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Many-valued Non-deterministic Semantics for First-order Logics of Formal (In)consistency

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Abstract. A *paraconsistent logic* is a logic which allows non-trivial inconsistent theories. One of the oldest and best known approaches to the problem of designing useful paraconsistent logics is da Costa's approach, which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. da Costa's approach has led to the family of Logics of Formal (In)consistency (LFIs). In this paper we provide non-deterministic semantics for a very large family of first-order LFIs (which includes da Costa's original system C_1^*, as well as thousands of other logics). We show that our semantics is effective and modular, and we use this effectiveness to derive some important properties of logics in this family.

1 Introduction

The concept of paraconsistency was introduced more than half a century ago, when several philosophers questioned the validity of classical logic with regard to its *ex contradictione quodlibet* (ECQ) principle. According to this counterintuitive principle, any proposition can be inferred from any inconsistent set of assumptions. Now the philosophical objections to this principle have recently been reinforced by practical considerations concerning information systems. Classical logic simply fails to capture the fact that information systems which contain some inconsistent pieces of information may produce useful answers to queries. The obvious conclusion from this state of affairs is that a more appropriate logic is needed for such systems. Thus [15] says:

> *Informally speaking, paraconsistency is the paradigm of reasoning in the presence of inconsistency. Classical logic intolerantly invalidates any useful reasoning if there is any inconsistency, no matter how irrelevant it may be. However, inconsistencies, as unpleasant and dangerous as they can be, are ubiquitous in information systems. For novel technology which often is not sufficiently mature before being launched on the market, the risk of inconsistencies is even higher. Hence, a thoroughly revised inconsistency-tolerant logic is needed for databases and information systems, also because many future applications (e.g., the self-organizing cognitive evolution of networked information systems, involving negotiation, argumentation, diagnosis, learning, etc.) are likely to deal directly with inconsistencies as inherent constituents of real-life situations.*

A paraconsistent logic is a logic that allows contradictory, yet non-trivial, theories. There are several approaches to the problem of designing useful paraconsistent logics (see, e.g. [6, 9, 7]). One of the best known is da Costa's approach ([12, 10, 11]), which has led to the family of *Logics of Formal Inconsistency* (LFIs). This family is based on two main ideas. First of all, propositions are divided into two sorts: the "normal" (or "consistent") and the "abnormal" (or "inconsistent") ones. The second idea is to express the meta-theoretical notions of consistency/inconsistency at the object language level, by including in it a (primitive or defined) connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". (Sometimes the dual connective $\bullet$, expressing inconsistency is used, see e.g. [8, 11]). Using the consistency operator, one can limit the applicability of the rule $\varphi, \neg\varphi \vdash \psi$ (capturing the ECQ principle) to the case when φ is consistent (i.e., $\varphi, \neg\varphi, \circ\varphi \vdash \psi$).

Although the syntactic formulations of the LFIs are relatively simple, already on the propositional level the problem of finding useful semantic interpretations for them is rather complicated. Thus the vast majority of the propositional LFIs *cannot* be characterized by means of finite multi-valued matrices. What is more, for almost all of them no useful infinite characteristic matrix is known either. Therefore other types of semantics, like bivaluations semantics and possible translations semantics, have been proposed for them ([10, 11]). However, it is not clear how to extend these types of semantics to the first-order level.

An alternative framework for providing semantics for propositional paraconsistent logics was introduced in [1] (and used in [2, 3, 4]). This framework uses a generalization of the standard multi-valued matrices, called *non-deterministic matrices* (*Nmatrices*). Nmatrices are multi-valued structures, in which the value assigned by a valuation to a complex formula can be chosen *non-deterministically* out of a certain nonempty set of options. The framework of Nmatrices has a number of attractive properties. First of all, the semantics provided by Nmatrices is *modular*: the main effect of each of the rules of a proof system is to reduce the degree of non-determinism of operations, by forbidding some options. The semantics of a proof system is obtained by combining in a rather straightforward way the semantic constraints imposed by its rules. Secondly, this semantics is *effective*. By this we mean that any legal *partial* valuation closed under subformulas can be extended to a full valuation. This property is crucial for the usefulness of semantics, in particular for constructing counterexamples [1].

This paper has two main goals. The first is to combine the results of [2] and [3] (which treat different families of propositional LFIs) into one unified framework. The second (and more important) goal is to extend this semantic framework (and to generalize the corresponding results) to the full first-order level. [2]

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[1] No general theorem of effectiveness is available for the semantics of bivaluations or for possible translations semantics. As a result, effectiveness has to be proven from scratch for any instance of these types of semantics.

[2] First steps in this direction have been taken in [20].

It turned out that one encounters severe complications when moving (in the context of LFIs) from the propositional level to the first-order one. They are mostly related to the lack of the IPE principle (intersubstitutability of provable equivalents) in LFIs. This is an important principle of classical logic, according to which $\psi(A) \leftrightarrow \psi(B)$ is provable whenever $A \leftrightarrow B$ is provable. Unfortunately this principle does not hold for the family of LFIs studied in this paper (see [10, 11]). For instance, already on the propositional level one usually cannot infer $\neg(A \wedge B) \leftrightarrow \neg(B \wedge A)$ from $A \wedge B \leftrightarrow B \wedge A$. This abnormality becomes really harmful on the first-order level. Even the α-conversion principle (identifying syntactic objects differing only in the names of their bound variables) does not hold in the first-order systems which are obtained from the propositional LFIs considered here by the addition of the usual rules and axioms for $\forall$ and $\exists$. Thus although $\forall xp(x) \leftrightarrow \forall yp(y)$ is provable in these systems, $\neg\forall xp(x) \leftrightarrow \neg\forall yp(y)$ is not. This is of course unacceptable in any reasonable logical system. A similar problem occurs concerning vacuous quantification: although $\forall x \forall yp(x) \leftrightarrow \forall xp(x)$ is provable, $\neg\forall x\forall yp(x) \leftrightarrow \neg\forall xp(x)$ is not.

The straightforward solution to this problem proposed by da Costa ([12, 13]) is to add an explicit axiom capturing the principles of α-equivalence and vacuous quantification. However, the non-deterministic semantics for systems with such axioms become more complicated. As a result, their effectiveness becomes less evident. Nevertheless, we shall be able to prove the effectiveness of our semantics for all the first-order LFIs studied in this paper. Then we show how this effectiveness can be used in order to prove important proof-theoretical properties of those LFIs.

2 Preliminaries

Notation: Given a first-order language L, Frm_L is its set of wffs, Frm_L^{cl} - its set of sentences and Trm_L^{cl} - its set of closed terms. $Fv[\psi]$ ($Fv[\mathbf{t}]$) is the set of variables occurring free in a formula ψ (a term $\mathbf{t}$). $\psi\{\mathbf{t}/x\}$ is the formula obtained from ψ by substituting the term $\mathbf{t}$ for every free occurrence of x in ψ. $P^{+}(\mathcal{V})$ denotes the set of all non-empty subsets of the set $\mathcal{V}$.

The following definition formalizes for first-order languages the notion of a *substitution of subformulas* in a sentence.

Definition 1 (Substitutable subformulas) *Given a sentence ψ of L, the set $SSF(\psi)$ of its substitutable subformulas is inductively defined as follows:*

- $SSF(p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)) = \{p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)\}$
- $SSF(\diamond(\psi_1, ..., \psi_n)) = \{\diamond(\psi_1, ..., \psi_n)\} \cup SSF(\psi_1) \cup ... \cup SSF(\psi_n)$
- *If $x \notin Fv[\psi]$, then $SSF(Qx\psi) = \{Qx\psi\} \cup SSF(\psi)$. Otherwise, $SSF(Qx\psi) = \{Qx\psi\}$.*

Denote by $\varphi(\psi)$ an L-sentence φ, such that $\psi \in SSF(\varphi)$. Let $\varphi(\psi)$ and θ be L-sentences. We denote by $\varphi(\theta)$ the result of substituting θ for ψ in φ.

For capturing the principles of α-conversion and void quantifiers, we need the notion of a *congruence relation*.

Definition 2 (Congruence relation) *Given a first-order language L, a binary relation $\sim$ between L-formulas is a* congruence relation *if (i) $\sim$ is an equivalence relation, (ii) If $\psi_1 \sim \varphi_1, ..., \psi_n \sim \varphi_n$ then $\diamond(\psi_1, ..., \psi_n) \sim \diamond(\varphi_1, ..., \varphi_n)$ for every n-ary connective $\diamond$ of L, and (iii) If $\psi \sim \varphi$ then $Qx\psi \sim Qx\varphi$ for $Q \in \{\forall, \exists\}$.*

2.1 A Taxonomy of First-order LFIs

Let $\mathcal{L}_{\text{cl}}^{+}$ be a first-order language with the propositional connectives $\{\wedge, \vee, \supset\}$ and the quantifiers $\{\forall, \exists\}$. $\mathcal{L}_{\text{cl}}$ is the language obtained from $\mathcal{L}_{\text{cl}}^{+}$ by extending its set of propositional connectives with the unary connective $\neg$. $\mathcal{L}_{\text{C}}$ is the language obtained from $\mathcal{L}_{\text{cl}}$ by the addition of the unary connective $\circ$.

Definition 3 *Let* $\mathbf{HCL}^{+}$ *be some propositional Hilbert-type system which has Modus Ponens as the sole inference rule, and is sound and strongly complete for the positive fragment of CPL (classical propositional logic). The first-order system* $\mathbf{HCL}^{+}_{FOL}$ *over* $\mathcal{L}_{\text{cl}}^{+}$ *is obtained from it by adding the following axioms and inference rules:*

$(\forall_{\mathbf{f}})$ $\forall x\psi \rightarrow \psi\{\boldsymbol{t}/x\}$
$(\exists_{\mathbf{t}})$ $\psi\{\boldsymbol{t}/x\} \rightarrow \exists x\psi$

$$\frac{(\varphi \rightarrow \psi)}{(\varphi \rightarrow \forall x\psi)}\ (\forall_{\mathbf{t}}) \qquad \frac{(\psi \rightarrow \varphi)}{(\exists x\psi \rightarrow \varphi)}\ (\exists_{\mathbf{f}})$$

where $\boldsymbol{t}$ is free for x in ψ, and $x \notin Fv[\varphi]$.

Remark: It can be shown that $\mathbf{HCL}^{+}_{FOL}$ is an axiomatization of the negation-free fragment of classical first-order logic (in fact, a proof of this can be extracted from the proof of theorem 24 below). It is also easy to see that the usual deduction theorem of classical first-order logic (If φ is a sentence then ψ is derivable from $T \cup \{\varphi\}$ iff $\varphi \rightarrow \psi$ is derivable from T) is true for any extension of $\mathbf{HCL}^{+}_{FOL}$ by axiom schemata.

Definition 4 *The system* $\mathbf{QB_0}$ *is obtained from* $\mathbf{HCL}^{+}_{FOL}$ *by adding the schemata:*

(t) $\neg\varphi \vee \varphi$
(p) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

Remark: It is not difficult to provide semantics for $\mathbf{QB_0}$. However, in this paper we concentrate on da Costa's systems, which include the additional explicit axiom (mentioned in the introduction) for capturing the principles of α-conversion and of vacuous quantifiers. For this purpose we define the following congruence relation between L-formulas:

Definition 5 ($\sim_L^{dc}$) *Given a first-order language L, $\sim_L^{dc}$ is the minimal congruence relation between L-formulas, which satisfies:*

- *If $\psi\{z/x\} \sim_L \psi'\{z/y\}$, where z is a fresh variable, then $Qx\psi \sim_L^{dc} Qy\psi'$ for $Q \in \{\forall, \exists\}$.*
- *If $\psi \sim_L^{dc} \psi'$ and $x \notin Fv[\psi]$, then $Qx\psi \sim_L^{dc} \psi'$ for $Q \in \{\forall, \exists\}$.*

In other words, $\psi \sim_L^{dc} \psi'$ if ψ' can be obtained from ψ by renaming of bound variables and deletion/addition of void quantifiers.

Definition 6 *The system* **QB** *is obtained from* $\mathbf{QB}_0$ *by adding the axiom schema* $\psi \supset \psi'$, *where* $\psi \sim_{\mathcal{L}_C}^{dc} \psi'$.

Next we obtain a large family of first-order systems by adding different combinations of the following schemata, studied in the literature of LFIs (see, e.g. [10, 11, 8]).

Definition 7 *Let Ax be the set consisting of the following schemata:* [3]

(c) $\neg\neg\varphi \supset \varphi$
(e) $\varphi \supset \neg\neg\varphi$
(w) $\circ(\neg\varphi)$
($\mathbf{i}_1$) $\neg\circ\varphi \supset \varphi$
($\mathbf{i}_2$) $\neg\circ\varphi \supset \neg\varphi$
($\mathbf{k}_1$) $\circ\varphi \vee \varphi$
($\mathbf{k}_2$) $\circ\varphi \vee \neg\varphi$
($\mathbf{a}_\neg$) $\circ\varphi \supset \circ(\neg\varphi)$
($\mathbf{a}_\sharp$) $(\circ\varphi \wedge \circ\psi) \supset (\circ(\varphi\sharp\psi))$ *for* $\sharp \in \{\wedge, \vee, \supset\}$
($\mathbf{o}_\sharp$) $(\circ\varphi \vee \circ\psi) \supset (\circ(\varphi\sharp\psi))$ *for* $\sharp \in \{\wedge, \vee, \supset\}$
($\mathbf{v}_\sharp$) $\circ(\varphi\sharp\psi)$ *for* $\sharp \in \{\wedge, \vee, \supset\}$
($\mathbf{a}_Q$) $\forall x \circ\varphi \supset (\circ(Qx\varphi))$ *for* $Q \in \{\forall, \exists\}$
($\mathbf{o}_Q$) $\exists x \circ\varphi \supset (\circ(Qx\varphi))$ *for* $Q \in \{\forall, \exists\}$
($\mathbf{v}_Q$) $\circ(Qx\psi)$ *for* $Q \in \{\forall, \exists\}$

For $\mathbf{X} \subseteq Ax$, $\mathbf{QB}[\mathbf{X}]$ *is the system obtained by adding the schemata in* $\mathbf{X}$ *to* **QB**.

The set Ax' consists of the following schemata:

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$
(d) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$
(b) $(\neg(\varphi \wedge \neg\varphi) \vee \neg(\neg\varphi \wedge \varphi)) \supset \circ\varphi$

[3] The schemata **(c)**, **(e)** , **($\mathbf{i}_1$)**, **($\mathbf{i}_2$)**, **($\mathbf{k}_1$)**, **($\mathbf{k}_2$)**, **($\mathbf{a}_\neg$)**, **($\mathbf{a}_\sharp$)** and **($\mathbf{o}_\sharp$)** were treated for the propositional case in [3] (**($\mathbf{k}_1$)** and **($\mathbf{k}_2$)** were called there **($\mathbf{d}_1$)** and **($\mathbf{d}_2$)**). The schemata **($\mathbf{a}_Q$)** and **($\mathbf{o}_Q$)** were treated in [20] (for the three-valued case). The schemata **(w)**, **($\mathbf{v}_\sharp$)** and **($\mathbf{v}_Q$)** are treated in the context of Nmatrices for the first time. It might have been more natural to refer to the schema (**w**) as (**$\mathbf{v}_\neg$**), but we keep the name used in [8].

For $y \in \{(\mathbf{l}), (\mathbf{d}), (\mathbf{b})\}$ and $\mathbf{X} \subseteq Ax$, $\mathbf{QB}y[\mathbf{X}]$ is the system obtained from $\mathbf{QB}[\mathbf{X}]$ by adding the schema y.

Notation: We shall usually denote $\mathbf{QB}[\mathbf{X}]$ ($\mathbf{QB}y[\mathbf{X}]$) by $\mathbf{QB}s$ ($\mathbf{QB}ys$), where s is a string consisting of the names of the axioms in $\mathbf{X}$. Thus we'll write **QBic** instead of $\mathbf{QB}[\{(\mathbf{i}), (\mathbf{c})\}]$ and **QBlic** instead of $\mathbf{QB}(\mathbf{l})[\{(\mathbf{i}), (\mathbf{c})\}]$. If both $(\mathbf{x_1})$ and $(\mathbf{x_2})$ are in $\mathbf{X}$ for $\mathbf{x} \in \{\mathbf{i}, \mathbf{k}\}$, we abbreviate it by $\mathbf{x}$. Also, if $\mathbf{x_y}$ is in $\mathbf{X}$ for every $\mathbf{y} \in \{\supset, \wedge, \vee\}$ and some $\mathbf{x} \in \{\mathbf{a}, \mathbf{o}, \mathbf{v}\}$, we shall write $\mathbf{x_p}$. Similarly, if $\mathbf{x_y}$ is in $\mathbf{X}$ for every $\mathbf{y} \in \{\forall, \exists\}$ and some $\mathbf{x} \in \{\mathbf{a}, \mathbf{o}, \mathbf{v}\}$, we shall write $\mathbf{x_Q}$. For both $\mathbf{x_p}$ and $\mathbf{x_Q}$ we shall write $\mathbf{x}$.

Remark: Denote by $\mathbf{QC}_1$ the system **QBlcia**. If we take $\circ\psi$ to be an *abbreviation* of $\neg(\psi \wedge \neg\psi)$, then $\mathbf{QC}_1$ becomes da Costa's original system C_1^* from [12, 13]. [4] Note that C_1^* is over the language of $\{\neg, \vee, \wedge, \supset, \forall, \exists\}$.

2.2 Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a multi-valued matrix given in [1, 2, 3, 21, 20].

Definition 8 (Non-deterministic matrix) *A non-deterministic matrix (Nmatrix) for a language L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where: $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$ and $\mathcal{O}$ includes the following interpretation functions:*

- $\tilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to P^+(\mathcal{V})$ *for every n-ary connective $\diamond$.*
- $\tilde{Q}_{\mathcal{M}} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$ *for every quantifier Q.*

Definition 9 (L-structure) *Let $\mathcal{M}$ be an Nmatrix. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I \rangle$ where D is a (non-empty) domain and I is a function interpreting constants, function symbols, and predicate symbols of L, satisfying the following conditions: $I[c] \in D$ if c is a constant, $I[f] : D^n \to D$ if f is an n-ary function, and $I[p] : D^n \to \mathcal{V}$ if p is an n-ary predicate.*
I is extended to interpret closed terms of L as follows:

$$I[f(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[f][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$$

Here a note on our treatment of quantification in the framework of Nmatrices is in order. The standard approach to interpreting first-order formulas is by using *objectual* (or referential) semantics, where the variable is thought of as ranging over a set of objects from the domain (see. e.g. [16, 17]). An alternative approach is *substitutional* quantification ([18]), where quantifiers are interpreted substitutionally, i.e. a universal (an existential) quantification is true if and only if every one (at least one) of its substitution instances is true (see. e.g. [19, 14]). [21] explains the motivation behind choosing the substitutional approach for the framework of Nmatrices, and points out the problems of the objectual approach

[4] The name C_1^* is used in [10] for another, different, first-order paraconsistent system.

in this context. The substitutional approach assumes that every element of the domain has a closed term referring to it. Thus given a structure $S = \langle D, I\rangle$, we extend the language L with *individual constants*, one for each element of D.

Definition 10 (L(D)) *Let S=$\langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from L by adding to it the set of* individual constants $\{\overline{a} \mid a \in D\}$. *$S' = \langle D, I'\rangle$ is the $L(D)$-structure, such that I' is the extension of I satisfying: $I'[\overline{a}] = a$.*

Given an L-structure $S = \langle D, I\rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I'\rangle$ as S and to I' as I when the meaning is clear from the context.

Next we define the congruence relation $\sim^S$, which is the semantic counterpart of the syntactic congruence relation $\sim^{dc}_L$ (see Definition 5).

Definition 11 **($\sim^S$)** *Let S be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ between terms of $L(D)$ is defined inductively as follows:*

- $x \sim^S x$
- *For closed terms $\boldsymbol{t}, \boldsymbol{t'}$ of $L(D)$: $\boldsymbol{t} \sim^S \boldsymbol{t'}$ when $I[\boldsymbol{t}] = I[\boldsymbol{t'}]$.*
- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $f(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S f(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*

The relation $\sim^S$ between formulas of $L(D)$ is the minimal congruence relation, satisfying:

- *If $\boldsymbol{t}_1 \sim^S \boldsymbol{t}'_1, \boldsymbol{t}_2 \sim^S \boldsymbol{t}'_2, ..., \boldsymbol{t}_n \sim^S \boldsymbol{t}'_n$, then $p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \sim^S p(\boldsymbol{t}'_1, ..., \boldsymbol{t}'_n)$.*
- *If $\psi\{z/x\} \sim^S \varphi\{z/y\}$, where x, y are distinct variables and z is a new variable, then $Qx\psi \sim^S Qy\varphi$ for $Q \in \{\forall, \exists\}$.*
- *If $\psi \sim^S \varphi$ and $x \notin Fv[\varphi]$, then $\psi \sim^S Qx\varphi$.*

The proofs of the following two easy lemmas are left for the reader:

Lemma 12 *Let S be an L-structure, and $\boldsymbol{t}_1, \boldsymbol{t}_2$ closed terms of $L(D)$, such that $\boldsymbol{t}_1 \sim^S \boldsymbol{t}_2$. Let ψ_1, ψ_2 be $L(D)$-formulas, such that $\psi_1 \sim^S \psi_2$. Then $\psi_1\{\boldsymbol{t}/x\} \sim^S \psi_2\{\boldsymbol{t}_2/x\}$.*

Lemma 13 *Let $S = \langle D, I\rangle$ be an L-structure.*

1. *Let A, B be two L-formulas. If $A \sim^{dc}_L B$, then $A \sim^S B$.*
2. *Let A, B be two L-formulas such that $I[\boldsymbol{t}_1] \neq I[\boldsymbol{t}_2]$ for any two closed terms $\boldsymbol{t}_1 \neq \boldsymbol{t}_2$ occurring in A and B respectively. Then $A \sim^{dc}_L B$ iff $A \sim^S B$.*

Remark: The difference between $\sim^{dc}_L$ and $\sim^S$ is as follows:

1. $\sim^{dc}_L$ is a relation between formulas of L, while $\sim^S$ is a relation between formulas of $L(D)$.
2. $\sim^S$ is defined with respect to some structure S, while $\sim^{dc}_L$ is purely syntactic.

3. Unlike $\sim^{dc}_L$, $\sim^S$ identifies two sentences ψ, ψ' such that ψ' is obtained from ψ by substituting any number of closed terms for closed terms with the same denotation in S. For instance, let S be an L-structure, such that $I[d] = I[c]$ for two constants $d \neq c$. Then $p(c) \not\sim^{dc}_L p(d)$, but $p(c) \sim^S p(d)$. The motivation for this is purely technical and is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose we have a closed term $\mathbf{t}$, such that $I[\mathbf{t}] = a \in D$. But a also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute $\mathbf{t}$ for $\overline{a}$ in every context, as will be shown in the sequel.

Definition 14 (S-valuation) *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. An S-valuation $v : Frm^{\mathsf{cl}}_L \to \mathcal{V}$ is* legal *in $\mathcal{M}$ if it satisfies the following conditions:*

- *v respects the $\sim^S$ relation, i.e. $v[\psi] = v[\psi']$ for every two L-sentences ψ, ψ', such that $\psi \sim^S \psi'$.*
- *$v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_n]]$.*
- *$v[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}_{\mathcal{M}}[v[\psi_1], ..., v[\psi_n]]$.*
- *$v[Qx\psi] \in \tilde{Q}_{\mathcal{M}}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$.*

Definition 15 *Let $S = \langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$.*

1. *An $\mathcal{M}$-legal S-valuation v is* a model *of a formula ψ* in *$\mathcal{M}$, denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi'] \in \mathcal{D}$ for every closed instance ψ' of ψ in $L(D)$.*
2. *A formula ψ is $\mathcal{M}$-valid in S if for every $\mathcal{M}$-legal S-valuation v, $S, v \models_{\mathcal{M}} \psi$. ψ is $\mathcal{M}$-valid if ψ is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$.*
3. *The consequence relation $\vdash_{\mathcal{M}}$ between sets of L-formulas and L-formulas is defined as follows: $\Gamma \vdash_{\mathcal{M}} \psi$ if for every L-structure S and every $\mathcal{M}$-legal S-valuation v: $S, v \models_{\mathcal{M}} \Gamma$ implies that $S, v \models_{\mathcal{M}} \psi$.*
4. *An Nmatrix $\mathcal{M}$ is* sound *for a proof system $\mathbf{S}$ if $\vdash_{\mathbf{S}} \subseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is* complete *for $\mathbf{S}$ if $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathbf{S}}$. $\mathcal{M}$ is a characteristic Nmatrix for $\mathbf{S}$ if it is sound and complete for $\mathbf{S}$.*

The following is an extension of Definition 2.9 and Theorem 2.10 from [3] to first-order languages:

Definition 16 (Reduction, refinement) *Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1\rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2\rangle$ be Nmatrices for L.*

1. *A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$, such that:*
   - *For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.*
   - *$F(y) \in \tilde{\diamond}_{\mathcal{M}_2}[F(x_1), ..., F(x_n)]$ for every n-ary connective $\diamond$ of L and every $x_1, ..., x_n, y \in \mathcal{V}_1$, such that $y \in \tilde{\diamond}_{\mathcal{M}_1}[x_1, ..., x_n]$.*
   - *$F(y) \in \tilde{Q}_{\mathcal{M}_2}[\{F(z) \mid z \in H\}]$ for $Q \in \{\forall, \exists\}$ and every $y \in \mathcal{V}_1$ and $H \subseteq P^+(\mathcal{V}_1)$, such that $y \in \tilde{Q}_{\mathcal{M}_1}[H]$.*
2. *$\mathcal{M}_1$ is a* refinement *of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.*

3. $\mathcal{M}_1$ *is a* simple refinement *of* $\mathcal{M}_2$ *if it is a refinement of* $\mathcal{M}_2$, $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, $\tilde{\diamond}_{\mathcal{M}_1}[\overrightarrow{x}] \subseteq \tilde{\diamond}_{\mathcal{M}_2}[\overrightarrow{x}]$ *for every* n*-ary connective* $\diamond$ *of* L *and every* $\overrightarrow{x} \in \mathcal{V}_1^n$, *and* $\tilde{Q}_{\mathcal{M}_1}[H] \subseteq \tilde{Q}_{\mathcal{M}_2}[H]$ *for* $Q \in \{\forall, \exists\}$ *and every* $H \subseteq P^+(\mathcal{V}_1)$.

Theorem 1. *If* $\mathcal{M}_1$ *is a refinement of* $\mathcal{M}_2$, *then* $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Proof: a straightforward extension of the proof of theorem 2.10 from [3].

3 Effectiveness of First-order Nmatrices

One of the most important properties of the semantic framework of Nmatrices is its *effectiveness*, in the sense that for determining whether $\Gamma \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on *subformulas* of $\Gamma \cup \{\varphi\}$.

Definition 17 *Let* S *be an* L*-structure. A set of sentences* $W_S \subseteq Frm^{\mathsf{cl}}_{L(D)}$ *is* closed under subformulas *if it satisfies the following conditions:*

- *For every* n*-ary connective* $\diamond$: $\psi_1, ..., \psi_n \in W_S$ *whenever* $\diamond(\psi_1, ..., \psi_n) \in W_S$.
- *For* $Q \in \{\forall, \exists\}$ *and every* $a \in D$: $\psi\{\overline{a}/x\} \in W_S$ *whenever* $Qx\psi \in W_S$.

Definition 18 *Let* S *be an* L*-structure and* $\mathcal{M}$ *- an Nmatrix for* L*. Let* $W_S \subseteq Frm^{\mathsf{cl}}_{L(D)}$ *be a set closed under subformulas. A partial* $\mathcal{M}$*-legal* S*-valuation on* W_S *is a function* $v : W_S \to \mathcal{V}$*, satisfying:*

- $\psi \sim^S \psi'$ *implies* $v[\psi] = v[\psi']$ *for every* $\psi, \psi' \in W_S$.
- $v[p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n)] = I[p][I[\boldsymbol{t}_1], ..., I[\boldsymbol{t}_1]]$ *whenever* $p(\boldsymbol{t}_1, ..., \boldsymbol{t}_n) \in W_S$.
- $v[\diamond(\psi_1, ..., \psi_n)] \in \tilde{\diamond}[v[\psi_1], ..., v[\psi_n]]$ *whenever* $\diamond(\psi_1, ..., \psi_n) \in W_S$.
- $v[Qx\psi] \in \tilde{Q}[\{v[\psi\{\overline{a}/x\}] \mid a \in D\}]$ *whenever* $Qx\psi \in W_S$.

Definition 19 *An Nmatrix* $\mathcal{M}$ *for* L *is* effective *if for every* L*-structure* S *and every set of* $L(D)$*-sentences* W_S *which is closed under subformulas: if* v_p *is a partial* $\mathcal{M}$*-legal* S*-valuation on* W_S*, then it can be extended to a full* $\mathcal{M}$*-legal* S*-valuation.*

For the propositional case, the proof of effectiveness of an Nmatrix $\mathcal{M}$ is very simple (see proposition 2 in [2]). However, in the first-order case effectiveness becomes less evident because any $\mathcal{M}$-legal S-valuation has to respect the $\sim^S$ relation. In fact, given an Nmatrix $\mathcal{M}$ for L and a partial $\mathcal{M}$-legal S-valuation v_p on some set $W_S \subseteq Frm^{\mathsf{cl}}_{L(D)}$ closed under subformulas, it is not necessarily guaranteed that v_p can be extended to a full S-valuation legal in $\mathcal{M}$. Consider, for instance, a first-order language L with a constant c and a unary predicate p. Let $\mathcal{M} = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ be an Nmatrix for L with the following non-standard interpretation of $\forall$: $\tilde{\forall}[H] = \{t\}$ for every $H \subseteq P^+(\{t, f\})$. Let $S = \langle \{a\}, I \rangle$ be the L-structure in which $I[c] = a$ and $I[p][a] = f$. Let $W = \{p(c)\}$ (obviously, W is closed under subformulas). Then no partial valuation on W can be extended to a full $\mathcal{M}$-legal valuation v, respecting both the $\sim^S$ relation and the interpretation of $\forall$, because such v should assign f to $p(c)$ and t to $\forall x p(c)$, while $\forall x p(c) \sim^S p(c)$. Thus in order to be effective, an Nmatrix has to satisfy a certain condition:

Definition 20 *An Nmatrix $\mathcal{M}$ for L is suitable for $\sim_L^{dc}$ if for every $a \in \mathcal{V}$ and every quantifier Q of L: $a \in \tilde{Q}[\{a\}]$.*

For instance, an Nmatrix $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ with the following natural interpretations of $\forall$ and $\exists$ is suitable for $\sim_L^{dc}$:

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{D}' & \text{if } H \subseteq \mathcal{D}' \\ \mathcal{V}' - \mathcal{D}' & \text{otherwise} \end{cases} \quad \widetilde{\exists}[H] = \begin{cases} \mathcal{D}' & \text{if } H \cap \mathcal{D}' \neq \emptyset \\ \mathcal{V}' - \mathcal{D}' & \text{otherwise} \end{cases}$$

Proposition 21 *Any Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for L which is suitable for $\sim_L^{dc}$, is effective.*

Proof: Let S be an L-structure and let W_S be a set of $L(D)$-sentences, closed under subformulas. Let v_p be some partial S-valuation on W_S which is $\mathcal{M}$-legal. We show that it can be extended to a full S-valuation v which is legal in $\mathcal{M}$.
For every n-ary connective $\diamond$ of L and every $a_1, ..., a_n \in \mathcal{V}$, choose a truth-value $\mathbf{b}^{\diamond}_{a_1,...,a_n} \in \tilde{\diamond}[a_1, ..., a_n]$. For $Q \in \{\forall, \exists\}$ of L and every $B \subseteq P^+(\mathcal{V})$, choose a truth-value $\mathbf{b}^Q_B \in \tilde{Q}[B]$, so that for every $a \in \mathcal{V}$: $\mathbf{b}^Q_{\{a\}} = a$ (this is possible, since $\mathcal{M}$ is suitable for $\sim_L^{dc}$).
Denote by $H_{\sim^S}$ the set of all equivalence classes of $Frm^{\mathsf{cl}}_{L(D)}$ under $\sim^S$. Denote by $[\![\psi]\!]$ the equivalence class of ψ. Define the function $\chi : H_{\sim^S} \to \mathcal{V}$ as follows:

$$\chi[[\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

$$\chi[[\![\diamond(\psi_1, ..., \psi_n)]\!]] = \begin{cases} v_p[\varphi] & \varphi \in [\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W_S \\ \mathbf{b}^{\diamond}_{\chi[[\![\psi_1]\!]],...,\chi[[\![\psi_n]\!]]} & [\![\diamond(\psi_1, ..., \psi_n)]\!] \cap W_S = \emptyset \end{cases}$$

$$\chi[[\![Qx\psi]\!]] = \begin{cases} v_p[\varphi] & \varphi \in [\![Qx\psi]\!] \cap W_S \\ \mathbf{b}^Q_{\{\chi[[\![\psi\{\overline{a}/x\}]\!]] \mid a \in D\}} & [\![Qx\psi]\!] \cap W_S = \emptyset \end{cases}$$

Note that because v_p is $\mathcal{M}$-legal, the value of $v_p[\varphi]$ in the above definition does not depend on the choice of φ (among those satisfying the relevant condition). Hence χ is well-defined. Next define

$$v[\psi] = \chi[[\![\psi]\!]]$$

The proof that v is $\mathcal{M}$-legal is not difficult and is left to the reader. Obviously, v is an extension of v_p. □

4 Non-deterministic Semantics for First-order LFIs

4.1 Finite Non-deterministic Semantics

In this section we provide five-valued (or less) non-deterministic semantics for first-order LFIs obtained from the basic system **QB** by adding various combinations of schemata from Ax (not including the schemata **(l)**, **(b)** and **(d)**. We deal with systems including these schemata in the next subsection). The semantics presented below is an extension to first-order languages of the semantics from [3].

The system **QB** treats the connectives $\wedge, \vee, \supset$ and the quantifiers $\forall, \exists$ similarly to classical logic. The treatment of $\circ$ and $\neg$ is different: intuitively, the truth/falsity of $\neg\psi$ or $\circ\psi$ is not completely determined by the truth/falsity of ψ. More data is needed for it. The central idea is to include all the relevant data concerning a sentence ψ in the truth-value from $\mathcal{V}$ which is assigned to ψ. In our case the relevant data beyond the truth/falsity of ψ is the truth/falsity of $\neg\psi$ and of $\circ\psi$. This leads to the use of elements from $\{0,1\}^3$ as truth-values, where the intended meaning of assigning $\langle x, y, z\rangle$ to ψ is as follows:

- $x = 1$ iff ψ is true
- $y = 1$ iff $\neg\psi$ is true
- $z = 1$ iff $\circ\psi$ is true

However, the axioms **(t)** and **(b)** rule out some of the truth-values. By **(t)**, at least one of the sentences $\psi, \neg\psi$ should be true, thus ruling out $\langle 0,0,1\rangle$ and $\langle 0,0,0\rangle$. Similarly, **(b)** rules out $\langle 1,1,1\rangle$. We are left with the following five truth-values:

- $t = \langle 1,0,1\rangle$
- $t_I = \langle 1,0,0\rangle$
- $I = \langle 1,1,0\rangle$
- $f = \langle 0,1,1\rangle$
- $f_I = \langle 0,1,0\rangle$

Note that since the first component of a truth-value assigned to a formula should indicate whether that formula is true, the designated truth-values should be those whose first component is 1. Thus we are led to the following definition (which is an extension to first-order languages of Definition 3.1 from [3]):

Definition 22 *The Nmatrix $\mathcal{QM}_5 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ for $\mathcal{L}_C$ is defined as follows:*

- $\mathcal{V} = \{t, t_I, I, f, f_I\}$, $\mathcal{D} = \{t, t_I, I\}$.
- *Let $\mathcal{F} = \mathcal{V} - \mathcal{D}$. The operations in $\mathcal{O}$ are defined as follows:*

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{if } a, b \in \mathcal{D} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \{t, t_I\} \\ \mathcal{D} & \text{if } a \in \{f, f_I, I\} \end{cases} \qquad \widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{t_I, f_I, I\} \end{cases}$$

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases} \qquad \widetilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

Note that the non-deterministic truth tables in $\mathcal{QM}_5$ corresponding to the operations $\neg$ and $\circ$ are:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

$\tilde{\circ}$	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
	$\{t, I, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$	$\{t, I, t_I\}$	$\{f, f_I\}$

Lemma 23 (Effectiveness of $\mathcal{QM}_5$) *$\mathcal{QM}_5$ is effective.*

Proof: This follows from the suitability of $\mathcal{QM}_5$ for $\sim_L^{dc}$, and proposition 21.

The following theorem is a generalization of theorem 3 of [20].

Theorem 24 (Soundness and completeness) *Let $\Gamma \cup \{\psi_0\}$ be a set of $\mathcal{L}_C$-formulas. $\Gamma \vdash_{\mathbf{QB}} \psi_0$ iff $\Gamma \vdash_{\mathcal{QM}_5} \psi_0$.*

The proof of soundness is not hard and is left to the reader.

For completeness, we first note that by definition of the interpretation of $\forall$ in $\mathcal{QM}_5$, $\forall x\varphi \vdash_{\mathcal{QM}_5} \varphi$ and $\varphi \vdash_{\mathcal{QM}_5} \forall x\varphi$ for every formula φ and every variable x. Obviously the same relations hold between φ and $\forall x\varphi$ in $\mathbf{HCL}^+_{FOL}$, and so in $\vdash_{\mathbf{QB}}$. It follows that we may assume that all formulas in $\Gamma \cup \{\psi_0\}$ are sentences. It is also easy to see that we may restrict ourselves to L_r, the subset of L consisting of all the constants, function, and predicate symbols occurring in $\Gamma \cup \{\psi_0\}$. Now suppose that $\Gamma \not\vdash_{\mathbf{QB}} \psi_0$. We will construct an $\mathcal{L}_C$-structure S and an $\mathcal{QM}_5$-legal S-valuation v, such that $S, v \models_{\mathcal{M}_5} \Gamma$, but $S, v \not\models_{\mathcal{QM}_5} \psi_0$. Let L' be the language obtained from L_r by adding a countably infinite set of new constants. It is a standard matter to show (using a usual Henkin-type construction) that Γ can be extended to a maximal set Γ^* of sentences in L', such that:

- $\Gamma^* \not\vdash_{\mathbf{QB}} \psi_0$.
- $\Gamma \subseteq \Gamma^*$.
- For every L'-sentence $\exists x\psi \in \Gamma^*$ there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \in \Gamma^*$.
- For every L'-sentence $\forall x\psi \notin \Gamma^*$, there is a constant $\mathbf{c}$ of L', such that $\psi\{\mathbf{c}/x\} \notin \Gamma^*$.

(The last property follows from property 3, the deduction theorem for **QB**, and the fact that for any $x \notin Fv[\varphi]$, $(\forall x\psi \supset \varphi) \supset \exists x(\psi \supset \varphi)$ is provable in the positive fragment of first-order classical logic, and so also in **QB**). It is now straightforward to show that Γ^* has the following properties for every L'-sentences ψ, φ, and $\forall x\theta$:

1. If $\psi \notin \Gamma^*$, then $\psi \supset \psi_0 \in \Gamma^*$.
2. $\psi \vee \varphi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$.
3. $\psi \wedge \varphi \in \Gamma^*$ iff both $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$.
4. $\varphi \supset \psi \in \Gamma^*$ iff either $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$.

5. Either $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$.
6. If ψ and $\neg\psi$ are both in Γ^*, then $\circ\psi \notin \Gamma^*$.
7. If $\psi \in \Gamma^*$, then for every L'-sentence ψ' such that $\psi' \sim_L^{dc} \psi$: $\psi' \in \Gamma^*$.
8. If $\forall x\theta \in \Gamma^*$, then for every closed L'-term $\mathbf{t}$: $\theta\{\mathbf{t}/x\} \in \Gamma^*$. If $\forall x\theta \notin \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L', such that $\theta\{\mathbf{t}_\theta/x\} \notin \Gamma^*$.
9. If $\exists x\theta \in \Gamma^*$, then there is some closed term $\mathbf{t}_\theta$ of L, such that $\theta\{\mathbf{t}_\theta/x\} \in \Gamma^*$. If $\exists x\theta \notin \Gamma^*$, then for every closed term $\mathbf{t}$ of L': $\theta\{\mathbf{t}/x\} \notin \Gamma^*$.

The L'-structure $S = \langle D, I\rangle$ is defined as follows:

- D is the set of all the closed terms of L'.
- For every constant c of L': $I[c] = c$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = \langle x, y, z\rangle$, where $x, y, z \in \{0, 1\}$ and:
  - $x = 1$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.
  - $y = 1$ iff $\neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.
  - $z = 1$ iff $\circ p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*$.

Lemma 25 *$I^*[\boldsymbol{t}] = \boldsymbol{t}$ for every $\boldsymbol{t} \in D$.*

Proof: by induction on the structure of $\mathbf{t}$.

Note that in the extended language $L'(D)$ we now have an individual constant $\overline{\mathbf{t}}$ for every term $\mathbf{t} \in D$. For any L'-term $\mathbf{t}$, define $\widetilde{\mathbf{t}}$ as follows:

$$\widetilde{\mathbf{t}} = \begin{cases} \mathbf{s} & \text{if } \mathbf{t} = \overline{\mathbf{s}} \text{ for some } \mathbf{s} \in D \\ \mathbf{t} & \text{otherwise} \end{cases}$$

Given an $L'(D)$-sentence ψ, define the sentence $\widetilde{\psi}$ inductively as follows:

- $\widetilde{p(\mathbf{t}_1, ..., \mathbf{t}_n)} = p(\widetilde{\mathbf{t}}_1, ..., \widetilde{\mathbf{t}}_n)$
- $\widetilde{\diamond(\psi_1, ..., \psi_n)} = \diamond(\widetilde{\psi_1}, ..., \widetilde{\psi_n})$
- $\widetilde{Qx\psi} = Qx\widetilde{\psi}$

In other words, $\widetilde{\psi}$ is obtained by replacing all individual constants $\overline{\mathbf{t}}$ occurring in ψ by the respective (closed) term $\mathbf{t}$.

Lemma 26 *1. For any $L'(D)$-sentence ψ, $\psi \sim^S \widetilde{\psi}$.*
2. For any $\psi, \varphi \in Frm^{\mathsf{cl}}_{L'(D)}$: if $\psi \sim^S \varphi$, then $\widetilde{\psi} \sim_L^{dc} \widetilde{\varphi}$.
3. For every $L'(D)$-sentence ψ and every $\boldsymbol{t} \in D$: $\widetilde{\psi\{\overline{\boldsymbol{t}}/x\}} = \widetilde{\psi}\{\boldsymbol{t}/x\}$.

Proof: The proofs of part 1 and 3 are straightforward. Part 2 follows from Lemma 13-2 and Lemma 25.

Next we define the refuting S-valuation $v : Frm^{\mathsf{cl}}_{L'(D)} \to \mathcal{V}$ as follows:

$$v[\psi] = \langle x_\psi, y_\psi, z_\psi\rangle$$

where $x_\psi, y_\psi, z_\psi \in \{0, 1\}$ and:

- $x_\psi = 1$ iff $\widetilde{\psi} \in \Gamma^*$.
- $y_\psi = 1$ iff $\widetilde{\neg\psi} \in \Gamma^*$.
- $z_\psi = 1$ iff $\widetilde{\circ\psi} \in \Gamma^*$.

Let ψ, ψ' be two $L'(D)$-sentences, such that $\psi \sim^S \psi'$. Then by lemma 26-2, $\widetilde{\psi} \sim_L^{dc} \widetilde{\psi}'$, and by property 7 of Γ^*, $\widetilde{\psi} \in \Gamma^*$ iff $\widetilde{\psi}' \in \Gamma^*$. Similarly, since $\neg\psi \sim^S \neg\psi'$ and $\circ\psi \sim^S \circ\psi'$, $(\neg\widetilde{\psi} =)\widetilde{\neg\psi} \sim_L^{dc} \widetilde{\neg\psi'}(= \neg\widetilde{\psi}')$ and $\widetilde{\circ\psi} \sim_L^{dc} \widetilde{\circ\psi'}$. Thus $\widetilde{\neg\psi} \in \Gamma^*$ iff $\widetilde{\neg\psi'} \in \Gamma^*$ and $\widetilde{\circ\psi} \in \Gamma^*$ iff $\widetilde{\circ\psi'} \in \Gamma^*$. Hence $v[\psi] = v[\psi']$ and so v respects the $\sim^S$ relation.
It remains to check that v respects the interpretations of the connectives and quantifiers in $\mathcal{QM}_5$. This is guaranteed by the properties of Γ^*. We prove this for the cases of $\circ$ and $\forall$:

- Let $v[\psi] \in \{t, f\}$. Then $\widetilde{\circ\psi} \in \Gamma^*$ and so $v[\circ\psi] \in \mathcal{D}$. Similarly for the case of $v[\psi] \in \{t_I, f_I, I\}$.
- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v[\psi\{\overline{a}/x\}] \mid a \in D\} \subseteq \mathcal{D}$. Suppose by contradiction that $v[\forall x\psi] \notin \mathcal{D}$. Then $\widetilde{\forall x\psi} = \forall x\widetilde{\psi} \notin \Gamma^*$. By property 8 of Γ^*, there exists some closed L'-term $\mathbf{t}$, such that $\widetilde{\psi}\{\mathbf{t}/x\} \notin \Gamma^*$. Then $v[\widetilde{\psi}\{\mathbf{t}/x\}] \notin \mathcal{D}$. Since $\psi \sim^S \widetilde{\psi}$, $\psi\{\mathbf{t}/x\} \sim^S \widetilde{\psi}\{\mathbf{t}/x\}$ by lemma 12. We have already shown that v respects the $\sim^S$ relation, and so $v[\psi\{\mathbf{t}/x\}] \notin \mathcal{D}$. By lemma 12 again, $\psi\{\mathbf{t}/x\} \sim^S \psi\{\overline{\mathbf{t}}/x\}$, and so $v[\psi\{\overline{\mathbf{t}}/x\}] \notin \mathcal{D}$. A contradiction.
- Let $\forall x\psi$ be an $L'(D)$-sentence, such that $\{v[\psi\{\overline{a}/x\}] \mid a \in D\} \cap \mathcal{F} \neq \emptyset$. Suppose by contradiction that $v[\forall x\psi] \notin \mathcal{F}$. Then $\forall x\widetilde{\psi} \in \Gamma^*$. By property 8 of Γ^*, for every closed L'-term $\mathbf{t}$: $\widetilde{\psi}\{\mathbf{t}/x\} \in \Gamma^*$. Then $v[\widetilde{\psi}\{\mathbf{t}/x\}] \in \mathcal{D}$. Similarly to the previous case, we get that $v[\psi\{\overline{a}/x\}] \in \mathcal{D}$ for every $a \in D$, in contradiction to our assumption.

Now for every L'-sentence ψ: $v[\psi] \in \mathcal{D}$ iff $\psi \in \Gamma^*$. So $S, v \models_{\mathcal{QM}_5} \Gamma$ (recall that $\Gamma \subseteq \Gamma^*$), but $S, v \not\models_{\mathcal{QM}_5} \psi_0$. $\square$

Next we turn to the semantics of the systems obtained from the basic system **QB** by adding various combinations of the schemata from Ax. As explained in the introduction, the main idea is *modularity*: each schema induces some semantic condition, leading to a certain refinement of the basic Nmatrix $\mathcal{QM}_5$.

Definition 27 *The refining conditions induced by the schemata from Ax are:*

Cond(c) : *if $x \in \{f, f_I\}$ then $\tilde{\neg}x \subseteq \{t, t_I\}$*
Cond(e) : *$\tilde{\neg}I = \{I\}$*
Cond(w) : *$\tilde{\neg}x \subseteq \{t, f\}$*
Cond(i$_1$) : *f_I should be deleted, and $\tilde{\circ}f \subseteq \{t, t_I\}$*
Cond(i$_2$) : *t_I should be deleted, and $\tilde{\circ}t = \{t\}$*
Cond(k$_1$) : *f_I should be deleted.*
Cond(k$_2$) : *t_I should be deleted.*
Cond(a$_\neg$) : *$\tilde{\neg}t = \{f\}$ and $\tilde{\neg}f = \{t\}$*
Cond(a$_\sharp$) : *if $a, b \in \{t, f\}$, then $a\tilde{\sharp}b \subseteq \{t, f\}$*

Cond(o$_\sharp$) : *if* $a \in \{t, f\}$ *or* $b \in \{t, f\}$*, then* $a\tilde{\sharp}b \subseteq \{t, f\}$
Cond(v$_\sharp$) : $x\tilde{\sharp}y \subseteq \{t, f\}$ *for every* $x, y \in \mathcal{V}$.
Cond(a$_Q$) : *for every* $H \subseteq \{t, f\}$*,* $\tilde{Q}[H] \subseteq \{t, f\}$
Cond(o$_Q$) : *if* $H \cap \{t, f\} \neq \emptyset$ *then* $\tilde{Q}[H] \subseteq \{t, f\}$
Cond(v$_Q$) : $\tilde{Q}[H] \subseteq \{t, f\}$ *for every* $H \subseteq \mathcal{V}$.

Definition 28 *For* $\mathbf{X} \subseteq Ax$*, let* $\mathcal{QM}_5(\mathbf{X})$ *be the weakest simple refinement (see Definition 16) of* $\mathcal{QM}_5$*, in which the conditions of the schemata from* $\mathbf{X}$ *are satisfied. In other words,* $\mathcal{QM}_5(\mathbf{X}) = \langle \mathcal{V}_X, \mathcal{D}_X, \mathcal{O}_X \rangle$*, where:*

- *If both* **(e)** *and* **(w)** *are in* $\mathbf{X}$*, then* I *is deleted.*
- $\mathcal{V}_X$ *is the set of values from* $\{t, f, t_I, f_I, I\}$ *which are not deleted either by a combination of both* **(e)** *and* **(w)***, or by any condition of a schema from* $\mathbf{X}$*.*
- $\mathcal{D}_X = \mathcal{V}_X \cap \{t, t_I, I\}$*.*
- *For any connective* $\diamond$ *and any* $a_1, ..., a_n \in \mathcal{V}_X$*,* $\tilde{\diamond}_{\mathcal{QM}_5(\mathbf{X})}$ *assigns to* $\overrightarrow{a}$ *the set of all truth-values in* $\tilde{\diamond}_{\mathcal{QM}_5}$ *which are not forbidden by any condition of a schema from* $\mathbf{X}$*.*
- *For* $Q \in \{\forall, \exists\}$ *and any* $H \subseteq P^+(\mathcal{V}_X)$*,* $\tilde{Q}_{\mathcal{QM}_5(\mathbf{X})}$ *assigns to* $\overrightarrow{a}$ *the set of all truth-values in* $\tilde{Q}_{\mathcal{QM}_5}$ *which are not forbidden by any condition of a schema from* $\mathbf{X}$*.*

Notation: We write $\mathcal{QM}_5\mathbf{s}$ instead of $\mathcal{QM}_5(\mathbf{X})$, where $\mathbf{s}$ is the string of all the names of the schemata from $\mathbf{X}$.

Remarks:

1. Assume that $\mathbf{X} \subseteq Ax$, and that either $(\mathbf{w}) \notin \mathbf{X}$ or $(\mathbf{e}) \notin \mathbf{X}$. It is not difficult to see that in this case $\{t, f, I\} \subseteq \mathcal{V}_X$, $\{t, I\} \subseteq \mathcal{D}_X$, and both $\tilde{\diamond}_{\mathcal{QM}_5(X)}[\overrightarrow{a}]$ and $\tilde{Q}_{\mathcal{QM}_5(\mathbf{X})}[H]$ are not empty (where $\diamond$ is an n-ary connective, $\overrightarrow{a} \in \mathcal{V}_X^n$, $Q \in \{\forall, \exists\}$, and $H \subseteq P^+(\mathcal{V}_X)$). The case when both **(w)** and **(e)** are in $\mathbf{X}$ is different, since these conditions are not coherent in the presence of I. It is easy to see that in this case $\mathbf{X}$ is equivalent to classical logic (and so it is not paraconsistent). An adequate semantics for it can be obtained simply by deleting I. Alternatively, one may delete all truth values except t and f.
2. Note the following dependencies between the conditions:
   (a) $(\mathbf{k_j})$ follows from $(\mathbf{i_j})$ for $\mathbf{j} \in \{1, 2\}$.
   (b) **(c)** follows from $(\mathbf{a}_\neg)$ and $(\mathbf{k}_1)$ (taken together).
   (c) $(\mathbf{a}_\neg)$ follows from **(c)**, $(\mathbf{k}_1)$ and $(\mathbf{k}_2)$ (taken together), and from **(w)**.
   (d) $(\mathbf{a_x})$ follows from $(\mathbf{o_x})$ and $(\mathbf{o_x})$ follows from $(\mathbf{v_x})$ for $\mathbf{x} \in \{\vee, \wedge, \supset, \forall, \exists\}$.

Examples:

1. The non-deterministic truth table for $\neg$ in $\mathcal{QM}_5\mathbf{c}$ is:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{f_I}$	$\mathbf{I}$	$\mathbf{t}$	$\mathbf{t_I}$
	$\{t, t_I\}$	$\{t, t_I\}$	$\{I, t, t_I\}$	$\{f, f_I\}$	$\{f, f_I\}$

2. The only truth-values which are retained in $\mathcal{QM}_5\mathbf{ci}$ are t, f, and I. The non-deterministic truth tables in this Nmatrix corresponding to the operations $\neg$, $\circ$, $\forall$, and $\exists$ are :

$\widetilde{\circ}$	**f**	**I**	**t**
	$\{t\}$	$\{f\}$	$\{t\}$

$\widetilde{\neg}$	**f**	**I**	**t**
	$\{t\}$	$\{I,t\}$	$\{f\}$

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t,I\}$	$\{t,I\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t,I\}$	$\{t,I\}$
$\{t,f\}$	$\{f\}$	$\{t,I\}$
$\{t,I\}$	$\{t,I\}$	$\{t,I\}$
$\{f,I\}$	$\{f\}$	$\{t,I\}$
$\{t,f,I\}$	$\{f\}$	$\{t,I\}$

3. In $\mathcal{QM}_5\mathbf{cio}$ the tables for $\forall, \exists$ change to:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t,I\}$	$\{t,I\}$
$\{t,f\}$	$\{f\}$	$\{t\}$
$\{t,I\}$	$\{t\}$	$\{t\}$
$\{f,I\}$	$\{f\}$	$\{t\}$
$\{t,f,I\}$	$\{f\}$	$\{t\}$

4. In $\mathcal{QM}_5\mathbf{cia}$ the tables for $\forall, \exists$ change to:

H	$\widetilde{\forall}[H]$	$\widetilde{\exists}[H]$
$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{f\}$	$\{f\}$
$\{I\}$	$\{t,I\}$	$\{t,I\}$
$\{t,f\}$	$\{f\}$	$\{t\}$
$\{t,I\}$	$\{t,I\}$	$\{t,I\}$
$\{f,I\}$	$\{f\}$	$\{t,I\}$
$\{t,f,I\}$	$\{f\}$	$\{t,I\}$

5. The truth table for $\wedge$ in $\mathcal{QM}_5\mathbf{v}_\wedge$ becomes fully deterministic:

$\widetilde{\wedge}$	**f**	$\mathbf{f_I}$	**I**	**t**	$\mathbf{t_I}$
f	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\mathbf{f_I}$	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$	$\{f\}$
I	$\{f\}$	$\{f\}$	$\{t\}$	$\{t\}$	$\{t\}$
t	$\{f\}$	$\{f\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\mathbf{t_I}$	$\{f\}$	$\{f\}$	$\{t\}$	$\{t\}$	$\{t\}$

Theorem 29 (Soundness and completeness) *Let* $\mathbf{X} \subseteq Ax$. *Let* $\Gamma \cup \{\psi_0\}$ *be a set of* $\mathcal{L}_C$*-formulas.* $\Gamma \vdash_{\mathbf{B}[\mathbf{X}]} \psi_0$ *iff* $\Gamma \vdash_{\mathcal{QM}_5(\mathbf{X})} \psi_0$.

Proof: a straightforward modification of the proof of theorem 24. We only have to check that the conditions imposed by the schemata in $\mathbf{X}$ are respected by the valuation v defined in the proof. We prove this for $(\mathbf{a}_Q)$ and $(\mathbf{o}_Q)$:

- Suppose that $(\mathbf{a}_Q) \in \mathbf{X}$. Then from the definition of Γ^* it follows that that $\forall x \circ \psi \notin \Gamma^*$ in case $\circ Qx\psi \notin \Gamma^*$. Let $Qx\psi$ be an $L'(D)$-sentence, such that $H_\psi = \{v[\psi\{\overline{a}/x\}] \mid a \in D\} \subseteq \{t, f\}$. Suppose by contradiction that $v[Qx\psi] \notin \{t, f\}$. Then $\widetilde{\circ Qx\psi} = \circ Qx(\widetilde{\psi}) \notin \Gamma^*$ and so $\forall x \circ (\widetilde{\psi}) = \forall x(\widetilde{\circ\psi}) \notin \Gamma^*$. By property 8 of Γ^*, there exists some closed term $\mathbf{t}$ of L', such that $(\widetilde{\circ\psi})\{\mathbf{t}/x\} \notin \Gamma^*$. By lemma 26-3, $(\widetilde{\circ\psi})\{\mathbf{t}/x\} = (\circ(\widetilde{\psi\{\mathbf{t}/x\}}))$. By definition of v, $v[\psi\{\mathbf{t}/x\}] \notin \{t, f\}$. By lemma 12, $\psi\{\mathbf{t}/x\} \sim^S \psi\{\overline{\mathbf{t}}/x\}$. Since v respects the $\sim^S$ relation (this is proved like in theorem 24), $v[\psi\{\overline{\mathbf{t}}/x\}] \notin \{t, f\}$, in contradiction to our assumption about H_ψ.
- Suppose that $(\mathbf{o}_Q) \in \mathbf{X}$. Then $\exists x \circ \psi \notin \Gamma^*$ in case $\circ Qx\psi \notin \Gamma^*$. Let $Qx\psi$ be an $L'(D)$-sentence, such that $H_\psi \cap \{t, f\} \neq \emptyset$, where $H_\psi = \{v[\psi\{\overline{a}/x\}] \mid a \in D\}$. Suppose by contradiction that $v[Qx\psi] \notin \{t, f\}$. Then $(\widetilde{\circ Qx\psi}) = \circ Qx(\widetilde{\psi}) \notin \Gamma^*$ and so $\exists x \circ (\widetilde{\psi}) = \exists x(\widetilde{\circ\psi}) \notin \Gamma^*$. By property 9 of Γ^*, for every closed term $\mathbf{t}$ of L', $\widetilde{\circ\psi}\{\mathbf{t}/x\} \notin \Gamma^*$. By lemma 26-3, $(\widetilde{\circ\psi})\{\mathbf{t}/x\} = (\circ(\widetilde{\psi\{\mathbf{t}/x\}}))$. By definition of v, $v[\psi\{\mathbf{t}/x\}] \notin \{t, f\}$. By lemma 12, $\psi\{\mathbf{t}/x\} \sim^S \psi\{\overline{\mathbf{t}}/x\}$. Since again v respects the $\sim^S$ relation, $v[\psi\{\overline{\mathbf{t}}/x\}] \notin \{t, f\}$ for every $\mathbf{t} \in D$, in contradiction to our assumption.

Lemma 30 (Effectiveness of $\mathcal{QM}_5(\mathbf{X})$) *For every* $\mathbf{X} \subseteq Ax$, $\mathcal{QM}_5(\mathbf{X})$ *is effective.*

Proof: This follows from proposition 21.

4.2 Infinite Non-deterministic Semantics

We turn now to the extensions of the systems handled in the previous section by the schemata **(l)**,**(d)** and **(b)** (see Definition 7). It is easy to see that any of these schemata entails in **QB** both $(\mathbf{k}_1)$ and $(\mathbf{k}_2)$. Recall that the semantic effect of the latter two axioms is to delete t_I and f_I from the basic Nmatrix $\mathcal{QM}_5$. Thus the infinite Nmatrices provided in this section are all refinements (see Definition 16) of the *three-valued* Nmatrix $\mathcal{M}_5\mathbf{k}$.

To provide some informal intuition about the infinite semantics, note that what all of the above schemata have in common is a conjunction of a formula with its negation. Consider for instance the schema **(l)** $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$. Its validity is guaranteed only if $v[\neg(\varphi \wedge \neg\varphi)] \notin \mathcal{D}$ whenever $v[\circ\varphi] \notin \mathcal{D}$. Informally, to ensure this, we need to be able to isolate a conjunction of an "inconsistent" formula ψ with its own negation from conjunctions of ψ with other formulas. This can be done by enforcing an intimate connection between the truth-value of an “inconsistent” formula and the truth-value of its negation. This, in turn, requires a supply of infinitely many truth-values.

The following definition is a generalization of Definition 8 in [2]:

Definition 31 Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. Define the following Nmatrices for the language $\mathcal{L}_C$:

$\mathcal{QM}_3$**l:** This is the Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where:

1. $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$
2. $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$
3. $\mathcal{O}$ is defined by:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\widetilde{\forall}[H] = \begin{cases} \mathcal{D} & \text{if } H \subseteq \mathcal{D} \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\exists}[H] = \begin{cases} \mathcal{D} & \text{if } H \cap \mathcal{D} \neq \emptyset \\ \mathcal{F} & \text{otherwise} \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$\mathcal{QM}_3$**d:** This is defined like $\mathcal{QM}_3$**l**, except that $\widetilde{\wedge}$ is defined as follows:

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$\mathcal{QM}_3$**b:** This is defined like $\mathcal{QM}_3$**l**, except that $\widetilde{\wedge}$ is defined as follows:

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & (\text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\}) \text{ or } (b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\}) \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Theorem 32 (Soundness and completeness) *Let $\Gamma \cup \{\psi_0\}$ be a set of $\mathcal{L}_C$-formulas. For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$, $\Gamma \vdash_{\mathbf{QBy}} \psi_0$ iff $\Gamma \vdash_{\mathcal{QM}_3 y} \psi_0$.*

Proof: We do the proof for the case of **QBl**. The proofs in the other two cases are similar.
Soundness: Define the function $F : \mathcal{T} \cup \mathcal{I} \cup \mathcal{F} \to \{t, I, f\}$ as follows:

$$F(x) = \begin{cases} f & x \in \mathcal{F} \\ t & x \in \mathcal{T} \\ I & x \in \mathcal{I} \end{cases}$$

It is easy to see that F is a reduction of $\mathcal{QM}_3\mathbf{l}$ to $\mathcal{QM}_5\mathbf{k}$, and so $\mathcal{QM}_3\mathbf{l}$ is a refinement of $\mathcal{QM}_5\mathbf{k}$. By theorem 1, $\vdash_{\mathcal{QM}_5\mathbf{k}} \subseteq \vdash_{\mathcal{QM}_3\mathbf{l}}$. To prove soundness, it remains to show that **(l)** is $\mathcal{QM}_3\mathbf{l}$-valid. Let S be an L-structure and v an $\mathcal{QM}_3\mathbf{l}$-legal S-valuation, such that $v[\circ\psi] \in \mathcal{F}$. Then $v[\psi] = I^i_j$ for some i, j. Hence $v[\neg\psi] \in \{I^{i+1}_j, t^{i+1}_j\}$ and so $v[\psi \wedge \neg\psi] \in \mathcal{T}$ and $v[\neg(\psi \wedge \neg\psi)] \in \mathcal{F}$.

Completeness: Assume that $\Gamma \not\vdash_{\mathbf{QBl}} \psi_0$. Again we may assume that all elements of $\Gamma \cup \psi_0$ are sentences. Like in the proof of theorem 24, we proceed with a Henkin construction to get a maximal theory Γ^*, such that $\Gamma^* \not\vdash_{\mathbf{QBl}} \psi_0$ over the extended language L', and Γ^* satisfies the properties from the proof of theorem 24. Let D be the set of all the closed terms of L', and let Cl be the set of all the equivalence classes of $L'(D)$-sentences under $\sim^S$. For every $\mathcal{E} \in Cl$, choose the *minimal representative* of $\mathcal{E}$, $Min(\mathcal{E})$, to be a sentence with the least number of quantifiers of all the sentences in $\mathcal{E}$. (For instance, the sentences $\forall x p(c)$ and $p(c)$ are in the same equivalence class, but $Min(\mathcal{E}) \neq \forall x p(c)$ since $p(c)$ has less quantifiers). Let $\lambda i.\alpha_i$ be an enumeration of all the equivalence classes of $\mathcal{L}_C(D)$-sentences under $\sim^S$, such that the minimal representatives of which do not begin with $\neg$ (for instance, the minimal representative of $[\![\forall x \neg p(c)]\!]$ begins with $\neg$). It is easy to see that for any equivalence class $[\![\psi]\!]$, there are unique $n_{[\![\psi]\!]}, k_{[\![\psi]\!]}$ such that for every $A \in [\![\psi]\!]$, $A = \overline{\neg}_{k_\psi}\varphi$ for some $\varphi \in \alpha_{n_{[\![\psi]\!]}}$, where $\overline{\neg}_k\theta$ is a sentence obtained from θ by adding k preceding negation symbols and any number of preceding void quantifiers (Note that for any atomic sentence $p(\mathbf{t}_1, ..., \mathbf{t}_n)$, $k([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!]) = 0$). An L'-structure $S = \langle D, I \rangle$, and an $L'(D)$ valuation v in $\mathcal{QM}_3\mathbf{l}$ are now defined as follows (where $\widetilde{\psi}$ is defined as in the proof of Lemma 25):

- For every constant c of L': $I[c] = c$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$: $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$.
- For every $\mathbf{t}_1, ..., \mathbf{t}_n \in D$:

$$I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = \begin{cases} f & p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ t^0_{n([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!])} & \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \notin \Gamma^* \\ I^0_{n([\![p(\mathbf{t}_1, ..., \mathbf{t}_n)]\!])} & p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^*, \neg p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma^* \end{cases}$$

$$v[\psi] = \begin{cases} f & \widetilde{\psi} \notin \Gamma^* \\ t^{k(\llbracket\psi\rrbracket)}_{n(\llbracket\psi\rrbracket)} & (\widetilde{\neg\psi}) \notin \Gamma^* \\ I^{k(\llbracket\psi\rrbracket)}_{n(\llbracket\psi\rrbracket)} & \widetilde{\psi} \in \Gamma^*, (\widetilde{\neg\psi}) \in \Gamma^* \end{cases}$$

It is easy to see that v is well-defined. Obviously, $v[\psi] \in \mathcal{D}$ for every $\psi \in \Gamma^*$, while $v[\psi_0] = f$. It remains to show that v is $\mathcal{QM}_3$**l**-legal.

Let A, B be $L'(D)$-formulas such that $A \sim^S B$. Then $n_{\llbracket A\rrbracket} = n_{\llbracket B\rrbracket}$, and $k_{\llbracket A\rrbracket} = k_{\llbracket B\rrbracket}$. Also, $\neg A \sim^S \neg B$, and by Lemma 26-2 $\widetilde{A} \sim^{dc}_L \widetilde{B}$ and $\widetilde{\neg A} \sim^{dc}_L \widetilde{\neg B}$. By property 7 of Γ^*, $\widetilde{A} \in \Gamma^*$ iff $\widetilde{B} \in \Gamma^*$ and $\widetilde{\neg A} \in \Gamma^*$ iff $\widetilde{\neg B} \in \Gamma^*$. Thus by definition of v, $v[A] = v[B]$ and so v respects the $\sim^S$ relation.
The proof that v respects the operations corresponding to $\vee$, $\supset$, $\forall$ and $\exists$ is like in the proof of Theorem 24. We consider next the cases of $\circ$, $\neg$ and $\wedge$:

$\circ$: That $v[\circ\psi] = f$ in case $v[\psi] \in \mathcal{I}$ is shown as in the proof of Theorem 24. Assume next that $v[\psi] \in \mathcal{T} \cup \mathcal{F}$. Then either $\widetilde{\psi} \notin \Gamma^*$, or $\widetilde{\neg\psi} \notin \Gamma^*$. It follows that $\widetilde{\psi \wedge \neg\psi} \notin \Gamma^*$, and so $\neg(\widetilde{\psi} \wedge \neg\widetilde{\psi}) \in \Gamma^*$. Hence $\circ\widetilde{\psi} \in \Gamma^*$ by **(l)**, and so $v[\circ\psi] \in \mathcal{D}$.

$\neg$: The proofs that $v[\psi] = f$ implies $v[\neg\psi] \in \mathcal{D}$ and that $v[\psi] \in \mathcal{T}$ implies $v[\neg\psi] = f$ are like in the proof of Theorem 24. Assume next that $v[\psi] = I^k_n$. Then both $\widetilde{\psi}$ and $\widetilde{\neg\psi}$ are in Γ^*, and $\psi = \neg_k\varphi$ where $\varphi \in \alpha_n$. Thus $\neg\psi = \neg_{k+1}\varphi$ for $\varphi \in \alpha_n$, and so $n_{\llbracket\neg\psi\rrbracket} = n$, $k_{\llbracket\neg\psi\rrbracket} = k+1$. It follows by definition of v that $v[\neg\psi]$ is either I^{k+1}_n or t^{k+1}_n (depending whether $\neg\neg\psi$ is in Γ^* or not).

$\wedge$: The proofs that if $v[\psi_1] = f$ or $v[\psi_2] = f$ then $v[\psi_1 \wedge \psi_2] = f$, and that $v[\psi_1 \wedge \psi_2] \in \mathcal{D}$ otherwise, are like in the proof of Theorem 24. Assume next that $v[\psi_1] = I^k_n$ and $v[\psi_2] \in \{I^{k+1}_n, t^{k+1}_n\}$. Then both $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are in Γ^*, and so $\widetilde{\psi_1 \wedge \psi_2} \in \Gamma^*$. Also, $\psi_1 = \neg_k\varphi_1$, $\psi_2 = \neg_{k+1}\varphi_2$ for $\varphi_1, \varphi_2 \in \alpha_n$. It follows that $\psi_2 \sim^S \neg\psi_1$ and $\psi_1 \wedge \psi_2 \sim^S \psi_1 \wedge \neg\psi_1$. By lemma 25-2, $\widetilde{\psi_1 \wedge \psi_2} \sim^{dc}_L \widetilde{\psi_1 \wedge \neg\psi_1}$. By property 7 of Γ^*, $\widetilde{\psi_1 \wedge \neg\psi_1} \in \Gamma^*$, and so $\widetilde{\psi}_1, \widetilde{\neg\psi_1} \in \Gamma^*$. This entails that $\widetilde{\circ\psi_1} \notin \Gamma^*$. Hence schema **(l)** implies that $\neg(\widetilde{\psi_1 \wedge \neg\psi_1}) \notin \Gamma^*$. Hence $v[\psi_1 \wedge \psi_2] \in \mathcal{T}$.

Obviously, $v[\psi] \in \mathcal{D}$ for every $\psi \in \Gamma$, while $v[\psi_0] = f$. Hence $\Gamma \not\vdash_{\mathcal{QM}_3\mathbf{l}} \psi_0$. $\square$

Definition 33 *For* $\mathbf{X} \subseteq Ax$*,* $\mathcal{QM}_3\mathbf{l}(\mathbf{X})$ *is obtained from* $\mathcal{QM}_3\mathbf{l}$ *through the following modifications:*

1. *If* $(\mathbf{i}_1) \in \mathbf{X}$*:* $\quad a \in \mathcal{F} \Rightarrow \widetilde{\circ}(a) = \mathcal{T}$
2. *If* $(\mathbf{i}_2) \in \mathbf{X}$*:* $\quad a \in \mathcal{T} \Rightarrow \widetilde{\circ}(a) = \mathcal{T}$
3. *If* $(\mathbf{c}) \in \mathbf{X}$ *or* $(\mathbf{a}_\neg) \in \mathbf{X}$*:* $\quad \widetilde{\neg} f = \mathcal{T}$
4. *If both* $(\mathbf{e})$ *and* $(\mathbf{w})$ *are in* $\mathbf{X}$*, delete all the truth-values in* $\mathcal{I}$*. Otherwise, if* $(\mathbf{e}) \in \mathbf{X}$*:* $\quad \widetilde{\neg} I^j_i = \{I^{j+1}_i\}$*. If* $(\mathbf{w}) \in \mathbf{X}$*:* $\quad a \in \mathcal{F} \Rightarrow \widetilde{\neg} a = \mathcal{T}$ *and* $\widetilde{\neg} I^i_j = \{t^i_j\}$
5. *If* $(\mathbf{a}_\wedge) \in \mathbf{X}$*:* $\quad a \in \mathcal{T}$ *and* $b \in \mathcal{T} \Rightarrow a \widetilde{\wedge} b = \mathcal{T}$
6. *If* $(\mathbf{a}_\vee) \in \mathbf{X}$*:* $\quad a \in \mathcal{T}, b \notin \mathcal{I}$ *or* $b \in \mathcal{T}, a \notin \mathcal{I} \Rightarrow a \widetilde{\vee} b = \mathcal{T}$

7. *If* $(\mathbf{a}_\supset)\in \mathbf{X}$*:* $a \in \mathcal{F}, b \notin \mathcal{I}$ *or* $b \in \mathcal{T}, a \notin \mathcal{I} \Rightarrow a\tilde{\supset}b = \mathcal{T}$
8. *If* $(\mathbf{o}_\wedge)\in \mathbf{X}$*:* $a \in \mathcal{T}$ *or* $b \in \mathcal{T}$ *and* $a, b \in \mathcal{D} \Rightarrow a\tilde{\wedge}b = \mathcal{T}$
9. *If* $(\mathbf{o}_\vee)\in \mathbf{X}$*:* $a \in \mathcal{T}$ *or* $b \in \mathcal{T} \Rightarrow a\tilde{\vee}b = \mathcal{T}$
10. *If* $(\mathbf{o}_\supset)\in \mathbf{X}$*:* $a \in \mathcal{F}$ *or* $b \in \mathcal{T} \Rightarrow a\tilde{\supset}b = \mathcal{T}$
11. *If* $(\mathbf{v}_\wedge)\in\mathbf{X}$*:* $a, b \in \mathcal{T} \cup \mathcal{I} \Rightarrow a\tilde{\wedge}b = \mathcal{T}$
12. *If* $(\mathbf{v}_\vee)\in\mathbf{X}$*:* $a \notin \mathcal{F}$ *or* $b \notin \mathcal{F} \Rightarrow a\tilde{\vee}b = \mathcal{T}$
13. *If* $(\mathbf{v}_\supset)\in\mathbf{X}$*:* $a \in \mathcal{F}$ *or* $b \in \mathcal{T} \cup \mathcal{I} \Rightarrow a\tilde{\supset}b = \mathcal{T}$
14. *If* $(\mathbf{a}_\forall)\in\mathbf{X}$*:* $H \subseteq \mathcal{T} \Rightarrow \tilde{Q}[H] = \mathcal{T}$
15. *If* $(\mathbf{a}_\exists)\in\mathbf{X}$*:* $H \subseteq \mathcal{T} \cup \mathcal{F}$ *and* $H \cap \mathcal{T} \neq \emptyset \Rightarrow \tilde{Q}[H] = \mathcal{T}$
16. *If* $(\mathbf{o}_\forall)\in\mathbf{X}$*:* $H \cap \mathcal{T} \neq \emptyset$ *and* $H \subseteq \mathcal{D} \Rightarrow \tilde{\forall}[H] = \mathcal{T}$
17. *If* $(\mathbf{o}_\exists)\in\mathbf{X}$*:* $H \cap \mathcal{T} \neq \emptyset \Rightarrow \tilde{Q}[\exists] = \mathcal{T}$
18. *If* $(\mathbf{v}_\forall)\in\mathbf{X}$*:* $H \subseteq \mathcal{T} \cup \mathcal{I} \Rightarrow \tilde{\forall}[H] = \mathcal{T}$
19. *If* $(\mathbf{v}_\exists)\in\mathbf{X}$*:* $(H \cap (\mathcal{T} \cup \mathcal{I})) \neq \emptyset \Rightarrow \tilde{\exists}[H] = \mathcal{T}$

The Nmatrices $\mathcal{QM}_3\mathbf{d}(\mathbf{X})$ *and* $\mathcal{QM}_3\mathbf{b}(\mathbf{X})$ *are defined similarly.*

Remark: it is easy to see that for any $\mathbf{X} \subseteq Ax$ and $y \in \{(\mathbf{l}), (\mathbf{d}), (\mathbf{b})\}$, the set of conditions in $\mathbf{X}$ is coherent, the interpretations of the connectives and the quantifiers of $\mathcal{QM}_3y(\mathbf{X})$ never return empty sets and so $\mathcal{QM}_3y(\mathbf{X})$ is well-defined.

Theorem 34 (Soundness and completeness) *Let* $\Gamma \cup \{\psi_0\}$ *be a set of* $\mathcal{L}_C$*-formulas. Let* $\mathbf{X} \subseteq Ax$ *and* $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$*. Then* $\Gamma \vdash_{\mathbf{QB}y[\mathbf{X}]} \psi_0$ *iff* $\Gamma \vdash_{\mathcal{QM}_3y(\mathbf{X})} \psi_0$*.*

Proof: It is easy to show that $\mathcal{QM}_3y(\mathbf{X})$ is a (simple) refinement of $\mathcal{QM}_3(\mathbf{X})$ and so by theorem 1, $\vdash_{\mathcal{QM}_3(\mathbf{X})} \subseteq \vdash_{\mathcal{QM}_3y(\mathbf{X})}$. It is also easy to check that for any schema in $\mathbf{X}$, the relevant condition guarantees its validity in $\mathcal{QM}_3y(\mathbf{X})$, and so soundness follows. The proof of completeness is a straightforward extension of the proof of theorem 32.

Corollary 35 *Let* $\Gamma \cup \psi$ *be a set of* L_C*-formulas, in which* $\circ$ *does not occur. Then* $\Gamma \vdash_{\mathbf{QBlca}} \psi$ *iff* $\Gamma \vdash_{\mathbf{QBlcia}} \psi$*.*

Proof: It can be easily checked that the only difference between the Nmatrices $\mathcal{QM}_3$**lcia** and $\mathcal{QM}_3$**lca** is in their interpretation of $\circ$.

Corollary 36 *Let the Nmatrix* $\mathcal{QM}_3C_1^*$ *for* $\mathcal{L}_{\text{cl}}$ *be obtained from the Nmatrix* $\mathcal{QM}_3$**lcia** *for* $\mathcal{L}_C$ *(or* $\mathcal{QM}_3$**lca***) by discarding the interpretation of* $\circ$*. Then* $\mathcal{QM}_3C_1^*$ *is a characteristic Nmatrix for* C_1^**.*

Proof: similar to the proof of theorem 34. (Another alternative is to use a translation of C_1^* to **QBlcia**, similar to the translation of the proof of theorem 107 of [11] for the propositional case.)

Remark: da Costa's C_1 is usually considered to be the $\circ$-free analogue of the propositional fragment of **QBlcia** (called **Cila** in [8, 11]). However, from the above corollaries it follows that it is equally justified to identify it with **Cla**, the propositional fragment of **QBlca**. A similar observation applies to C_1^*.

Lemma 37 (Effectiveness) *For every* $\mathbf{X} \subseteq Ax$ *and every* $y \in \{(\mathbf{l}), (\mathbf{d}), (\mathbf{b})\}$, $\mathcal{QM}_3\mathbf{y}(\mathbf{X})$ *is effective.*

Proof: This follows from proposition 21, and the suitability of $\mathcal{QM}_3\mathbf{y}(\mathbf{X})$ for $\sim_L^{dc}$.

5 Logical Indistinguishability in First-order LFIs

In this section we apply the framework of Nmatrices and in particular their effectiveness to prove a very important proof-theoretical property of the first-order LFIs investigated here.

Definition 38 *Let* $\mathbf{S}$ *be a system which includes positive classical logic. Two sentences* A *and* B *are* logically indistinguishable *in* $\mathbf{S}$ *if* $\varphi(A) \vdash_{\mathbf{S}} \varphi(B)$ *and* $\varphi(B) \vdash_{\mathbf{S}} \varphi(A)$ *for every sentence* $\varphi(\psi)$ *in the language of* $\mathbf{S}$.

Theorem 39 *Let* $\mathbf{S}$ *be a system over a first-order language* L *which includes* $\{\neg, \supset\}$*, and assume that* $A \vdash_{\mathbf{S}} B$ *whenever* $A \sim_L^{dc} B$*. If one of the following holds, then two sentences* A, B *are logically indistinguishable in* $\mathbf{S}$ *iff* $A \sim_L^{dc} B$*:*

1. $\mathbf{QBbcia_pwv_Q}$ *is an extension of* $\mathbf{S}$.
2. $\mathbf{QBbcia_pev_Q}$ *is an extension of* $\mathbf{S}$.
3. $\mathbf{QBbive}$ *is an extension of* $\mathbf{S}$

Proof: For all the parts one direction is trivial: assume that $A \sim_L^{dc} B$. Then since $\sim_L^{dc}$ is a congruence relation, $\psi(A) \sim_L^{dc} \psi(B)$ for every ψ and so A, B are logically indistinguishable by our assumption about S.

For the converse, let A, B be two sentences, such that $A \not\sim_L^{dc} B$.
For the first and the second parts, let q be an atomic propositional sentence[5], such that q does not occur in A or B. Let $S = \langle D, I \rangle$ be some L-structure, such that $I[q] = I_0^0$, and for every two closed terms $\mathbf{t}_1 \neq \mathbf{t}_2$ occurring in A and B respectively, $I[\mathbf{t}_1] \neq I[\mathbf{t}_2]$. Let W_S be the minimal set of $L(D)$-sentences closed under subformulas, such that $A, B, q \in W_S$. Let v be some partial S-valuation on W_S, satisfying: $v[q] = I_0^0$, $v[q \supset (B \supset B)] = I_0^0$, $v[\circ(q \supset (B \supset B))] = f$, $v[q \supset (A \supset A)] = t_0^0$, and $v[\circ(q \supset (A \supset A))] = t_0^0$ (such v exists, since both $v[A \supset A]$ and $v[B \supset B]$ are in $\mathcal{D}$, and by lemma 13, $q \supset (A \supset A) \not\sim_L^{dc} q \supset (B \supset B)$ iff $q \supset (A \supset A) \not\sim^S q \supset (B \supset B)$). It is easy to check that v is legal in $\mathcal{QM}_3\mathbf{bcia_pwv_Q}$ and in $\mathcal{QM}_3\mathbf{bcia_pev_Q}$. By lemma 37 it follows that $\circ(q \supset (A \supset A)) \nvdash_{\mathbf{S}} \circ(q \supset (B \supset B))$. Hence A and B are not logically indistinguishable in $\mathbf{S}$.

For the third part, assume without a loss in generality that $A \supset A$ is not a subformula of $B \supset B$. Let $S = \langle D, I \rangle$ be an L-structure, such that for every two closed terms $\mathbf{t}_1 \neq \mathbf{t}_2$ occurring in A and B respectively, $I[\mathbf{t}_1] \neq I[\mathbf{t}_2]$. Let W_S be the minimal set of $L(D)$-sentences closed under subformulas, such that $\neg\neg\neg(B \supset$

[5] For simplicity we assume that we have propositional sentences in L, but it is not difficult to replace q by a suitable first-order sentence.

$B) \in W_S$. Let v be some partial S-valuation on W_S, satisfying: $v[B \supset B] = t_0^0$, $v[\neg(B \supset B)] = f$, $v[\neg\neg(B \supset B)] = I_0^0$, $v[\neg\neg\neg(B \supset B)] = I_0^1$. Extend v to a partial valuation defined also on the subformulas of $\neg\neg\neg(A \supset A)$, which satisfies: $v[A \supset A] = t_0^0$, $v[\neg(A \supset A)] = f$, $v[\neg\neg(A \supset A)] = t_0^0$, $v(\neg\neg\neg(A \supset A)) = f$. Again this is possible since by lemma 13. It is easy to see that v is legal in $\mathcal{QM}_3$**bive**. By lemma 37, it follows that $\neg\neg\neg(B \supset B) \nvdash_{\mathbf{S}} \neg\neg\neg(A \supset A)$. Hence A and B are not logically indistinguishable in **S**. □

Remarks:

1. This theorem extends similar theorems from [2] and [20]. In [2] it is proved for propositional systems weaker than the propositional fragments of $\mathbf{QBbcia_pe}$ and $\mathbf{QBbio_pe}$. In [20] a similar theorem for the first-order case is proved for systems weaker than $\mathbf{QBbcia_pe}$. This theorem extends these results in the following aspects:
   - Covering first-order systems stronger than $\mathbf{QBbcia_p}$ and weaker than $\mathbf{QBbcia_pwv_Q}$.
   - Covering first-order systems stronger than $\mathbf{QBbcia_pe}$ and weaker than $\mathbf{QBbcia_pev_Q}$.
   - Extending to the first-order case the propositional results of [2] for systems weaker than $\mathbf{QBbio_pe}$ and generalizing them to systems weaker than **QBbive**.
2. Extensions of **QBcio** do not have the property described above. In fact, it can be shown that $\circ(A \supset A)$ and $\circ(B \supset B)$ are logically indistinguishable in **QBcio** for any two sentences A and B (it is shown in [11] for the propositional case).
3. Extensions of **QBiew** also do not have the above property. In fact, it is easy to see that **QBiew** collapses into classical logic, where any two equivalent formulas are logically indistinguishable.

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