Many-Valued Modal Logics

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Abstract. Two families of many-valued modal logics are investigated. Semantically, one family is characterized using Kripke models that allow formulas to take values in a finite many-valued logic, at each possible world. The second family generalizes this to allow the accessibility relation between worlds also to be many-valued. Gentzen sequent calculi are given for both versions, and soundness and completeness are established.

1 Introduction

The logics that have appeared in artificial intelligence form a rich and varied collection. While classical (and maybe intuitionistic) logic suffices for the formal development of mathematics, artificial intelligence has found uses for modal, temporal, relevant, and many-valued logics, among others. Indeed, I take it as a basic principle that an application should find (or create) an appropriate logic, if it needs one, rather than reshape the application to fit some narrow class of 'established' logics. In this paper I want to enlarge the variety of logics available by introducing natural blendings of modal and many-valued logics.

Many-valued modal logics have been considered before [12, 13, 11, 6, 8, 7], but perhaps in too narrow a sense. The basic idea was to retain the general notion of possible world semantics, while allowing formulas to have values in a many-valued space, at each possible world. What seems not to have been considered is allowing the accessibility relation between possible worlds itself to be many-valued. But many-valued accessibility is a very natural notion. After all, some worlds alternative to this one are more relevant, others less, as one intuitively thinks of these things.

The general plan of the paper is as follows. In order to have a suitable proof-theoretic framework on which to build, I begin with a many-valued sequent calculus, allowing a uniform approach to a rich family of finitely many-valued logics. This calculus is new, quite different from the approach in [10], and may be of some independent interest. Completeness is established. Then two families of many-valued modal logics are introduced. In semantic terms, one family requires the accessibility relation to be classical; the other allows it to be many-valued. For both families, sequent calculus formulations are presented, and completeness is shown.

2 Basic Syntax and semantics

Before modal issues can be discussed, a multiple-valued framework must be erected on which to build. I want to keep things as general as I can manage, and so I begin with a space of truth values that constitutes a finite lattice — nothing more. This means there are natural notions of conjunction and disjunction, but not necessarily of negation. Implication plays an intermediate role: it is allowed, and interpreted by the partial ordering of the lattice of truth values, but in general it can not be nested. However, for certain special categories of lattices a natural notion of nestable implication is possible — I treat such things separately, after dealing with the more general case.

Notation For the rest of this paper, $\mathcal{T}$ is a finite lattice. Its members are referred to as *truth values*. The lattice ordering is denoted $\leq$, and the meet and join operations by $\wedge$ and $\vee$. The bottom and the top of $\mathcal{T}$ are denoted *false* and *true* respectively, and it is assumed that *false* $\neq$ *true*.

Next a logical language, relative to $\mathcal{T}$, is specified. It was observed above that implications will not generally be nestable, so there are two categories of compound expressions: formulas, and implications between formulas. In classical two-valued logic, $P \wedge \neg P$ and $P \vee \neg P$ serve to define within the language counterparts of *false* and *true*. In multiple-valued logics as general as those being considered now, the basic machinery may not be enough to define counterparts of all truth values, and so propositional constants are explicitly added to the language. To keep things simple, just assume members of $\mathcal{T}$ themselves can be used as atomic formulas. Now for the details.

Definition 2.1 The propositional language $L_{\mathcal{T}}$ is specified as follows:

- *Atomic formulas* of $L_{\mathcal{T}}$ are *propositional variables*, denoted P, Q, etc., and *propositional constants*, which are members of $\mathcal{T}$.
- *Formulas* of $L_{\mathcal{T}}$ are built up as follows. Atomic formulas of $L_{\mathcal{T}}$ are formulas of $L_{\mathcal{T}}$. If A and B are formulas of $L_{\mathcal{T}}$, so are $(A \wedge B)$ and $(A \vee B)$.
- *Implications* of $L_{\mathcal{T}}$ are expressions of the form $(A \supset B)$ where A and B are formulas.

For reading convenience: *formulas* are denoted A, B, etc.; *implications* are denoted X, Y, etc.; *sets of implications* are denoted Γ, Δ, etc. Also, Γ, X will be used to abbreviate $\Gamma \cup \{X\}$, and similarly for other set-theoretic combinations. Finally, parentheses in formulas and implications will often be omitted.

In later sections the set of formulas will sometimes be closed under implication. But whether this is the case or not, implications are the basic unit of currency. Now for semantic issues.

Definition 2.2 A *valuation* is a mapping from atomic formulas of $L_{\mathcal{T}}$ to $\mathcal{T}$ that maps each member of $\mathcal{T}$ to itself.

If v is a valuation, its action is extended to all formulas in the obvious way: $v(A \wedge B) = v(A) \wedge v(B)$, where the $\wedge$ on the right is the meet of $\mathcal{T}$. Similarly for $\vee$. Then $v(A) \in \mathcal{T}$ if A is a formula of $L_{\mathcal{T}}$. Finally, implications of $L_{\mathcal{T}}$ are mapped to $\mathcal{T}$ so that $v(A \supset B) = \textit{true}$ if and only if $v(A) \leq v(B)$. (If $v(A) \not\leq v(B)$, the exact value of $v(A \supset B)$ will not matter for now — anything except *true* will do. Something more explicit will be said when nestable implication is considered.)

3 A Many-valued proof procedure

Most many-valued proof procedures in the literature have been axiomatic, [10] for instance. Here a somewhat unusual Gentzen-type sequent calculus will be introduced. Unlike most sequent calculi, to the left and the right of the sequent arrow will be sets of *implications*. Still, the underlying idea is the usual one.

Definition 3.1 $\Gamma \rightarrow \Delta$ is *valid* if, for every valuation v, either some member of Γ is not true under v, or some member of Δ is true under v.

In most cases it is straightforward to check that each of the axioms below is valid, and the rules preserve validity. Proofs are omitted. The usual structural rules come first.

Identity Axiom

$$X \rightarrow X$$

Thinning

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \cup \Gamma' \rightarrow \Delta \cup \Delta'}$$

Cut

$$\frac{\Gamma \rightarrow \Delta, X \quad \Gamma, X \rightarrow \Delta}{\Gamma \rightarrow \Delta}$$

Cut applications will generally be combined with Thinning, without comment. Next are the rules for implication. The first is a transitivity axiom.

Axiom of Transitivity

$$X \supset Y, Y \supset Z \rightarrow X \supset Z$$

Proposition 3.2 *The following are derived rules:*

$$\frac{\Gamma \rightarrow \Delta, C \supset A \quad \Gamma, C \supset B \rightarrow \Delta}{\Gamma, A \supset B \rightarrow \Delta}$$

$$\frac{\Gamma \rightarrow \Delta, B \supset C \quad \Gamma, A \supset C \rightarrow \Delta}{\Gamma, A \supset B \rightarrow \Delta}$$

Proof An argument for the first rule is given; the second is similar.

$$\dfrac{\dfrac{\Gamma \rightarrow \Delta, C \supset A \quad C \supset A, A \supset B \rightarrow C \supset B}{\Gamma, A \supset B \rightarrow \Delta, C \supset B}\,\text{Cut} \quad \Gamma, C \supset B \rightarrow \Delta}{\Gamma, A \supset B \rightarrow \Delta}\,\text{Cut}$$

■

Next are two important rules for implication, making use of the truth values of $\mathcal{T}$.

Rule $RI\supset$

$$\frac{\Gamma, t \supset A \rightarrow \Delta, t \supset B \quad \text{(for every } t \in \mathcal{T}\text{)}}{\Gamma \rightarrow \Delta, A \supset B}$$

Rule $\supset RI$

$$\frac{\Gamma, B \supset t \to \Delta, A \supset t \quad (\text{for every } t \in \mathcal{T})}{\Gamma \to \Delta, A \supset B}$$

Here is the simple justification for the first of these rules; the second is treated in a similar way. Suppose the sequent below the line is not valid. Say the valuation v maps every member of Γ to *true*, but does not map any member of $\Delta, A \supset B$ to *true*. In particular, $v(A) \not\leq v(B)$. Let $t \in \mathcal{T}$ be $v(A)$. Then v maps every member of $\Gamma, t \supset A$ to *true*, but does not map any member of $\Delta, t \supset B$ to *true*, so one of the sequents above the line is not valid.

It is interesting to observe what the rules above say for classical logic. Here there are only two truth values. In Rule $RI \supset$, when t is *false*, the sequent $\Gamma, t \supset A \to \Delta, t \supset B$ is trivially obtainable, assuming $\to$ *false* $\supset B$. Thus only the $t =$ *true* case is significant. If A is identified with *true* $\supset A$, Rule $RI \supset$ degenerates, in the classical case, to

$$\frac{\Gamma, A \to \Delta, B}{\Gamma \to \Delta, A \supset B}$$

which is one of the usual classical sequent rules for implication. Rule $\supset RI$ can be interpreted in a similar way, yielding a kind of contrapositive version of the rule.

There are more rules and axioms to come, but there are some results that can be established now.

Proposition 3.3 *For every formula A, $\to A \supset A$*

Proof Let t be an arbitrary member of $\mathcal{T}$. Then $t \supset A \to t \supset A$, by Identity, so $\to A \supset A$, by $RI \supset$. ∎

Proposition 3.4 *With Rules $RI \supset$ and $\supset RI$ available, the Axiom of Transitivity follows from either of the Rules of Proposition 3.2.*

Proof Transitivity is shown to follow from the first of the rules; the other is similar. Let t be an arbitrary member of $\mathcal{T}$.

$$\dfrac{\dfrac{t \supset A \to t \supset A}{t \supset A \to t \supset C, t \supset B, t \supset A}\ \text{thinning} \qquad \dfrac{t \supset B \to t \supset B}{t \supset A, t \supset B \to t \supset C, t \supset B}\ \text{thinning}}{t \supset A, A \supset B \to t \supset C, t \supset B \quad \text{(sequent one)}}\ \text{Prop 3.2}$$

$$\dfrac{\dfrac{t \supset C \to t \supset C}{t \supset A, t \supset C \to t \supset C, t \supset A}\ \text{thinning} \qquad \dfrac{t \supset C \to t \supset C}{t \supset A, t \supset C, t \supset B \to t \supset CA}\ \text{thinning}}{A \supset B, t \supset A, t \supset C \to t \supset C \quad \text{(sequent two)}}\ \text{Prop 3.2}$$

$$\dfrac{\dfrac{\text{(sequent one)} \quad \text{(sequent two)}}{A \supset B, B \supset C, t \supset A \to t \supset C}\ \text{Prop 3.2}}{A \supset B, B \supset C \to A \supset C}\ RI \supset$$

∎

Next are the expected lattice-theoretic rules for the lattice-theoretic connectives.

Conjunction Axioms

$$\rightarrow A \wedge B \supset A$$

$$\rightarrow A \wedge B \supset B$$

$$C \supset A, C \supset B \rightarrow C \supset A \wedge B$$

Disjunction Axioms

$$\rightarrow A \supset A \vee B$$

$$\rightarrow B \supset A \vee B$$

$$A \supset C, B \supset C \rightarrow A \vee B \supset C$$

From the axioms above, the 'usual' properties of the connectives follow; details are omitted. Finally, rules that reflect the properties of $\mathcal{T}$ itself.

Propositional Constant Axioms

$$\rightarrow a \supset b \quad \text{if } a \leq b$$

$$a \supset b \rightarrow \quad \text{if } a \not\leq b$$

Proposition 3.5 *For any formula A, $\rightarrow A \supset true$ and $\rightarrow false \supset A$.*

Proof Again only the first is proved. Let $t \in \mathcal{T}$ be arbitrary.

$$\cfrac{\cfrac{\rightarrow t \supset true}{t \supset A \rightarrow t \supset true}\ \text{thinning}}{\rightarrow A \supset true}\ RI\supset$$

■

Finally suppose an upward closed set of *designated values*, say $\{d_1, \ldots, d_k\}$, is specified in the usual multiple-valued logic style. Then the Gentzen calculus equivalent of A being a tautology of the multiple-valued logic is the provability of the sequent $\rightarrow d_1 \supset A, \ldots, d_k \supset A$.

4 Many-valued completeness

Soundness of the sequent calculus is immediate. Completeness depends critically on a derived rule, which we state below as Theorem 4.1. Proof of this theorem can be found in the Appendix.

Theorem 4.1 *The following is a derived rule:*

$$\frac{\Gamma, A \supset t, t \supset A \rightarrow \Delta \quad (\text{for all } t \in \mathcal{T})}{\Gamma \rightarrow \Delta}$$

Using Theorem 4.1, completeness can be established by a fairly standard Lindenbaum-style argument.

Definition 4.2 Let X be an implication, and let S be a set of implications. Say S is X-*inconsistent* if, for some finite $\Gamma \subseteq S$, there is a proof of the sequent $\Gamma \to X$. Also S is X-*consistent* if it is not X-inconsistent.

In the usual way, an X-consistent set can be extended to a *maximal* X-consistent one. And, as usual, it follows from the Cut Rule that if S is maximal X-consistent and $\Gamma \to Y$ for some $\Gamma \subseteq S$, then $Y \in S$.

Proposition 4.3 *Suppose S is a maximal X-consistent set. Then, for each formula A there is exactly one $a \in \mathcal{T}$ such that both $a \supset A$ and $A \supset a$ are in S.*

Proof First, it can not happen that $a \supset A$, $A \supset a$, $b \supset A$ and $A \supset b$ are all in S, for $a \neq b$, because if $a \not\leq b$ say, then

$$\frac{a \supset A, A \supset b \to a \supset b \quad a \supset b \to}{a \supset A, A \supset b \to} \text{Cut}$$

from which the X-inconsistency of S follows by Thinning.

Second, suppose that for no $a \in \mathcal{T}$ do we have $a \supset A$ and $A \supset a$ in S. Then (using the finiteness of $\mathcal{T}$) there must be a finite set $\Gamma \subseteq S$ such that $\Gamma, A \supset a, a \supset A \to X$ is provable, for every $a \in \mathcal{T}$. Then by Theorem 4.1, $\Gamma \to X$ is provable, contradicting the X-consistency of S. ■

Now, suppose X is not provable. Then $\emptyset$ is X-consistent — extend it to a maximal X-consistent set, S. Define a valuation v as follows: for an *atomic* formula A set $v(A)$ to be that unique $a \in \mathcal{T}$ such that both $A \supset a$ and $a \supset A$ are in S. It is straightforward to check that, for *every* formula F, $v(F)$ is the unique member $f \in \mathcal{T}$ such that $F \supset f$ and $f \supset F$ are in S.

Say X is the implication $A \supset B$, where $v(A) = a$ and $v(B) = b$. If $a \leq b$ then S would be X-inconsistent, by the following:

$$\frac{A \supset a, b \supset B, a \supset b \to A \supset B \quad \to a \supset b}{A \supset a, b \supset B \to A \supset B} \text{Cut}$$

Consequently $a \not\leq b$, so $v(X) = v(A \supset B)$ is not true. The following has now been established.

Theorem 4.4 *If an implication X is true under every valuation, X is provable in the sequent calculus.*

5 Nested implication

Up till now implication has been interpreted by a lattice ordering, and so an implication is either *true* or it isn't, and that's all that has mattered. If implication is to be nestable, something more elaborate is needed, and this requires more structure than just that of a lattice. Here one way of dealing with this is presented — other ways may be possible.

Definition 5.1 $L_{\mathcal{T}}[\supset]$ is the language defined like $L_{\mathcal{T}}$, but with the extra formation rule: if A and B are formulas, so is $A \supset B$ (note that now implications are special kinds of formulas).

There are several notions of implication that have been studied in multiple-valued logics. Here one particular family that has nice mathematical properties is considered. No claim is made that it is best for all applications. Proof theoretically, the laws of *importation* and *exportation* are assumed. Semantically, the basic notion is that of *relative pseudo-complement* from the algebraic semantics of intuitionistic logic, [9].

Definition 5.2 An element $c \in \mathcal{T}$ is the *pseudo-complement of a relative to b* if c is the greatest member of $\mathcal{T}$ such that $a \wedge c \leq b$. If the pseudo-complement of a relative to b exists, it is denoted by $a \Rightarrow b$. If relative pseudo-complements always exist, $\mathcal{T}$ is said to have a relative pseudo-complement operation.

For finite lattices, having a relative pseudo-complement operation is equivalent to being distributive. Thus the class with such an operation is a large and natural one. If $\mathcal{T}$ has a relative pseudo-complement operation, valuations can be extended to formulas of $L_{\mathcal{T}}[\supset]$ by setting $v(A \supset B) = v(A) \Rightarrow v(B)$. This is compatible with the earlier interpretation of implication that could not be nested, since one can show that when the relative pseudo-complement operation is meaningful, $a \Rightarrow b = \mathit{true}$ if and only if $a \leq b$, see [9].

Since finite lattices are the only ones considered here, a bottom element always exists; it has been denoted *false* throughout. A relatively pseudo-complemented lattice with a bottom is called a *pseudo-boolean algebra*. Such algebras play a significant role in the algebraic semantics of intuitionistic logic, though this is not an important issue here.

Proof-theoretically, the sequent calculus is extended by adding the following rules.

Implication Axioms

$$(A \wedge B) \supset C \rightarrow A \supset (B \supset C)$$

$$A \supset (B \supset C) \rightarrow (A \wedge B) \supset C$$

The completeness proof extends easily; details are omitted.

6 An Example

Two versions of many-valued modal logic will be presented. The first version is the most direct generalization of conventional modal logic. Before presenting the technical details, a special case of considerable importance may help to motivate what follows.

Suppose $\langle \mathcal{G}, \mathcal{R} \rangle$ is a *frame* in the usual modal sense: $\mathcal{G}$ is a non-empty set of possible worlds, and $\mathcal{R}$ is a binary accessibility relation on $\mathcal{G}$. Generally this is converted into a model by adding an interpretation v, mapping possible worlds and atomic formulas to (classical) truth values. Then v is extended to all formulas in the customary way, in particular setting $v(w, \Box A)$ to be *true* just in case $v(w', A) = \mathit{true}$ for all $w' \in \mathcal{G}$ for which $w\mathcal{R}w'$. It follows that $v(w, \Box A) = \mathit{false}$ if $v(w', A) = \mathit{false}$ for some $w' \in \mathcal{G}$ with $w\mathcal{R}w'$.

Now imagine that the interpretation v is allowed to be *partial*; at each world it maps some atomic formulas to *true*, others to *false*, and is undefined on still others. What is a natural way of extending the incomplete information contained in v to all formulas?

For the classical connectives this is straightforward. $A \wedge B$ should be taken to be *true* at a world if both A and B are *true* there, and *false* if one of them is *false*. This leaves $A \wedge B$ without a truth value in all other cases; for instance, if A is *true* but B lacks a truth value. See [1] for an extensive treatment of this partial logic.

The modal case can be dealt with along the same lines. Set $v(w, \Box A) = \mathit{true}$ if $v(w', A) = \mathit{true}$ for all $w' \in \mathcal{G}$ such that $w\mathcal{R}w'$. Set $v(w, \Box A) = \mathit{false}$ if $v(w', A) = \mathit{false}$ for some $w' \in \mathcal{G}$ with $w\mathcal{R}w'$. Otherwise $v(w, \Box A)$ is undefined. Then, for instance, if there is some $w' \in \mathcal{G}$ with $w\mathcal{R}w'$ such that $v(w', A)$ is undefined, and at all other worlds alternate to w, A evaluates to *true*, then $v(w, \Box A)$ will be undefined. A treatment of partial modal logic along these lines was suggested by Kripke in [5], where it was observed that his theory of truth could be extended naturally to it.

In order to fit partial modal logic into the present treatment the obvious approach is to treat the undefined case as if it were a third truth value. For the propositional connectives this yields a well known three-valued logic, generally called *Kleene's strong three-valued logic*, [4]. This logic can be presented most simply here by taking $\mathcal{T} = \{\mathit{false}, \bot, \mathit{true}\}$, with the ordering $\mathit{false} \leq \bot \leq \mathit{true}$. The third value, $\bot$, can be read as 'unknown' or 'undetermined.'

Incidentally, in Kleene's logic negation plays a major role, though it has not been discussed here so far. The negation operation switches *false* and *true*, and leaves $\bot$ unchanged. It can be introduced into the Gentzen system here very simply. Add a unary operator, $\neg$, and require the set of formulas to be closed under it. And add the following six obvious axioms.

$$
\begin{array}{ll}
\rightarrow \neg\mathit{true} \supset \mathit{false} & \rightarrow \mathit{false} \supset \neg\mathit{true} \\
\rightarrow \neg\bot \supset \bot & \rightarrow \bot \supset \neg\bot \\
\rightarrow \neg\mathit{false} \supset \mathit{true} & \rightarrow \mathit{true} \supset \neg\mathit{false}
\end{array}
$$

Now, what about the modal operator? The following is the three-valued version of the partial semantics sketched above. Take a valuation, v, to be a mapping from possible worlds and atomic formulas to the set $\{\mathit{false}, \bot, \mathit{true}\}$. Extend it to all formulas by evaluating $\wedge$, $\vee$ and $\neg$ at each world according to Kleene's logic, and using the following for the modal case:

$$v(w, \Box A) = \bigwedge\{v(w', A) \mid w\mathcal{R}w'\}.$$

Such an approach has been considered before, for instance in [13, 11] (in the latter, however, the underlying three-valued logic is one due to Łukasiewicz). In the next section we present the semantics in detail, and give an appropriate corresponding modification to the Gentzen system.

7 Many-valued modal logic, version I

In this section the technical details of a family of many-valued modal logics are presented, both via semantics and a Gentzen system. For starters, the syntax is extended in the obvious way. Throughout this section $\mathcal{T}$ is an arbitrary finite lattice, possibly distributive.

Definition 7.1 $L_{\mathcal{T}}[\Box]$ is the language defined like $L_{\mathcal{T}}$, but with the extra formation rule: if A is a formula, so is $\Box A$. Likewise $L_{\mathcal{T}}[\Box, \supset]$ allows closure of the set of formulas under both $\Box$ and $\supset$.

As was remarked earlier, the semantics is a direct modification of conventional possible world models.

Definition 7.2 A *binary modal model* is a structure $\langle \mathcal{G}, \mathcal{R}, v \rangle$ where $\mathcal{G}$ is a non-empty set of possible worlds, $\mathcal{R}$ is a (classical, two-valued) relation on $\mathcal{G}$, and v maps worlds and atomic formulas to $\mathcal{T}$, subject to the usual condition that members of $\mathcal{T}$ map to themselves.

The mapping v is extended to all formulas in the usual way, with the special condition:

$$v(w, \Box A) = \bigwedge \{v(w', A) \mid w \mathcal{R} w'\}$$

It is easy to see that if $\mathcal{T}$ is the two-valued lattice of classical logic, this semantics collapses to the usual Kripke version. Also, the necessity operator has been considered only — a possibility operator could also be introduced, with the semantic characterization:

$$v(w, A) = \bigvee \{v(w', A) \mid w \mathcal{R} w'\}.$$

This operator will not be investigated here.

Corresponding to the notion of possible world model is the following rule of inference.

Binary Necessitation Rule For $a_1, \ldots, a_n, b \in \mathcal{T}$, and formulas $A_1, \ldots, A_n, B$ (n may be 0):

$$\frac{a_1 \supset A_1, \ldots, a_n \supset A_n \rightarrow b \supset B}{a_1 \supset \Box A_1, \ldots, a_n \supset \Box A_n \rightarrow b \supset \Box B}$$

The validity of this rule will be discussed in the next section. Here are a few of the formal consequences of the rule.

Proposition 7.3

1. *for each* $t \in \mathcal{T}$, $\rightarrow t \supset \Box t$
2. $\dfrac{\rightarrow A \supset B}{\rightarrow \Box A \supset \Box B}$
3. $\rightarrow (\Box A \wedge \Box B) \supset \Box (A \wedge B)$
4. $\rightarrow \Box (A \wedge B) \supset (\Box A \wedge \Box B)$

Proof Item 1 is immediate from the Binary Necessitation Rule and $\rightarrow t \supset t$. For item 2 take t to be an arbitrary member of $\mathcal{T}$. Then:

$$\dfrac{\dfrac{\dfrac{t \supset A, A \supset B \rightarrow t \supset B \quad \rightarrow A \supset B}{t \supset A \rightarrow t \supset B}\text{ Cut}}{t \supset \Box A \rightarrow t \supset \Box B}\text{ Binary Necessitation}}{\rightarrow \Box A \supset \Box B}\text{ RI} \supset$$

Proofs of items 3 and 4 are omitted. ■

Now suppose nested implications are allowed, using the language $L_{\mathcal{T}}[\Box, \supset]$, and the rules from Section 5. Then several additional items of interest can be proved.

Proposition 7.4 *Let a be a member of $\mathcal{T}$ and A be an arbitrary formula. Then:*

1. $\rightarrow (a \supset \Box A) \supset \Box(a \supset A)$;
2. $\rightarrow \Box(a \supset A) \supset (a \supset \Box A)$.

Proof The proof of item 1, slightly abbreviated, is as follows. Let $t \in \mathcal{T}$ be arbitrary. Then beginning with an Implication Axiom:

$$\dfrac{\dfrac{\dfrac{(t \wedge a) \supset A \rightarrow t \supset (a \supset A)}{(t \wedge a) \supset \Box A \rightarrow t \supset \Box(a \supset A)}\ \text{Binary Necessitation}}{t \supset (a \supset \Box A) \rightarrow t \supset \Box(a \supset A)}\ \text{Implication Axiom}}{\rightarrow (a \supset \Box A) \supset \Box(a \supset A)}\ RI \supset$$

The proof of part 2 is simpler, uses items from Proposition 7.3, and is omitted. ∎

Next is a result that appears as a variation on the Binary Necessitation Rule — its significance will appear in a later section.

Theorem 7.5 *For $a_1, \ldots, a_n, b \in \mathcal{T}$, and formulas $A_1, \ldots, A_n, B$:*

$$\dfrac{\rightarrow (a_1 \supset A_1 \wedge \ldots \wedge a_n \supset A_n) \supset (b \supset B)}{\rightarrow (a_1 \supset \Box A_1 \wedge \ldots \wedge a_n \supset \Box A_n) \supset (b \supset \Box B)}$$

Proof This follows easily from the preceeding two propositions. ∎

8 Soundness and Completeness

The soundness of the Binary Necessitation Rule is straightforward. Suppose

$$a_1 \supset A_1, \ldots, a_n \supset A_n \rightarrow b \supset B$$

is valid in all binary modal models. And suppose $\langle \mathcal{G}, \mathcal{R}, v \rangle$ is a model, and $w \in \mathcal{G}$ is a world such that for $i = 1, \ldots, n$, $v(w, a_i \supset \Box A_i) = \mathit{true}$. Then $a_i = v(a_i) \leq v(\Box A_i)$, and by the conditions on $\Box$ in binary modal models, for each $w' \in \mathcal{T}$ with $w\mathcal{R}w'$, $a_i \leq v(w', A_i)$, so that $v(w', a_i \supset A_i) = \mathit{true}$. Then by the assumption above, $v(w', b \supset B) = \mathit{true}$, so $b \leq v(w', B)$. Since this is the case for each such w', it follows that $b \leq v(w, \Box B)$, and so $v(w, b \supset \Box B) = \mathit{true}$. Thus the validity of the following has been established

$$a_1 \supset \Box A_1, \ldots a_n \supset \Box A_n \rightarrow b \supset \Box B.$$

Completeness, as usual, is more work, though the non-modal many-valued argument extends directly.

Theorem 8.1 *The modal system with the Binary Necessitation Rule is complete with respect to binary modal models.*

Proof Most of the details are omitted. Take $\mathcal{G}$ to be the collection of all maximal X-consistent sets, for all implications X. For each $w \in \mathcal{G}$ and each formula A, set $v_0(w, A)$ to be the unique $a \in \mathcal{T}$ such that $A \supset a$ and $a \supset A$ are in w. For $w, w' \in \mathcal{G}$, set $w\mathcal{R}w'$ if $v_0(w, \Box A) \leq v_0(w', A)$ for all formulas A. Finally, set $v(w, A) = v_0(w, A)$ for atomic A.

The key item to be verified is that v and v_0 agree on all formulas, not just on the atomic ones. This is done by an induction on degree. Only one case is presented: consider the formula $\Box A$, and assume the result is known for A.

1) If $w\mathcal{R}w'$ then $v_0(w, \Box A) \leq v_0(w', A)$. So

$$\begin{array}{rcl} v_0(w, \Box A) & \leq & \bigwedge\{v_0(w', A) \mid w\mathcal{R}w'\} \\ & = & \bigwedge\{v(w', A) \mid w\mathcal{R}w'\} \\ & = & v(w, \Box A) \end{array}$$

2) Suppose $v(w, \Box A) \not\leq v_0(w, \Box A)$. Set $w_0' = \{v_0(w, \Box C) \supset C \mid \text{all formulas } C\}$. Then w_0' is $v(\Box A) \supset A$-consistent, for otherwise, for some $v_0(w, \Box C_i) \supset C_i$ in w_0', we would have:

$$v_0(w, \Box C_1) \supset C_1, \ldots, v_0(w, \Box C_n) \supset C_n \rightarrow v(w, \Box A) \supset A$$

so by Binary Necessitation,

$$v_0(w, \Box C_1) \supset \Box C_1, \ldots, v_0(w, \Box C_n) \supset \Box C_n \rightarrow v(w, \Box A) \supset \Box A$$

but all implications on the left are members of w, so $v(\Box A) \supset \Box A \in w$, which implies that $v(w, \Box A) \leq v_0(w, \Box A)$, contrary to assumption.

Now extend w_0' to a maximal $v(\Box A) \supset A$-consistent set, w'. Then $w' \in \mathcal{T}$. Also, for an arbitrary formula C, $v_0(w, \Box C) \supset C \in w'$, and so $v_0(w, \Box C) \leq v_0(w', C)$, and thus $w\mathcal{R}w'$. Then:

$$\begin{array}{rcl} v(w, \Box A) & = & \bigwedge\{v(w', A) \mid w\mathcal{R}w'\} \\ & \leq & v(w', A) \\ & = & v_0(w', A) \end{array}$$

But, by construction, $v(w, \Box A) \supset A \notin w'$, so $v(w, \Box A) \not\leq v_0(w', A)$, a contradiction.

3) Now, the completeness proof is finished in the usual way. If an implication, X, is not provable, $\emptyset$ is X-consistent. Extend it to a maximal X-consistent set, w. Then $w \in \mathcal{G}$, and it follows from the argument above that $v(w, X)$ will not be true. ■

The proof above extends readily to allow nested implications, provided $\mathcal{T}$ has pseudo-complements, that is, provided $\mathcal{T}$ is distributive.

9 A Different Example

In the next section a system of modal logic is presented that allows models to have many-valued accessibility relations. Before getting to the technical details, an example is in order.

Suppose there are two experts, Rosencrantz and Guildenstern (R and G), who are being asked to pass judgement on the truth of various statements, in various situations. Then a natural truth-value space to work with is a four-valued one: neither says true; R says true but G does not; G says true but R does not; and both say true. Truth values can be identified with subsets of $\{R, G\}$. Now, two kinds of judgements are possible: 1) A is true in situation w; and 2) w is a situation that should be considered. The first kind of judgement can be thought of as assigning a truth value to A at the possible world w. The second kind of judgement amounts to a many-valued accessibility relation.

Suppose, for instance, that there are three situations; 'this world,' and two others, w_1 and w_2. Suppose both R and G say w_1 should be considered, but only R says w_2 should be. Suppose also that only G says A would be true in situation w_1, and nobody says A would be true in situation w_2. The question is, how should "A is true no matter what," that is, $\Box A$, be evaluated in this world?

It seems clear that the value of $\Box A$ in this world should be the meet of values calculated for each accessible world; in a sense, it should be what is common to all alternative situations. As far as w_1 goes, everybody says it should be considered, but only G says A is true there. Intuitively, from R we get a no, and from G a yes, as far as the world w_1 is concerned. Thus w_1 contributes $\{G\}$. For w_2, G does not say it should be considered at all. Then for G it can not serve as a counter-example, so in effect w_2 counts as a 'yes' for G. For R, on the other hand, w_2 should be considered, but A is false there, so w_2 counts as a 'no' for R. Thus w_2 also contributes $\{G\}$, and so $\Box A$ in this world should be given the value $\{G\}$.

On closer examination, the informal evaluation above really amounts to using the following rule. The truth value of $\Box A$ is the intersection, over all worlds, of the truth value of A at an alternative world union the complement of the accessibility value of that alternative world. If we write $\mathcal{R}(w, w')$ for the truth value of the accessibility relation, $\overline{\mathcal{R}}(w, w')$ for its complement, and $v(w, A)$ for the truth value of A at w, the rule amounts to this.

$$v(w, \Box A) = \bigwedge\{\overline{\mathcal{R}}(w, w') \vee v(w', A) \mid \text{ all } w'\}.$$

To place this in context, note that the truth-value space above is a powerset algebra, and hence is distributive. Distributive finite lattices have relative pseudo-complement operations. Further, a natural notion of negation can be defined in any pseudo-boolean algebra: $-a = (a \Rightarrow \mathit{false})$. In this case it is easy to check that the negation operation is the same as set-theoretic complementation. Finally, it is a standard result about pseudo-boolean algebras that when negation meets the condition that $a \vee -a = \mathit{true}$ is an identity, then $a \Rightarrow b = -a \vee b$, [9]. Consequently, the evaluation rule given above is equivalent to the following:

$$v(w, \Box A) = \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', A) \mid \text{ all } w'\}.$$

Finally it is this that is taken as definitive in what follows.

10 Many-valued modal logic, version II

For this section, assume $\mathcal{T}$ is a finite distributive lattice (hence with a relative pseudo-complement operation), and the language is $L[\Box, \supset]$, allowing nested implications.

Definition 10.1 An *implicational modal model* is a structure $\langle \mathcal{G}, \mathcal{R}, v\rangle$ where $\mathcal{G}$ is a non-empty set of possible worlds, $\mathcal{R}$ is a mapping from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{T}$, and v maps worlds and atomic formulas to $\mathcal{T}$, again subject to the usual condition that members of $\mathcal{T}$ map to themselves.

The mapping $\mathcal{R}$ can be thought of as a many-valued relation between possible worlds. The mapping v is extended to all formulas as usual, with the following special condition:

$$v(w, \Box A) = \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', A) \mid w' \in \mathcal{G}\}$$

It is straightforward to verify that if $\mathcal{R}$ only takes on the values *true* and *false*, an implicational modal model can be identified with a binary modal model in the obvious way. It follows that any sequent that is valid in all implicational modal models is also valid in all binary ones. The converse is not true, at least for sequents involving constants. For instance, if $c \in \mathcal{T}$ is different from *true* or *false*, the sequent $\rightarrow true \supset \Box c, \Box c \supset c$ will be valid in all binary modal models, but examples can be produced to show it is not valid in all implicational ones.

The next job is to modify the basic Gentzen system to reflect validity in implicational models. For this purpose, in place of the Binary Necessitation Rule, use the following.

Implicational Necessitation Rule For $a_1, \ldots, a_n, b \in \mathcal{T}$, and formulas $A_1, \ldots, A_n$, B:

$$\frac{\rightarrow (a_1 \supset A_1 \wedge \ldots \wedge a_n \supset A_n) \supset (b \supset B)}{\rightarrow (a_1 \supset \Box A_1 \wedge \ldots \wedge a_n \supset \Box A_n) \supset (b \supset \Box B)}$$

The number n is allowed to be 0, in which case the rule is taken as

$$\frac{\rightarrow b \supset B}{\rightarrow b \supset \Box B}$$

Notice that Theorem 7.5 says Implicational Necessitation is a derived rule in the system allowing the Binary Necessitation Rule.

Soundness of the Implicational Necessitation Rule, with respect to implicational modal models, is most easily shown indirectly. It is an easy consequence of the following.

Proposition 10.2 *The following are sound rules, or valid sequents, with respect to implicational modal models.*

1. $\dfrac{\rightarrow A \supset B}{\rightarrow \Box A \supset \Box B}$
2. *for each* $a \in \mathcal{T}$, $\rightarrow \Box(a \supset A) \supset (a \supset \Box A)$
3. *for each* $a \in \mathcal{T}$, $\rightarrow (a \supset \Box A) \supset \Box(a \supset A)$
4. $\rightarrow (\Box A \wedge \Box B) \supset \Box(A \wedge B)$

Proof We show two of the items simultaneously, the other two are straightforward. The following makes use of fundamental properties of relative pseudo-complements, see [9]. Let $a \in \mathcal{T}$; then:

$$\begin{aligned}
v(w, \Box(a \supset A)) &= \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', a \supset A) \mid w' \in \mathcal{G}\} \\
&= \bigwedge\{\mathcal{R}(w, w') \Rightarrow (v(w', a) \Rightarrow v(w', A)) \mid w' \in \mathcal{G}\} \\
&= \bigwedge\{v(w', a) \Rightarrow (\mathcal{R}(w, w') \Rightarrow v(w', A)) \mid w' \in \mathcal{G}\} \\
&= \bigwedge\{a \Rightarrow (\mathcal{R}(w, w') \Rightarrow v(w', A)) \mid w' \in \mathcal{G}\} \\
&= a \Rightarrow \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', A) \mid w' \in \mathcal{G}\} \\
&= v(w, a) \Rightarrow v(w, \Box A) \\
&= v(w, a \supset \Box A)
\end{aligned}$$

■

Completeness is more work, naturally. Only the basic ideas will be sketched here — much detail is omitted. First, a somewhat different notion of consistency is needed. The following definition is suitable for this section, and replaces the earlier notion of consistency.

Definition 10.3 A set S is X-*inconsistent* if, for some finite $\{Y_1, \ldots, Y_n\} \subseteq S$, the sequent $\longrightarrow (Y_1 \wedge \ldots \wedge Y_n) \supset X$ is provable. S is X-*consistent* if it is not X-inconsistent.

A set S that is X-consistent can be extended to a maximal one, S', and it will be the case that, for each for each formula Z there will be exactly one member $t \in \mathcal{T}$ such that $Z \supset t$ and $t \supset Z$ are both in S'. The proof of this is via a reduction to Theorem 4.1 but is omitted here since it is both technical and simple.

Theorem 10.4 *The modal system with the Implicational Necessitation Rule is complete with respect to implicational modal models.*

Proof Again take $\mathcal{G}$ to be the collection of all maximal X-consistent sets, for all X (but with the notion of consistency as defined for this section). For $w \in \mathcal{G}$, set $v_0(w, A)$ to be the unique $a \in \mathcal{T}$ such that $A \supset a$ and $a \supset A$ are both in w. And for $w, w' \in \mathcal{G}$, set $\mathcal{R}(w, w') = \bigwedge\{v_0(w, \Box A) \Rightarrow v_0(w', A) \mid \text{all formulas } A\}$. Finally, set v to be the same as v_0 on atomic formulas. This determines an implicational modal model.

Following the usual style of completeness proof, the heart of the argument is to show that v and v_0 agree on all formulas. Once this is shown, the remaining parts are routine. It is this part alone that is presented here. And in fact, only one case is considered, that of necessitation.

Assume that, for some formula A, and for all worlds $w \in \mathcal{G}$, $v(w, A) = v_0(w, A)$. We show $v(w, \Box A) = v_0(w, \Box A)$.

First, by definition, $\mathcal{R}(w, w') \leq v_0(w, \Box A) \Rightarrow v_0(w', A)$. It follows by standard properties of relative pseudo-complements that $v_0(w, \Box A) \leq \mathcal{R}(w, w') \Rightarrow v_0(w', A)$. So

$$\begin{aligned} v(w, \Box A) &= \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', A) \mid w' \in \mathcal{G}\} \\ &= \bigwedge\{\mathcal{R}(w, w') \Rightarrow v_0(w', A) \mid w' \in \mathcal{G}\} \\ &\geq v_0(w, \Box A) \end{aligned}$$

That is, $v_0(w, \Box A) \leq v(w, \Box A)$. Now suppose $v(w, \Box A) \not\leq v_0(w, \Box A)$; we derive a contradiction.

Let $w' = \{v_0(w, \Box B) \supset B \mid \text{ all formulas } B\}$. The claim is, w' is $v(w, \Box A) \supset A$-consistent. For if not, there are formulas $B_1, \ldots, B_n$ such that

$$\longrightarrow [(v_0(w, \Box B_1) \supset B_1) \wedge \ldots \wedge (v_0(w, \Box B_n) \supset B_n)] \supset [v(w, \Box A) \supset A]$$

but then by the Implicational Necessitation Rule,

$$\longrightarrow [(v_0(w, \Box B_1) \supset \Box B_1) \wedge \ldots \wedge (v_0(w, \Box B_n) \supset \Box B_n)] \supset [v(w, \Box A) \supset \Box A]$$

But each implication in the hypothesis part is in w, and it follows that $v(w, \Box A) \supset \Box A \in w$. But this implies that $v(w, \Box A) \leq v_0(w, \Box A)$, contradicting the assumption. Thus w' is $v(w, \Box A) \supset A$-consistent.

Extend w' to w'', maximal $v(w, \Box A) \supset A$-consistent. Now, $v_0(w, \Box B) \supset B$ is in w'' for each B, so $v_0(w, \Box B) \leq v_0(w'', B)$, and hence by standard properties of relative pseudo-complement, $v_0(w, \Box B) \Rightarrow v_0(w'', B) = \mathit{true}$. Since B is arbitrary, $\mathcal{R}(w, w'') = \mathit{true}$.

Further, $v(w, \Box A) \supset A$ is not in w'', so $v(w, \Box A) \not\leq v_0(w'', A)$. But,

$$\begin{array}{rcl} v(w, \Box A) & = & \bigwedge\{\mathcal{R}(w, w') \Rightarrow v(w', A) \mid w' \in \mathcal{G}\} \\ & \leq & \mathcal{R}(w, w'') \Rightarrow v(w'', A) \\ & = & \mathcal{R}(w, w'') \Rightarrow v_0(w'', A) \\ & = & v_0(w'', A) \end{array}$$

and this is a contradiction.

We have shown that $v(w, \Box A) \leq v_0(w, \Box A)$, and so $v(w, \Box A) = v_0(w, \Box A)$. ■

11 Conclusion

The material presented in this paper is only a beginning on what is really needed. The most glaring omission is that not enough intuitive feeling for these logics has been presented. What are their distinguishing characteristics? How do they differ from each other? Frankly, I wish I could say more about this. Investigation of many-valued modal logics, especially allowing many-valued accessibility, is at an early stage, and answers to important questions like these must wait on more experience. I invite others to investigate here.

All the logics considered above are analogs of the modal logic K. No special conditions are placed on the accessibility relations. For the binary version, conditions like transitivity, reflexivity, and so on, are straightforward. For the version allowing accessibility to be many-valued, though, analogs of these conditions are available. For instance, a requirement that $\mathcal{R}(w, w') \wedge \mathcal{R}(w', w'') \leq \mathcal{R}(w, w'')$ is a natural generalization of transitivity. I do not know what effect imposing conditions like these will have. Work remains here too.

Finally the proof procedures given above, though based on Gentzen systems, are far from automatable. It seems clear that the Cut Rule is not eliminable, for instance. Still, automatable many-valued proof procedures based on different principles are possible, see [2, 3]. I have made some progress on extending these notions to the modal case, but presentation of the results must wait on further research.

Appendix

In this appendix are presented the results needed to establish completeness of the non-modal many-valued sequent calculus (Section 4). The material is divided into two sections, the first containing general lattice-theoretic results, the second a derived rule of the sequent calculus.

12 Spanning sets

Completeness will follow primarily from a derived rule given in the next section. Since all that is being assumed about $\mathcal{T}$ is that it is a finite lattice, some way is needed for grouping members so that results about $\mathcal{T}$ can be proved uniformly. That is the purpose of this section. First, some convenient general terminology.

Definition 12.1 For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$, $\mathcal{A} \leq \mathcal{B}$ means: for every $a \in \mathcal{A}$ there is some $b \in \mathcal{B}$ with $a \leq b$, and for every $b \in \mathcal{B}$ there is some $a \in \mathcal{A}$ with $a \leq b$. (This is sometimes called the Egli-Milnor ordering.) The notation $\mathcal{A} < \mathcal{B}$ means $\mathcal{A} \leq \mathcal{B}$ but not $\mathcal{B} \leq \mathcal{A}$.

It is easy to check that this gives a transitive relation on subsets of $\mathcal{T}$, but not an antisymmetric one. Also $\{\mathit{false}\}$ is smallest in this relation, and $\{\mathit{true}\}$ is biggest.

Definition 12.2 Two members $a, b \in \mathcal{T}$ are *comparable* if $a \leq b$ or $b \leq a$; a and b are *incomparable* if they are not comparable. A set $\mathcal{A} \subseteq \mathcal{T}$ is an *antichain* if any two distinct members are incomparable.

It is straightforward to prove that, if $\mathcal{A}$ and $\mathcal{B}$ are antichains and $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$, then $\mathcal{A} = \mathcal{B}$. Thus on antichains of $\mathcal{T}$ we have a partial ordering.

Definition 12.3 A set $\mathcal{A} \subseteq \mathcal{T}$ is a *spanning set* if $\mathcal{A}$ is an antichain and every member of $\mathcal{T}$ is comparable with some member of $\mathcal{A}$.

There are many spanning sets. The sets $\{\mathit{false}\}$ and $\{\mathit{true}\}$ are spanning sets. Also, any *maximal* antichain (maximal under the subset relation) is a spanning set. Consequently, every antichain can be extended to a spanning set.

Definition 12.4 If $\mathcal{A} \subseteq \mathcal{T}$ is a spanning set, b is *below* $\mathcal{A}$ if $b \leq a$ for some $a \in \mathcal{A}$, and b is *above* $\mathcal{A}$ if $a \leq b$ for some $a \in \mathcal{A}$. Likewise, b is *strictly below* $\mathcal{A}$ if b is below $\mathcal{A}$ but $b \notin \mathcal{A}$; similarly for *strictly above*.

For a spanning set $\mathcal{A}$, by the comparability condition, every member of $\mathcal{T}$ is either above or below $\mathcal{A}$. And it follows from the independence condition that if a member of $\mathcal{T}$ is both above and below $\mathcal{A}$, it is in $\mathcal{A}$. Thus, for each $a \in \mathcal{T}$, *exactly one of*: a is in $\mathcal{A}$, a is strictly above $\mathcal{A}$, a is strictly below $\mathcal{A}$.

Proposition 12.5 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are spanning sets in $\mathcal{T}$ with $\mathcal{A} \leq \mathcal{B}$. And suppose some member of $\mathcal{T}$ is strictly above $\mathcal{A}$ and strictly below $\mathcal{B}$. Then there is a spanning set $\mathcal{C}$ such that $\mathcal{A} < \mathcal{C} < \mathcal{B}$.*

Proof A member of $\mathcal{T}$ that is strictly above $\mathcal{A}$ and strictly below $\mathcal{B}$ will be referred to as being *strictly between* $\mathcal{A}$ and $\mathcal{B}$. Now, construct a set $\mathcal{C}$ in stages, as follows. Let $\mathcal{C}_0$ be a maximal antichain in $\mathcal{T}$ whose members are strictly between $\mathcal{A}$ and $\mathcal{B}$. By hypothesis, $\mathcal{C}_0$ is not empty. Next, extend $\mathcal{C}_0$ by adding members of $\mathcal{A}$, one at a time, in any order, until no more can be added while preserving the antichain property; call the resulting set $\mathcal{C}_1$. Finally, extend $\mathcal{C}_1$ by adding members of $\mathcal{B}$, again until no more can be added while preserving the antichain property. Call the resulting set $\mathcal{C}$.

By construction, $\mathcal{C}$ is an antichain. Every member of $\mathcal{T}$ is comparable with $\mathcal{C}$, by the following argument. Suppose $x \in \mathcal{T}$. By remarks above, if x is not strictly between $\mathcal{A}$ and $\mathcal{B}$, x must be below $\mathcal{A}$ or above $\mathcal{B}$. Consequently the argument divides into three parts.

1) Suppose x is strictly between $\mathcal{A}$ and $\mathcal{B}$. If x is in $\mathcal{C}_0$ we are done. If not, it must be comparable with some member of $\mathcal{C}_0$, since $\mathcal{C}_0$ was a *maximal antichain* in $\mathcal{T}$. Either way, x is comparable with $\mathcal{C}$.

2) Suppose x is below $\mathcal{A}$. Then $x \leq a$ for some $a \in \mathcal{A}$. If $a \in \mathcal{C}_1$, x is below $\mathcal{C}_1$ and hence below $\mathcal{C}$. If $a \notin \mathcal{C}_1$, adding it would yield a set that is not an antichain.

Since members of $\mathcal{A}$ are incomparable, it must be that a and c are comparable for some $c \in \mathcal{C}_0$. If $a \le c$ then $x \le c$, and hence x is below $\mathcal{C}$. If $c \le a$ a contradiction can be derived, as follows. Since c is strictly above $\mathcal{A}$, $a' \le c$ for some $a' \in \mathcal{A}$. Then $a' \le a$, and from the fact that $\mathcal{A}$ is an antichain it follows that $a' = a$. Then $c = a \in \mathcal{A}$, contradicting the fact that c is *strictly* above $\mathcal{A}$.

3) Finally, suppose x is above $\mathcal{B}$. Then $b \le x$ for some $b \in \mathcal{B}$. If $b \in \mathcal{C}$, x is above $\mathcal{C}$ and we are done. If $b \notin \mathcal{C}$, b is comparable with some member of $\mathcal{C}$. Since $\mathcal{B}$ is an antichain, b must be comparable with some member of $\mathcal{C}_1$. Now the case divides in two.

3a) Suppose b is comparable with some member c of $\mathcal{C}_0$. If $c \le b$ then $c \le x$, so x is above $\mathcal{C}$. The alternative, $b \le c$, is impossible, by an argument similar to that given in case *2)*.

3b) Suppose b is comparable with some member c of $\mathcal{C}_1 - \mathcal{C}_0$. If $c \le b$ then $c \le x$ so again x is above $\mathcal{C}$. Now suppose $b \le c$. Since $c \in \mathcal{C}_1 - \mathcal{C}_0$, $c \in \mathcal{A}$, and since $\mathcal{A} \le \mathcal{B}$, for some $b' \in \mathcal{B}$ we have $c \le b'$. Then $b \le c \le b'$, and since $\mathcal{B}$ is an antichain, it follows that $b = c = b'$. Since $b \le x$, $c \le x$, so x is above $\mathcal{C}$ in this case also.

Finally it must be shown that $\mathcal{A} < \mathcal{C} < \mathcal{B}$. An argument to establish that $\mathcal{A} \le \mathcal{C} \le \mathcal{B}$ is along lines similar to that above. If $\mathcal{C} \le \mathcal{A}$, then we would have $\mathcal{A} = \mathcal{C}$, which is impossible since $\mathcal{C}$ contains members strictly above $\mathcal{A}$. Thus $\mathcal{A} < \mathcal{C}$. Similarly $\mathcal{C} < \mathcal{B}$. ■

Now, suppose $\mathcal{B}$ is a spanning set, and $\mathcal{B} \ne \{\textit{false}\}$. Then there are spanning sets $< \mathcal{B}$, in particular, $\{\textit{false}\}$. Since $\mathcal{T}$ is finite, there must be some spanning set $\mathcal{A} < \mathcal{B}$ that is maximal (in the $\le$ ordering) among spanning sets that are $< \mathcal{B}$; such an $\mathcal{A}$ is said to be *immediately beneath* $\mathcal{B}$. By Proposition 12.5, if $\mathcal{A}$ is immediately beneath $\mathcal{B}$, there can be no members of $\mathcal{T}$ strictly between $\mathcal{A}$ and $\mathcal{B}$. Thus the following has been established, a result which plays a fundamental role in the next section.

Proposition 12.6 *If $\mathcal{B}$ is a spanning set different from $\{\textit{false}\}$, there is a spanning set $\mathcal{A}$ immediately beneath $\mathcal{B}$ and, for every $x \in \mathcal{T}$, either x is below $\mathcal{A}$ or x is above $\mathcal{B}$.*

13 A Fundamental derived rule

This section is devoted to the statement and proof of a derived rule that plays a basic role in the completeness proof of Section 4. The first item is a Lemma about spanning sets, amounting to a formalized version of part of Proposition 12.6.

Lemma 13.1 *Suppose $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_k\}$ are two spanning sets in $\mathcal{T}$, with $\mathcal{A}$ immediately beneath $\mathcal{B}$. Then the following sequent is provable, for any formula A:*

$$\rightarrow A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A.$$

Proof It is simplest to present the proof backwards. We want to prove

$$\rightarrow A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A.$$

Using Rule $RI \supset$, this would follow if we could show, for every $c_1 \in \mathcal{T}$,

$$c_1 \supset A \rightarrow c_1 \supset a_1, A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A.$$

If c_1 is above $\mathcal{B}$, then for some i, $b_i \leq c_1$, and then $\rightarrow b_i \supset c_1$. But $b_i \supset c_1, c_1 \supset A \rightarrow b_i \supset A$, from which the desired sequent is obtained by Cut and Thinning. Thus it remains to show the sequent for c_1 not above $\mathcal{B}$, which implies that c_1 is below $\mathcal{A}$. If $c_1 \leq a_1$ we easily get the sequent by Thinning, from $\rightarrow c_1 \supset a_1$. We are still left with the remaining cases. To summarize: we are done if we can show

$$c_1 \supset A \rightarrow c_1 \supset a_1, A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A$$

under the assumption that c_1 is below $\mathcal{A}$ but $c_1 \not\leq a_1$.

The same argument can be repeated using Rule $RI \supset$ with each of $A \supset a_2, \ldots, A \supset a_n$. This reduces the problem to showing:

$$c_1 \supset A, \ldots, c_n \supset A \rightarrow c_1 \supset a_1, \ldots, c_n \supset a_n, A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A$$

where each of $c_1, \ldots, c_n$ is below $\mathcal{A}$, but for each i, $c_i \not\leq a_i$.

Next, a similar procedure can be applied to each of $b_1 \supset A, \ldots, b_k \supset A$, this time using Rule $\supset RI$, reducing the problem to showing:

$$\begin{array}{lll} c_1 \supset A, \ldots, c_n \supset A, A \supset d_1, \ldots, A \supset d_k & \rightarrow & c_1 \supset a_1, \ldots, c_n \supset a_n, \\ & & b_1 \supset d_1, \ldots, b_k \supset d_k, \\ & & A \supset a_1, \ldots, A \supset a_n, \\ & & b_1 \supset A, \ldots, b_k \supset A \end{array}$$

where each of $c_1, \ldots, c_n$ is below $\mathcal{A}$, but for each i, $c_i \not\leq a_i$, and each of $d_1, \ldots, d_k$ is above $\mathcal{B}$, but for each j, $b_j \not\leq d_j$. In fact, this will be established by showing that, under these restrictions on c_i and d_j,

$$c_1 \supset A, \ldots, c_n \supset A, A \supset d_1, \ldots, A \supset d_k \rightarrow .$$

Note that, using Conjunction and Disjunction Axioms,

$$c_1 \supset A, \ldots, c_n \supset A, A \supset d_1, \ldots, A \supset d_k \rightarrow c_1 \vee \ldots \vee c_n \supset d_1 \wedge \ldots \wedge d_k$$

so it is enough to prove

$$c_1 \vee \ldots \vee c_n \supset d_1 \wedge \ldots \wedge d_k \rightarrow$$

and by a Propositional Constant Axiom, this will follow provided

$$c_1 \vee \ldots \vee c_n \not\leq d_1 \wedge \ldots \wedge d_k$$

so it is this, finally, that must be established.

Suppose $c_1 \vee \ldots \vee c_n$ were below $\mathcal{A}$. Then, for some i, $c_1 \vee \ldots \vee c_n \leq a_i$. But then, $c_i \leq a_i$, contrary to the conditions above. Consequently $c_1 \vee \ldots \vee c_n$ is not below $\mathcal{A}$ so, by Proposition 12.6, $c_1 \vee \ldots \vee c_n$ must be above $\mathcal{B}$. Then $b_j \leq c_1 \vee \ldots \vee c_n$ for some j. Now, if we had $c_1 \vee \ldots \vee c_n \leq d_1 \wedge \ldots \wedge d_k$, it would follow that $b_j \leq d_j$, again contrary to the conditions above. Consequently we do not have $c_1 \vee \ldots \vee c_n \leq d_1 \wedge \ldots \wedge d_k$, which concludes the proof. ■

Now it is possible to prove Theorem 4.1, which is restated here for convenience.

Theorem 4.1 *The following is a derived rule:*

$$\frac{\Gamma, A \supset t, t \supset A \rightarrow \Delta \quad (\text{for all } t \in \mathcal{T})}{\Gamma \rightarrow \Delta}$$

Proof Assume, for the rest of this proof, that $\Gamma, A \supset t, t \supset A \to \Delta$ is provable, for each $t \in \mathcal{T}$. It will be shown that $\Gamma \to \Delta$ also has a proof. Spanning sets are the chief tool here.

For this proof only, call a spanning set $\mathcal{A}$ *good* if $\Gamma, a \supset A \to \Delta$ has a proof, for each $a \in \mathcal{A}$.

Using the partial ordering, $\leq$, on spanning sets, defined in Section 12, the biggest spanning set is $\{true\}$. It is good, a fact that follows from our hypothesis, and from Proposition 3.5 as follows:

$$\frac{\Gamma, A \supset true, true \supset A \to \Delta \quad \to A \supset true}{\Gamma, true \supset A \to \Delta}\ \text{Cut}$$

Next, suppose $\mathcal{B} = \{b_1, b_2, \ldots, b_k\}$ is a good spanning set, and $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ is a spanning set that is immediately beneath $\mathcal{B}$. Then $\mathcal{A}$ is also good, by the following argument. First, by Lemma 13.1 the following sequent is provable:

$$\to A \supset a_1, \ldots, A \supset a_n, b_1 \supset A, \ldots, b_k \supset A.$$

Since $\mathcal{B}$ is good, for each $j = 1, 2, \ldots, k$ the following sequent is provable:

$$\Gamma, b_j \supset A \to \Delta.$$

Then k applications of Cut yield:

$$\Gamma \to \Delta, A \supset a_1, \ldots, A \supset a_n.$$

Next, using the proof hypothesis,

$$\frac{\Gamma, A \supset a_1, a_1 \supset A \to \Delta \quad \Gamma \to \Delta, A \supset a_1, \ldots, A \supset a_n}{\Gamma, a_1 \supset A \to \Delta, A \supset a_2, \ldots, A \supset a_n}\ \text{Cut}$$

Also members of $\mathcal{A}$ are incomparable, so we have the following:

$$\frac{a_1 \supset A, A \supset a_2 \to a_1 \supset a_2 \quad a_1 \supset a_2 \to}{a_1 \supset A, A \supset a_2 \to}\ \text{Cut}$$

Combining these,

$$\frac{\Gamma, a_1 \supset A \to \Delta, A \supset a_2, \ldots, A \supset a_n \quad a_1 \supset A, A \supset a_2 \to}{\Gamma, a_1 \supset A \to \Delta, A \supset a_3, \ldots, A \supset a_n}\ \text{Cut}$$

Proceeding in this way, each of a_3, ..., a_n can be 'eliminated,' thus establishing the provability of:

$$\Gamma, a_1 \supset A \to \Delta.$$

But the choice of a_1 was arbitrary. A similar argument shows the provability, for each $i = 1, 2, \ldots, n$ of

$$\Gamma, a_i \supset A \to \Delta$$

and thus $\mathcal{A}$ is good.

Now, start with $\{true\}$. Choose a spanning set immediately beneath it, one immediately beneath that, and so on. In this way a descending sequence of spanning sets is produced. But since $\mathcal{T}$ is finite, such a sequence must terminate; a spanning set must be reached with no other immediately beneath it. By Proposition 12.6, the only such spanning set is $\{false\}$, so this is where the sequence terminates. Now, the first member, $\{true\}$, of the sequence is good, as was shown earlier. Then each member of the sequence must be good, by the argument above, and so $\{false\}$ must be good, and thus $\Gamma, false \supset A \rightarrow \Delta$ is provable. This, combined with Proposition 3.5, concludes the argument.

$$\frac{\Gamma, false \supset A \rightarrow \Delta \quad \rightarrow false \supset A}{\Gamma \rightarrow \Delta}\ \text{Cut}$$

■

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