic is a common platform for studying both multistate and vague systems but, to do so, it is necessary to lean on several principles of statistical inference.

Key words: Consistency profile; likelihood function; membership functions; reliability; probability; maintenance management; natural language; degradation modeling; decision making and utility.

1 Introduction and Overview

The calculus of coherent systems, innovated by Birnbaum *et al*. (1961) has served as a mathematical foundation for a theory of systems. Here, one explores the effect that a system's components have on the system. The bulk of the effort, however, has been devoted to the case of binary states with precise classification. That is, the components and the system can (at any point in time) be in one of two unambiguously defined states, functioning or failed. In actuality, items can function in degraded states, and these could be a discrete set or a continuum of states. An example of the former is a load-sharing system, like a transmission line for power with r strands. As the strands break, the rope transitions from its ideal load carrying capability to its complete disintegration (Smith, 1983). An example of the latter is a precipitator for reducing air pollution whose cleaning efficiency ranges from (almost) 100 to 0% (Matland & Singpurwalla, 1981). Systems that can exist in more than two states are called *multistate systems*.

There are two interrelated aims to this paper. The first is to contribute to the mathematics of multistate systems with precise classification via many-valued logic. To set the stage for this, we overview some key notions and results in the reliability theory of binary systems.

Section 1.2 is archival; however, Section 1.3 is current in the sense that it incorporates the view that, when discussing system reliability, one needs to distinguish between probability (which is personal) and propensity (which is physical), and that the assumption of the independence is conditional upon propensities. The second aim of this paper is to argue that multivalued logic also provides a framework for assessing the reliability of binary or multistate systems with imprecise classification. Imprecision (or vagueness) is articulated in Section 1.4; Section 1.5 is a guide to the rest of this paper.

1.1 Preamble: Notation and Terminology

Consider a system with n components. The system and each of its components can exist in several states in $\mathcal{S} \subseteq [0, 1]$. Let X_i, $i = 1, \ldots, n$ denote the state of component i at time $\tau \geq 0$, and denote $\mathbf{X} = (X_1, \ldots, X_n)$. Binary systems are those for which $\mathcal{S} = \{0, 1\}$, where 1 (0) denotes a functioning (failed) state. The state of the system is a function of $\mathbf{X}$, called the 'structure function'. We denote by $\phi(\mathbf{X})$ the structure function for a binary system. The structure function for a system with multiple states will be denoted by $\psi(\mathbf{X})$. We assume that the component and system states belong to the same set $\mathcal{S}$; e.g. $X_i \in \mathcal{S}$ and $\phi(\mathbf{X}) \in \mathcal{S}$. However, it is possible that the X_i's belong to [0,1] whereas $\phi(\mathbf{X})$ can only take values in $\{0,1\}$.

1.2 The Calculus of Binary Systems with Precise Classification

The following is an overview of the calculus of binary systems (Barlow & Proschan, 1975); we generalize this construction in Sections 3 and 4. Let $\mathcal{S} = \{0, 1\}$ with $X_i = 1$ (0) if component i functions (fails), $i = 1, \ldots, n$; similarly, $\phi(\mathbf{X}) : \mathcal{S}^n \to \mathcal{S}$ equals 1 (0) if the system functions (fails). ϕ is a binary coherent system if (1) ϕ is non-decreasing in each argument of $\mathbf{X}$, and (2) each component is relevant. Examples of binary coherent systems are a series system, a parallel redundant system, and a k-out-of-n system. The dual of a binary coherent system $\phi(\mathbf{X})$ is defined as $\phi^D(\mathbf{X}) = 1 - \phi(\mathbf{1} - \mathbf{X})$, where $\mathbf{1} - \mathbf{X} = (1 - X_1, 1 - X_2, \ldots, 1 - X_n)$. Any binary structure function ϕ with n components can be decomposed as $\phi(\mathbf{X}) = X_i\, \phi(1_i, \mathbf{X}) + (1 - X_i)\, \phi(0_i, \mathbf{X})$, for all $\mathbf{X}$, $i = 1, \ldots, n$; this is later referred to as the *pivotal decomposition*. The following notation, definitions and theorems are conventional (Barlow & Proschan, 1975):

$$\mathbf{X} \cdot \mathbf{Y} = (X_1 \cdot Y_1,\ X_2 \cdot Y_2,\ \ldots,\ X_n \cdot Y_n),$$
$$\mathbf{X} \amalg \mathbf{Y} = (X_1 \amalg Y_1,\ X_2 \amalg Y_2,\ \ldots,\ X_n \amalg Y_n),$$

where $X_i \amalg Y_i = 1 - (1 - X_i)(1 - Y_i), i = 1, 2, \ldots, n$.

THEOREM 1: *For any binary coherent system* $\phi, \phi_S(\mathbf{X}) \stackrel{\text{def}}{=} \prod_{i=1}^n X_i \leq \phi(\mathbf{X}) \leq \coprod_{i=1}^n X_i \stackrel{\text{def}}{=} \phi_P(\mathbf{X})$.

THEOREM 2: *For any binary coherent system* ϕ,

$$\phi(\mathbf{X} \amalg \mathbf{Y}) \geq \phi(\mathbf{X}) \amalg \phi(\mathbf{Y}) \tag{1}$$

and

$$\phi(\mathbf{X} \cdot \mathbf{Y}) \leq \phi(\mathbf{X}) \cdot \phi(\mathbf{Y}), \tag{2}$$

with equality holding in equation (1) *(equation* 2*) if and only if the structure function* ϕ *is* $\phi_P(\phi_S)$. *Proofs of Theorems 1 and 2 can be found in Barlow & Proschan* (1975).

1.3 Reliability of Binary Systems

Suppose that the X_i's are *exchangeable*, and that p_i is the *propensity* of X_i being 1; that is, $p_i = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n}$ [cf. Lindley & Singpurwalla (2002) or Spizzichino (2001)]. Then, conditional on p_i, our subjective probability that $X_i = 1$ is p_i, $i = 1, \ldots, n$. Unconditionally, $P(X_i = 1) = \int_0^1 p_i \pi(p_i) \mathrm{d}p_i = \mathrm{E}(p_i)$, where $\pi(p_i)$ encapsulates our uncertainty about the propensity p_i; i.e. $\pi(p_i)$ is our subjective probability of p_i. The notions of propensity and subjective probability are articulated in de Finetti's theorem on exchangeable Bernoulli sequences; see Lindley & Phillips (1976).

Much of the literature on the reliability of binary coherent systems is conditional on p_i. An exception is Lynn *et al*. (1998), in which the analysis is based on averaging out $p_1, \ldots, p_n$ with respect to a joint distribution.

Conditional on $\mathbf{p} = (p_1, \ldots, p_n)$, the reliability of the system is a function of $\mathbf{p}$, say $h(\mathbf{p})$, but only if the X_i's are (conditionally) independent; i.e. (1) given $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, X_i and X_j are independent, $\forall\, i \neq j$, and (2) given p_i, X_i is independent of p_j, $\forall\, j \neq i$. Consequently, $P(\phi(\mathbf{X}) = 1 \mid \mathbf{p}) = E(\phi(\mathbf{X}) \mid \mathbf{p}) = h(\mathbf{p})$.

Analogues of the pivotal decomposition and Theorems 1 and 2 follow, asserting that the reliability of any binary coherent system is bounded below (above) by that of a series (parallel) system, if the X_i's are conditionally (given $\mathbf{p}$) independent, and redundancy at the component level is superior to redundancy at the system level when the systems are connected in parallel; vice versa if in series; see Barlow & Proschan (1975).

1.4 Vagueness or Imprecision

For purposes of discussion, consider a generic element of $\mathcal{S} = [0, 1]$, say x. At any point, we may be able to inspect the system and declare that $\psi(\mathbf{X}) = x$. If we are able to place this x in a well-defined subset of $\mathcal{S}$, then we say that the states of the system can be *classified with precision*. There are scenarios, however, where the identification of a state can be done unambiguously, but the classification cannot; this is the case of *classification with 'vagueness'*.

In the context of coherent systems, vagueness is not synonymous with uncertainty of performance. Uncertainty of performance is lack of knowledge about the future state of the system, e.g. will the system be functioning 5 hours from now? Vagueness pertains to uncertainty about classification, i.e. an inability to place any outcome x in a subset of $\mathcal{S}$ because the boundaries of the subset cannot be sharply delineated. Some examples illustrate this point.

Suppose that $\mathcal{S} = \{0, 1, \ldots, 10\}$, with each element representing a state in which the system can exist, ranging from the ideal at 10, to the undesirable at 0. Then what is the subset of 'good states' in $\mathcal{S}$? This subset is not well defined; for example, is 7 a good state? If $\mathcal{S}$ were to be partitioned into 'good' and 'bad' states, such partitioning being a feature of *natural language* (Zadeh, 1965), would 5 qualify as a good state or a bad state? More likely, 5 qualifies as both a good state and a bad state. Thus if $\psi(\mathbf{X}) = 5$, then the state of the system is simultaneously good and bad. As another scenario, consider an automobile that has 3000 miles on it. Should this automobile be classified as a 'new' or a 'used' car? The question of classification arises in the contexts of setting insurance rates, taxation and warranties. The subset of miles that go into classifying a car as being 'new' is not sharply defined; it is *imprecise*. Most cars sold as being new have anywhere from 20 to 100 miles—perhaps even more—on them. In actual practice, decisions are often made on the basis of vague knowledge that is relevant, e.g. decisions about health care, maintenance and replacements (see Section 6). As another illustration, medical treatments are based on classification of 'high blood pressure' or 'bad cholesterol,' and such classifications fluctuate due to the subjectivity of interpretation between 'good' and 'bad'. The philosopher Black (1939) gives examples from other sciences. Of historical note is the famous example of Schrodinger's Cat [cf. Pagels (1982), p. 125] from quantum physics. Schrodinger's thought experiment pertains to a cat in a sealed radioactive box in outer space which, according to one school of thought, is simultaneously alive and dead. Examples from the statistical sciences wherein vague knowledge is relevant are most likely to arise from the behavioral and social contexts, such as inferences based on political polling, and medical decisions based on a quality of life questionnaire (Cox *et al*., 1992), wherein responses almost always tend to be vague.

The existing theory of both binary and multistate coherent systems with precise classification as its underlying premise is unable to deal with the types of scenarios mentioned above. Some other concerns have been voiced by Marshall (1994). One idea, namely to classify states by more than one criterion, precedes ours and we applaud him for this foresight; it makes a case for the viewpoint espoused here.

1.5 Overview of Paper

In Section 2, we give a synopsis of many-valued logic to include its connectives of negation, conjunction, disjunction, implication, and equivalence. In Section 3, we extend the material of Section 1.2 to the case of multistate systems; i.e. for those components and systems where $\mathcal{S}$ consists of more than two elements. Here, we invoke Lukasiewicz's (1930) many-valued logic to define the structure function of multistate systems, and arrive upon results that are in agreement with those currently available. The material of Section 3 serves two purposes. One, it shows how many-valued logic provides a common platform via which the material on multistate systems can be seen. Second, it sets the stage for developing the material of Sections 4 and 5, which is entirely new. A use of many-valued logic is unlike that used by Baxter (1984), El-Neweihi *et al*. (1978) and Griffith (1980), whose development centres around binary logic.

Sections 4 and 5 pertain to the scenarios wherein the classification of component and system states is vague. In both sections, $\mathcal{S}$ consists of two vague subsets, and these serve as an analogue to binary state systems with precise classification. A key tool here is the 'consistency profile' introduced by Black (1939). Zadeh's (1965) 'membership function' parallels the notion of a consistency profile. The harnessing of Lukasiewicz's many-valued logic with Black's consistency profile provides a vehicle for the treatment of vague coherent systems. To do so, however, we need to lean on aspects of statistical inference and the statistical treatment of expert testimonies.

Section 6 relates the material of Sections 4 and 5 to decision making in maintenance management using natural language. Section 7 concludes the paper.

2 Many-valued Logic: An Overview

Binary logic, upon whose foundation the theory of coherent structures has been developed, pertains to propositions that adhere to the 'Law of Bivalence' (or the 'Law of the Excluded Middle'): all propositions are either true or false. Lukasiewicz (1930) recognized the existence of propositions that can be both true and false simultaneously, and thus modified the calculus of binary propositions to develop a calculus of three-valued propositions. Alternatives exist to Lukasiewicz's three-valued logic; however, for us, Lukasiewicz's proposal is most appealing.

It is important to distinguish between the calculus of probability and the calculus of three-valued logic. Probability pertains to the quantification of uncertainty about events (or propositions) that adhere to the Law of Bivalence. Thus we have, as a part of the calculus of probability, the axiom of additivity. On the other hand, the calculus of many-valued logic is based on a rejection of the Law of Bivalence. The two are therefore different constructs.

Table 1

(a) *Truth Table for Lukasiewicz's $Y \wedge Z$.*

$Y \wedge Z$		Values of Proposition Z		
		0	1/2	1
Values of Proposition Y	0	0	0	0
	1/2	0	1/2	1/2
	1	0	1/2	1

(b) *Truth Table for Lukasiewicz's $Y \vee Z$.*

$Y \vee Z$		Values of Proposition Z		
		0	1/2	1
Values of Proposition Y	0	0	1/2	1
	1/2	1/2	1/2	1
	1	1	1	1

Consider two propositions Y and Z, each taking one of three values: 0, $\frac{1}{2}$ and 1. The negation of Y is $Y' = 1 - Y$, as proposed by Lukasiewicz (1930). When the proposition Y takes the value 1 (0) in a truth table, it signals the fact that the proposition is true (false) with certainty. Values of Y intermediate to 1 and 0 signal an uncertainty about the truth or the falsity of Y. The value $\frac{1}{2}$ is chosen arbitrarily for convenience; any value between 0 and 1 could have been chosen. The other logical connectives in the three-valued logic of Lukasiewicz are conjunction, disjunction, implication and equivalence, denoted $(Y \wedge Z)$, $(Y \vee Z)$, $(Y \rightarrow Z)$ and $(Y \equiv Z)$, respectively. The truth tables for the first two are given in Table 1, and we refer the interested reader to Malinowski (1993) for further details. Generalizations from the three-valued to the many-valued case to incorporate propositions that are true or false with various degrees of uncertainty are straightforward.

3 Invoking Many-Valued Logic for Multistate Systems

3.1 Introduction

The aim of this section is to generalize the case of binary systems with precise classification to systems that can exist in multiple ($m + 1$ with $m > 1$) states. The states are labeled $\frac{j}{m}$, $j = 0, 1, 2, \ldots, m$, with 1 representing a perfect state and 0, the state of total collapse. The intermittent states of degradation range from $\frac{m-1}{m}$ to $\frac{1}{m}$, where $\frac{1}{m}$ is the state which is penultimate to the total failure of the system. Thus, the range of states now takes the form $\mathcal{S} = \{\frac{j}{m}; j = 0, 1, 2, \ldots, m\}$ and, by allowing m to be infinite, we are able to consider a continuum of degraded states, in which case, $\mathcal{S} \subseteq [0, 1]$. With $\mathcal{S}$ so defined for both the components and the system, what would be the meaningful choices for the structure function when the system has a series, parallel, or k-out-of-n architecture?

In the past, several proposed definitions of multistate systems have been made. An overview of these is in El-Neweihi *et al*. (1978) and in Baxter (1984), which to the best of our knowledge represents the latest endeavors. Considering the fact that these papers appeared over 20 years ago, one may sense that a satisfactory answer to the above question is available. This may not be true, however, because all the proposed approaches reduce to a representation in terms of binary states and, thus, an adherence to binary logic. As an example, Baxter (1984), following Barlow & Proschan (1975), defines the structure function of a multistate system in terms of the system's 'min-path' and 'min-cut' sets, notions which can have an interpretation only within the context of binary systems. By contrast, our proposal here is to use Lukasiewicz's many-valued logic as a basis for defining the structure function of multistate systems.

Lukasiewicz's motivation for introducing a third value, namely $\frac{1}{2}$, and his calculus of three-valued logic was prompted by an uncertainty about the truth or the falsity of a proposition. The number $\frac{1}{2}$ did not reflect—in any sense—a degree of uncertainty. Whereas Lukasiewicz did not appear to have any motivation for his many-valued logic other than the need to generalize, the degree of uncertainty interpretation provides a vehicle for extending the three-valued logic. With this in mind, we may ask whether Lukasiewicz's calculus can be directly imported to the scenario of multistate systems when the degraded states can be specified with precision? Our examples of Table 1 illustrating the three-valued logic suggest that this can be done. More importantly, our results are consistent with those given in El-Neweihi *et al.* (1978). Consequently, the Lukasiewicz logic can be seen as providing a rationale for the existing results on multistate systems, a rationale that has been missing.

3.2 Definition and Structural Properties

Let X_i denote the state of component $i, i = 1, \ldots, n$, and $\psi = \psi(\mathbf{X})$ the state of the multistate system ; $\mathbf{X} = (X_1, \ldots, X_n)$. The X_i's and $\psi(\mathbf{X})$ take values in $\mathcal{S} = \{\frac{j}{m}, j = 0, 1, \ldots, m\}$.

Definition 1: (Griffith, 1980) ψ is a multistate coherent system if

1. ψ is non-decreasing in each argument of $\mathbf{X}$,
2. for each $i = 1, 2, \ldots, n$, there exist states $0 \leq a_i < b_i \leq m$ and a state vector $(\bullet_i, \mathbf{X})$ such that
$$\psi\left(\frac{a_i}{m}, \mathbf{X}\right) < \psi\left(\frac{b_i}{m}, \mathbf{X}\right);$$
that is, each component is relevant, and
3. $\psi(\frac{\boldsymbol{j}}{\boldsymbol{m}}) = \frac{j}{m}$ where $\frac{\boldsymbol{j}}{\boldsymbol{m}} = (\frac{j}{m}, \frac{j}{m}, \ldots, \frac{j}{m})$.

Properties 1 and 3 of Definition 1 are consistent with those of Barlow & Wu (1978), El-Neweihi *et al.* (1978) and Natvig (1982). Property 2 generalizes the notion of relevance.

To use the logic of many-valued propositions for multistate systems, it is necessary to order the state vector $\mathbf{X}$. Since each $X_i \in \{\frac{j}{m}, j = 0, 1, \ldots, m\}$, we order the X_i's by the values they take. Specifically, let $0 \leq X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(i:n)} \leq \cdots \leq X_{(n:n)} \leq 1$ denote the ordered vectors, i.e. $X_{(1:n)}$ is the weakest of all the n components and $X_{(n:n)}$ the strongest. Consequently, from Table 1(a), the structure function of a series system is $\psi_S = \min_i X_i = X_{(1:n)}$; that is, the performance of a multistate series system is no better than the performance of its weakest component. If $n = 2$, and if each X_i can take only three values $\{0, \frac{1}{2}, 1\}$ with $\frac{1}{2}$ denoting the degraded state, then Table 1(a) with $Y \wedge Z$ replaced by $\psi_S(\mathbf{X})$ and Y (Z) replaced by $X_1(X_2)$ gives us a table for the states of the system, given the states of the components. Figure 1(a) displays the state of $\phi_S(\mathbf{X}) = \phi_S(X_1, X_2)$ when X_1 and X_2 take binary values, 0 and 1. In contrast, Figure 1(b) shows the behaviour of $\psi_S(\mathbf{X})$ when X_1 and X_2 are allowed to take all values in the unit interval, showing the effect of continuously degrading components on the structure function. Clearly, $\psi_S(\mathbf{X})$ provides more granularity than $\phi_S(\mathbf{X})$.

For a parallel redundant system, $\psi_P(\mathbf{X}) = \max_i X_i = X_{(n:n)}$; see Table 1(b). This suggests that the performance of a multistate parallel system is no worse than the performance of its strongest component. In the three-valued case, the entries of Table 1(b) provide us with a table for the states of the system given the states of the components, when $n = 2$. The state of $\phi_P(\mathbf{X})$ when X_1 and X_2 take binary values, 0 and 1, is displayed in Figure 2(a). In contrast, Figure 2(b) shows the behaviour of $\psi_P(\mathbf{X})$ when X_1 and X_2 take all values in [0,1]. Again, $\psi_P(\mathbf{X})$ provides more granularity than $\phi_P(\mathbf{X})$.

For multistate k-out-of-n systems, we define $\psi_K(\mathbf{X}) = X_{(n-k+1:n)}$; this definition ensures consistency among systems, i.e. n-out-of-n systems are denoted $\psi_S(\mathbf{X})$ and 1-out-of-n systems are denoted $\psi_P(\mathbf{X})$. Interestingly, our set-up and definition of a multistate coherent system

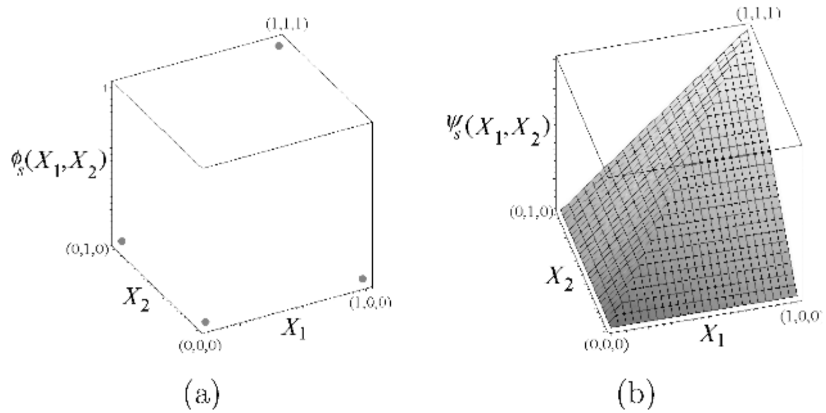

Figure 1. *(a) Two-component binary system, $\phi_S(\mathbf{X})$. (b) Two-component system, $\psi_S(\mathbf{X})$, with continuously degrading components. The coordinates are labeled $(X_1, X_2, \phi_S(\mathbf{X}))$ and $(X_1, X_2, \psi_S(\mathbf{X}))$, respectively.*

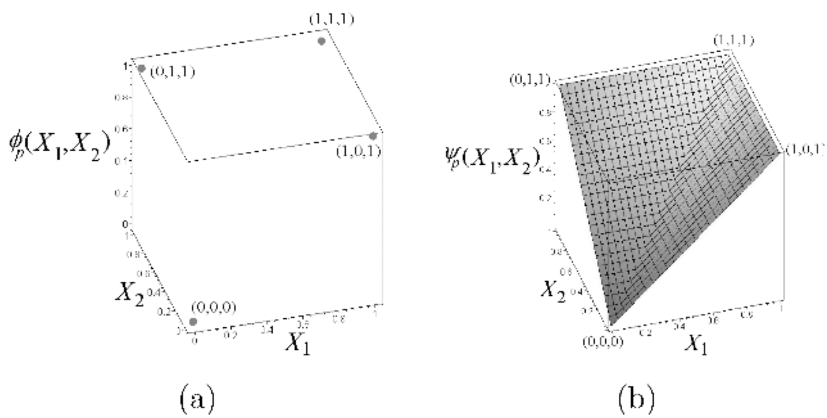

Figure 2. *(a) Two-component binary system, $\phi_P(\mathbf{X})$. (b) Two-component system, $\psi_P(\mathbf{X})$, with continuously degrading components. The coordinates are labeled $(X_1, X_2, \phi_P(\mathbf{X}))$ and $(X_1, X_2, \psi_P(\mathbf{X}))$*, respectively.

permits the definition of a dual of a binary coherent system to hold. The dual of a k-out-of-n system is $\psi_K^D(\mathbf{X}) = \psi_{(n-k+1:n)}(\mathbf{X})$, an $(n-k+1)$-out-of-n system.

In Lemma 1, the pivotal decomposition for binary structure functions is generalized for $(m+1)$ precise categories through consideration of their associated indicator variables.

LEMMA 1: *The following identity holds for every n-component multistate structure function ψ with precise classification: $\psi(\mathbf{X}) = \sum_{j=0}^{m} \psi[(\frac{j}{m})_i, \mathbf{X}] I_{[X_i=\frac{j}{m}]}$, for $i = 1, \dots, n$ where $I_{[X_i=\frac{j}{m}]} =$* $1(0)$ if $X_i = \frac{j}{m} (X_i \neq \frac{j}{m})$.

Proof. Any multistate structure function, $\psi(\mathbf{X})$ can be decomposed into a representation that considers the i-th component separately from the remaining $(n-1)$ components. In particular for the multistate component, X_i takes only one value from $\{0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}, 1\}$. The result follows.

Theorems 1 and 2 of Section 1 can be generalized for multistate coherent systems. To do so, we introduce the following additional notation. For $\mathbf{X} = \{X_1, \dots, X_n\}$ and $\mathbf{Y} = \{Y_1, \dots, Y_n\}$, $\mathbf{X} \leq \mathbf{Y}$ if $X_i \leq Y_i$ for each $i = 1, \dots, n$. As a generalization of Theorem 1, we have:

THEOREM 3: *Let ψ be a multistate coherent system of order n; i.e. ψ has n components. Then* $X_{(1:n)} \leq \psi(\mathbf{X}) \leq X_{(n:n)}$.

THEOREM 4: *Let ψ be a multistate coherent system of order n. Then*

$$\psi(\mathbf{X} \vee \mathbf{Y}) \geq \psi(\mathbf{X}) \vee \psi(\mathbf{Y}), \tag{3}$$

and

$$\psi(\mathbf{X} \wedge \mathbf{Y}) \leq \psi(\mathbf{X}) \wedge \psi(\mathbf{Y}). \tag{4}$$

The equality in (3) and (4) hold for all $\mathbf{X}$ and $\mathbf{Y}$ if and only if the system's architecture is parallel and series, respectively.

Thus, for a multistate coherent system, equation (3) reiterates the result that, structurally, component-level redundancy is superior to system level redundancy, and vice versa in equation (4). Theorems 3 and 4 and Lemma 1 are also in El-Neweihi & Proschan (1984). They are stated here for completeness.

Since $X_{(1:n)} = \psi_S(\mathbf{X})$ and $X_{(n:n)} = \psi_P(\mathbf{X})$, we have the result that the structure function of any multistate coherent structure is bounded by the structure functions of multistate series and parallel systems.

3.3 Multistate System Reliability under Precise Classification

Suppose that the component state vectors $X_1, \ldots, X_n$ are (conditionally) independent and identically distributed with $P(X_i = \frac{j}{m} \mid \tilde{p}_{j+1}) = \tilde{p}_{j+1}$, for $i = 1, \ldots, n$ and $j = 0, \ldots, m$, where $\tilde{p}_{j+1} \geq 0$ and $\sum_{j=0}^{m} \tilde{p}_{j+1} = 1$. That is, each X_i has a multinomial distribution over $\{\frac{j}{m}; j = 0, 1, 2, \ldots, m\}$ with parameter $\tilde{p}_{j+1}$, $j = 0, \ldots, m$. Let $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_{m+1})$. Clearly for each j, $P(\psi(\mathbf{X}) = \frac{j}{m})$ depends on $\tilde{\mathbf{p}}$ alone, since the X_i's are assumed to be conditionally (given $\tilde{\mathbf{p}}$) independent. Thus, we let $P(\psi(\mathbf{X}) = \frac{j}{m} \mid \tilde{\mathbf{p}}) = h_j(\tilde{\mathbf{p}})$, where h_j is some function of $\tilde{\mathbf{p}}$. Suppose that the architecture of ψ is a $(n-k+1)$-out-of-n system. Then

$$\begin{aligned} h_j(\tilde{\mathbf{p}}) &= P\left(\psi_{n-k+1}(\mathbf{X}) = \frac{j}{m} \middle| \tilde{\mathbf{p}}\right) \\ &= \sum_{a=k}^{n} \binom{n}{a} \left\{ \left(\sum_{b=1}^{j+1} \tilde{p}_b\right)^a \left(\sum_{b=j+2}^{m+1} \tilde{p}_b\right)^{n-a} - \left(\sum_{b=1}^{j} \tilde{p}_b\right)^a \left(\sum_{b=j+1}^{m+1} \tilde{p}_b\right)^{n-a} \right\}. \end{aligned}$$

Example 1: Let $m, n = 2$. Therefore, we consider a two-component system with three possible states: total failure (0), degradation ($\frac{1}{2}$), and perfect functioning (1), with associated probabilities $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}_3$, respectively. Then, the probability that the parallel system is totally failed is $h_0(\tilde{\mathbf{p}}) = P(\psi_P(\mathbf{X}) = 0 \mid \tilde{\mathbf{p}}) = \tilde{p}_1^2$. i.e. the parallel system is totally failed when all its components are totally failed. The probability that a series system totally fails is $h_0(\tilde{\mathbf{p}}) = P(\psi_S(\mathbf{X}) = 0 \mid \tilde{\mathbf{p}}) = 2\tilde{p}_1\tilde{p}_2 + 2\tilde{p}_1\tilde{p}_3 + \tilde{p}_1^2$; thus, a series system fails completely when at least one component is totally failed.

When $X_1, \ldots, X_n$ are independent but not identically distributed, we may generalize the above properties by introducing $P(X_i = \frac{j}{m} \mid p_{i_{j+1}}) = p_{i_{j+1}}$, $j = 0, \ldots, m$ where for each i, $p_{i_{j+1}} \geq 0$ and $\sum_{j=0}^{m} p_{i_{j+1}} = 1$. We define $\mathbf{p_i} = (p_{i_1}, \ldots, p_{i_{m+1}})$ to be the reliability vector associated with the i-th component and $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$. Given the conditional independence of the X_i's, a

$(n-k+1)$-out-of-n system has

$$h_j(\mathbf{p}) = P\left(\psi_{n-k+1}(\mathbf{X}) = \frac{j}{m} \,\middle|\, \mathbf{p}\right)$$
$$= \sum_a \left(\prod_{i\in J_a}\sum_{b=1}^{j+1} p_{i_b}\right)\left(\prod_{i\in J'_a}\sum_{b=j+2}^{m+1} p_{i_b}\right) - \sum_a \left(\prod_{i\in(J-1)_a}\sum_{b=1}^{j} p_{i_b}\right)\left(\prod_{i\in(J-1)'_a}\sum_{b=j+1}^{m+1} p_{i_b}\right),$$

where J_a is the subset of $(1, 2, \ldots, n)$ where at least k components are performing within level $\frac{j}{m}$ and J'_a is the complement of J_a. Similarly, $(J-1)_a$ is the subset of $(1, 2, \ldots, n)$ where at least k components function within level $\frac{j-1}{m}$ and $(J-1)'_a{}'_a$ is the complement of $(J-1)_a$.

Lemma 2 provides the pivotal decomposition for the reliability function, $h_j(\mathbf{p})$.

LEMMA 2: *The following identity holds for the pivotal decomposition of* $h_j(\mathbf{p})$:

$$h_j(\mathbf{p}) = \sum_{a=0}^{m} h_j\left[\left(\frac{a}{m}\right)_i, \mathbf{p}\right]\cdot p_{i_{a+1}}, \quad for\ j = 0, \ldots, m; i = 1, \ldots, n, \tag{5}$$

where $h_j[(\frac{a}{m})_i, \mathbf{p}] = P(\psi(\mathbf{X}) = \frac{j}{m} \mid X_i = \frac{a}{m}, \mathbf{p})$.

Proof. Follows from the Law of Total Probability.

4 Components with Imprecise State Classification

Binary state systems with precise classification were overviewed in Section 1.2, and the concept of vagueness introduced in Section 1.4. Sections 4 and 5 serve to combine these two notions to develop a mechanism for the treatment of vague coherent systems, with Section 4 devoted to the case of components in vague states, and Section 5 to the case of coherent systems in vague states.

The terms ‘coherence’ and ‘vagueness’ may seem contradictory; however, they do not pertain to the same object. The first is associated with the truth values of logical connectives, whereas the second pertains to the partitioning of a set into subsets. We start with some background on vagueness and then discuss approaches for quantifying it.

4.1 Vagueness: General Background

Vagueness has been discussed by philosophers like Bertrand Russell, and by physicists like Albert Einstein. To Russell (1923), ‘all language is more or less vague’ so that the Law of the Excluded Middle ‘is true when precise symbols are employed but it is not true when symbols are vague, as, in fact, all symbols are.’ Black (1939) recognized the inability of binary logic to satisfactorily represent propositions that are neither perfectly true nor false. He attempted to rectify this by analyzing the concept of vagueness in order to establish an ‘appropriate symbolism’ by which binary logic can be viewed as a special case. Unlike Lukasiewicz (1930), who was also concerned about the Law of the Excluded Middle, Black did not introduce three-valued propositions. Rather, he defined a vague proposition as one where the possible states of the proposition are not clearly defined with respect to inclusion, and introduced the mechanism of ‘consistency profiles’ as a way of treating vagueness. Black’s consistency profile is a graphical portrayal of the degree of membership of some proposition in a set of imprecisely defined states, with 1 representing absolute membership in a state and 0 an absolute lack of membership. Precise propositions are treated via step functions as consistency profiles, and vague propositions

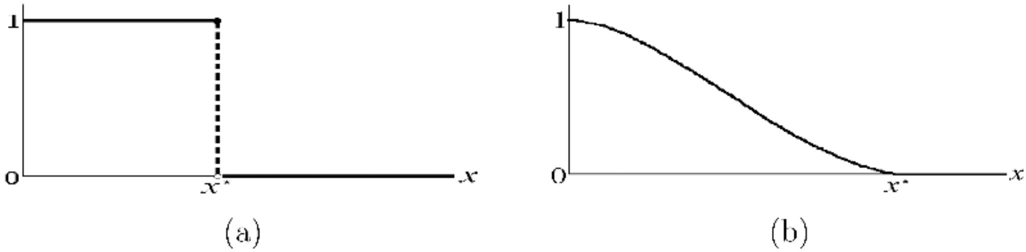

Figure 3. *Example of Consistency Profiles: (a) for a precise set. (b) for a vague set. The consistency profile is 0 after x^*.*

Table 2
Membership table for precise set, A_1, versus fuzzy set, A_2.

x	1	2	3	4	5	6	7	8	9	10
$\mu_{A_1}(x)$	0	0	0	0	0	0	1	1	1	1
$\mu_{A_2}(x)$	0	0	0	0	0	0.2	0.5	0.9	1	1

by consistency profiles that tend gradually from one extreme to another; see Figure 3. The scaling between 0 and 1 is arbitrary; other convenient limits could have been used. Further, the consistency profile which is specified by an individual, or a group of individuals, need not be unique.

4.2 Membership Functions and Probabilities of Fuzzy Sets

Black's (1939) consistency profile precedes Zadeh's (1965) membership function. For each x, a normalized membership function $0 \le \mu_A(x) \le 1$ describes a belief of containment of x in a set A. When $\mu_A(x) = 1$ or 0, A is a crisp (or precise) set; when $0 \le \mu_A(x) \le 1$, A is a *fuzzy set*. To illustrate the concept of a fuzzy set, consider

Example 2: Let $A_1 = \{x \in \{1, 2, \ldots, 10\} \mid x \ge 7\}$. For any specified x, there is no ambiguity as to whether x belongs to A_1 or not. By definition, $\mu_{A_1}(x) = 1$ when $x = 7$, 8, 9, or 10; otherwise, it is zero (see Table 2). Thus A_1 is a precise set, since $\mu_{A_1}(x) = 1$ or 0. By contrast, consider the set $A_2 = \{x \in \{1, 2, \ldots, 10\} \mid x \text{ is large}\}$. The term 'large' is vague; thus, we cannot precisely ascertain the containment of any x in A_2. A possible membership function for A_2, $\mu_{A_2}(x)$, is given in Table 2; this assignment is not unique.

For fuzzy sets, A and B in a basic set M, with membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively, Zadeh (1965) defined set operations that parallel those of precise sets. For any x in a given basic set M,

1. $\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$,
2. $\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)]$,
3. $\mu_{A'}(x) = 1 - \mu_A(x)$,
4. $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$, and
5. $A \equiv B \Leftrightarrow \mu_A(x) = \mu_B(x)$.

Thus, the union of fuzzy sets A and B is the fuzzy set $A \cup B$, whose membership function is max $[\mu_A(x), \mu_B(x)]$; similarly for the intersection and the complement. There is a parallel between operations with fuzzy sets and the conjunction and disjunction connectives of Lukasiewicz (1930). In Section 5.1, we use these operations to define structure functions of vague binary state systems. Thus, we claim that Lukasiewicz's logic provides a unifying framework via which both multistate as well as vague systems can be studied.

4.2.1 Probabilities of fuzzy sets

In the context of this paper, statistical inference plays a key role. This role comes into effect when we endow a probability measure for a fuzzy set, say A. There are two key ideas that drive this development, namely that (1) vague sets are a consequence of one's uncertainty about the boundaries of sharp sets, and (2) the membership function $\mu_A(x)$ is to be interpreted as data (or information) whose role is to help induce a likelihood function, just like the role of an observation in traditional statistical inference. The above ideas can be best exposited by envisioning the scenario of expert testimonies and information integration that has gained current popularity in statistical practice (cf. Reese *et al*. 2004).

Accordingly, we consider the actions of D, an assessor of probabilities (or a decision maker), who quantifies his (her) uncertainty about any outcome of X, say x, being classified in A via a prior probability $\pi_D(x \in A)$. The thesis here is that all uncertainties, including those of classification, be quantified via probability. In order to sharpen the prior probability, D consults an expert, say Z, and elicits from Z a membership function $\mu_A(x)$. This $\mu_A(x)$ can be seen as additional information about the nature of x's membership in A, and de facto serves a role analogous to that of observed data in statistical inference about outcomes. In essence, observed data are evidence about outcomes whereas membership functions are evidence about classification. In principle, D may consult several experts and elicit from each membership functions as a way to further sharpen the analysis.

With $\mu_A(x)$ at hand, D constructs his (her) likelihood function that $x \in A$; we denote this likelihood by $\mathcal{L}[x \in A; \mu_A(x)]$. The construction of this likelihood follows standard statistical procedures for formally incorporating expert testimonies, and should include things such as D's view of the expertise of Z and, in the case of several experts, correlations between them (cf. Lindley, 1991; Clarotti & Lindley, 1988). Since $\mathcal{L}[x \in A; \mu_A(x)]$ is D's likelihood that Z declares $\mu_A(x)$ when $x \in A$, the specification of this likelihood is a subjective exercise on the part of D. Conventionally, in statistical inference, likelihoods for unknown parameters are prescribed via probability models (for outcomes) using the observed data as fixed quantities. By contrast, what we have done here is prescribed a likelihood about classification using the membership as a fixed entity, but without the benefit of a probability model. In so doing, we have interpreted the likelihood in a broader sense, namely as a weighting function (Basu, 1975). In addition to $\mathcal{L}[x \in A; \mu_A(x)]$, D also needs to specify $\mathcal{L}[x \notin A; \mu_A(x)]$, which is D's likelihood that $x \notin A$ when Z declares a $\mu_A(x)$, and $P_D(x)$ which is D's subjective probability that an outcome x will occur. Thus D needs to specify two probability measures $\pi_D(x)$ and $\pi_D(x \in A)$, one for outcomes and one for classification, and two likelihoods, $\mathcal{L}[x \in A; \mu_A(x)]$ and $\mathcal{L}[x \notin A; \mu_A(x)]$.

With the above in place, D uses standard statistical methodology involving Bayes' Law, Bayes' Factors, and prior to posterior odds (cf. Kass, 1993) to obtain a probability measure for a fuzzy set A (cf. Singpurwalla & Booker, 2004) as

$$P_D[X \in A; \mu_A(x)] = \sum_x \left[1 + \frac{\mathcal{L}[x \notin A; \mu_A(x)]}{\mathcal{L} \in A; \mu_A(x)} \cdot \frac{\pi_D(x \notin A)}{\pi_D(x \in A)}\right]^{-1} P_D(x). \tag{6}$$

Equation (6) above is the essence of the material of this section; it is to play a key role in what is to follow. In obtaining the above, we have leaned heavily on the statistical notion of likelihood and the likelihood ratio. Equation (6) simplifies if D chooses to use Z's declared $\mu_A(x)$ as the sole basis for constructing his (her) likelihood, so that $\mathcal{L}[x \in A; \mu_A(x)] = \mu_A(x)$, and $\mathcal{L}[x \notin A; \mu_A(x)] = 1 - \mu_A(x)$. In this case,

$$P_D[X \in A; \mu_A(x)] = \sum_x \left[1 - \left(1 - \frac{1}{\mu_A(x)}\right) \cdot \frac{\pi_D(x \notin A)}{\pi_D(x \in A)}\right]^{-1} P_D(x). \tag{7}$$

4.2.2 *The role of precise and fuzzy data in vague systems*

In equations (6) and (7), $P_D(x)$ encapsulates D's prior uncertainty about an outcome x. Were D to have at his (her) disposal $\mathbf{x} = (x_1, \ldots, x_n)$, data on X, then $P_D(x)$ would get replaced by a posterior probability, say $P_D(x; \mathbf{x})$. The calculation of this posterior would be a routine exercise were D to invoke a probability model for outcomes, and were the actual observations $x_1, \ldots, x_n$ sharp (i.e. precisely stated). What must D do to update $P_D(x)$ if the data $\mathbf{x}$ is itself fuzzy?

To address this question, we first need to clarify as to what one means by *fuzzy data*, a term that has appeared in several book and article titles; see, for example Bertoluzza *et al*. (2002), and Viertl (2006). If by fuzzy data, we mean imprecision of observation (i.e. observation error), then the treatment of such data can be routinely handled via standard statistical technology, provided that an error distribution can be specified. The literature on 'calibration' adequately deals with this issue; see, for example, Huang (2002). If by fuzzy data, we mean a statement such as 'the outcome does or does not belong to the fuzzy set A', then the incorporation of such information for updating $P_D(x)$ is no more a standard matter. In other words, when the actual value taken by X, say x_i, is not declared, but what is declared is whether the actual value belongs or not to A, an assessment of $P_D(x$; observed value belongs (does not belong) to A) poses a challenge. This can be addressed if a likelihood for $X = x_i$ with the knowledge that the 'observed value belongs (does not belong) to A' can be specified by D. The specification of such a likelihood will entail several issues such as who provides D the said knowledge, Z or someone other than Z. If it is Z, then $\mu_A(x)$ provides some guidance to D about specifying the likelihood. If it is someone other than Z, then D needs to contemplate the knowledge provider's actions. These and other issues remain to be addressed, including the matter of calibrating Z and updating membership functions.

4.3 *Components in Vague Binary States*

The notion that units can exist in states that are vaguely defined was introduced in Section 1.4. Specifically, let X denote the state of a component at some time $\tau > 0$, and let X take values in $\mathcal{S} = \{x; 0 \leq x \leq 1\}$, with one representing the perfectly functioning state. Consider $\mathcal{G} \subset \mathcal{S}$, where $\mathcal{G} = \{x; x$ is a 'desirable' state$\}$. Suppose that interest centres around $X \in \mathcal{G}$. Suppose also that we are unable to specify an x^* such that $X \geq x^*$ implies that $X \in \mathcal{G}$ and, otherwise, $X \notin \mathcal{G}$. Thus, the boundary of $\mathcal{G}$ is not sharp; i.e. $\mathcal{G}$ is a fuzzy set. Let $\mu_{\mathcal{G}}(x)$ be the membership function of $\mathcal{G}$. Figure 4 illustrates plausible forms for $\mu_{\mathcal{G}}(x)$. Interest may centre around $\mathcal{G}$ for several reasons, a relevant one being a desire to use 'natural language' for communication with others on matters such as repair and replacement. Another possibility is that it may not be possible to observe the actual value of x, but one may be able to make a general statement about the state of the component.

The complement of $\mathcal{G}$, say $\mathcal{G}^C$, is that fuzzy set whose membership function is $1 - \mu_{\mathcal{G}}(x)$. It is important to note that, if another subset $\mathcal{B} \subset \mathcal{S}$ was defined as $\mathcal{B} = \{x; x$ is an 'undesirable' state$\}$, then $\mathcal{G}^C$ may or may not be $\mathcal{B}$ unless $\mu_{\mathcal{B}}(x)$, the membership function of $\mathcal{B}$, was such that $\mu_{\mathcal{B}}(x) = 1 - \mu_{\mathcal{G}}(x)$. In principle, one is free to choose a $\mu_{\mathcal{B}}(x)$ that need not bear a relationship to $\mu_{\mathcal{G}}(x)$. For example, in Figure 4(a), $\mu_{\mathcal{B}}(x)$ is symmetric to $\mu_{\mathcal{G}}(x)$, whereas in Figure 4(b), $\mu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{G}}(x)$ are not symmetric. There is precedent in the statistical sciences for choosing asymmetric likelihood functions. For example, one need not specify likelihood functions that are symmetrical for competing hypotheses.

Example 3: An assessor D wants to assess the probability that a component will be in a 'desirable' state $\mathcal{G}$ at some future time τ. That is, D wishes to specify $P_D[X \in \mathcal{G}; \mu_{\mathcal{G}}(x)]$, where a membership function of the form $\mu_{\mathcal{G}}(x) = x^4$, $0 \leq x \leq 1$ has been elicited by D from an expert,

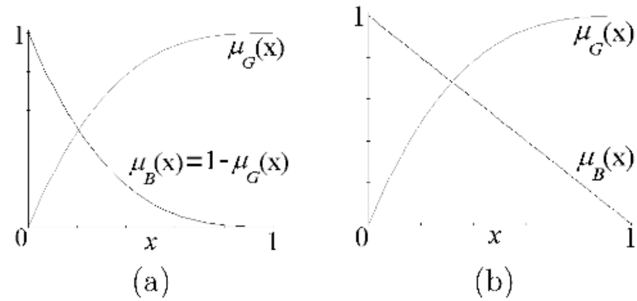

Figure 4. *Membership functions of $\mathcal{G}$ and $\mathcal{B}$: (a) Symmetric case. (b) Asymmetric case.*

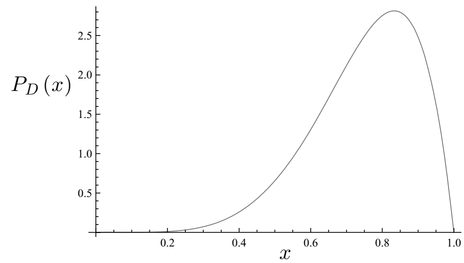

Figure 5. *Component state at time τ, $P_D(x)$.*

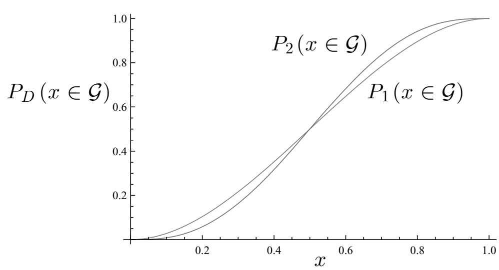

Figure 6. *Two possible prior forms of classifying $X = x$, $P_1(x \in \mathcal{G})$ and $P_2(x \in \mathcal{G})$, supplied by the assessor D.*

Z. Suppose that $P_D(x)$, D's personal probability that the state of the component at time τ will be x is of the form given in Figure 5; it is a Beta(6,2) density. Furthermore, suppose that D's belief that nature will classify any x in $\mathcal{G}$, namely $P_D(x \in \mathcal{G})$, is of the general form illustrated in Figure 6 with the label, $P_1(X \in \mathcal{G})$. Then, it can be seen—via equation (7)—that $P_D[X \in \mathcal{G}; \mu_{\mathcal{G}}(x)] = 0.6605$. As a consequence, $P_D[X \notin \mathcal{G}; \mu_{\mathcal{G}}(x)] = 1 - 0.6605 = 0.3395$. By contrast, suppose now that, if D were to specify $P_D(x \in \mathcal{G})$ via the label $P_2(x \in \mathcal{G})$ of Figure 6 and keep everything else the same; then $P_D[X \in \mathcal{G}; \mu_{\mathcal{G}}(x)]$ would increase to 0.7486. Thus, even a small change in the form of $P_D(x \in \mathcal{G})$ produces a noticeable change in D's final answer.

4.4 Reliability of Components in Vague Binary States

We say that a component's state is 'vague and binary' if interest centres around a single vague set of the kind $\mathcal{G}$ or $\mathcal{B}$ in our illustrations. As was mentioned before, we should bear in mind that, in general, $\mathcal{G}^C$ need not be $\mathcal{B}$ and vice versa, unless of course $\mathcal{G}$ and $\mathcal{B}$ are precise sets. For $\mathcal{G} = \{x; x \text{ is a 'desirable' state}\}$ and $\mu_{\mathcal{G}}(x)$ specified, it is reasonable to define the *reliability* of the component as $P_D[X \in \mathcal{G}; \mu_{\mathcal{G}}(x)]$. Equation (6) can now be used to evaluate this probability. With $\mathcal{B} = \{x; x \text{ is an 'undesirable' state}\}$, and $\mu_{\mathcal{B}}(x)$ specified, we may define the *unreliability* of the component as $P_D[X \in \mathcal{B}; \mu_{\mathcal{B}}(x)]$. We could have also defined the unreliability of the component as $P_D[X \in \mathcal{G}^C; \mu_{\mathcal{G}}(x)]$, where $\mathcal{G}^C$ is that fuzzy set whose membership function equals $1 - \mu_{\mathcal{G}}(x)$. With either choice for the definition of unreliability, we see that, when a component's state is vague and binary, *its unreliability is not necessarily the complement of its reliability*! This result is in contrast to that of binary coherent systems.

Example 4: The case of components that can exist in precise binary states can be encompassed within the above framework; $\mu_{\mathcal{G}}(x) = 1$ for $x \geq x^*$ and $\pi_D(x \in \mathcal{G}) = 1$ if $x \geq x^*$, and zero otherwise. Furthermore, $\mathcal{B} = \mathcal{G}^C$, thus $P_D[X \in \mathcal{G}; \mu_{\mathcal{G}}(x)] = 1 - F_D(x^*)$ and $P_D[X \in \mathcal{B}; \mu_{\mathcal{G}}(x)] = P_D[X \notin \mathcal{G}; \mu_{\mathcal{G}}(x)] = F_D(x^*)$, where $F_D(x^*)$ is the cumulative distribution function (cdf) associated with $p_D(x)$ evaluated at x^*.

5 Binary State Systems with Imprecise Classification

The purpose of this section is to extend the development of Section 4.3 on binary state components with imprecise classification to the case of binary state, n-component systems with imprecise classification. By 'binary state systems with imprecise classification', we mean those systems whose component states are vague and binary, and whose structure functions satisfy the logical connectives of Lukasiewicz; see Section 2. Our motivation for choosing this as a definition of structure functions is that the structure functions of binary state coherent systems with precise classification are exactly the membership functions of certain precise sets. The case of multistate systems with imprecise classification, though not discussed here, follows by analogy.

5.1 Structure Functions as Membership Functions of Precise Sets

Let X_i be the state of component i taking a particular value x_i, $i = 1, \ldots, n$. Suppose that each X_i can take values in $\mathcal{S} = \{x;\ 0 \leq x \leq 1\}$. Let $\mathcal{G}_i = \{x_i;\ x_i \text{ is a 'desirable' state}\}$, $\mathcal{G}_i \subset \mathcal{S}$. Let $\mu_{\mathcal{G}_i}(x_i)$ denote the membership function of $\mathcal{G}_i$, $i = 1, \ldots, n$. For now, suppose that $\mathcal{G}_i$ is precise for all i. That is, for each i, there exists an x_i^* such that $\mu_{\mathcal{G}_i}(x_i) = 1(0)$ when $x_i \geq x_i^*$ $(x_i < x_i^*)$. For ease of notation, this section focuses solely on the subspace $\mathcal{G}_i$; therefore, we use $\mu_i(x_i)$ to denote the representation of the above membership functions, with the understanding that the membership function assigned is dependent on the fuzzy classification, $\mathcal{G}_i$, which itself depends on component i. For the remainder of this paper, we let $\mathcal{L}[X \notin \mathcal{G}_i; \mu_i(x)] = 1 - \mu_i(x)$ and $\mathcal{L}[X \notin \mathcal{G}_{\phi(\mathbf{X})}; \mu_{\phi(\mathbf{X})}(x)] = 1 - \mu_{\phi(\mathbf{X})}(x)$, where $\phi(\mathbf{X})$ is as defined in Section 1.2.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ and suppose that the n components are in series. Thus the system's structure function is 1 if and only if $x_i \geq x_i^*$ for all $i = 1, \ldots, n$. However, $x_i \geq x_i^*$ implies that $\mu_i(x_i) = 1$ for each i. Thus we may write

$$\phi_S(\mathbf{X}) = \prod_{i=1}^{n} \mu_i(X_i) = \min_i[\mu_i(X_i)] \doteq \mu_{(1:n)}(\mathbf{X}), \tag{8}$$

where $\mu_{(1:n)}(\mathbf{X})$ is the membership function of the intersection of the n precise sets $\mathcal{G}_i$, $i = 1, \ldots, n$. Thus, the structure function of a series system with precise classification can also be interpreted as the membership function of the intersection of n precise sets. Similarly, if the n components were to be connected in parallel redundancy, then the structure function of the system would be

$$\phi_P(\mathbf{X}) = \coprod_{i=1}^{n} \mu_i(X_i) = \max_i[\mu_i(X_i)] \doteq \mu_{(n:n)}(\mathbf{X}), \tag{9}$$

which is the membership function of the union of $\mathcal{G}_i, i = 1, \ldots, n$. Finally, for a k-out-of-n system, we could write

$$\phi_K(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} \mu_i(X_i) \geq k \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Whereas the relationships of equations (8) and (9) have an interpretation within the calculus of fuzzy sets, equation (10) does not. Sums of membership functions are not a part of the calculus of fuzzy sets. We therefore seek an alternate way of expressing $\phi_K(\mathbf{X})$. We do this as follows.

Suppose that the $\mu_i(X_i)$ terms are relabeled so that $\mu_{(1:n)}(\mathbf{X})$ is the minimum and $\mu_{(n:n)}(\mathbf{X})$ is the maximum; i.e. $\mu_{(1:n)}(\mathbf{X}) \leq \mu_{(2:n)}(\mathbf{X}) \leq \cdots \leq \mu_{(n-k+1:n)}(\mathbf{X}) \leq \cdots \leq \mu_{(n:n)}(\mathbf{X})$. Since each $\mu_i(X_i)$ is either zero or one, the above ordering will result in equalities for many of the above terms. Once the above is done, we see that $\phi_K(\mathbf{X}) = \mu_{(n-k+1:n)}(\mathbf{X})$. Thus, in general, the structure function of a k-out-of-n system is the membership function of the precise set intersecting the k smallest $\mathcal{G}_i$ sets.

5.2 Structure Functions of Vague Binary State Systems

Motivated by the material of the previous section, we define the structure function of series, parallel, and k-out-of-n systems whose component states are vague and binary as

$$\begin{aligned} \phi_S(\mathbf{X}) &= \min_i[\mu_i(X_i)] = \mu_{(1:n)}(\mathbf{X}), \\ \phi_P(\mathbf{X}) &= \max_i[\mu_i(X_i)] = \mu_{(n:n)}(\mathbf{X}), \text{ and} \\ \phi_K(\mathbf{X}) &= \mu_{(n-k+1:n)}(\mathbf{X}). \end{aligned}$$

These structure functions are identical to those for the case of binary precise sets, except that now, $\mu_i(X_i)$ is a membership function of an associated vague set $\mathcal{G}_i, i = 1, \ldots, n$.

Finally, if $\pi_D(x_i \in \mathcal{G}_i)$ denotes D's probability that a particular x_i gets classified in $\mathcal{G}_i$, then by analogy with equation (7), we have

$$P_D[X_i \in \mathcal{G}_i; \mu_i(x_i)] = \int_{x_i} \left[1 - \left(1 - \frac{1}{\mu_i(x_i)}\right) \cdot \frac{\pi_D(x_i \notin \mathcal{G}_i)}{\pi_D(x_i \in \mathcal{G}_i)}\right]^{-1} \mathrm{d}P_D(x_i), \tag{11}$$

where $P_D(x_i)$ is D's probability that $X_i \leq x_i$.

Our development thus far has assumed that the membership functions $\mu_i(x_i), i = 1, \ldots, n$, are all distinct. Simplification occurs if $\mu_i(x_i) = \mu(x)$ for $i = 1, \ldots, n$. We limit our attention to the case of series and parallel systems because more complicated systems, such as networks can be represented as a combination of series-parallel systems.

5.3 Reliability of Vague Binary State Systems

If the state of each component in a system is a desirable state, will the system itself be in a desirable state? The answer to this question need not be in the affirmative. This is because requirements on the system could be more stringent than those on each component of the system. This is unlike the case of binary state systems with precise classification wherein a series system is judged to be reliable if all its components are reliable. Thus, there are two possible ways in which the reliability of a vague coherent system can be defined. The first is to assume that a series system is reliable if all its components are in a desirable state. The second is to require that for a system to be judged reliable, its state—say x—be a desirable state. Specifically, we require that $x \in \mathcal{G}_{\phi(\mathbf{X})}$, where $\mathcal{G}_{\phi(\mathbf{X})} = \{x; x \text{ is a 'desirable' system state}\}$ and $\mathcal{G}_{\phi(\mathbf{X})} \subset S$. Associated with $\mathcal{G}_{\phi(\mathbf{X})}$ is its membership function, $\mu_{\mathcal{G}_{\phi(\mathbf{X})}}(x)$. Similarly, in the case of a parallel system, we have two possibilities for defining reliability—the first one being that the system is reliable if at least one of its components is in a desirable state, and the second being the requirement that its state $x \in \mathcal{G}_{\phi(\mathbf{X})}$. We simplify notation by letting $\mu_{\phi(\mathbf{X})}(x) = \mu_{\mathcal{G}_{\phi(\mathbf{X})}}(x)$ and focusing the discussion on the subspace $\mathcal{G}_{(\cdot)}$.

For assessing reliability, let us consider the first case for series and parallel systems. Assuming the X_i's independent, the reliability of a series system would be $\prod_{i=1}^{n}[P_D[X_i \in \mathcal{G}_i; \mu_i(x_i)]]$ where $P_D[X_i \in \mathcal{G}_i; \mu_i(x_i)]$ is given by equation (11). The reliability of a parallel redundant system is $P_D(\bigcup_{i=1}^{n}\{X_i \in \mathcal{G}_i\}; \mu_i(x_i),\ i = 1, \ldots, n)$; it can be evaluated by the Inclusion-Exclusion formula of probability (Feller, 1968). The computations simplify when the X_i's are assumed identically distributed. The case of k-out-of-n systems follows along similar lines.

With regard to the above, a question arises as to what we mean by independence of the X_i's, when the X_i's take values in a vague set. In the context of equation (11), X_i and X_j, $i \neq j$, will be judged independent if

$$P_D(X_i \le x_i, X_j \le x_j) = P_D(X_i \le x_i) \cdot P_D(X_j \le x_j), \text{ and if}$$
$$P_D(x_i \in \mathcal{G}_i, x_j \in \mathcal{G}_j) = P_D(x_i \in \mathcal{G}_i) \cdot P_D(x_j \in \mathcal{G}_j) \text{ and}$$
$$P_D(x_i \notin \mathcal{G}_i, x_j \notin \mathcal{G}_j) = P_D(x_i \notin \mathcal{G}_i) \cdot P_D(x_j \notin \mathcal{G}_j).$$

The more interesting case is the second one, wherein a system is reliable if the state in which it resides is a desirable one. We start with the case of a series system with structure function $\phi_S(\mathbf{X})$. Its reliability is $P_D(\phi_S(\mathbf{X}) \in \mathcal{G}_{\phi_S(\mathbf{X})}; \mu_{\phi_S[\mathbf{X})}(x)]$ which, from equation (11), is of the form

$$P_D(\phi_S(\mathbf{X}) \in \mathcal{G}_{\phi_S(\mathbf{X})}; \mu_{\phi_S[\mathbf{X})}(x)] = \int_x \left[1 - \left(1 - \frac{1}{\mu_{\phi_S(\mathbf{X})}(x)}\right) \cdot \frac{\pi_D(x \notin \mathcal{G}_{\phi_S(\mathbf{X})})}{\pi_D(x \in \mathcal{G}_{\phi_S(\mathbf{X})})}\right]^{-1} \mathrm{d}P_D(x), \tag{12}$$

where $\pi_D(x \in \mathcal{G}_{\phi_S(\mathbf{X})})$ is D's probability that x is classified in $\mathcal{G}_{\phi_S(\mathbf{X})}$ were $\phi_S(X) = x$, and $P_D(x)$ is D's probability that $\phi_S(\mathbf{X}) \le x$.

Since $\phi_S(\mathbf{X}) = \min_i \mu_i(X_i) = \mu_{(1:n)}(\mathbf{X})$, we obtain $P_D(x)$ as follows:

$$\begin{aligned} P_D(\phi_S(\mathbf{X}) \ge x) &= P_D(\mu_{(1)}(\mathbf{X}) \ge x) \\ &= P_D(\mu_i(X_i) \ge x,\ i = 1, \ldots, n) \\ &= P_D(X_i \ge \mu_i^{-1}(x),\ i = 1, \ldots, n), \\ &= \prod_{i=1}^{n} P_D[X_i \ge \mu_i^{-1}(x)], \text{ if } X_i\text{'s are assumed independent,} \end{aligned} \tag{13}$$

where $\mu_i^{-1}(\cdot)$ denotes the inverse of $\mu_i(\cdot)$. Subsequently, $\mathrm{dP_D}(x)$ can be obtained. If the X_i's cannot be judged independent with respect to D's distribution for the X_i's, we need to specify a joint distribution for these, such as Marshall & Olkin's (1967) multivariate exponential, or any of its variants. In the case of parallel systems, the development will proceed along similar lines, save that now $P_D(x)$ will be obtained via $\prod_{i=1}^{n} P_D[X_i \le \mu_i^{-1}(x)]$. Finally, the case of $(n-k+1)$-out-of-n would follow by considering the distribution of the k-th order membership function, $\mu_{(k:n)}(x)$.

Example 5: Consider a two-component series system where the component performances are independent and identically distributed. D wishes to assess $P_D[\phi_S(\mathbf{X}) \in \mathcal{G}_{\phi_S(\mathbf{X})}; \mu_{\phi_S(\mathbf{X})}(x)]$. The first option is to compute the product of the component probabilities. Let $\mu_{\mathcal{G}_i}(x) = x^2$, and $P_D(x)$ and $P_D(x \in \mathcal{G}_i)$ be as shown in Figures 5 and 6, respectively, for $i = 1, 2$. Then, $P_D[\phi_S(\mathbf{X}) \in \mathcal{G}_{\phi_S(\mathbf{X})}; \mu_{\phi_S(\mathbf{X})}(x)] = 0.6232$. The second option is to compute the system reliability directly, through the use of Z's membership function for the entire system. Supposing that the expert holds a stronger standard for the system to be in a desirable state than that for the components, we let $\mu_{\phi_S(\mathbf{X})}(x) = x^{10}$. Meanwhile, D considers $P_D(x)$ and $P_D(x \in \mathcal{G}_i)$ as specified in Figures 5 and 6 for $\phi_S(\mathbf{X})$, implying that $P_D[\phi_S(\mathbf{X}) \in \mathcal{G}_{\phi_S(\mathbf{X})}; \mu_{\phi_S(\mathbf{X})}(x)] = 0.4321$. Thus, by holding the system to a more stringent standard, D's assessment of the system reliability is lower when considered directly, as opposed to that when using a more relaxed membership function to represent belief at the component level.

6 Maintenance Management in a Vague Environment

Examples 3–5 illustrate how D is able to assess the probability that the state of a unit will be in a 'desirable' state, or its complement. Why would D be interested in such a probability instead of the probability that the state of the unit will be x, $0 \le x \le 1$? Reasons were given in Section 4.3, the one pertaining to communication using 'natural language' being the most relevant. This point is best underscored via the scenario of maintenance wherein one must decide whether to repair, replace, or simply continue to monitor the unit. In practice, judgments about maintenance are not based on assessments of uncertainty about x; they are based on conjectures about whether or not the unit will be in a 'desirable' state.

Consider the following: a unit is required to perform service for some time period. The unit can exist in one of three states: $\mathcal{G}$ (for good), $\mathcal{B}$ (for bad), and $\mathcal{A}$ (for acceptable). When the unit is in state $\mathcal{G}$, the utility to D provided by the unit is $\mathcal{U}(\mathcal{G})$; analogously, we define $\mathcal{U}(\mathcal{A})$ and $\mathcal{U}(\mathcal{B})$. It is reasonable to suppose that $\mathcal{U}(\mathcal{A}) < \mathcal{U}(\mathcal{G})$ and, in principle, $-\mathcal{U}(\mathcal{B})$ could be greater than $\mathcal{U}(\mathcal{G})$, i.e. the cost for being in state $\mathcal{B}$ could dominate the reward for being in state $\mathcal{G}$. With the above in place, D's problem is to make a decision whether to replace the unit, denoted $\mathcal{R}$, or to repair the unit, denoted $\mathcal{M}$, or do nothing, denoted $\mathcal{N}$. There is a cost associated with each of these three actions, and these are denoted $-\mathcal{U}(\mathcal{R})$, $-\mathcal{U}(\mathcal{M})$, and $-\mathcal{U}(\mathcal{N})$, respectively. Presumably, $-\mathcal{U}(\mathcal{N}) < -\mathcal{U}(\mathcal{M}) < -\mathcal{U}(\mathcal{R})$. Which of the above three actions should D take?

The problem is solved by using *maximization of expected utility* (MEU) [cf. Lindley (1991), p. 58]. The decision tree of Figure 7 facilitates an implementation of this recipe; the rectangle represents D's decision node and the three circles denoted R_1, R_2, and R_3 represent the three nodes corresponding to the three actions $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{N}$, respectively. Each (random) node results in one of three outcomes, $\star = \mathcal{G}$, $\mathcal{A}$ or $\mathcal{B}$, and these are portrayed in Figure 7 only for the node R_3. At the terminus of the tree are the utilities. For example, $\mathcal{U}(\mathcal{N}, \mathcal{G})$ denotes the utility to D, when D's decision is to monitor the unit and the outcome is $\mathcal{G}$.

The MEU principle requires that, at each random node, D compute an expected utility of an action that leads to that node. For this, D needs to assess the probabilities that at τ, the state of the unit will be in $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{B}$, respectively. These probabilities would depend on three ingredients: membership functions of the kind $\mu_\star(x)$, $\mu_{\mathcal{A}}(x)$, and $\mu_{\mathcal{B}}(x)$; D's prior probability that an x is classified (by nature) in $\mathcal{G}$, $\mathcal{A}$, and $\mathcal{B}$ (i.e. $P_D(x \in \star)$, $\star = \mathcal{G}, \mathcal{A}, \mathcal{B}$), and $P_D(x)$, D's subjective probability that the state of the unit will be x. Since $\sum_{\star=\mathcal{G},A,B} P_D(x \in \star) = 1$, D need only specify any two probabilities. Once these are at hand, D invokes equation (7) to obtain the required probabilities. All of the above is straightforward except that $P_D(x)$ depends on the action that D takes. Both repair and replacement actions tend to right-skew the form of $P_D(x)$ toward one. Thus, with respect to the illustration of Figure 5, a repair action will tend to shift the probability mass closer to one, and moreso with replacement. To summarize, the impact of D's

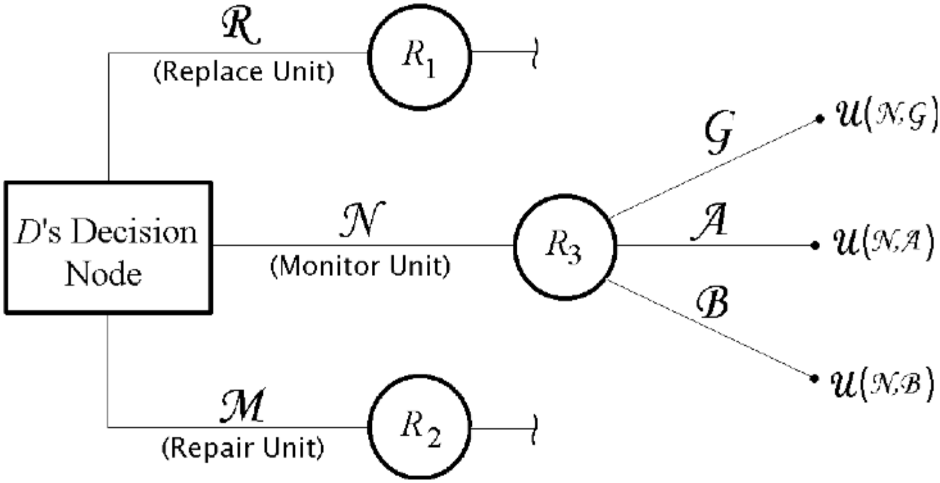

Figure 7. *D's decision tree for maintenance actions.*

actions on D's probabilities of the state of the unit are reflected only in $P_D(x)$. The membership functions and the classification probabilities are unaffected. To denote such a dependence, we shall replace the $P_D(x)$ of equation (7) by $P_D(x;\bullet)$, and $P_D[X \in \star;\ \mu_\star(x)]$ by $P_D(X \in \star;\ \mu_\star(x), \bullet)$ for $\bullet = \mathcal{R}, \mathcal{M}$ and $\mathcal{N}$; $\star = \mathcal{G}, A, B$.

Whereas the development in Sections 4 and 5 pertained to the binary case involving two vague sets $\mathcal{B}$ and $\mathcal{G}$, our example here involves three vague sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{G}$, and their respective membership functions, $\mu_\bullet(x)$, $\bullet = \mathcal{A}, \mathcal{B}$ and $\mathcal{G}$. Of these, only $\mu_\mathcal{A}(x)$ warrants comment since the general nature of the other two has been discussed before; see Figures 4(a) and (b). It is reasonable to suppose that the general form of $\mu_\mathcal{A}(x)$ is either bell-shaped or an inverted U.

Finally, a question arises as to whether $\mu_\mathcal{A}(x)$, $\mu_\mathcal{B}(x)$, and $\mu_\mathcal{G}(x)$ can take any arbitrary form independent of each other. The answer to this question is in the negative because the membership functions go to determine the quantities $P_D[X \in \mathcal{A}; \mu_\mathcal{A}(x)]$, $P_D[X \in \mathcal{B}; \mu_\mathcal{B}(x)]$ and $P_D[X \in \mathcal{G}; \mu_\mathcal{G}(x)]$, and these must sum to one. Thus, D needs to ensure coherence of the membership functions just like how D needs to ensure a coherence of the classification and state probabilities. Since D elicits membership functions from Z, it is incumbent on D to ensure that membership functions do not lead to results that violate the countable additivity axiom of probability. This important point has not been addressed in Singpurwalla & Booker (2004).

The utilities at the terminus of a tree, $\mathcal{U}(\mathcal{N}, \mathcal{G})$, $\mathcal{U}(\mathcal{N}, \mathcal{A})$ and $\mathcal{U}(\mathcal{N}, \mathcal{B})$ are straightforward to write out. Thus, for example, $\mathcal{U}(\mathcal{N}, \mathcal{G}) = \mathcal{U}(\mathcal{N}) + \mathcal{U}(\mathcal{G})$, which is the sum of the disutility due to monitoring and the utility of the unit being in state $\mathcal{G}$. Similarly, $\mathcal{U}(\mathcal{N}, \mathcal{B}) = \mathcal{U}(\mathcal{N}) + \mathcal{U}(\mathcal{B})$, and $\mathcal{U}(\mathcal{N}, \mathcal{A}) = \mathcal{U}(\mathcal{N}) + \mathcal{U}(\mathcal{A})$. With this in place, we compute the expected utility at each node. Thus, for example, $\mathcal{U}(\mathcal{N})$, the *expected utility* at node R_3 is $\mathcal{U}(N) = \sum_{\star=\mathcal{G},A,B} \mathcal{U}(N, \star) \cdot P_D(X \in \star; \mu_\star(x), \mathcal{N})$, where $P_D(X \in \star; \mu_\star(x), \mathcal{N})$ is the right-hand side of equation (7) with $P_D(x)$ replaced by $P_D(x; \mathcal{N})$; similarly, the other terms of $\mathcal{U}(\mathcal{N})$. The expected utilities at nodes R_1 and R_3 are analogously computed as $\mathcal{U}(\mathcal{R})$ and $\mathcal{U}(\mathcal{M})$, respectively, *mutatis mutandis*. Once the above are done, D's maintenance decision is to choose that action for which the expected utility is a maximum. Thus, for example, if $\mathcal{U}(\mathcal{N}) > \mathcal{U}(\mathcal{R}) > \mathcal{U}(\mathcal{M})$, then D's decision would be simply to do nothing.

How does the above material differ from that which is currently available in the literature on maintenance planning? The current literature would require each node to be binary and, to compute the expected utility at each node, all we need is the probability that $x \geq x^*$. This probability can be had once D specifies $P_D(x; \bullet)$, $\bullet = \mathcal{R}, \mathcal{N}$ and $\mathcal{M}$. By contrast, we allow an x to exist in three vaguely defined sets, and allow x to simultaneously exist in more than one of these. The advantage is flexibility and a facility to entertain an analysis that facilitates natural language communication. Further, in the existing literature, uncertainties are assessed about times to failure via probabilistic failure models, and failure is viewed as a sharply defined event. Consequently, the analysis is forced into a binary framework. By contrast, our uncertainties are focused on x which can encapsulate degradation of a unit.

7 Summary and Conclusions

The term 'complex stochastic systems' is well entrenched into the vocabulary of statisticians, though it generally pertains to a use of the Markov Chain Monte Carlo method. This paper takes a broader view of this term by embedding within it the theory of vague coherent structures. This theory, which is generally associated with work in applied probability and reliability is germane to statisticians, especially those whose focus is on biostatistics, genetics, graphical models, and neural nets. With that in mind, we have devoted Section 1 to an overview of the key notions and ideas of binary state systems whose two states can be precisely delineated. The mathematics which drives the development of results for such systems is binary logic. In Section 1, we also set the stage for the material of Sections 4 and 5 by introducing the idea of imprecise or vague sets. The need for such sets has been acknowledged by physicists, philosophers, and logicians. More recently, their need has also been recognized by those involved in decision making and natural language processing. Section 2 is devoted to multivalued logic in the context of multivalued propositions. The focus here is on the connectives of conjunction and disjunction; these connectives can be used to define the structure function of multistate systems, a topic treated in Section 3. In Section 3, it is assumed that the classification of states is precise. This topic has been covered before via the literature on multistate reliability; however, what is new here is the departure from binary logic to multivalued logic.

Sections 4 and 5 impart to this paper a feature that is novel. Specifically, they pertain to the development of reliability for components and systems whose state space is vague. In actuality, vague state spaces are more realistic than the usual zero-one states, which are an idealization. In Sections 4 and 5, we also show that the usual notions of reliability do not always hold when the state space is vague. For example, the unreliability of a unit is not one minus its reliability, and that there is more than one way to define system reliability.

There is another aspect of this paper that warrants comment. In the existing theory of coherent structures with precise classification, statistical principles have no role to play. All that is needed is the calculus of probability. By contrast, when dealing with vague systems, membership functions and consistency profiles create a role for the likelihood function and, in so doing, mandate a consideration of the principles of Bayesian statistical inference.

The illustrative examples of Sections 4 and 5, and the maintenance management architecture of Section 6 should give the reader an inkling of the practical import of the material here. For example, in maintenance and replacement actions pertaining to decision making uncertainty, the usual strategy is to assume that the state space is binary—functioning and failed. In actuality, functioning can occur at different levels whose boundaries cannot be sharply delineated. Thus, it makes more sense to study maintenance and replacement when the state space is vague for, in actuality, this is how such decisions are made.

Acknowledgements

The authors wish to thank the reviewers for their helpful comments. Nozer Singpurwalla was supported by the Office of Naval Research Contract N00014-06-1-0037 and by the Army Research Office Grant W911NF-05-1-0209 with The George Washington University.

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Résumé

L'état de l'art dans la théorie de structure cohérente est guidé par deux assertions qui sont tous deux limitants : (1) toutes les unités d'un système peuvent exister dans un de deux états, défaillant ou fonctionnant; et (2) à n'importe quel moment, chaque unité peut seulement exister dans un des susdits états. En réalité, les unités peuvent exister dans plus de deux états et c'est possible qu'une unité puisse *simultanément* exister dans plus d'un état. Cette dernière caractéristique est une conséquence de l'opinion qu'il ne soit peut-être pas possible de définir avec précision les sous-ensembles d'un ensemble d'états; on appelle de tels sous-ensembles vagues. La première restriction a été adressée par les méthodes appelées "systèmes multi-états"; pourtant, ces méthodes n'ont pas pris avantage des mathématiques sur les propositions multivalues en logique. Ici, nous invoquons ses tables de vérité pour définir la fonction des systémes multi-états et exploiter ensuite nos résultats dans le contexte d'ambiguïté. Une contribution clé de ce papier est d'argumenter que la logique de plusieurs values est une plateforme commune pour étudier tant les systèmes multi-états que les systémes vagues, mais pour faire ceci, il est nécessaire de se baser sur plusieurs principes d'inférence statistique.

[*Received June 2007, accepted March 2008*]