*Map Cohomology Operations and Enumeration of Vector Bundles**

LAWRENCE L. LARMORE

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1. Introduction. The present paper is a continuation of the work of Dold–Whitney [6] on the classification of vector bundles over a finite dimensional space. In the course of the present work, a new tool, map-cohomology operations, is described. This concept proves useful in the classification of maps by homotopies.

Basic to this work is the fact that the oriented vector bundles over a space X are in a $1 - 1$ correspondence with $[X; BSO(n)]$, the homotopy classes of maps from X to $BSO(n)$ [4], while the stable classes of oriented bundles over X correspond to $[X; BSO]$.

Some work has been done toward classifying bundles over X using Postnikov systems. $[X; BSO]$ is an Abelian group given to us by K-theory; if $\dim X \leqq 7$, we have an exact sequence:

$$0 \to H^4(X; Z) \xrightarrow{d} [X; BSO] \xrightarrow{w} w_2(X) \to 0,$$

where $w_2(X) = \{x \in H^2(X; Z_2) \mid \beta_2 x^2 = 0\}$, where $w[f] = f^* w_2$ and where $dz = [g]$, where $g^* w_4 = \pi z$. In [6], Dold and Whitney classify all oriented vector bundles over a 4-dimensional complex. In [7], James and Thomas classify n-dimensional realizations of a stable vector bundle over an n-dimensional complex if the bundle is oriented or if n is odd. In the present paper we arrive at formulas for counting realizations by a slightly different method.

Generally, if X is a finite dimensional complex and if ξ is a stable oriented bundle over X, and if n is sufficiently large, the number of realizations of ξ as an n-bundle (*i.e.*, n-plane bundle), given that such a realization exists, may be put into a $1 - 1$ correspondence, in a non-canonical way, with K/L, where K is a cohomology group of X and L is a subgroup. We shall derive specific formulae for L in terms of ξ in certain cases (*cf.* 5.8, 5.10).

Throughout, we let β_2 denote the Bokstein of the coefficient sequence $Z \to$

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$Z \to Z_2$ and we let π: $H^*(\ ; Z) \to H^*(\ ; Z_2)$ be the map induced by $Z \to Z_2$. We assume that all spaces have base points and that all homotopies are base-point preserving. We let $p_i: X_1 \times X_2 \to X_i$, for $i = 1$ or 2, denote the projection and $\triangle: X \to X \times X$ denote the diagonal map.

2. Map-cohomology operations.

2.1 Let B be any space with base point, let p and q be integers and let M and N be Abelian groups. Let Map $(X, A; B)$ be the set of maps from X to B which send A to the basepoint. A map-cohomology operation of type $\{B, p, q, M, N\}$ is a collection of functions θ from Map $(X, A; B) \times H^p(X, A; M)$ to the set of subsets of $H^q(X, A; N)$. There is one such function for each cellular pair (X, A), and they are subject to the following conditions:

(i) If $f \sim f'$ rel A, then $\theta(f, x) = \theta(f', x)$.
(ii) If g: $(X', A') \to (X, A)$ is any map, then $g^*\theta(f, x) \subset \theta(fg, g^*x)$.
We say θ is *homogeneous* if
(iii) for any map f, $0 \in \theta(f, 0)$.

We define $D\theta$, the *domain* of θ, to be the set of pairs (f, x) where $\theta(f, x)$ is non-empty. If $\theta(f, x)$ is a one-element set for all (f, x), we say that θ is *primary*. If ψ is a $\{B, p, r, M, G\}$ map-cohomology operation, we say that ψ is *secondary* to θ provided $(f, x) \in D\psi$ if and only if $0 \in \theta(f, x)$.

2.2 If θ is a $\{B, p, q, M, N\}$ map-cohomology operation, and if e: $C \to B$ is any map, then we define a $\{C, p, q, M, N\}$ map-cohomology operation $e^*\theta$ by the formula $e^*\theta(f, x) = \theta(ef, x)$.

2.3 If θ is a map-cohomology operation, we say that β is a *choice function subsidiary to* θ if β is defined on $D\theta$, and $\phi \neq \beta(f, x) \subset \theta(f, x)$ for all $(f, x) \in D\theta$.

3. Classification of liftings. Throughout this section we let $\{E, \pi, B, K, K\}$ be a principal K-bundle in the sense of Steenrod [4], where $K = K(G, n)$ is an Eilenberg–MacLane space and G is an Abelian group. We let y: $E \to BK = K(G, n + 1)$ be the classifying map for the bundle, and let $k \in H^{n+1}(B, G)$ be the k-invariant, *i.e.* the image of the fundamental class of BK under y^*.

If (X, A) is any cellular pair, and if f and g are maps from X to E which send A to the basepoint of E, and if F is a homotopy rel A of πf with πg, we let $\delta^n(f, g; F) \in H^n(X, A; G)$ denote the obstruction to lifting F to a homotopy rel A of f with g. If $\{K, 0, *, K, K\}$ is the K-bundle over a one-point space $*$, and if h: $(X, A) \to (K, k_0)$ is any map, (where k_0 is the basepoint of K) then $\delta^n(h, c; 0) = h^*\alpha$, where $c: X \to K$ is the constant map, $0: X \times I \to *$ is the trivial homotopy, and α is the fundamental class of K.

We shall now let f: $X \to B$ be any map, where X is some cell complex with basepoint. Now f may be lifted to E if and only if $f^*k = 0$; we shall be concerned with the classification of these liftings up to homotopy.

3.1 Let g: $X \to E$ be a lifting of f. We define a subgroup L_f of $H^n(X; G)$ by

$L_f = \{\delta^n(g, g; F) \mid F \text{ is a homotopy of } f \text{ with itself}\}$. If $h\colon X \to E$ is any other lifting of f, then:

$$\delta^n(h, h; F) = \delta^n(h, g; fp_1) + \delta^n(g, g; F) + \delta^n(g, h; fp_1) = \delta^n(g, g; F),$$

thus L_f does not depend on the choice of g.

3.2 Theorem. Let $g\colon X \to E$ be a lifting of f. Let W be the set of equivalence classes of liftings of f, where two liftings are equivalent iff they are homotopic maps. Let $\Phi\colon W \to H^n(X; G)/L_f$ be defined as follows: $\Phi[h] = \delta^n(h, g; fp_1) + L_f$. Then Φ is one-to-one and onto.

Proof. If H is any homotopy of h_1 with h_2, then $\delta^n(h_1, h_2; \pi H) = 0$. Thus

$$\delta^n(h_2, g; fp_1) - \delta^n(h_1, g; fp_1)$$
$$= \delta^n(h_1, g; fp_1) + \delta^n(h_1, h_2; \pi H) = \delta^n(h_2, h_2; \pi H) \in L_f,$$

whence Φ is well-defined. If $\Phi[h_1] = \Phi[h_2]$, then $\delta^n(h_2, g; fp_1) - \delta^n(h_1, g; fp_1) = \delta^n(h_2, h_2; F)$ for some homotopy F of f with itself. We have

$$\delta^n(h_1, h_2; F) = \delta^n(h_1, g; fp_1) + \delta^n(g, h_2; fp_1) + \delta^n(h_2, h_2; F) = 0.$$

Thus F may be lifted to a homotopy H of h_1 with h_2. This shows that Φ is one-to-one.

Finally, if $a \in H^n(X; G)$, we shall find a map $h\colon X \to E$ such that $\delta^n(h, g; fp_1) = a$. The bundle induced over E by the map $\pi\colon E \to B$ is simply a product bundle $p_1\colon E \times K \to E$. Let $\mu\colon E \times K \to E$ be the bundle map induced by π; the following diagram is commutative:

$$\begin{array}{ccc} E\times K & \xrightarrow{\mu} & E \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & B \end{array}$$

Let $q\colon X \to K$ be a map such that $q^*\alpha = a (\alpha = \text{fundamental class})$. Let $h = \mu(g \times q)\triangle\colon X \to E$; then $\delta^n(h, g; fp_1) = \delta^n((g \times q)\triangle, (g \times c)\triangle; gp_1) = q^*\alpha = a$, where $c\colon X \to K$ is the constant map. Thus Φ is onto.

4. Classification of liftings by map-cohomology operations. In this section, we describe the group L_f in terms of map-cohomology operations, where $f\colon X \to B$ is a map from a cell complex to a space with finitely many non-zero homotopy groups. Throughout, assume that m is a positive integer, and that we are given:

(i) a positive integer $n(k)$ for each $0 < k \leqq m$,
(ii) an Abelian group G_k for each k $(0 < k \leqq m)$,
(iii) a cell complex with basepoint E_k for each k, where E_0 is a one-point space,
(iv) a map $y_k\colon E_{k-1} \to BK_k$ for each $0 < k \leqq m$, where BK_k is an Eilenberg-MacLane space of type $(G_k, n(k) + 1)$ which is a topological group [3],

(v) a map $\pi_k : E_k \to E_{k-1}$ for each k $(0 < k \leqq m)$ such that $\{E_k, \pi_k, E_{k-1}, K_k, K_k\}$ is the principle K_k-bundle induced by the universal K_k-bundle over BK_k and the map y_k, where $K_k = \Omega BK_k$, the loops on BK_k.

For all $k(0 < k \leqq m)$, let $\lambda_k : K_k \to E_k$ be the inclusion map. We let $\mu_k : E_k \times K_k \to E$ be the bundle map induced by π_k as described in 3.2. We now let $f: X \to E_m$ be any map which sends A to the basepoint of E_m, where (X, A) is a cellular pair, and we let $f_k = \pi_{k+1} \cdots \pi_m f$ be the composite of f with the appropriate fiberings, for all $0 \leqq k \leqq m$. Let $(K, M) = (X \times I, X \times \partial I \cup A \times I)$. We shall compute $L = L_{f_{m-1}} \subset H^{n(m)}(X, A; G_m)$.

4.1 We say that a homotopy H of f_k with itself is an (f_k, z) homotopy for some $z \in H^{n(k)-1}(X, A; G_k)$ if $H = \mu_k(f_k p_1 \times F)\triangle$ (where $\triangle: S \to S \times S$ is always the diagonal map, for any space S) for some map $F: (K, M) \to K_k$ where $F^*\alpha = sz$ (s = suspension). Clearly any two (f_k, z) homotopies are homotopic rel M rel $f_{k-1}p_1$. (In general, if s_0 and s_1 are maps from any space X to any space Y, and if A is a subset of X, and if $g: Y \to Z$ is any map, where Z is any space, then we say that s_0 is homotopic to s_1 rel A rel g if there is a homotopy $\{s_t\}$ connecting s_0 with s_1 such that $s_t | A = s_0 | A$ and $gs_t = gs_0$ for all $0 \leqq t \leqq 1$.) We say that any homotopy R of f_p with itself is (f, p) admissible; if, in addition, the composite $\pi_{k+1} \cdots \pi_p R$ is an (f_k, z) homotopy of f_k with itself, for some $z \in H^{n(k)-1}(X, A; G_k)$, we say that R is (f, p, k, z) admissible. For any (f, p) admissible homotopy R, let $D(f_{p+1}, R)$ denote $\delta^{n(p+1)}(f_{p+1}, f_{p+1}; R) \in H^{n(p+1)}(X, A; G_{p+1})$.

4.2 For each $k(0 < k < p \leqq m)$ and each $z \in H^{n(k)-1}(X; G_k)$ we let $\theta_{p,k}(f, z) = \{D(f_p, H) \mid H$ is an $(f, p-1, k, z)$ admissible homotopy$\}$.

Proposition. *$\theta_{p,k}$ is a map-cohomology operation of type $\{E_m, n(k) - 1, n(p), G_k, G_p\}$.*

Proof. Let H be a homotopy rel A of h^0 with h^1, where h^0 and h^1 are any maps from X to E_m which send A to the basepoint. Let $H_r = \pi_{r+1} \cdots \pi_m H$, a homotopy of h_r^0 with h_r^1, for each r $(0 < r \leqq m)$. Let z be any element of $H^{n(k)-1}(X, A; G_k)$, and let a be any element of $\theta_{p,k}(h^0, z)$. Now $a = D(h_p^0, R_0)$ for some $(h^0, p-1, k, z)$ admissible homotopy, R_0. Let $S_0 = \mu_k(h_k^0 p_1 \times F)\triangle$ where $F: (K, M) \to K_k$ is any map such that $F^*\alpha = sz$. For each $t \in I$, let $S_t = \mu_k(h_k^t p \times F)\triangle$, a homotopy of h_k^t with itself, where $h_k^t(x)$ is defined to be $H(x, t)$ for all $x \in X$. Now the composition $\pi_{k+1} \cdots \pi_p : E_p \to E_k$ has the Covering Homotopy Extension Property for K relative to M[1, *III*, 2], hence we may find a homotopy $\{R_t\}$ of R_0 such that $\pi_{k+1} \cdots \pi_{p-1} R_t = S_t$ for all $t \in I$, and each R_t is a homotopy of h_{p-1}^t with itself. Now R_1 is an $(h^1, p-1, k, z)$ admissible homotopy, hence $a = d(h_p^0, R_0) = D(h_p^1, R_1) \in \theta_{p,k}(h^1, z)$. Thus $\theta_{p,k}(h, z)$ does not depend on the homotopy class of h.

Let now $g': (X', A') \to (X, A)$ be any map, where (X', A') is another cellular pair. Let $z \in H^{n(k)-1}(X; G_k)$ and $a \in \theta_{p,k}(f, z)$. Then $a = D(f_p, H)$ for some $(f, p-1, k, z)$ admissible homotopy H. Let R be the composite $\pi_{k+1} \cdots \pi_{p-1} H$: then $R = \mu_k(f_k p_1 \times F)\triangle$ where $F: (K, M) \to K_k$ is a map such that $F^*\alpha = sz$. Now

$\pi_{k+1} \cdots \pi_{p-1}(H(g \times 1)) = \mu_k(f_k p_1 \times F)\triangle(g \times 1) = \mu_k(f_k g p_1 \times F(g \times 1))\triangle$, a homotopy of $f_k g$ with itself, and $(F(g \times 1))^*\alpha = sg^*z$, whence $H(g \times 1)$ is an $(fg, p - 1, k, g^*z)$ admissable homotopy.

Thus $g^*a = D(f_p g, H(g \times 1)) \in \theta_{p,k}(fg, g^*z)$. We conclude that $g^*\theta_{p,k}(f, k) \subset \theta_{p,k}(fg, g^*z)$.

4.3 Proposition. *For all* $0 < k < m$, $\theta_{k+1,k}$ *is a primary map-cohomology operation.*

Proof. For any $z \in H^{n(k)-1}(X; G_k)$, let $a_1, a_2 \in \theta_{k+1,k}(f, z)$. For $i = 1$ or 2, $a_i = \delta^{n(k+1)}(f_{k+1}, f_{k+1}; H_i)$, where $H_i = \mu_k(f_k p_1 \times F_i)\triangle$, $F_i^*\alpha = sz$. $F_1 \sim F_2$ rel L, whence $H_1 \sim H_2$ rel L. Thus $a_1 = a_2$.

4.4 Proposition. *For all* $0 < k < p < m$, $\theta_{p+1,k}$ *is secondary to* $\theta_{p,k}$.

Proof. Let $z \in H^{n(k)-1}(X; G_k)$. We need only show that $\theta_{p+1,k}(f, z)$ is non-empty iff $0 \in \theta_{p,k}(f, z)$. Note that $\theta_{p+1,k}(f, z)$ is non-empty iff there exists some (f, p, k, z) admissible homotopy.

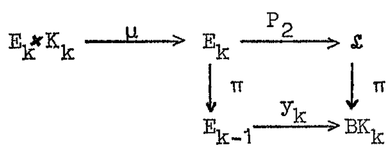
Suppose H is (f, p, k, z) admissible. Then $\pi_p H$ is $(f, p - 1, k, z)$ admissible. Now $0 = \delta^{n(p)}(f_p, f_p; \pi_p \mathrm{H}) \in \theta_{p,k}(f, z)$.

On the other hand suppose $0 \in \theta_{p,k}(f, z)$. Then $0 = \delta^{n(p)}(f_p, f_p; S)$ for some $(f, p - 1, k, z)$ admissible homotopy S. S may be lifted to a homotopy H which is (f, p, k, z) admissible.

4.5 Proposition. *If* H *is any homotopy of* f_k *with itself where* $\pi_k H = f_{k-1}p_1$, *then* H *is an* (f_k, z) *homotopy for some* $z \in H^{n(k)-1}(X; G_k)$.

Proof. We recall that $E_k = \{(e, \sigma) \in E_k \times \mathfrak{L} \mid y_k e = \Pi\sigma = \sigma(1) \in BK_k$, where $\mathfrak{L}$ is the path space of $BK_k\}$, that is, the space of all maps $\sigma: I \to BK_k$ such that $\sigma(0)$ is the base point $*$ of BK_k. We consider $\mathfrak{L}$ to be a topological group under pointwise multiplication in BK_k. Then $K_k = \Omega BK_k \subset \mathfrak{L}$ is a subgroup.

We may define $\mu = \mu_k : E_k \times K_k \to E_k$ as follows: if $((e, \sigma), \tau) \in E_k \times K_k$, let $\mu((e, \sigma), \tau) = (x, \sigma \cdot \tau)$.



Let $F: (K, M) \to K_k$ be the map given by $F(x, t) = (p_2 fx)^{-1} \cdot (p_2 H(x, t))$ for all $(x, t) \in K$, and let $z = F^*\alpha$. Then $\mu(fp_1 \times F)\triangle(x, t) = \mu(fx, F(x, t)) = \mu((\pi fx, p_2 fx), (p_2 fx)^{-1}(p_2 H(x, t))) = (\pi fx, p_2 H(x, t)) = (\pi H(x, t), p_2 H(x, t)) = H(x, t)$ for all $(x, t) \in K$. Thus H is an (f_k, z) homotopy.

4.6 Theorem. *For any* $1 < k < p \leqq m$ *and for any* $z \in H^{n(k-1)-1}(X; G_{k-1})$, *we have that* $\theta_{p,k-1}(f, z)$ *is a coset of the union of* $\theta_{p,k}(f, x)$ *over all* $x \in H^{n(k)-1}(X; G_k)$.

Proof. Suppose that $a_1, a_2 \in \theta_{p,k-1}(f, z)$. Then $a_i = D(f_p, A_i)$, where A_i

is an $(f, p-1, k-1, z)$ admissible homotopy, for $i = 0$ or 1. Then $A_1 - A_2$ is $(f, p-1, k-1, 0)$ admissible; hence $\pi_k \cdots \pi_p(A_1 - A_2) \sim fp_1$. Thus $A_1 - A_2$ is homotopic to an $(f, p-1, k, x)$ admissible homotopy (*cf.* 4.5) H; $a_1 - a_2 = D(f_p, H) \in \theta_{p,k}(f, x)$.

Suppose $a = D(f_p, A)$ and $b = D(f_p, B)$ for some $(f, p-1, k-1, z)$ admissible homotopy A and some $(f, p-1, k, x)$ admissible homotopy B, for some $x \in H^{n(k)-1}(X; G_k)$. Then B is $(f, p-1, k-1, 0)$ admissible; $A + B$ is $(f, p-1, k-1, z)$ admissible and then $a + b = D(f_p, A + B) \in \theta_{p,k-1}(f, z)$.

4.7 Theorem. *Let $f\colon X \to E_m$ be a map. Then if $m = 1$, $L_f = 0$. If $m > 1$, L_f is the union of $\theta_{m,1}(f, z)$ over all $z \in H^{n(1)-1}(X; G_1)$.*

Proof. If $m = 1$, any $(f, 0)$ admissible homotopy is trivial. If $m > 1$, and if $a \in L_f$, then $a = \delta^{n(m)}(f, f; R)$ for some homotopy R of f_{m-1} with itself. Now $\pi_2 \cdots \pi_{m-1}$ is an (f_1, z) homotopy for some $z \in H^{n(1)-1}(X; G)$ (*cf.* 4.5), thus R is $(f, m-1, 1, z)$ admissible and $a \in \theta_{m,1}(f, z)$.

If, on the other hand, $a \in \theta_{m,1}(f, z)$ for some z, then $a = D(f, R)$ for some $(f, m-1, 1, z)$ admissible homotopy R. R is $(f, m-1)$ admissible, thus $a \in L_f$.

5. Specific computations of L_f and application to enumeration of vector bundles. Through 5.1 and 5.2, we shall let X be a cell complex with base-point x_0. Let QX be the space obtained from $X \times I$ by collapsing $x_0 \times I$ to a point and by identifying $(x, 0)$ with $(x, 1)$ for all $x \in X$. Let $J\colon X \times I \in QX$ be the identification map, and let $k\colon X \to QX$ be the inclusion map, where $kx = J(x, 0)$ for all $x \in X$. If $f\colon X \to Y$ is any map, where Y is any space, and if F is a homotopy rel x_0 of f with itself, let $UF\colon QX \to Y$ denote the unique map such that $(UF)J = F$.

5.1 Let $(K, M) = (X \times I, x_0 \times I \cup X \times \partial I)$. Let $j\colon QX \to (QX, kX)$ be the identity and let $J\colon (K, M) \to (QX, kX)$ be the collapsing map. With coefficients in a commutative ring, we have a diagram:

$$\begin{array}{ccccccc}
0 \to H^p(QX, kX) & \xrightarrow{j^*} & H^p(QX) & \underset{(Up_1)^*}{\overset{k^*}{\rightleftarrows}} & H^p(X) & \longrightarrow & 0 \\
J^* \downarrow \simeq & & & & & & \\
H^p(K, M) & \underset{\simeq}{\overset{s}{\longleftarrow}} & H^{p-1}(X) & & & &
\end{array}$$

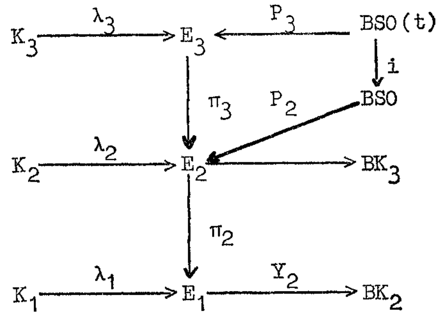
where the top row is an exact sequence split by $(Up_1)^*$.

For any $a \in H^*(X)$, let $Qa = j^*(J^*)^{-1}sa \in H^*(QX)$ and let $qa = (Up_1)^*a \in H^*(QX)$. We then have that $(qa) \cup (qb) = q(a \cup b)$, $(Qa) \cup (Qb) = 0$, and $(qa) \cup (Qb) = Q(a \cup b)$, for all $a, b \in H^*(X)$.

5.2 Lemma. *Let $\eta = \{E, \pi, B, K, K\}$ be a principal K-bundle, where $K = K(G, n)$, and let $y\colon B \to BK$ be the classifying map for η. Let $f\colon X \to E$ be any map and let $F\colon f \sim f$ be a homotopy. Then $(UF)^*y^*\alpha = -Q\delta^n(f, f; F)$, where α is the fundamental class of BK.*

Proof. Let $x_0 \in X$ be the base point. Let $g\colon kX \to E$ be the unique map where $gkx = fx$ for all $x \in X$, and let $0\colon kx_0 \to E$ be the trivial map ($kx_0 \in kX \subset QX$). Now $s\delta^n(f, f; F) = \gamma^{n+1}(gJ; F)$, the obstruction to lifting F to a map on $X \times I$ which agrees with gJ on $X \times \partial I \cup x_0 \times I$ (s = suspension). Thus $(UF)^*y^*\alpha = -\gamma^{n+1}(0; UF) = -j^*\gamma^{n+1}(g; UF) = -j^*(J^*)^{-1}\gamma^{n+1}(gJ; F) = -j^*(J^*)^{-1}s\delta^n(f, f; F) = -Q\delta^n(f, f; F)$.

5.3 For $t = 3, 4, 5$ or 6 we construct a Postnikov Tower for $BSO(t)$ over BSO in the manner of Mahowald [9].



The map i is the classifying map for the universal t-bundle over $BSO(t)$. $K_1 = K(Z_2, 2)$, $K_2 = K(Z, 4)$, and $K_3 = K(G_3, n(3))$. G_3, $n(3)$ and $P_2^*y_3^*\alpha$ (α = fundamental class of BK_3) are given as:

t	G_3	$n(3)$	$P_2^*y_3^*\alpha$
3	Z_2	3	w_4
4	Z	4	$\beta_2 w_4$
5	Z_2	5	w_6
6	Z	6	$\beta_2 w_6$.

G_3 is simply $\pi_{n(3)}(V_t)$, the first non-zero homotopy group of V_t, the space of t-planes in R^∞. P_2 is a 7-W.H.E. (weak homotopy equivalence) since $\pi_k(BSO) = 0$ for all $4 < k < 8$. P_3 is an $n(3)$-W.H.E., and, if $t = 3$, a 4-W.H.E. since $\pi_4(V_3) = 0$.

We shall let σ_2 denote $(\lambda_1^*)^{-1}u_2 \in H^2(E_1; Z_2)$; $(\pi_2 P_2)^*\sigma_2 = w_2$, since $\pi_2 P_2$ is a 2-W.H.E. We have that $y_2^*v_5 = 2\beta_4\mathfrak{p}\sigma_2 = \beta_2\sigma_2^2$ [6], where $\mathfrak{p}\colon H^2(\ ; Z_2) \to H^4(\ ; Z_4)$ is the Pontrjagin square, and β_4 is the Bokstein of the sequence $Z \to Z \to Z_4$. For each $2 \leqq i \leqq 7$, we shall let o_i denote $(P_2^*)^{-1}w_i$.

5.4 Let $\nu_1 = \mu_1(1 \times \lambda_1^{-1})\colon E_1 \times E_1 \to E_1$. We may find a multiplication $\nu_2 : E_2 \times E_2 \to E_2$ consistent with ν_1, *i.e.* such that $\nu_2 \mid E_2 \vee E_2 = 1 \vee 1$ and $\pi_2\nu_2 = \nu_1(\pi_2 \times \pi_2)$.

Proof. The k-invariant, $\beta_2\sigma_2^2$ is primitive, that is, $\nu_1^*\beta_2\sigma_2^2 = \beta_2\sigma_2^2 \times 1 + 1 \times \beta_2\sigma_2^2$. Thus we may find a multiplication ν_2 as above [8, Thm. 1.3].

5.5 Let $(U, V) = (P_\infty \times I, P_\infty \times \partial I \cup p_0 \times I)$, where $p_0 \in P_\infty = K(Z_2, 1)$ is the base point. There is an embedding $i: P_\infty \to SO$ where $i^*\{k+1\} = u_1^k$ for all $k(\{k+1\}$ is defined in [5, p. 48]). ΩSO is of the homotopy type of BSO, hence i corresponds to a map $R: (U, V) \to BSO$. The transgression of $\{k+1\}$ is w_{k+1}, hence $R^* w_{k+1} = su_1^k$ for all $k \geqq 1$. Let $S: (U, V) \to K_1$ be the map where $S^* u_2 = su_1$ (s = suspension homomorphism).

Since $(\pi_2 P_2 R)^* \sigma_2 = su_1$, $\pi_2 P_2 R$ is homotopic to $\lambda_1 S$; by the Covering Homotopic Extension Property [1, *III*, 2] we may choose a map $H: (U, V) \to E_2$ homotopic to $P_2 R$ such that $\pi_2 H = \lambda_1 S$. For each $2 \leqq k+1 \leqq 7$, $H^* o_{k+1} = su_1^k$.

5.6 Proposition. *Let $g: X \to E_3$ be any map and let $f = \pi g$. Let $z \in H^3(X; Z)$ and let $\theta_{3,2}(g, z) = \{\delta^{n(3)}(g, g; R) \mid R$ is $(g, 2, 2, z)$ admissible$\}$ (cf. 4.2). Then $\theta_{3,2}(f, z)$ consists of the element*

$$\begin{array}{ll} \pi z & \text{if } t = 3, \\ 0 & \text{if } t = 4, \\ Sq^2 \pi z + \pi z f^* o_2 & \text{if } t = 5, \\ \beta_2(Sq^2 \pi z + \pi z f^* o_2) & \text{if } t = 6. \end{array}$$

Proof. First we note that $\theta_{3,2}$ is a primary map-cohomology operation (*cf.* 4.3). Let $F: (U, V) \to K_2$ be a map where $F^* v_4 = sz$, and let $R = \mu_2(fp_1 \times F)\triangle: f \sim f$, an $(f, 2, 2, z)$ admissible homotopy. $F^* \pi v_4 = s\pi z$, whence $(UF)^* \pi v_4 = Q\pi z$. $UR = \mu_2(fUp_1 \times UF)\triangle: QX \to E_2$ and $\mu_2 o_4 = o_4 \times 1 + 1 \times \pi v_4$. Thus $(UR)^* o_4 = qf^* o_4 + Q\pi z$. Now $F^* o_2 = 0$, whence $(UF)^* o_2 = 0$. Since $\mu_2^* o_2 = o_2 \times 1$, we have $(UR)^* o_2 = qf^* o_2$.

Using the Wu formula for $Sq^i w_j$, we conclude:

If $t = 3$, then $f^* o_4 = 0$, whence $(UR)^* o_4 = Q\pi z$.

If $t = 4$, then $f^* \beta_2 o_4 = 0$, whence $(UR)^* \beta_2 o_4 = 0$.

If $t = 5$, then $f^* o_6 = 0$, whence $(UR)^* o_6 = Q(\pi z f^* o_2 + Sq^2 \pi z)$.

If $t = 6$, then $f^* \beta_2 o_6 = 0$, whence $(UR)^* \beta_2 o_6 = Q\beta_2(\pi z f^* o_2 + Sq^2 \pi z)$.

Apply 5.2 and we are done.

5.7 Proposition. *For any map $g: X \to E_3$ and any $x \in H^1(X; Z_2)$, let*

$$\begin{array}{rll} \beta_{3,1}(g, x) & = x^3 + xf^* o_2 & \text{if } t = 3, \\ & = \beta_2(x^3 + xf^* o_2) & \text{if } t = 4, \\ & = x^5 + x^3 f^* o_2 + x^2 f^* o_3 + xf^* o_4 & \text{if } t = 5, \\ & = \beta_2(x^5 + x^3 f^* o_2 + x^2 f^* o_3 + xf^* o_4) & \text{if } t = 6, \end{array}$$

where $f = \pi g$. Then $\beta_{3,1}$ is a choice function subsidiary to $\theta_{3,1}$ (cf. 4.2, 2.3).

Proof. Let $H: (U, V) \to E_2$ and $\nu_2: E_2 \times E_2 \to E_2$ be as given in 5.5 and

5.4 respectively. Let $h: X \to P_\infty$ be a map where $h^*u_1 = x$, and let $R = \nu_2(fp_1 \times H(h \times 1))\Delta: f \sim f$. A routine computation shows that $\pi_2 R = \mu_1(\pi_2 f p_1 \times S(h \times 1))\Delta$, where $S: (P_\infty \times I, P_\infty \times \partial I \cup p_0 \times I) \to K_1$ is the suspension. R is $(g, 2, 1, x)$ admissible, since $(S(h \times 1))^* u_2 = sx$.

Now for all $2 \leqq i \leqq 7$, $\nu_2^* o_i = \sum_{j=0}^{i} \sigma_j \times \sigma_{i-j}$, where $o_0 = 1$ and $o_1 = 0$. Routine computation shows that $(UR)^* o_4 = qf^* o_4 + Q(x^3 + xf^* o_2)$ and $(UR)^* o_6 = qf^* o_6 + Q(x^5 + x^3 f^* o_2 + x^2 f^* o_3 + xf^* o_4)$. We note that $qf^* y_3^* \alpha = 0$, where α is the fundamental class of BK_3; the theorem follows now from 5.2.

5.8 Theorem. *If X is a finite complex of dimension $k(t)$, where $k(3) = 4$ and $k(t) = t$ for $t = 4, 5$ and 6, and if ξ is a stable oriented vector bundle over X, then ξ may be represented as a t-bundle (unless $t = 3$ and $w_4 = 0$), and these representations may be put into a $1 - 1$ correspondence with $H^t(X; G)/L$, where $G = Z_2$ if $t = 3$ or 5, $G = Z$ if $t = 4$ or 6 and where L consists of all*

$$x^3 + xw_2 + \pi y \qquad \text{if} \quad t = 3,$$

$$\beta_2(x^3 + xw_2) \qquad \text{if} \quad t = 4,$$

$$x^5 + x^3 w_2 + x^2 w_3 + xw_4 + Sq^2 \pi y + \pi y w_2 \qquad \text{if} \quad t = 5,$$

$$\beta_2(x^5 + x^3 w_2 + x^2 w_3 + xw_4 + S_q^2 \pi y + yw_2) \qquad \text{if} \quad t = 6,$$

where $x \in H^1(X; Z_2)$, $y \in H^3(X; Z)$ and w_i is the i^{th} Stiefel Whitney class of ξ.

Proof. $P_3 : BSO(t) \to E_3$ is a $k(t)$-W.H.E.; thus $[X: BSO(t)] \simeq [X; E_3]$. The theorem follows from 4.6, 4.7, 5.6 and 5.7.

5.9 Let E_1 and E_2 be as in 5.3, and let E_3 be the next stage in a Postnikov system for BSO. $G_3 = Z$, $n(3) = 8$, and the k-invariant, $y^* v_9$, is $\beta_2 o_4^2 \in H^9(E_2; Z)$. Let $P_3: BSO \to E_3$ be an 8-W.H.E. We let $\{E_4, \pi_4, E_3, K_4, K_4\}$ be a $K_4 = K(Z_2, 7)$ bundle induced by the map $y_4 : E_3 \to BK_4$, where $P_3^* y_4^* u_8 = w_8$. We let $P_4 : BSO(7) \to E_4$ be a 7-W.H.E.; $\pi_8(V_7) = 0$, *so* P_4 is an 8-W.H.E. For each $2 \leqq i \leqq 8$, let $s_i = (P_3^*)^{-1} w_i$.

Let now $g: X \to E_4$ be any map, let $f = \pi g$, and let $(K, M) = (X \times I, X \times \partial I \cup x_0 \times I)$, where $x_0 \in X$ is the basepoint. If $x \in H^1(X; Z_2)$, we may find a $(g, 3, 1, x)$ admissible homotopy $R_1 = \nu_3(fp_1 \times F_1)\Delta$, where $\nu_3 : E_3 \times E_3 \to E_3$ is a multiplication consistent with ν_2 (*cf.* 5.4) and $F_1 : (K, M) \to E_3$ satisfies the condition $F_1^* s_{i+1} = sx^i$ for all $2 \leqq i + 1 \leqq 8$ (*cf.* 5.5). We conclude that $\delta^7(g, g; R_1) = x^7 + x^5 f^* s_2 + x^4 f^* s_3 + x^3 f^* s_4 + x^2 f^* s_5 + xf^* s_6$.

If $y \in H^3(x; Z)$, let $T: (K, M) \to K_2$ be a map where $T^* v_4 = sy$. Then $\lambda_2 T$ has a lifting $F_2 : (K, M) \to E_3$, since $T^* \lambda_2^* \beta_2 o_4 = T^*(\beta_2 \pi v_4) = 0$. Let $R_2 = \nu_3(fp_1 \times F_2)\Delta: f \sim f$; computation gives us that R_2 is $(g, 3, 2, y)$ admissible and $\delta^7(g, g; R_2) = S_q^2 \pi y f^* s_2 + \pi y f^* s_4$.

If $z \in H^7(X; Z)$, let $F: (K, M) \to K_3$ be a map with $F^* v_8 = sz$. Then $R_3 = \nu_3(fp_1 \times F)\Delta: f \sim f$ is $(g, 3, 3, z)$ admissible and $\delta^7(g, g; R_3) = \pi z$.

5.10 We may apply 4.6 and 5.9 to obtain:

Theorem. *If X is a finite complex of dimension 8, and if ξ is a stable oriented* *bundle over X, then ξ may be represented as a 7-bundle if and only if $w_8 = 0$, in which case the representations may be put into a $1-1$ correspondence with $H^7(X; Z_2)/L$, where L is the subgroup of all elements of the form:*

$$x^7 + x^5w_2 + x^4w_3 + x^3w_4 + x^2w_5 + xw_6 + Sq^2\pi yw_2 + \pi yw_4 + \pi z,$$

where $x \in H^1(X; Z_2)$, $y \in H^3(X; Z)$, and $z \in H^7(X; Z)$, and w_i is the i^{th} Stiefel Whitney class of ξ.

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University of Illinois, Chicago Circle

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