Mapping class group and a global Torelli theorem for hyperkähler manifolds

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Abstract

A mapping class group of an oriented manifold is a quotient of its diffeomorphism group by the isotopies. We compute a mapping class group of a hypekähler manifold M, showing that it is commensurable to an arithmetic lattice in $SO(3, b_2 - 3)$. A Teichmüller space of M is a space of complex structures on M up to isotopies. We define **a birational Teichmüller space** by identifying certain points corresponding to bimeromorphically equivalent manifolds. We show that the period map gives the isomorphism between connected components of the birational Teichmüller space and the corresponding period space $SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$. We use this result to obtain a Torelli theorem identifying each connected component of the birational moduli space with a quotient of a period space by an arithmetic group. When M is a Hilbert scheme of n points on a K3 surface, with $n - 1$ a prime power, our Torelli theorem implies the usual Hodge-theoretic birational Torelli theorem (for other examples of hyperkähler manifolds, the Hodge-theoretic Torelli theorem is known to be false).

Contents

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1 Introduction

1.1 Hyperkähler manifolds and their moduli

Throughout this paper, **a hyperkähler manifold** is a compact, holomorphically symplectic manifold of Kähler type, simply connected and with $H^{2,0}(M) = \mathbb{C}$. In the literature, such manifolds are often called **simple**, or **irreducible**. For an explanation of this term and an introduction to hyperkähler structures, please see Subsection 2.1.

We shall say that a complex structure I on M is **of hyperkähler type** if (M, I) is a hyperkähler manifold.

There are many different ways to define the moduli of complex structures. In this paper we use the earliest one, which is due to Kodaira-Spencer and Kuranishi. Let M be an oriented manifold, $\mathfrak{I}$ the space of all complex structures of hyperkähler type, compatible with orientation, and $\mathcal{M} := \mathfrak{I}/\operatorname{\mathsf{Diff}}$ its quotient by the group of oriented diffeomorphisms.[1] We call $\mathcal{M}$ **the moduli space** of complex structures of hyperkähler type (or just "the moduli space") of M. This space is usually non-Hausdorff.

For a hyperkähler manifold, the non-Hausdorff points of $\mathcal{M}$ are easy to control, due to a theorem of D. Huybrechts (Theorem 4.24). If $I_1, I_2 \in \mathcal{M}$ are inseparable points in $\mathcal{M}$, then the corresponding hyperkähler manifolds are bimeromorphic (Proposition 4.25).

In many cases, the moduli of complex structures on M can be described in terms of Hodge structures on the cohomology of M. Such results are called *Torelli theorems*. In this paper, we state a Torelli theorem for hyperkähler manifolds, using the language of mapping class group and Teichmüller spaces.

This approach to the Torelli-type problems was pioneered by A. Todorov in several important preprints and papers ([T1], [T2]; see also [LTYZ]).

1.2 Teichmüller space of a hyperkähler manifold

To define the period space for hyperkähler manifolds, one uses the so-called Bogomolov-Beauville-Fujiki (BBF) form on the second cohomology. Historically, it was the BBF form which was defined in terms of the period space, and not vice versa, but the other way around is more convenient.

Let Ω be a holomorphic symplectic form on M. Bogomolov and Beauville ([Bo2], [Bea1]) defined the following bilinear symmetric 2-form on $H^2(M)$:

$$\begin{aligned}\widetilde{q}(\eta,\eta') :=& 2\int_M \eta\wedge\eta'\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ &- \frac{n-1}{n}\frac{\left(\int_M \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_M \eta'\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)}{\int_M \Omega^n\wedge\overline{\Omega}^n}\end{aligned} \tag{1.1}$$

where $n = \dim_{\mathbb{H}} M$.

Remark 1.1: The form $\widetilde{q}$ is compatible with the Hodge decomposition, which is seen immediately from its definition. Also, $\widetilde{q}(\Omega,\overline{\Omega}) > 0$.

The form $\widetilde{q}$ is topological by its nature.

Theorem 1.2: ([F]) Let be a simple hyperkähler manifold of real dimension $4n$. Then there exists a bilinear, symmetric, primitive non-degenerate integral

[1] Throughout this paper, we speak of oriented diffeomorphisms, but the reasons for this assumption are purely historical. We could omit the mention of orientation, and most of the results will remain valid.

2-form $q: H^2(M,\mathbb{Z}) \otimes H^2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$ and a constant $c \in \mathbb{Z}$ such that

$$\int_M \eta^{2n} = cq(\eta,\eta)^n, \tag{1.2}$$

for all $\eta \in H^2(M)$. Moreover, q is proportional to the form $\widetilde{q}$ of (1.1), and has signature $(+,+,+,-,-,-,...)$.
■

Remark 1.3: If n is odd, the equation (1.2) determines q uniquely, otherwise – up to a sign. To choose a sign, we use (1.1).

Definition 1.4: Let M be a hyperkähler manifold, and Ω a holomorphic symplectic form on M. **Beauville-Bogomolov-Fujiki form** on M is a form $q: H^2(M,\mathbb{Q}) \otimes H^2(M,\mathbb{Q}) \longrightarrow \mathbb{Q}$ which satisfies (1.2), and has $q(\Omega,\overline{\Omega}) > 0$.

Definition 1.5: Let (M,I) be a compact hyperkähler manifold, $\mathfrak{I}$ the set of oriented complex structures of hyperkähler type on M, and $\mathsf{Diff}_0(M)$ the group of isotopies. The quotient space $\mathsf{Teich} := \mathfrak{I}/\mathsf{Diff}_0(M)$ is called **the Teichmüller space** of (M,I), and the quotient of Teich over a whole oriented diffeomorphism group **the coarse moduli space of** (M,I).

Remark 1.6: In a similar way one defines the moduli of Kähler structures or of complex structures on a given Kähler or complex manifold. This approach was originaly suggested by Kodaira and Spencer in their fundamental work on deformation theory ([KS1], [KS3]). Kodaira and Spencer constructed a local moduli space for the special cases when obstructions to deformation vanish. The results of Kodaira-Spencer were obtained in full generality by M. Kuranishi, who constructed a finite-dimensional complex-analytic slice of the action of diffeomorphism group on the space of all integrable almost complex structures ([Ku1], [Ku2]). The precise correspondence between the results of Kuranishi and the action of the diffeomorphism group was spelled out by A. Douady in his Bourbaki talk, [Dou].

Remark 1.7: As shown by F. Catanese [C, Proposition 15], for Kähler manifolds with trivial canonical bundle, e.g. for the hyperkähler manifolds, the Teichmüller space is locally isomorphic to the Kuranishi moduli space.

Definition 1.8: Let (M,I) be a simple hyperkähler manifold, and Teich its Teichmüller space. For any $J \in \mathsf{Teich}$, (M,J) is also a simple hyperkähler manifold, as seen from Lemma 2.6 below, hence $H^{2,0}(M,J)$ is one-dimensional. Consider a map $\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M,\mathbb{C})$, sending J to the line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. Clearly, Per maps Teich into an open subset of the quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0\}. \tag{1.3}$$

The map $\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ is called **the period map**, and the set $\mathbb{P}\mathsf{er}$ **the period space**.

The following fundamental theorem is due to F. Bogomolov [Bo2].

Theorem 1.9: ([Bo2]) Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then the period map $\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ is locally an unramified covering (that is, an etale map). ■

Remark 1.10: Bogomolov's theorem implies that Teich is smooth. However, it is not necessarily Hausdorff (and it is non-Hausdorff even in the simplest examples).

Remark 1.11: D. Huybrechts has shown that Per is surjective ([H1], Theorem 8.1).

Remark 1.12: Using the boundedness results of Kollar and Matsusaka ([KM]), D. Huybrechts has shown that the space Teich has only a finite number of connected components ([H5], Theorem 2.1).

The moduli space $\mathcal{M}$ of complex structures of hyperkähler type on M is a quotient of Teich by the action of the mapping class group $\Gamma := \mathsf{Diff}/\mathsf{Diff}_0$ of diffeomorphisms up to isotopies. There is an interesting intermediate group Diff_H of all diffeomorphisms acting trivially on $H^2(M)$. One has $\mathsf{Diff}_0 \subset \mathsf{Diff}_H \subset \mathsf{Diff}$. The corresponding quotient $\mathsf{Teich}/\mathsf{Diff}_H$ is called **the coarse, marked moduli space** of complex structures, and its points – **marked hyperkähler manifolds**. To choose a marking it means to choose a basis in the cohomology of M. The period map is well defined on $\mathsf{Teich}/\mathsf{Diff}_H$.

We don't use the marked moduli space in this paper, because the Teichmüller space serves the same purpose. In the literature on moduli spaces, the marked moduli space is used throughout, but these results are easy to translate to the Teichmüller spaces' language using the known facts about the mapping class group.

For a K3 surface, the Teichmüller space is not Hausdorff. However, a quotient of polarized moduli space by the mapping class group *is* Hausdorff and quasi-projective ([Vi]). Moreover, a version of Torelli theorem is valid, providing an isomorphism between Teich/Γ and $\mathbb{P}\mathsf{er}/O^+(H^2(M,\mathbb{Z}))$.[2] This result has a long history, with many people contributing to different sides of the picture, but its conclusion could be found in [BR] and [Si].

One could state this Torelli theorem as a result about the Hodge structures, as follows. The Torelli theorem claims that there is a bijective correspondence between isomorphism classes of K3 surfaces and the set of isomorphism classes of appropriate Hodge structures on a 22-dimensional space equipped with an integer lattice, a spin orientation (Remark 7.15) and an integer quadratic form.

[2] For an explanation of O^+, please see Definition 7.9.

It is natural to expect that this last result would be generalized to other hyperkähler manifolds, but such a straightforward generalization is invalid. In [De], O. Debarre has shown that there exist birational hyperkähler manifolds which are non-isomorphic, but have the same periods. A hope to have a Hodge theoretic Torelli theorem for birational moduli was extinguished in early 2000-ies. As shown by Yo. Namikawa in a beautiful (and very short) paper [Na], there exist hyperkähler manifolds M, M' which are not bimeromorphically equivalent, but their second cohomology have equivalent Hodge structures.

For the benefit of the reader, we give here a brief reprise of the Namikawa's construction. Let T be a compact, complex, 2-dimensional torus, and $T^{[n]}$ its Hilbert scheme. The Albanese map $T^{[n]} \stackrel{\text{Alb}}{\longrightarrow} T$ is a locally trivial fibration. Denote by $T^{[n]}_T$ **the generalized Kummer manifold**, $T^{[n]}_T := \text{Alb}^{-1}(0)$. When $n = 2$, it is a K3 surface obtained from the torus using the Kummer construction. For $n > 2$, the Hodge structure on $H^2(T^{[n]}_T)$ is easy to describe. One has

$$H^2(T^{[n]}_T) \cong \text{Sym}^2(H^1(T)) \oplus \mathbb{R}\eta,$$

where η is the fundamental class of the exceptional divisor of $M := T^{[n]}_T$. Therefore, $H^2(M)$ has the same Hodge structure as $M' = (T^*)^{[n]}_{T^*}$, where T^* is the dual torus. However, the manifolds M and M' are not bimeromorphically equivalent, when T is generic. This is easy to see, for instance, for $n = 3$, because the exceptional divisor of $M = T^{[3]}_T$ is a trivial $\mathbb{C}P^1$-fibration over T, and the exceptional divisor of $M' = (T^*)^{[3]}_{T^*}$ is fibered over T^* likewise. Since bimeromorphic maps of holomorphic symplectic varieties are non-singular in codimension 2, any bimeromorphic isomorphism between M and M' would bring a bimeromorphic isomorphism between these divisors, and therefore between T and T^*, which is impossible for general T.

A less elementary construction, due to E. Markman, gives a counterexample to the Hodge-theoretic global Torelli theorem when $M = K3^{[n]}$ is the Hilbert scheme of points on a K3 surface, and $n-1$ is not a prime power ([M2]). When $n-1$ is a prime power, a Hodge-theoretic birational Torelli theorem holds true (Subsection 7.2).

We are going to prove a different version of Torelli theorem, using the language of Teichmüller spaces and the mapping class groups.

1.3 The birational Teichmüller space

The Teichmüller space approach allows one to state the Torelli theorem for hyperkähler manifolds as it is done for curves. However, before any theorems can be stated, we need to resolve the issue of non-Hausdorff points.

Definition 1.13: Let M be a topological space. We say that points $x, y \in M$ are **inseparable** (denoted $x \sim y$) if for any open subsets $U \ni x, V \ni y$, one has $U \cap V \neq \emptyset$.

Remark 1.14: As follows from Proposition 4.25 and Theorem 4.24, inseparable points on a Teichmüller space correspond to bimeromorphically equivalent hyperkähler manifolds.

Theorem 1.15: Let Teich be a Teichmüller space of a hyperkähler manifold, and $\sim$ the inseparability relation defined above. Then $\sim$ is an equivalence relation. Moreover, the quotient $\mathsf{Teich}_b := \mathsf{Teich}/_\sim$ is a smooth, Hausdorff complex analytic manifold.

Proof: Theorem 4.17, Theorem 4.22. ■

We call the quotient $\mathsf{Teich}/_\sim$ **the birational Teichmüller space**, denoting it as Teich_b. The operation of taking the quotient $.../_\sim$ as above has good properties in many situations, and brings similar results quite often. We call $W/_\sim$ **the Hausdorff reduction** of W whenever it is Hausdorff (see Subsection 4.3 for a detailed exposé).

1.4 The mapping class group of a hyperkähler manifold

Define the mapping class group $\Gamma := \mathsf{Diff}/\mathsf{Diff}_0$ of a manifold M as a quotient of the group of oriented diffeomorphisms of M by isotopies. Clearly, Γ acts on $H^2(M,\mathbb{R})$ perserving the integral structure. We are able to determine the group Γ up to commensurability, proving that it is commensurable to an arithmetic group $O(H^2(M,\mathbb{Z}),q)$ of finite covolume in $O(3,b_2(M)-3)$.

Theorem 1.16: Let M be a compact, simple hyperkähler manifold, and $\Gamma = \mathsf{Diff}/\mathsf{Diff}_0$ its mapping class group. Then Γ acts on $H^2(M,\mathbb{R})$ preserving the Bogomolov-Beauville-Fujiki form. Moreover, the corresponding homomorphism $\Gamma \longrightarrow O(H^2(M,\mathbb{Z}),q)$ has finite kernel, and its image has finite index in $O(H^2(M,\mathbb{Z}),q)$.

Proof: This is Theorem 3.5. ■

Using results of E. Markman ([M2]), it is possible to compute the mapping class group for a Hilbert scheme of points on a K3 surface $M = K3^{[n]}$, when $n-1$ is a prime power (Theorem 7.8).

1.5 Teichmüller space and Torelli-type theorems

The following version of the Torelli theorem is proven in Section 6.

Theorem 1.17: Let M be a compact, simple hyperkähler manifold, and Teich_b its birational Teichmüller space. Consider the period map $\mathsf{Per}: \mathsf{Teich}_b : \longrightarrow \mathbb{P}\mathsf{er}$, where $\mathbb{P}\mathsf{er}$ is the period space defined as in (1.3). Then Per is a diffeomorphism, for each connected component of Teich_b.

Proof: This is Theorem 4.29. ■

The proof of Theorem 1.17 is obtained by using the quaternionic structures, associated with holomorphic symplectic structures by the Calabi-Yau theorem, and the corresponding rational lines in Teich and $\mathbb{P}\mathsf{er}$.

If one wants to obtain a more traditional Torelli-type theorem, one should consider the set of equivalence classes of complex structures up to birational equivalence. This set can be interpreted in terms of the Teichmüller space as follows.

Consider the action of the mapping class group Γ on the Teichmüller space Teich, and let Teich^I be a connected component of Teich containing a given complex structure I. Denote by $\Gamma_I \subset \Gamma$ a subgroup of Γ preserving Teich^I. Since Teich has only a finite number of connected components ([H5], Theorem 2.1), Γ_I has a finite index in Γ. The coarse moduli space of complex structures on M is $\mathsf{Teich}^I/\Gamma_I$, and the birational moduli is $\mathsf{Teich}_b^I/\Gamma_I$, where Teich_b^I is the appropriate connected component of Teich_b. Theorem 1.17 immediately implies the following Torelli-type result.

Theorem 1.18: Let M be a compact, simple hyperkähler manifold, $\mathcal{M}_b := \mathsf{Teich}_b^I/\Gamma_I$ a connected component of the birational moduli space defined above, and

$$\mathcal{M}_b \xrightarrow{\mathsf{Per}} \mathbb{P}\mathsf{er}/\Gamma_I \tag{1.4}$$

the corresponding period map. Then (1.4) is a bijection. ■

Remark 1.19: The image $i(\Gamma_I)$ of Γ_I in $O(H^2(M,\mathbb{Z}),q)$ has finite index (Theorem 1.16). Therefore, it is an arithmetic sugbroup of finite covolume.

Comparing this with Theorem 1.18, we immediately obtain the following corollary.

Corollary 1.20: Let M be a compact, simple hyperkähler manifold, and $\mathcal{M}_b$ a connected component of its birational moduli space, obtained as above. Then $\mathcal{M}_b$ is isomorphic to a quotient of a homogeneous space

$$\mathbb{P}\mathsf{er} = \frac{O(b_2-3,3)}{SO(2)\times O(b_2-3,1)}$$

by an action of an arithmetic subgroup $i(\Gamma_I) \subset O(H^2(M,\mathbb{Z}),q)$.[3] ■

In a traditional version of Torelli theorem, one takes a quotient of $\mathbb{P}\mathsf{er}$ by $O^+(H^2(M,\mathbb{Z}),q)$ instead of $i(\Gamma_I) \subset O^+(H^2(M,\mathbb{Z}),q)$.[4] However, such a result cannot be valid, as shown by Namikawa. Corollary 1.20 explains why this occurs: for Namikawa's examples, the group $i(\Gamma_I)$ is a proper subgroup in $O^+(H^2(M,\mathbb{Z}),q)$, and the composition

$$\mathcal{M}_b \longrightarrow \mathbb{P}\mathsf{er}/\Gamma_I \longrightarrow \mathbb{P}\mathsf{er}/O^+(H^2(M,\mathbb{Z}),q) \tag{1.5}$$

[3] For this interpretation of $\mathbb{P}\mathsf{er}$, please see Subsection 2.4.

[4] $O^+(H^2(M,\mathbb{Z}),q)$ is a group of orthogonal maps with positive spin norms (Definition 7.9).

is a finite quotient map. We obtained the following corollary.

Corollary 1.21: Let M be a compact, simple hyperkähler manifold, $\mathcal{M}_b$ a connected component of its birational moduli space, and

$$\mathcal{M}_b \longrightarrow \mathbb{P}\mathsf{er}/O^+(H^2(M,\mathbb{Z}),q) \tag{1.6}$$

the corresponding period map. Then (1.6) is a finite quotient. ■

Remark 1.22: Please notice that the space $\mathbb{P}\mathsf{er}/O^+(H^2(M,\mathbb{Z}),q)$ is usually non-Hausdorff. However, it can be made Hausdorff if one introduces additional structures (such as a polarization), and then Corollary 1.21 becomes more useful.

For the Hilbert scheme of n points on a K3 surface, the image of Γ_I in $O^+(H^2(M,\mathbb{Z}))$ was computed by E. Markman in [M2] (see Theorem 7.8). When $n-1$ is a prime power, $i(\Gamma_I) = O^+(H^2(M,\mathbb{Z}))$, and the composition (1.6) is an isomorphism, which is used to obtain the usual (Hodge-theoretic) version of Torelli theorem.

1.6 A Hodge-theoretic Torelli theorem for $K3^{[n]}$

In [M1], [M2], E. Markman has proved many vital results on the way to compute the mapping class group of a Hilbert scheme of points on K3 (denoted by $K3^{[n]}$). Markman's starting point was the notion of a monodromy group of a hyperkähler manifold. A monodromy group of M is the group generated by monodromy of the Gauss-Manin local systems for all deformations of M (see Subsection 7.1 for a more detailed description). In Subsection 7.1, we relate the monodromy group Mon to the mapping class group Γ_I, showing that Mon is isomorphic to an image of Γ_I in $PGL(H^2(M,\mathbb{C}))$. For $M = K3^{[n]}$, Markman has computed the monodromy group, using the action of Fourier-Mukai transform in the derived category of coherent sheaves. He used this computation to show that the standard (Hodge-theoretic) global Torelli theorem fails on $K3^{[n]}$, unless $n-1$ is a prime power. We complete Markman's analysis of global Torelli problem for $K3^{[n]}$, proving the following.

Theorem 1.23: Let $M = K3^{[n]}$ be a Hilbert scheme of points on a K3 surface, where $n-1$ is a prime power, and I_1, I_2 deformations of complex structures on M. Assume that the Hodge structures on $H^2(M,I_1)$ and $H^2(M,I_2)$ are isomorphic, and this isomorphism is compatible with the Bogomolov-Beauville-Fujiki form and the natural spin orientation on $H^2(M,I_1)$ and $H^2(M,I_2)$. (Remark 7.15). Then (M,I_1) is bimeromorphic to (M,I_2).

Proof: This is Theorem 7.19. ■

Remark 1.24: E. Markman in [M2] constructed counterexamples to the Hodge-theoretic global Torelli problem for $K3^{[n]}$, where $n-1$ is not a prime power.

1.7 Moduli of polarized hyperkähler varieties

For another application of Corollary 1.20, fix an integral class $\eta \in H^2(M,\mathbb{Z})$, $q(\eta,\eta) > 0$, and let $\mathsf{Teich}_{\pm,\eta}$ be a divisor in the connected component of the Teichmüller space consisting of all I with $\eta \in H^{1,1}(M,I)$. For a general $I \in \mathsf{Teich}_{\pm,\eta}$, η or $-\eta$ is a Kähler class on (M,I) ([H3]; see also Theorem 2.7). However, there could be special points where $\pm\eta$ is not Kähler.

Let Teich^I_η be a connected component of $\mathsf{Teich}_{\pm,\eta}$, containing $I \in \mathsf{Teich}$, in such a way that η is a Kähler class on (M,I). Denote by $\overline{\mathcal{M}}_\eta$ the quotient of Teich^I_η by the subgroup Γ^I_η of the mapping class group fixing η and preserving the component Teich^I_η. The same argument as above can be used to show that Γ^I_η is commensurable to an arithmetic subgroup in $SO(\eta^\perp)$, where $\eta^\perp \subset H^2(M,\mathbb{R})$ is an orthogonal complement to η.

We call $\overline{\mathcal{M}}_\eta$ **a connected component of the moduli space of weakly polarized hyperkähler manifolds.** A corresponding component $\mathcal{M}_\eta$ of the moduli of polarized hyperkähler manifolds is an open subset of $\overline{\mathcal{M}}_\eta$ consisting of all I for which η is Kähler. Since $\pm\eta$ is a Kähler class whenever $\operatorname{Pic}(M) = \langle\eta\rangle$ (Theorem 2.7), $\mathcal{M}_\eta$ is dense in $\overline{\mathcal{M}}_\eta$. It is known (due to the general theory which goes back to Viehweg, Grothendieck and Kodaira-Spencer) that $\mathcal{M}_\eta$ is Hausdorff and quasiprojective (see e.g. [Vi], [GHS2]).

Recall that for any simple hyperkähler manifold M, the space $H^2(M,\mathbb{R})$ has signature $(+,+,+,-,...,-)$, with a fixed orientation on each of the positive 3-planes. The period space for weakly polarized hyperkähler manifolds is defined as

$$\mathbb{P}\mathsf{er}_{\pm,\eta} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid\ q(l,l) = 0,\ q(\eta,l) = 0,\ q(l,\bar{l}) > 0\}, \tag{1.7}$$

where the 3-plane $\langle\eta, \operatorname{Re} l, \operatorname{Im} l\rangle$ has the same orientation as for a hyperkähler structure. Then the corresponding period map $\mathsf{Teich}_\eta \longrightarrow \mathbb{P}\mathsf{er}_\eta$ induces an isomorphism from the Hausdorff reduction $\mathsf{Teich}^I_{\eta,b}$ of Teich^I_η to $\mathbb{P}\mathsf{er}_\eta$, as follows from Theorem 1.17.

Notice that, unless one fixes the orientation of the 3-plane $\langle\eta, \operatorname{Re} l, \operatorname{Im} l\rangle$, the space $\{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid\ q(l,l) = 0,\ q(\eta,l) = 0,\ q(l,\bar{l}) > 0\}$ would have *two* connected components.

We define **a connected component of the birational moduli space of weakly polarized hyperkähler manifolds** $\overline{\mathcal{M}}_{b,\eta}$ as a quotient of the component $\mathsf{Teich}^I_{b,\eta}$ by the corresponding mapping class group Γ^I_η. It is obtained from $\overline{\mathcal{M}}_\eta$ by identifying inseparable points.

Just as in Subsection 2.4, we may identify the period space $\mathbb{P}\mathsf{er}_\eta$ with the Grassmannian of positive 2-planes in $\eta^\perp$. This gives

$$\mathbb{P}\mathsf{er}_\eta \cong SO(b_2-3,2)/SO(2) \times SO(b_2-3).$$

This is significant, because $\mathbb{P}\mathsf{er}_\eta$ (unlike $\mathbb{P}\mathsf{er}$) is a symmetric space. The corresponding result for the moduli spaces can be stated as follows.

Corollary 1.25: Let (M, η) be a compact, simple, polarized hyperkähler manifold, $\overline{\mathcal{M}}_{b,\eta}$ a connected component of the weakly polarized birational moduli space, defined above, G the group of integral orthogonal automorphisms of the lattice $\eta^\perp$ of primitive elements in $H^2(M)$, and

$$\overline{\mathcal{M}}_{b,\eta} \longrightarrow \mathbb{P}\mathsf{er}_\eta / G \tag{1.8}$$

the corresponding period map. Then (1.8) is a finite quotient. Moreover, $\overline{\mathcal{M}}_{b,\eta}$ is isomorphic to a quotient of a symmetric domain $\mathbb{P}\mathsf{er}_\eta$ by an arithmetic group Γ_η^I acting as above. ■

The quotients of such symmetric spaces by arithmetic lattices were much studied by Gritsenko, Hulek, Nikulin, Sankaran and many others (see e.g. [GHS1], [GHS2] and references therein). The geometry of $\mathbb{P}\mathsf{er}_\eta / G$ is in many cases well understood. Using the theory of automorphic forms, many sections of pluricanonial (or, in some cases, plurianticanonical) class can be found, depending on $q(\eta, \eta)$ and other properties of the lattice $\eta^\perp$. In such cases, Corollary 1.25 can be used to show that the weakly polarized birational moduli space has ample (or antiample) canonical class.[5]

The automorphic forms on polarized moduli were also used to show nonexistence of complete families of polarized K3 surfaces ([BKPS]). This program was proposed by J. Jorgensen and A. Todorov in 1990-ies, in a string of influential (but, sometimes, flawed) preprints, culminating with [JT].

2 Hyperkähler manifolds

In this Section, we recall a number of results about hyperkähler manifolds, used further on in this paper. For more details and reference, please see [Bes].

2.1 Hyperkähler structures

Definition 2.1: Let (M, g) be a Riemannian manifold, and I, J, K endomorphisms of the tangent bundle TM satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}.$$

The triple (I, J, K) together with the metric g is called **a hyperkähler structure** if I, J and K are integrable and Kähler with respect to g.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type $(2,0)$ on (M, I). This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

[5] See [DV] for an alternative approach to the same problem.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture ([Bes], [Bea1]).

Theorem 2.2: Let M be a compact, Kähler, holomorphically symplectic manifold, ω its Kähler form, $\dim_{\mathbb{C}} M = 2n$. Denote by Ω the holomorphic symplectic form on M. Suppose that $\int_M \omega^{2n} = \int_M (\operatorname{Re}\Omega)^{2n}$. Then there exists a unique hyperkähler metric g with the same Kähler class as ω, and a unique hyperkähler structure (I, J, K, g), with $\omega_J = \operatorname{Re}\Omega$, $\omega_K = \operatorname{Im}\Omega$. ■

Further on, we shall speak of "hyperkähler manifolds" meaning "holomorphic symplectic manifolds of Kähler type", and "hyperkähler structures" meaning the quaternionic triples together with a metric.

Every hyperkähler structure induces a whole 2-dimensional sphere of complex structures on M, as follows. Consider a triple $a, b, c \in R$, $a^2 + b^2 + c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponging quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence L is an almost complex structure. Since I, J, K are Kähler, they are parallel with respect to the Levi-Civita connection. Therefore, L is also parallel. Any parallel complex structure is integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ **a complex structure induced by a hyperkähler structure**. There is a 2-dimensional holomorphic family of induced complex structures, and the total space of this family is called **the twistor space** of a hyperkähler manifold.

2.2 Bogomolov's decomposition theorem

The modern approach to Bogomolov's decomposition is based on Calabi-Yau theorem (Theorem 2.2), Berger's classification of irreducible holonomy and de Rham's splitting theorem for holonomy reduction ([Bea1], [Bes]). It is worth mentioning that the original proof of decomposition theorem (in [Bo1]) was much more elementary.

Theorem 2.3: Let (M, I, J, K) be a compact hyperkähler manifold. Then there exists a finite covering $\widetilde{M} \longrightarrow M$, such that $\widetilde{M}$ is decomposed, as a hyperkähler manifold, into a product

$$\widetilde{M} = M_1 \times M_2 \times \ldots M_n \times T,$$

where (M_i, I, J, K) satisfy $\pi_1(M_i) = 0$, $H^{2,0}(M_i, I) = \mathbb{C}$, and T is a hyperkähler torus. Moreover, M_i are uniquely determined by M and simply connected, and T is unique up to isogeny.

Proof: See [Bea1], [Bes]. ■

Definition 2.4: Let (M, I, J, K) be a compact hyperkähler manifold which satisfies $\pi_1(M) = 0$, $H^{2,0}(M, I) = \mathbb{C}$. Then M is called **a simple hyperkähler manifold**, or **an irreducible hyperkähler manifold**

Remark 2.5: Notice that Theorem 2.3 implies that irreducible hyperkähler manifolds are simply connected. In particular, they do not admit a further decomposition. This explains the term "irreducible".

As we mentioned in the Introduction, all hyperkähler manifolds considered further on are assumed to be simple. Since the Hodge numbers are invariant under Kähler deformations, the deformations of simple manifolds are always simple. However, the irreducibility is a topological property, as implied by the following lemma.

Lemma 2.6: Let M be a compact hyperkähler manifold, which is homotopy equivalent to a simple hyperkähler manifold. Then M is also simple.

Proof: Let A^* be the part of the rational cohomology of M generated by $H^2(M)$. It is well known (see [V2] and [V3]) that A^* is up to the middle dimension a symmetric algebra. Since M is simply connected, it is diffeomorphic to a product of simple hyperkähler manifolds. Denote by A_i^* the corresponding subalgebras in cohomology generated by $H^2(M_i)$. These subalgebras are described in a similar way, and are symmetric up to the middle. Then $A^* \cong \bigotimes A_i^*$ by Künneth formula. Since the algebras A^*, A_i^* are symmetric up to the middle, this is impossible, as follows from an easy algebraic computation. ■

2.3 Kähler cone for hyperkähler manifolds

The following theorem is implied by results of S. Boucksom, using the characterization of a Kähler cone due to J.-P. Demailly and M. Paun (see also [H3]).

Notice that the Beauville-Bogomolov-Fujiki form q on

$$H^{1,1}(M, \mathbb{R}) := H^{1,1}(M) \cap H^2(M, \mathbb{R})$$

has signature $(+, -, -, -, ...)$, hence the set of vectors $\nu \in H^{1,1}(M, \mathbb{R})$ with $q(\nu, \nu) > 0$ has two connected components.

Theorem 2.7: Let M be a simple hyperkähler manifold such that all integer $(1,1)$-classes satisfy $q(\nu, \nu) \geqslant 0$. Then its Kähler cone is one of two components K_+ of the set $K := \{\nu \in H^{1,1}(M, \mathbb{R}) \mid q(\nu, \nu) > 0\}$.

Proof: This is [V6], Corollary 2.6. ■

For us, the case of trivial Neron-Severi lattice is of most interest.

Corollary 2.8: Let M be a compact, simple hyperkähler manifold such that $H^{1,1}(M) \cap H^2(M,\mathbb{Q}) = 0$. Then its Kähler cone is one of two components of a set $K := \{\nu \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ q(\nu,\nu) > 0\}$. ■

2.4 The structure of the period space

Let M be a hyperkähler manifold, and $b_2 = \dim H^2(M)$. It is well known that its period space $\mathbb{P}\mathsf{er}$ (see (1.3)) is diffeomorphic to the Grassmann space $Gr_{++}(H^2(M,\mathbb{R})) := O(b_2-3,3)/SO(2)\times O(b_2-3,1)$ of 2-dimensional oriented planes $V \subset H^2(M,\mathbb{R})$ with $q\big|_V$ positive definite. Indeed, for any line

$$l \in \mathbb{P}\mathsf{er} \subset \mathbb{P}H^2(M,\mathbb{C}),$$

let V_l be the span of $\langle \operatorname{Re} l, \operatorname{Im} l\rangle$. From (1.3) it follows that $l \cap H^2(M,\mathbb{R}) = 0$, hence V_l is an oriented 2-dimensional plane. Since $q(l,\bar{l}) > 0$, the restriction $q\big|_V$ is positive definite. This gives a map from $\mathbb{P}\mathsf{er}$ to $Gr_{++}(H^2(M,\mathbb{R}))$. To construct the inverse map, we start from a 2-dimensional plane $V \subset H^2(M,\mathbb{R})$ and consider the quadric $\{v \in \mathbb{P}(V\otimes\mathbb{C}) \ \ | \ \ q(v,v) = 0\}$. This quadric is actually a union of 2 points in $\mathbb{P}(V\otimes\mathbb{C}) \cong \mathbb{C}P^1$, with each of these points corresponding to a different choice of orientation on V. This gives an inverse map from $Gr_{++}(H^2(M,\mathbb{R}))$ to $\mathbb{P}\mathsf{er}$.

The following simple claim is well known. For the convenience of the reader, we recall its proof here.

Claim 2.9: The period space $\mathbb{P}\mathsf{er}$ is connected and simply connected.

Proof: We represent $\mathbb{P}\mathsf{er}$ as $Gr_{++}(H^2(M,\mathbb{R})) = O(b_2-3,3)/SO(2)\times O(b_2-3,1)$. The group $O(b_2-3,3)$ is disconnected, but $O(b_2-3,1)$ is also disconnected, hence the connected components cancel each other, and $Gr_{++}(H^2(M,\mathbb{R}))$ is naturally isomorphic to $SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$.

To see that it is simply connected, we take a long exact sequence of homotopy groups

$$\begin{aligned} ... \longrightarrow \pi_2(Gr_{++}(H^2(M,\mathbb{R}))) \longrightarrow \pi_1(SO(2)\times SO(b_2-3,1)) \stackrel{(*)}{\longrightarrow} \\ \stackrel{(*)}{\longrightarrow} \pi_1(SO(b_2-3,3)) \longrightarrow \pi_1(Gr_{++}(H^2(M,\mathbb{R}))) \longrightarrow 0, \end{aligned}$$

and notice that the map (*) above is surjective (it is easy to see from the corresponding maps of spinor groups and Clifford algebras). ■

3 Mapping class group of a hyperkähler manifold

Definition 3.1: A connected CW-space M is called **nilpotent** if its fundamental group $\pi_1(M)$ is nilpotent, acting nilpotently on homotopy groups of M.

Definition 3.2: Let M be an oriented manifold, Diff the group of oriented diffeomorphisms, and Diff_0 the group of isotopies, that is, the connected component of the group Diff. Then the quotient $\mathsf{Diff}/\mathsf{Diff}_0$ is called **the mapping class group of** M (see e.g. [LTYZ]).

Definition 3.3: Let A, A' be subgroups in a group B. Recall that A is **commensurable with** A' if $A \cap A'$ has finite index in A and A'. Let $G_{\mathbb{Z}}$ a group scheme over $\mathbb{Z}$, and $G_{\mathbb{R}} = G_{\mathbb{Q}} \otimes \operatorname{Spec}\mathbb{R}$ be the corresponding real algebraic group. A subgroup $\Gamma \subset G_{\mathbb{R}}$ is called **arithmetic** if Γ is commensurable with the group of integer points in $G_{\mathbb{R}}$.

Using rational homotopy theory, formality of Deligne-Griffiths-Morgan-Sullivan and Smale's h-cobordism, D. Sullivan proved the following general result.

Theorem 3.4: Let M be a compact simply connected (or nilpotent) Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map $\mathsf{Diff}/\mathsf{Diff}_0 \longrightarrow \Gamma$ has finite kernel, and its image has finite index in Γ. Finally, Γ is an arithmetic subgroup in the group $\Gamma_{\mathbb{Q}}$ of automorphisms of $H^*(M,\mathbb{Q})$ preserving $p_i(M)$.

Proof: Theorem 13.3 of [Su] is stated for general smooth manifolds of $\dim_{\mathbb{R}} \geqslant 5$; to apply it to Kähler manifolds, one needs to use [Su, Theorem 12.1]. The final statement is [Su, Theorem 10.3]. ■

For hyperkähler manifolds, the group $\operatorname{Aut}(H^*(M,\mathbb{Q}))$ is determined (up to commensurability), which leads to the following application of Sullivan's theorem.

Theorem 3.5: Let M be a compact, simple hyperkähler manifold, its dimension $\dim_{\mathbb{C}} M = 2n$, and Γ_A the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Consider the action of Γ_A on $H^2(M,\mathbb{Q})$ and let Γ_2 be an image of Γ_A in $GL(H^2(M,\mathbb{Q}))$. Then

(i) Γ_2 preserves the Bogomolov-Beauville-Fujiki form q on $H^2(M,\mathbb{Q})$.

(ii) Γ_2 is an arithmetic subgroup of $O(H^2(M,\mathbb{Q}),q)$.

(iii) The natural projection $\Gamma_A \longrightarrow \Gamma_2$ has finite kernel.

(iv) The mapping class group $\mathsf{Diff}/\mathsf{Diff}_0$ acts on $H^*(M,\mathbb{Z})$ with finite kernel, and the image of $\mathsf{Diff}/\mathsf{Diff}_0$ in Γ_2 has finite index.

Proof: From the Fujiki formula $v^{2n} = q(v,v)^n$, it is clear that Γ_A preserves the Bogomolov-Beauville-Fujiki, up to a sign. For n odd, the Fujiki formula fixes the sign, For n even, the sign is also fixed, because Γ_A preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant

c is positive, because the degree of $c_2(B)$ is positive for any Yang-Mills bundle with $c_1(B) = 0$ (this argument is based on [H5], section 4; see also [Ni]).

In [V2, Theorem 13.1] (see also [V3, Theorem 2.3]) it was shown that the group $Spin(H^2(M,\mathbb{Q}), q)$ acts on the cohomology algebra $H^*(M,\mathbb{Q})$ by automorphisms, preserving the Pontryagin classes.[1] Therefore, the group $\Gamma_2 \subset O(H^2(M,\mathbb{Q}), q)$ is an arithmetic subgroup in $O(H^2(M,\mathbb{Q}), q)$. This gives Theorem 3.5, (ii).

To see that the kernel K of a map $\Gamma_A \longrightarrow \Gamma_2$ is finite, we notice that the subgroup $K \subset \operatorname{Aut}(H^*(M,\mathbb{Q}))$ acts trivially on $H^2(M)$, hence preserves all Lefschetz $\mathfrak{sl}(2)$-triples $(L_\omega, \Lambda_\omega, H)$ associated with different $\omega \in H^{1,1}(M)$. The commutators of $[L_\omega, \Lambda_\omega]$ generate the Lie algebra $\mathfrak{so}(H^2(M,\mathbb{Q}), q)$ acting by derivations on $H^*(M,\mathbb{Q})$, as shown in [V2] (see also [V3]), hence K centralizes $Spin(H^2(M,\mathbb{Q}), q)$. The complexification of this group contains the complex structure operators associated with any complex, hyperkähler structure on M (see [V2], [V3]). Since K centralizes $Spin(H^2(M,\mathbb{Q}), q)$, K preserves the Hodge decomposition, for any complex structure I on M of hyperkähler type. Using the Hodge decomposition and the Lefschetz $\mathfrak{sl}(2)$-action, one defines the Riemann-Hodge pairing, writing down the Riemann-Hodge formulas as usual; it is positive definite. Since K commutes with the $\mathfrak{sl}(2)$-triples and the Hodge decomposition, it preserves the Riemann-Hodge pairing h. Therefore, K is an intersection of a lattice and a compact group $\operatorname{Spin}(H^*(M), h)$, hence finite. We proved Theorem 3.5, (iii). Theorem 3.5, (iv) follows directly from (iii) and Theorem 3.4. ■

Remark 3.6: Let $V_\mathbb{Q}$ be a rational vector space equipped with a quadratic form q, and $V_\mathbb{R} := V_\mathbb{Q} \otimes \mathbb{R}$. By [VGO], Example 7.5, the following conditions are equivalent:

(i) For any arithmetic subgroup $\Gamma \subset SO(V_\mathbb{R}, q)$, Γ has finite covolume (that is, the quotient $SO(V_\mathbb{R}, q)/\Gamma$ has finite Haar measure).

(ii) The algebraic group $SO(V_\mathbb{Q}, q)$ has no non-trivial homomorphisms to the multiplicative group $\mathbb{Q}^{>0}$ of rational numbers (in this case we say that $SO(V_\mathbb{Q}, q)$ has no non-trivial rational characters).

For $V_\mathbb{Q} = H^2(M,\mathbb{Q})$ with the Beauville-Bogomolov-Fujiki form, the latter condition always holds, hence the mapping class group is mapped to a discrete subgroup of finite covolume $\Gamma_2 \subset SO(H^2(M,\mathbb{R}), q)$.

[1] In these two papers, the action of the corresponding Lie algebra was obtained, giving a $\operatorname{Spin}(H^2(M,\mathbb{Q}), q)$-action by the general Lie group theory. In [V4, Corollary 8.2] the action of the centre of $\operatorname{Spin}(H^2(M,\mathbb{Q}), q)$ was computed. It was shown that it acts as -1 on odd cohomology and trivially on even cohomology. This Lie algebra, by its construction, preserves all cohomology classes which are of type (p,p) for all complex deformations of M; therefore, the group $Spin(H^2(M,\mathbb{Q}), q)$ acts trivially on Pontryagin's classes.

4 Weakly Hausdorff manifolds and Hausdorff reduction

4.1 Weakly Hausdorff manifolds

Definition 4.1: Let M be a topological space, and $x \in M$ a point. Suppose that for each $y \neq x$, there exist non-intersecting open neighbourhoods $U \ni x, V \ni y$. Then x is called **a Hausdorff point** of M.

Remark 4.2: The topology induced on the set of all Hausdorff points in M is clearly Hausdorff.

Definition 4.3: Let M be an n-dimensional real analytic manifold, not necessarily Hausdorff. Suppose that the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties of codim $\geqslant 2$. Suppose, moreover, that the following assumption (called "assumption **S**" in the sequel) is satisfied.

(S) For every $x \in M$, there is a *closed* neighbourhood $B \subset M$ of x and a continuous map $\Psi: B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, inducing a homeomorphism from an open neighbourhood of x in B onto an open neighbourhood of $\Psi(x)$ in $\mathbb{R}^n$.

Then M is called **a weakly Hausdorff manifold**.

Definition 4.4: Two points $x, y \in M$ are **inseparable** (denoted $x \sim y$) if for any open subsets $U \ni x, V \ni y$, one has $U \cap V \neq \emptyset$.

Remark 4.5: A closure of an open set U contains all points which are inseparable from some $x \in U$. To extend a homeomorphism from $\Psi_0: B_0 \longrightarrow \mathbb{R}^n$ from an open neighbourhood B_0 to its closure B in order to fulfill the assertion of **S** above, we need to extend Ψ_0 to all points which are inseparable from some $x \in B$.

Remark 4.6: Throughout this paper, we could work in much weaker assumptions. Instead of real analytic, we could demand that M is a Lipschitz manifold, and Z has Hausdorff codimension > 1. All the proofs in the sequel would remain valid in this general situation. Also, the assumption **S** seems to be unnecessary, though convenient. In fact, counterexamples to **S** are hard to find, and it might possibly follow from the rest of assumptions.

Example 4.7: Let Teich be a Teichmüller space of a hyperkähler manifold M, and $Z \subset M$ the set of all $I \in \mathsf{Teich}$ such that the corresponding Neron-Severi lattice $H^{1,1}(M, I) \cap H^2(M, \mathbb{Z})$ has rank $\geqslant 1$. Clearly, $Z = \bigcup_\eta Z_\eta$, with the union taken over all elements $\eta \in H^2(M, \mathbb{Z})$,[1] and

$$Z_\eta = \{I \in \mathsf{Teich} \ \ | \ \ \eta \in H^{1,1}(M, I)\}.$$

As follows from [H1] (see Remark 4.28 below), the complement $\mathsf{Teich}\backslash Z$ is Hausdorff. The period map $\mathsf{Per}: \ \mathsf{Teich} \longrightarrow \mathbb{P}\mathsf{er}$ is locally a diffeomorphism, hence the assumption **S** is also satisfied. Therefore, Teich is weakly Hausdorff.

The following definition is straightforward; it is a non-Hausdorff version of a notion of a manifold with smooth boundary. We have to give it in precise detail, because the notion of a "boundary" is ambiguous in non-Hausdorff situation.

Definition 4.8: Denote by $\overline{U}$ the closure of U in M, and by $\overline{U}^\circ$ the set of interior points of $\overline{U}$. Define **the boundary** as $\partial_M U := \overline{U}\backslash\overline{U}^\circ$. We say that an open subset $U \subset M$ of a smooth manifold M has **smooth boundary,** if each point in M has a neighbourhood V and a map mapping the closure of V to $\mathbb{R}^n$, inducing a diffeomorphism on V, and mapping the set $\overline{V}\cap\partial_M U$ to $[0,\infty]\times\mathbb{R}^{n-1}$; The closure $\overline{U}$ for such U is called **a smooth submanifold with boundary**. In this case, $\partial_M U$ is a smooth codimension 1 submanifold of M.

Further on, we shall need the following claim. It can be (roughly) stated as follows. Take a subset B in a weakly Hausdorff n-manifold, diffeomorphic to a closed ball in $U \cong \mathbb{R}^n$ with smooth boundary $\partial_U B$. Then its closure $\overline{B}$ in M is obtained by adding two kinds of extra points: those in the closure $\overline{\partial_U B}$ of $\partial_U B$ in M and those which are interior to $\overline{B}$.

Claim 4.9: Let M be a weakly Hausdorff manifold, $U \subset M$ a subset diffeomorphic to $\mathbb{R}^n$, and $B \subset U$ a connected, open subset of U which has compact closure in U with smooth boundary $\partial_U B \subset U$. Consider the set $\overline{B}\backslash\overline{B}^\circ$ of all points in the closure $\overline{B}$ of B in M which are not interior in $\overline{B}$. Then $\overline{B}\backslash\overline{B}^\circ$ coincides with the closure $\overline{\partial_U B}$ of $\partial_U B$ in M.

Proof: Clearly, $\partial_U B$ contains no interior points of $\overline{B}$. Therefore,

$$\overline{\partial_U B} \subset \overline{B}\backslash\overline{B}^\circ.$$

We need only to prove the opposite inclusion.

Denote by W the set of Hausdorff points of M. Since $M\backslash W$ has codimension $\geqslant 2$, $W \cap \partial_U B$ is dense in $\partial_U B$. The boundary $W \cap \partial_U B$ separates W onto two disjoint open subsets, $W_1 := W \cap B$ and $W_2 := W\backslash\overline{B}$. Since W is dense, $\overline{B} = \overline{W}_1$, and $\overline{B}\backslash\overline{B}^\circ \subset \overline{W}_2$. Therefore, Claim 4.9 would follow if we prove an inclusion

$$\overline{W}_1 \cap \overline{W}_2 \subset \overline{\partial_U B}. \tag{4.1}$$

[1]The group $H^2(M, \mathbb{Z})$ is torsion-free, by the Universal Coefficients Theorem, because M is simply connected.

Let $z \in \overline{W}_1 \cap \overline{W}_2$. Then in any neighbourhood of z there are points of W_1 and W_2. Since W is a smooth manifold with countably many codimension $\geqslant 2$ subvarieties removed, and W_1, W_2 are disjoint open subsets of W separated by a smooth boundary $W \cap \partial_U B$, this implies that any neighbourhood of z contains a point in $W \cap \partial_U B$. Indeed, by Lemma 4.10 below, for any path connected open subset $D \subset M$, the intersection $W \cap D$ is also connected. Unless $(W \cap \partial B) \cap D$ is non-empty, the open set $W \cap D$ is represented as a union of two non-empty disjoint open subsets $W_1 \cap D$ and $W_2 \cap D$, which is impossible, because it is connected. This implies (4.1), and finishes the proof of Claim 4.9. ■

The following trivial lemma, used in the proof of Claim 4.9, is well-known; we include it here for completeness.

Lemma 4.10: Let M be a path connected real analytic manifold, and $W = M \backslash \bigcup Z_i$, where $\bigcup Z_i$ is a union of countably many real analytic manifolds of codimension at least 2. Then W is path connected.

Proof: This result is clearly local. Therefore, we may assume that M is isomorphic to $\mathbb{R}^n$. Given two points $x, y \in W$, we shall prove that there is $z \in W$ such that a straight segment of a line connecting z to x and the one connecting z to y belong to W. Let $P^x \cong \mathbb{R}P^n$ be the set of all lines passing through x, and P^x_W the set of these lines which belong to W. Clearly, the set $P^x_{Z_i}$ of lines $l \in P$ intersecting Z_i, being a projection of Z_i to P, has real codimension 1 in P. Therefore, the complement to a set P^x_W is of measure 0 in P^x. Similarly one defines P^y_W and proves that it is dense. Let now Q be the set of all pairs of lines $l^x \in P^x, l^y \in P^y$ which intersect. Clearly, Q is equipped with smooth projections π_x, π_y to P^x and P^y, with 1-dimensional fibers. Since the complements to P^x_W and P^y_W in P^x and P^y have measure 0, the intersection $\pi_x^{-1}(P^x_W) \cap \pi_y^{-1}(P^y_W)$ is non-empty. For each pair of lines

$$(l^x, l^y) \in \pi_x^{-1}(P^x_W) \cap \pi_y^{-1}(P^y_W) \subset Q,$$

l^x and l^y are lines which are contained in W, intersect and connect x to y. ■

4.2 Inseparable points in weakly Hausdorff manifolds

Lemma 4.11: Let M be a weakly Hausdorff manifold, $x, y \in M$ inseparable points, and $U \ni x, V \ni y$ open sets. Then x and y are interior points of $\overline{U} \cap \overline{V}$, where $\overline{U}, \overline{V}$ denotes the closure of U, V.

Remark 4.12: This statement is false without the weak Hausdorff assumption. Indeed, take as M the union of two real lines, with $t < 0$ identified, x the 0 of the first line, y the 0 of the second line. Choose a neighbourhood U of x and V of y. The points x and y are clearly inseparable, but the intersection of $\overline{U} \cap \overline{V}$ is a closure of an interval $[-a, 0[$, with $a > 0$ a positive number, hence x and y are not interior points of $\overline{U} \cap \overline{V}$.

Proof of Lemma 4.11: Consider an open ball $B \subset U$ with smooth boundary $\partial_U B$ containing x. Since x and y are inseparable, y belongs to a closure $\overline{B}$ of B. Then either y is interior in $\overline{B}$, or y lies in the closure of its boundary $\partial_U B$, as follows from Claim 4.9. To prove Lemma 4.11 it remains to show that the second option is impossible. Using the assumption "**S**" of the definition of weakly Hausdorff manifolds, we obtain that $\Psi(y) = \Psi(x)$, where $\Psi: \ B \longrightarrow \mathbb{R}^n$ is the map defined in "**S**". Choosing B sufficiently small, we can always assume that $\Psi\big|_B$ is a homeomorphism. Then $\Psi(x) = \Psi(y)$ is in the interior of $\Psi(B)$, hence $\Psi(y) \notin \Psi(\partial_U B)$. Since Ψ is continuous, $\Psi^{-1}(\Psi(\partial_U B))$ contains the closure of $\partial_U B$. Therefore, $y \notin \overline{\partial_U B}$ by Claim 4.9. We proved Lemma 4.11. ■

We shall also need the following trivial lemma.

Lemma 4.13: In assumptions of Claim 4.9, let $W \subset M$ the set of Hausdorff points of M. Then the intersection $W \cap \overline{B}^\circ$ lies in B.

Proof: Denote by B_{cl} the union of B and $\partial_U B$ (Definition 4.8). Let $x \in W \cap \overline{B}^\circ$. Then x is a limit of a sequence $\{x_i\} \in B$. Since B_{cl} is compact, $\{x_i\}$ has a limit point x' in B_{cl}. Since x is Hausdorff, and $x \sim x'$, one has $x = x'$. Therefore, $x \in B_{cl}$. By Claim 4.9, one and only one of two things happens: either x is interior in $\overline{B}$, or it belongs to the closure $\overline{\partial_U B}$ of the smooth sphere $\partial_U B = B_{cl} \backslash B$. The later case is impossible, because x is interior in $\overline{B}$. Therefore, x is interior in B_{cl}. ■

Proposition 4.14: Let M be a weakly Hausdorff manifold, and $\sim$ be inseparability relation defined above. Then $\sim$ is an equivalence relation.

Remark 4.15: Without the weak Hausdorff assumption, $\sim$ is not an equivalence relation. Indeed, consider for example a union $\mathbb{R} \coprod \mathbb{R} \coprod \mathbb{R}$ of three real lines and glue $t < 0$ for the first two lines, and $t > 0$ for the second two. Then 0_1 (the zero on the first line) is inseparable from 0_2, and 0_2 from 0_3, but $0_1 \not\sim 0_3$.

Proof of Proposition 4.14: Only transitivity needs to be proven. Let $x_1 \sim x_2$, $x_2 \sim x_3$ be points in M, $U_1 \ni x_1$, $U_3 \ni x_3$ their neighbourhoods. By Lemma 4.11, x_2 is an interior point of $\overline{U}_1$ and $\overline{U}_3$. Therefore, $\overline{U}_1 \cap \overline{U}_3$ is non-empty, and contains an open subset $A := \overline{U}_1^\circ \cap \overline{U}_3^\circ$, where $\overline{U}_i^\circ$ be the set of interior points of $\overline{U}_i$. The intersection $A \cap W$ of A with the set of Hausdorff points in non-empty, because W is dense. The intersection $\overline{U}_i^\circ \cap W$ lies in U_i, as follows from Lemma 4.13, hence $A \cap W$ lies in U_1 and U_3, and these two open sets have non-trivial intersection. ■

Further on, we shall be interested in the quotient $M/_\sim$, equipped with a quotient topology. By definition, a subset $U \subset M/_\sim$ is open if its preimage in M is open, and closed if its preimage in M is closed.

Claim 4.16: Let M be a weakly Hausdorff manifold, and $B \subset M$ an open subset with smooth boundary. Consider its closure $\overline{B}$, and let $\overline{B}^\circ$ be the set of its interior points. Then $\overline{B}^\circ$ is the set of all points $y \in M$ which are inseparable from some $x \in B$.

Proof: Let $x \in B$ be any point, and $y \in M$ a point inseparable from x. By Lemma 4.11, for any neighbourhood $U \ni y$, y is an interior point of $\overline{U} \cap \overline{B}$. Therefore, y is an interior point of $\overline{B}$.

To finish the proof of Claim 4.16, it remains to show that any interior point $z \in \overline{B}$ is inseparable from some $z' \in B$. This statement is local in B, hence we may assume that B is diffeomorphic to an open ball satisfying the assumptions of the property **S** of Definition 4.3.

Choose a diffeomorphism $B \stackrel{\Psi}{\longrightarrow} B^{\mathbb{R}^n}$ to an open ball $B^{\mathbb{R}^n} \subset \mathbb{R}^n$. Using the property **S** of Definition 4.3, we may assume that Ψ can be extended to a continuous map from the closure $\overline{B}$ to the closed ball $\overline{B}^{\mathbb{R}^n}$.

Any interior point $z \in \overline{B}$ can be obtained as a limit of a sequence of points $\{z_i\} \subset B$. Let $\zeta \in \overline{B}^{\mathbb{R}^n}$ be a limit of $\{\Psi(z_i)\}$ in $\overline{B}^{\mathbb{R}^n}$, which exists because $\overline{B}^{\mathbb{R}^n}$ is compact. Choosing a subsequence, we may also assume that $\lim\{\Psi(z_i)\}$ is unique. Then $\zeta = \Psi(z)$, and it is an interior point of $\overline{B}^{\mathbb{R}^n}$, as follows from Claim 4.9. Since $B \stackrel{\Psi}{\longrightarrow} B^{\mathbb{R}^n}$ is a diffeomorphism, the sequence $\{z_i\}$ has a limit $z' \in B$. Since $\Psi(z) = \Psi(z') = \lim\{\Psi(z_i)\}$, the point z is inseparable from z'. ■

Theorem 4.17: Let M be a weakly Hausdorff manifold, and $\sim$ the inseparability relation. Consider the quotient space $M/_\sim$ equipped with a natural quotient topology. Then $M/_\sim$ is Hausdorff, and the projection map $M \stackrel{\varphi}{\longrightarrow} M/_\sim$ is open.

Proof: Since M is a manifold, we can choose a base of open subsets $U \subset M$ with smooth boundary. By Claim 4.16, $\varphi^{-1}(\varphi(U)) = \overline{U}^\circ$, where $\overline{U}^\circ$ is the set of all interior points of the closure $\overline{U}$. Therefore, the image of U is open in $M/_\sim$, and φ is an open map.

Denote by $\Gamma_\sim \subset M \times M$ the graph of $\sim$. It is well known that a topological space X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$. Since the projection $M \times M \stackrel{\varphi\times\varphi}{\longrightarrow} M/_\sim \times M/_\sim$ is open, and

$$\varphi(M \times M \backslash \Gamma_\sim) = (M/_\sim \times M/_\sim) \backslash \Delta,$$

to prove that $M/_\sim$ is Hausdorff it remains to show that $\Gamma_\sim$ is closed in $M \times M$.

Let $(x, y) \notin \Gamma_\sim$, equivalently, $x \not\sim y$. Choose open neighbourhoods $U \ni x, V \ni y$, $U \cap V = \emptyset$. Then $U \times V \cap \Gamma_\sim = \emptyset$. This implies that $\Gamma_\sim$ is closed. We proved that $M/_\sim$ is Hausdorff. ■

4.3 Hausdorff reduction for weakly Hausdorff manifolds

Definition 4.18: Let $X \stackrel{\varphi}{\longrightarrow} Y$ be a surjective morphism of topological spaces, with Y Hausdorff. Suppose that for any map $X \xrightarrow{\varphi'} Y'$, with Y' Hausdorff, the map φ' is factorized through φ. Then φ is called **the Hausdorff reduction map**, and Y **the Hausdorff reduction of** X. Being an initial object in the category of diagrams $X \xrightarrow{\varphi'} Y'$ (with Y' Hausdorff), the Hausdorff reduction if obviously unique, if it exists.

Remark 4.19: If $x \sim y$ are inseparable points of M, any morphism $M \xrightarrow{\varphi} M'$ to a Hausdorff space M' satisfies $\varphi(x) = \varphi(y)$. Therefore, whenever the quotient $M/_\sim$ is Hausdorff, it is a Hausdorff reduction of M.

Example 4.20: By Theorem 4.17, for any weakly Hausdorff manifold M, the quotient $M/_\sim$ is its Hausdorff reduction.

Definition 4.21: A **local homeomorphism** is a continuous map $X \xrightarrow{\psi} Y$ such that for all $x \in X$ there is a neighbourhood $U \ni x$ such that $\psi\big|_U$ is a homeomorphism onto its image, which is open in Y. If ψ is also a smooth, it is called **a local diffeomorphism**, or **etale map**.

Theorem 4.22: Let M be a weakly Hausdorff manifold, and

$$\varphi: \ M \longrightarrow M/_\sim$$

its Hausdorff reduction. Then φ is etale, and $M/_\sim$ is a Hausdorff manifold.

Proof: Let $U \subset M$ be an open neighbourhood of a given point x, diffeomorphic to $\mathbb{R}^n$, and $B \subset U$ a closed neighbourhood diffeomorphic to a closed ball. Since U is Hausdorff, the restriction $\varphi\big|_U$ is injective. An injective map from a compact B to a Hausdorff space is a homeomorphism to its image. Then the restriction of φ to interior of B is a homeomorphism. ■

4.4 Inseparable points in the marked moduli and the Teichmüller space

Definition 4.23: Let $K \subset \Gamma$ be a subgroup of the mapping class group acting trivially on $H^2(M)$. It is a finite group by Theorem 3.5 (iv). Recall that **the marked moduli space** Teich_{H^2} is a quotient of the Teichmüller space by K.

The following result is due to D. Huybrechts.

Theorem 4.24: ([H3]) Let M be a hyperkähler manifold, Teich_{H^2} its marked moduli space, and $x, y \in \mathsf{Teich}_{H^2}$ points corresponding to hyperkähler manifolds M_x and M_y. Suppose that x and y are inseparable, in the sense of Definition 1.13. Then the manifolds M_x and M_y are bimeromorphically equivalent. Conversely, if M_1 and M_2 are bimeromorphically equivalent, they can be realised as inseparable points on the Teichmüller space. ■

We extend this result to a Teichmüller space.

Proposition 4.25: Let $I \in \mathsf{Teich}$ be a non-Hausdorff point. Then its image in Teich_{H^2} is also non-Hausdorff.

Proposition 4.25 is proven at the end of this section. Its proof easily follows from

Theorem 4.26: Let $I \in \mathsf{Teich}$, and $K_I \subset K$ be a stabilizer of I in the group K, defined as a subgroup of the mapping class group acting trivially on $H^2(M)$. Then

(i) Denote by $G_I \subset K$ be a subgroup fixing a connected component $\mathsf{Teich}^I \ni I$ of Teich. Then K_I is normal in G_I.

(ii) The group K_I is independent from the choice of I in the component Teich^I of the Teichmüller space.

(iii) Moreover, the natural projection $\mathsf{Teich}^I \longrightarrow \mathsf{Teich}^I_{H^2}$ is a finite covering, with $\mathsf{Teich}^I_{H^2} = \mathsf{Teich}^I/(G_I/K_I)$. Here, $\mathsf{Teich}^I_{H^2}$ denotes the connected component of Teich_{H^2} containing I.

Proof: Part (i) clearly follows from (ii). To prove (ii), consider the action of K_I on $T_I \mathsf{Teich}$. We shall prove that a stabilizer $St(K_I) \subset \mathsf{Teich}^I$ is open and closed in Teich^I.

By Bogomolov's theorem (Theorem 1.9), $T_I \mathsf{Teich}$ is naturally identified with $H^{1,1}_I(M)$. However, the action of K on $H^2(M)$ is trivial, hence any $\alpha \in K_I$ acts trivially on $T_I \mathsf{Teich}$. For any finite order diffeomorphism of Teich, dimension of its fixed point set passing through I is equal to the dimension of the corresponding unit eigenspace in $T_I \mathsf{Teich}$. Therefore, K_I acts as identity on an open subset $St(K_I) \subset \mathsf{Teich}^I$. This subset is also closed, which can be seen from the following argument.

Let $J \in \mathsf{Teich}^I$ be a point, and $\{J_k\} \in St(K_I) \subset \mathsf{Teich}^I$ a sequence converging to $J \in \mathsf{Teich}^I$. To prove that $St(K_I)$ is closed, it would suffice to show that $J \in St(K_I)$.

Consider some $h \in K_I$, and let $\widetilde{h}$ be the lift of h to $\mathsf{Diff}(M)$. To prove that $J \in St(K_I)$, it would suffice to find an isotopy ν such that $\nu^* J = \widetilde{h}^* J$.

Consider at infity-dimensional space of integrable almost complex structures Comp, equipped with topology of Fréchet convergence (it is a Fréchet manifold: see [Ham]). The space Teich is defined as a quotient $\mathsf{Teich} := \mathsf{Comp} / \mathsf{Diff}^0$, with factor topology ([Ham], [Ku1]).

Chose representatives $\widetilde{J}$, $\widetilde{J}_k \in \mathsf{Comp}$. From the definition of the topology on Teich it is obvious that we can chose $\{\widetilde{J}_k\}$ converging to $\widetilde{J}$ in the Fréchet topology on Comp.

Choose a sequence $[\omega_k] \in H^2(M, J_k)$ of Kähler classes converging to a Kähler class $[\omega]$ on (M, J). This is possible to do by Kodaira's deformational stability of Kähler structures. Let ω_k, ω be the corresponding Ricci-flat Kähler metrics on $(M, \widetilde{J}_k)$ and $(M, \widetilde{J})$. Since the solutions of Calabi-Yau problem continuously depend on the data (complex structure and the Kähler class), the tensors ω_k converge to ω.

Chose isotopies $\nu_k \in \mathsf{Diff}_0$ in such a way that $\nu_k^* \widetilde{J}_k = \widetilde{h}^* \widetilde{J}_k$. This is possible to do, because $\widetilde{h}^* \widetilde{J}_k$ is equivalent to $\widetilde{J}_k$ in the Teichmüller space. The map $(\widetilde{h}^{-1}\nu_k)^*$ is holomorphic on $(M, \widetilde{J}_k)$ and preserves the complex structure, hence induces an isometry of Calabi-Yau metrics. Since the group of isometries is compact, $\widetilde{h}^{-1}\nu_k$ converges to a holomorphic isometry N of $(M, \widetilde{J})$. Let $\nu := \widetilde{h}N$. By construction, $\nu = \lim \nu_k$ in Fréchet topology. Since Diff^0 is closed in Diff, and all ν_k lie in Diff^0, this implies that $\nu \in \mathsf{Diff}^0$. We have just shown that $\nu^* J = \widetilde{h}^* J$, where ν is an isotopy.

Now, Teich^I is a union of subsets $St(K_{I'})$, for various $I' \in \mathsf{Teich}^I$, which are all open and closed in Teich^I. Since Teich^I is connected, this implies $\mathsf{Teich}^I = St(K_I)$. This proves Theorem 4.26 (ii).

To prove Theorem 4.26 (iii), recall that K is defined as a kernel of the natural map $\Gamma \longrightarrow GL(H^2(M,\mathbb{R}))$, where $\Gamma := \mathsf{Diff}^+(M)/\mathsf{Diff}_0(M)$ is the mapping class group. From Theorem 4.26 (ii) we obtain that $\mathsf{Teich}^I \longrightarrow \mathsf{Teich}^I_{H^2}$ is a quotient of Teich^I by $G_I \subset K$. Then Theorem 4.26 (iii) is implied by the following trivial lemma, applied to $X = \mathsf{Teich}^I$ and $G = G_I/K_I$.

Lemma 4.27: Let G be a finite group freely acting on a manifold X (possibly non-Hausdorff). Assume that X/G is also a manifold, and the projection $X \longrightarrow X/G$ is locally a homeomorphism. Then the quotient map $X \stackrel{\pi}{\longrightarrow} X/G$ is a covering.

Proof: It would suffice to prove that G acts properly, that is, to show that each point $x \in X$ has a neighbourhood $U \ni x$ which is disjoint from gU, for all non-trivial $g \in G$.

Since the projection map $X \stackrel{\pi}{\longrightarrow} X/G$ is etale, there exists a neighbourhood $U \ni x$ such that $\pi : X \longrightarrow X/G$ is a homeomorphism. The group G freely acts on $\bigcup_{g\in G} gU$, and for each $g \in G$ the restriction $\pi : gU \longrightarrow \pi(U)$ is a homeomorphism. Then the sets gU never intersect, for different $g \in G$. Therefore, U is a neighbourhood of x which satisfies $\forall g \in G, g \neq e, U \cap gU = \emptyset$. ■

Proof of Proposition 4.25: Proposition 4.25 follows from Theorem 4.26, because G_I/K_I is a finite group which properly acts on Teich^I, hence the projection $\mathsf{Teich}^I \longrightarrow \mathsf{Teich}^I_{H^2}$ is a finite covering, which maps non-Hausdorff points to non-Hausdorff points. ■

4.5 The birational Teichmüller space for a hyperkähler manifold

Remark 4.28: Let M_1, M_2 be bimeromorphically equivalent hyperkähler manifolds. By [H1, Proposition 9.2] and Proposition 4.25, the Neron-Severi lattice

$\mathsf{NS}(M_i) = H^{1,1}(M,\mathbb{Z})$ has rank $\geqslant 1$, unless the bimeromorphism $M_1 \rightsquigarrow M_2$ is biregular. Therefore, a point $I \in \mathsf{Teich}$ with $\operatorname{rk} \mathsf{NS}(M,I) = 0$ must be separable. This argument was used earlier in Example 4.7 to prove that Teich is weakly Hausdorff.

Clearly, the map $\mathsf{Per}: \mathsf{Teich}_b \longrightarrow \mathbb{P}\mathsf{er}$ is well defined (it follows directly from the definition of the Hausdorff reduction). Indeed, the birational Teichmüller space Teich_b is obtained as a Hausdorff reduction of the Teichmüller space. The main result of this paper is the following theorem

Theorem 4.29: (global Torelli theorem) Let M be a simple hyperkähler manifold, Teich_b its birational Teichmüller space, and

$$\mathsf{Per}: \mathsf{Teich}_b \longrightarrow \mathbb{P}\mathsf{er} \tag{4.2}$$

the period map defined as above. Then (4.2) is a diffeomorphism, for each connected component of Teich_b.

Theorem 4.29 follows from Proposition 4.30, because $\mathbb{P}\mathsf{er}$ is simply connected (Claim 2.9).

Proposition 4.30: Consider the map $\mathsf{Per}: \mathsf{Teich}_b \longrightarrow \mathbb{P}\mathsf{er}$ defined as in Theorem 4.29. Then Per is a covering.

Proposition 4.30 is proven in Remark 6.16 below.

As an immediate corollary, we obtain the following result

Corollary 4.31: Let M be a hyperkähler manifold, Teich_{H^2} its marked moduli space, and $\mathsf{Teich} \stackrel{\Psi}{\longrightarrow} \mathsf{Teich}_{H^2}$ the natural projection. Then Ψ is a diffeomorphism on each connected component.

Proof: By Theorem 4.26, Ψ is a finite covering, hence it induces a finite covering of the corresponding Hausdorff reductions. However, Ψ induces an isomorphism of Hausdorff reductions, because each component of $\mathsf{Teich}_{H^2} / \sim$ and $\mathsf{Teich} / \sim$ is isomorphic to $\mathbb{P}\mathsf{er}$. ■

Remark 4.32: The Hausdorff reduction $\mathsf{Teich}/_\sim$ classifies complex structures on M up to "bimeromorphic equivalence" and the action of the isotopy group. We call $\mathsf{Teich}/_\sim$ **the birational Teichmüller space**, denoting it as Teich_b. However, the term "bimeromorphic equivalence" is vague. Clearly, there are distinct points in $\mathsf{Teich}/_\sim$ which represent bimeromorphic (and biholomorphic) hyperkähler manifolds. A better description of this equivalence might be gleaned from [H3] and [Bou] (I am grateful to Eyal Markman for this observation, also found in [M3]). Consider the Hodge isometry $f: H^2(M_1,\mathbb{Z}) \longrightarrow H^2(M_2,\mathbb{Z})$ between the second cohomology corresponding to two inseparable points in Teich.

In the language of Boucksom, f maps the Kähler cone to one of the "rational chambers" of the positive cone. As shown in [Bou, Theorem 4.3], there are three possibilities:

(i) f could map the Kähler cone to the Kähler cone, which means that f is induced by an isomorphism. In this case M_1 and M_2 correspond to the same points of the marked moduli space.

(ii) f could map the Kähler cone onto a different rational chamber, which belongs to the *fundamental uniruled chamber.* In this case f is induced by a graph of a bimeromorphic morphism.

(iii) f could map the Kähler cone onto a different rational chamber, which does **not** belong to the fundamental uniruled chamber. In this case f is induced by a **reducible** correspondence. One of its irreducible components is a graph of a birational morphism. Other components, which necessarily exist, will appear as certain fiber products of uniruled divisors.

5 Hyperkähler lines in the moduli space

In this Section, we introduce generic hyperkähler lines (GHK lines), and prove that every two points of the Teichmüller space of hyperkähler manifold are connected by a sequence of 5 GHK lines (Proposition 5.8)

5.1 Hyperkähler lines and hyperkähler structures

Definition 5.1: Let M be a simple hyperkähler manifold, $\mathbb{P}\mathsf{er}$ its period space, and $W \subset H^2(M,\mathbb{R})$ an oriented 3-dimensional subspace, such that $q\big|_W$ is positive definite. Consider a 2-dimensional sphere $S_W \subset \mathbb{P}\mathsf{er}$ consisting of all oriented 2-dimensional planes $V \subset W$. Using an isomorphism

$$\mathbb{P}\mathsf{er} \cong Gr_{+,+}(H^2(M,\mathbb{R}))$$

constructed in Subsection 2.4, we can consider S_W as a subvariety in $\mathbb{P}\mathsf{er}$. This subvariety is called **a hyperkähler line associated with a 3-dimensional plane** $W \subset H^2(M,\mathbb{R})$.

Remark 5.2: Let (M,g,I,J,K) be a hyperkähler structure, $S \subset \mathsf{Teich}$ the sphere of induced complex structures defined as in Subsection 2.1, and $W := \langle\omega_I,\omega_J,\omega_K\rangle \subset H^2(M,\mathbb{R})$ the corresponding 3-dimensional plane. It is easy to see that the sphere $\mathsf{Per}(S) \subset \mathbb{P}\mathsf{er}$ coincides with the hyperkähler line S_W defined as above. This explains the term.

Definition 5.3: Let $S_W \subset \mathbb{P}\mathsf{er}$ be a hyperkähler line associated with a 3-dimensional subspace $W \subset H^2(M,\mathbb{R})$. We say that S_W is **a generic hyperkähler line** if the orthogonal complement to W has no rational points:

$$W^\perp \cap H^2(M,\mathbb{Q}) = 0.$$

Often, we shall abbreviate "generic hyperkähler line" to "GHK line"

5.2 Generic hyperkähler lines and the Teichmüller space

Let (M, I) be a hyperkähler manifold. The Hodge structure on $H^2(M, I)$ is determined from the Bogomolov-Beauville-Fujiki form q and the corresponding 1-dimensional space $l = \mathsf{Per}(I) \subset H^2(M, \mathbb{C})$: one has $H^{2,0}(M, I) = l$, $H^{0,2}(M, I) = \bar{l}$, and $H^{1,1}(M, I) = \langle l, \bar{l}\rangle^\perp$, where $\perp$ denotes the orthogonal complement. We define the Neron-Severi lattice of (M, I) as $\mathsf{NS}(M, I) := H^{1,1}(M, I) \cap H^2(M, \mathbb{Z})$. Since $H^{1,1}(M, I) = \langle l, \bar{l}\rangle^\perp$, the lattice $\mathsf{NS}(M, I)$ depends only on the point $\mathsf{Per}(I) \in \mathbb{P}\mathsf{er}$. We shall often consider the Neron-Severi lattice of a point $l \in \mathbb{P}\mathsf{er}$, defined as above. Since a simple hyperkähler manifold is simply connected, $\mathsf{NS}(M, I) = \operatorname{Pic}(M, I)$. This allows us to define the Picard group $\operatorname{Pic}(l)$ for $l \in \mathbb{P}\mathsf{er}$:

$$\operatorname{Pic}(l) = NS(l) = \langle l, \bar{l}\rangle^\perp \cap H^2(M, \mathbb{Z}).$$

Claim 5.4: Let $S \subset \mathbb{P}\mathsf{er}$ be a hyperkähler line, associated with a 3-dimensional subspace $W \subset H^2(M, \mathbb{R})$. Then the following assumptions are equivalent.

(i) S is a GHK line

(ii) For some $l \in S$, the corresponding Neron-Severi lattice $\mathsf{NS}(M, l)$ is trivial.

(iii) For some $w \in W$, its orthogonal complement $w^\perp \subset H^2(M, \mathbb{R})$ has no non-zero rational points.

Proof: The points of S are parametrized by oriented 2-dimensional planes $V \subset W$, and the corresponding Neron-Severi lattice $\mathsf{NS}(M, V)$ is $V^\perp \cap H^2(M, \mathbb{Z})$. Now, the chain of inclusions

$$W^\perp \cap H^2(M, \mathbb{Q}) \subset V^\perp \cap H^2(M, \mathbb{Q}) \subset w^\perp \cap H^2(M, \mathbb{Q})$$

immediately brings the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). To finish the proof, it remains to deduce (iii) from (i). Let

$$R := \bigcup_{\substack{\eta \in H^2(M, \mathbb{Q}) \\ \eta \neq 0}} \eta^\perp$$

be the union of all hyperplanes orthogonal to non-zero rational vectors. Since $W^\perp \cap H^2(M, \mathbb{Q}) = 0$, W does not lie in R. Therefore, $W \cap R$ is a countable union of planes of positive codimension. Take $w \in W \backslash R$. Clearly, $w^\perp \cap H^2(M, \mathbb{Q}) = 0$. ■

Remark 5.5: The same proof also implies that for any generic hyperkähler line, the set of all $I \in S$ with $\mathsf{NS}(M, I) \neq 0$ is countable. Indeed, it is a countable union of closed complex subvarieties of positive codimension in $\mathbb{C}P^1$.

5.3 Connected sequences of GHK lines

Further on in this subsection, we shall use the following trivial linear-algebraic lemma.

Lemma 5.6: Let A be a real vector space, equipped with a non-degenerate scalar product q, $W \subset A$ a d-dimensional subspace on which q is positive definite,[1] and $W' \subset A$ a positive subspace of dimension $d' < d$. Then there exists a non-zero vector $b \in W$, such that the subspace generated by b and W' is also positive.

Proof: Assume that $W \cap W' = 0$ (otherwise, we could choose $b \in W \cap W'$). Then $\dim W'^{\perp} \cap W > 0$. Choose $b \in W'^{\perp} \cap W$. ■

Remark 5.7: Of course, the set of such b is open in W.

Proposition 5.8: Let $x, y \in \mathbb{P}\mathsf{er}$. Then x can be connected to y by a sequence of 5 sequentially intersecting GHK lines.

Proof. Step 0: Using the identification between $\mathbb{P}\mathsf{er}$ and the Grassmann space $Gr_{+,+}(H^2(M,\mathbb{R}))$ (Subsection 2.4), we shall consider points of $\mathbb{P}\mathsf{er}$ as 2-dimensional subspaces $V \subset H^2(M,\mathbb{R})$ with $q\big|_V$ positive definite. The hyperkähler lines are understood as 3-dimensional spaces $W \subset H^2(M,\mathbb{R})$ with $q\big|_W$ positive definite. Under this identification, the incidence relation is translated into $V \subset W$.

Step 1: Let $x, y \in \mathbb{P}\mathsf{er}$ be distinct points, and $V_x, V_y \subset H^2(M,\mathbb{R})$ the associated 2-planes. Then x and y belong to the same hyperkähler line S if and only if $V_x \cap V_y$ is non-zero, and the space $\langle V_x, V_y\rangle$ generated by V_x, V_y is positive. This is an immediate consequence of Step 0.

Step 2: Let $x \in \mathbb{P}\mathsf{er}$ be a point, and $V_x \subset H^2(M,\mathbb{R})$ the corresponding 2-plane. A vector $\omega \in V_x^{\perp}$ in the positive cone of V_x defines a 3-dimensional plane $\langle \omega, V_x\rangle$. This gives a hyperkähler line $C_\omega \subset \mathsf{Teich}$ passing through x, whenever $q(\omega,\omega) > 0$. Clearly, for generic $\omega \in V_x^{\perp}$, all rational points of $\omega^{\perp}$ lie in $(H^{2,0} \oplus H^{0,2}) \cap H^2(M,\mathbb{Q})$. Therefore, the orthogonal complement to $H^{2,0} \oplus H^{0,2} \oplus \mathbb{R}\omega$ has no rational points (see also Claim 5.4).

Step 3: Let W_1 and W_2 be 3-dimensional positive subspaces in the space $H^2(M,\mathbb{R})$, containing $a \in H^2(M,\mathbb{R})$. Assume that $a^{\perp} \cap H^2(M,\mathbb{Q}) = 0$. By Claim 5.4 this implies, in particular, that the subspaces W_i correspond to GHK lines S_{W_1}, S_{W_2}. Then there exists a GHK line intersecting S_{W_1} and S_{W_2}. Indeed, from Lemma 5.6 it follows that there exists a positive 2-dimensional plane $V := \langle a, z\rangle \subset H^2(M,\mathbb{R})$ generated by a, z, with $z \in W_1$. Applying Lemma 5.6,

[1] Further on, such spaces will be called **positive**.

again, we find a positive 3-dimensional plane $W := \langle a, z, z' \rangle \subset H^2(M, \mathbb{R})$, with $z' \in W_2$. By Claim 5.4, W corresponds to a GHK line S_W. Now, Step 1 immediately implies that S_W intersects S_{W_1} and S_{W_2}.

Step 4: Let $x, t \in \mathbb{P}\mathrm{er}$. Using Step 2, we find GHK lines passing through x and t. Denote by W_x, $W_t \subset H^2(M, \mathbb{R})$ the corresponding 3-planes, and let $a \in W_x$ be a vector which satisfies $a^\perp \cap H^2(M, \mathbb{Q}) = 0$ (such a exists by Claim 5.4.) Using Lemma 5.6, we choose a non-zero $b \in W_t$, in such a way that $q|_{\langle a,b\rangle}$ is positive definite. Now, let W be a positive 3-plane in $H^2(M, \mathbb{R})$ containing a and b. By Step 3, there exist a GHK line intersecting S_{W_x} and S_W, and another GHK line intersecting S_{W_t} and S_W. We proved Proposition 5.8. ■

6 GHK lines and exceptional sets

6.1 Lifting the GHK lines to the Teichmüller space

The following proposition ensures that GHK lines are in some sense "liftable" to the Teichmüller space. This is a key idea used to prove that the period map is a covering.

Proposition 6.1: Let $I \in \mathsf{Teich}$ be a point in the Teichmüller space of a hyperkähler manifold, $\mathsf{NS}(M, I) = 0$, and $S \subset \mathbb{P}\mathrm{er}$ a hyperkähler line passing through $\mathsf{Per}(I)$.[1] Then there exists a holomorphic curve $S_I \subset \mathsf{Teich}$ passing through I and satisfying $\mathsf{Per}(S_I) = S$.

Proof: Denote by $W \subset H^2(M, \mathbb{R})$ the 3-dimensional space used to define S. Let Ω be the holomorphic symplectic form of (M, I), and $V := \langle \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega \rangle \subset H^2(M, \mathbb{R})$ the corresponding 2-dimensional space. Then $V \subset W$, and the 1-dimensional orthogonal complement $V^\perp \cap W$ intersects both components of the cone $\{x \in H^{1,1}_I(M, \mathbb{R}) \mid q(x, x) > 0\}$. One of these components coincides with the Kähler cone (Corollary 2.8). Choose a Kähler form $\omega \in V^\perp_W$, normalize it in such a way that

$$q(\mathrm{Re}\,\Omega, \mathrm{Re}\,\Omega) = q(\mathrm{Im}\,\Omega, \mathrm{Im}\,\Omega) = q(\omega, \omega),$$

and let (M, I, J, K) be the hyperkähler structure associated with ω as in Theorem 2.2. Denote by S_I the line of complex structures associated with this hyperkähler structure. As shown above (Remark 5.2), the period map Per maps S_I isomorphically to S. ■

Abusing the language, we call a $\mathbb{C}P^1$ of induced complex structures associated with a hyperkähler structure "a hyperkähler line" as well. These "hyperkähler lines" lie in the Teichmüller space, and the hyperkähler lines defined previously lie in the period space. Then Proposition 6.1 can be restated saying

[1] Such a hyperkähler line is necessarily generic, by Claim 5.4.

that a GHK line passing through a point $l \in \mathbb{P}\mathsf{er}$, satisfying $\mathsf{NS}(M, l) = 0$, can be always lifted to a hyperkähler line $S \subset \mathsf{Teich}$ for each $I \in \mathsf{Teich}$ such that $\mathsf{Per}(I) = l$.

Definition 6.2: Let $\mathbb{P}\mathsf{er}$ be a period space for a hyperkähler manifold M, and $\psi: D \longrightarrow \mathbb{P}\mathsf{er}$ an etale map from a Hausdorff manifold D. Given a hyperkähler line $S \subset \mathbb{P}\mathsf{er}$, denote by $S_{\mathrm{Pic}>0}$ the set of all $I \in S$ satisfying $\operatorname{rk}\operatorname{Pic}(M, I) > 0$. We say that ψ is **compatible with generic hyperkähler lines** if for any GHK line $S \subset \mathbb{P}\mathsf{er}$, the space $X := \psi^{-1}(S)$ is a union of several disjoint copies of S, which are closed and open in X, and another subset $Y \subset X$, which satisfies $\psi(Y) \subset S_{\mathrm{Pic}>0}$.

Proposition 6.3: Let M be a hyperkähler manifold, and

$$\mathsf{Per}: \ \mathsf{Teich}_b \longrightarrow \mathbb{P}\mathsf{er}$$

its period map. Then Per is compatible with generic hyperkähler lines.

Proof: Let $S \subset \mathbb{P}\mathsf{er}$ be a GHK line, $l \in S$ a point with $\mathsf{NS}(M, l) = 0$, and $I \in \mathsf{Teich}$ a point in the fiber $\mathsf{Per}^{-1}(l)$. By Proposition 6.1, S can be lifted to a hyperkähler line $S_I \subset \mathsf{Teich}$ passing through I. Since Per is etale, the restriction $\mathsf{Per}: \ S_I \longrightarrow S$ is a diffeomorphism. By Claim 6.4 below, S_I is a connected component of $\mathsf{Per}^{-1}(S)$. ■

The following claim is completely trivial.

Claim 6.4: Let $X \xrightarrow{\psi} Y$ be a local homeomorphism of Hausdorff spaces, $S \subset Y$ a compact subset, and $S_1 \subset X$ a subset of $\psi^{-1}(S)$, with $\psi\big|_{S_1}: S_1 \longrightarrow S$ a homeomorphism. Then S_1 is closed and open in $\psi^{-1}(S)$

Proof: The set S_1 is closed because it is homeomorphic to S which is compact, and X is Hausdorff. Suppose that S_1 is not open in $\psi^{-1}(S)$; then, there exists a sequence of points $\{x_i\} \subset \psi^{-1}(S)\backslash S_1$ converging to $x \in S_1$. Choose a neighbourhood $U \ni x$ such that $\psi\big|_U$ is a homeomorphism. Replacing $\{x_i\}$ by a subsequence, we may assume that $\{x_i\} \subset U$. Then $\psi\big|_{S_1 \cap U}$ is a homeomorphism onto its image S_U, which is a neighbourhood of $\psi(x)$ in S. Replacing $\{x_i\}$ by a subsequence again, we may assume that all $\psi(x_i)$ lie in S_U. Since $\psi\big|_U$ is bijective onto its image, this map induces a bijection from $S_1 \cap U$ to S_U. Therefore, $\{x_i\} \subset S_1 \cap U$. We obtained a contradiction, proving that S_1 is open in $\psi^{-1}(S)$. ■

6.2 Exceptional sets of etale maps

In [Br], F. Browder has discovered several criteria which can be used to prove that a given etale map is a covering. Unfortunately, in our case neither of his theorems can be applied, and we are forced to devise a new criterion, which is then applied to the period map.

Definition 6.5: Let $X \xrightarrow{\psi} Y$ be a local homeomorphism of Hausdorff topological spaces, *e.g.* an etale map. Consider a connected, simply connected subset $R \subset Y$, and let $\{R_\alpha\}$ be the set of connected components of $\psi^{-1}(R)$. **An exceptional set** of (ψ, R) is $R\backslash\psi(R_\alpha)$.

Remark 6.6: The following topological criterion is the main technical engine of this section. Its proof is complicated, but completely abstract, and we hope that this result might have independent uses outside of hyperkähler geometry. We include an alternative proof of this proposition in the Appendix by Eyal Markman (Section 8).

Proposition 6.7: Let $X \xrightarrow{\psi} Y$ be a local diffeomorphism of Hausdorff manifolds. Assume that for any open subset $U \subset Y$, the closure $\overline{U} \subset Y$ has empty exceptional sets, provided that U has smooth boundary. Then ψ is a covering.

Proof: Proposition 6.7 is local in Y, hence it will suffice to prove it when Y is diffeomorphic to $\mathbb{R}^n$. Choose a flat Riemannian metric on $Y \cong \mathbb{R}^n$. Lifting the corresponding Riemannian metric to X, we can consider X as a Riemannian manifold, also flat. The Riemannian structure defines a metric on Y and X as usual. For a point x in a metric space M, **a closed ε-ball with center in x** is the set

$$\overline{B}_\varepsilon(x) := \{m \in M \ \mid \ d(x,m) \leqslant \varepsilon\}.$$

Taking strict inequality, we obtain **an open ball**,

$$B_\varepsilon(x) := \{m \in M \ \mid \ d(x,m) < \varepsilon\}.$$

Clearly, $\overline{B}_\varepsilon(x)$ is closed, $B_\varepsilon(x)$ is open, and $\overline{B}_\varepsilon(x)$ is the closure of $B_\varepsilon(x)$, and its completion, in the sense of metric geometry.

For any $x \in X$, $y = \psi(x)$, let $D_x \subset \mathbb{R}^{>0}$ be the set of all $\varepsilon \in \mathbb{R}^{>0}$ such that the corresponding ε-ball $\overline{B}_\varepsilon(x)$ is mapped to $\overline{B}_\varepsilon(y)$ bijectively. Clearly, D_x is an initial interval of $\mathbb{R}^{>0}$. We are going to show that D_x is open and closed in $\mathbb{R}^{>0}$.

Step 1: The interval D_x is open, for any etale map $X \xrightarrow{\psi} \mathbb{R}^n$. Indeed, for any $\varepsilon \in D_x$, the corresponding ε-ball $\overline{B}_\varepsilon(x)$ is compact, because it is isometric to $\overline{B}_\varepsilon(y)$. Every point of $\overline{B}_\varepsilon(x)$ has a neighbourhood which is isometrically mapped to its image in Y. Take a covering $\{B_\varepsilon(x), U_1, U_2, ...\}$ of $\overline{B}_\varepsilon(x)$ where U_i are open balls with this property, centered in a point on the boundary of $\overline{B}_\varepsilon(x)$. Since $\overline{B}_\varepsilon(x)$ is compact, $\{B_\varepsilon(x), U_i\}$ has a finite subcovering $U_1, ..., U_n$. By construction, for each point $z \in W := B_\varepsilon(x) \cup \bigcup_i U_i$, the set W contains a straight line (geodesic) from x to z. Indeed, W is a union of an open ball $B_\varepsilon(x)$ and several open balls centered on its boundary, and all these balls are isometric to open balls in $\mathbb{R}^n$. Since ψ maps straight lines to straight lines, it maps $B_{\varepsilon'}(x)$ surjectively to $B_{\varepsilon'}(y)$. To show that this map is also injective, consider two points $a_1, a_2 \in B_{\varepsilon'}(x)$, mapped to $b \in B_{\varepsilon'}(y)$, and let $[x, a_1]$ and $[x, a_2]$ be the corresponding intervals of a straight line. Since $\psi(a_1) = \psi(a_2) = b$, one has $\psi([x, a_1]) = \psi([x, a_2])$, and these intervals have the same length. Also, $[x, a_1] \cap B_\varepsilon(x) = [x, a_2] \cap B_\varepsilon(x)$, because $\psi\Big|_{B_\varepsilon(x)}$ is injective. Therefore, the intervals $[x, a_1]$ and $[x, a_2]$ coincide, and $a_1 = a_2$.

Step 2: Let $\psi: X \longrightarrow \mathbb{R}^n$ be an etale map, $y = \psi(x)$, and suppose that $\psi: B_s(x) \longrightarrow B_s(y)$ is bijective, for some $s > 0$. Then $\varphi: B_s(x) \longrightarrow B_s(y)$ is an isometry, with respect to the metric on B_s induced from the ambient manifold. Indeed, ψ is etale, hence any piecewise geodesic path in X is projected to such one in $\mathbb{R}^n$. Therefore, ψ does not increase distance: $d(a, b) \geqslant d(\psi(a), \psi(b))$. The open ball $B_s(y)$ is geodesically convex, hence for any $y_1, y_2 \in B_s(y)$, the geodesic interval $[y_1, y_2]$ can be lifted to a geodesic in $B_s(x)$. This implies an inverse inequality: $d(a, b) \leqslant d(\psi(a), \psi(b))$. We proved that $\varphi: B_s(x) \longrightarrow B_s(y)$ is an isometry. This implies that the map $\psi: \overline{B}_s(x) \longrightarrow \overline{B}_s(y)$ of their metric completions is also an isometry. In particular, this map is injective.

Step 3. In the assumptions of Step 2, we prove that $\overline{B}_s(x)$ is a connected component of $\psi^{-1}(\overline{B}_s(y))$. Notice that $\overline{B}_s(x)$ is a closure of $B_s(x)$, which is homeomorphic to a ball in $\mathbb{R}^n$, hence $\overline{B}_s(x)$ is connected. To prove that it is a connected component, we need only to show that it is open in $\psi^{-1}(\overline{B}_s(y))$.

The corresponding map of open balls $\psi: B_s(x) \longrightarrow B_s(y)$ is by definition bijective. The closed ball $\overline{B}_s(x)$ is closed in $\psi^{-1}(\overline{B}_s(y))$. For any $z \in \partial\overline{B}_s(x)$ on the boundary of $\overline{B}_s(x)$, an open ball S centered in z is split by the boundary

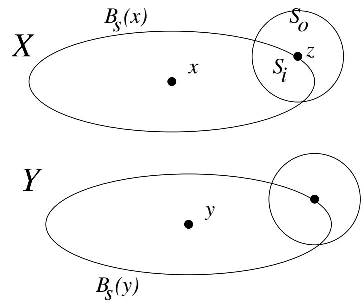
$$\partial\overline{B}_s(x) = \{x' \in X \mid d(x, x') = s\}$$

onto two open components, $S_o := \{x' \in X \mid d(x, x') > s\}$ and $S_1 := \{x' \in X \mid d(x, x') < s\}$, with S_1 mapping to $B_s(y)$, $\partial\overline{B}_s(x)$ mapping to its boundary, and S_o to $Y \backslash \overline{B}_s(y)$.[2] This implies that

$$\psi^{-1}(\overline{B}_s(y) \cap \psi(S)) = S \cap \overline{B}_s(x).$$

Therefore, $\overline{B}_s(x)$ is open in $\psi^{-1}(\overline{B}_s(y))$.

[2] Here we use the fact that $\psi\Big|_S$ is a bijection, for S sufficiently small, hence the image of S cannot wrap on itself.



Step 4. Now we can show that D_x is closed. This argument uses the triviality of exceptional sets (the first time in this proof, the rest follows just from the etaleness of ψ). Let $s := \sup D_x$, and $\overline{B}_s(x)$ the corresponding closed ball. We prove that $\psi : \overline{B}_s(x) \longrightarrow \overline{B}_s(y)$ is a homeomorphism.

From Step 3, it follows that $\overline{B}_s(x)$ is a connected component of the preimage $\psi^{-1}(\overline{B}_s(y))$. Since the exceptional sets of $\overline{B}_s(y)$ are all empty, the map $\psi : \overline{B}_s(x) \longrightarrow \overline{B}_s(y)$ is surjective. It is injective as follows from Step 2.

We proved that D_x is open and closed, hence $D_x = \mathbb{R}^{>0}$, and ψ maps any connected component of X bijectively to Y. ■

Remark 6.8: An exceptional set of (ψ, U) is always closed in U.

Lemma 6.9: Let M be a Hausdorff manifold, $M \stackrel{\psi}{\longrightarrow} \mathbb{P}\mathsf{er}$ a local diffeomorphism, compatible with GHK lines, $U \subset \mathbb{P}\mathsf{er}$ an open, simply connected subset, U_α a component of $\psi^{-1}(U)$, and K_α the corresponding exceptional set. Consider a GHK line $C \subset \mathbb{P}\mathsf{er}$, and let C_1 be a connected component of $C \cap U$. Then $C_1 \subset K_\alpha$, or $C_1 \cap K_\alpha = \emptyset$.

Proof: Suppose that $D := C_1 \cap (U \backslash K_\alpha)$ is non-empty. Since K_α is closed in U, D is open in C_1. Then D contains points $l \in D$ with $\mathsf{NS}(M, l) = \emptyset$ (Remark 5.5). The set $\psi^{-1}(l)$ is non-empty, because $l \notin K_\alpha$. Since ψ is compatible with GHK lines, for any $I \in \psi^{-1}(l)$, there is a curve $C_I \subset M$ passing through I and projecting bijectively to C. Clearly, the connected component of $C_I \cap \psi^{-1}(U) \ni I$ is bijectively mapped to C_1, hence $C_1 \cap K_\alpha = \emptyset$. ■

Remark 6.10: A version of Lemma 6.9 is also true if $\overline{U}$ is a closed set, obtained as a closure of an open subset $U \subset \mathbb{P}\mathsf{er}$, and C_1 a connected component of $\overline{U} \cap C$, for a GHK curve C. If C_1 contains interior points, the same argument as above can be used to show that $C_1 \subset K_\alpha$, or $C_1 \cap K_\alpha = \emptyset$.

6.3 Subsets covered by GHK lines

Let $U \subset \mathbb{P}\mathsf{er}$ be an open subset, or a closure of an open subset with smooth boundary, and $K \subset U$ a subset of U. Given a GHK line $C \subset \mathbb{P}\mathsf{er}$, denote by C_U a connected component of $C \cap U$. This component is non-unique for some C

and U. Denote by $\Omega_U(K)$ the union of all segments $C_U \subset U$ intersecting K, for all GHK lines $C \subset \mathbb{P}\mathsf{er}$. In other words, $\Omega_U(K)$ is the set of all points connected to K by a connected segment of $C \cap U$, with $C \subset \mathbb{P}\mathsf{er}$ a GHK line. Let $\Omega_U^*(X)$ be the union of $\Omega_U(X), \Omega_U(\Omega_U(X)), \Omega_U(\Omega_U(\Omega_U(X))), ...$

Proposition 6.11: Let $U \subset \mathbb{P}\mathsf{er}$ be an open subset, and $x \in U$ a point. Then $\Omega_U^*(x)$ is open in U.

Proof: We give two alternative proofs of this proposition. The first one is based on [Bea2] and [V3, Theorem 3.1, Theorem 5.1], where essentially the same argument was used to show that the category of polystable holomorphic vector bundles on a generic hyperkähler manifold M is independent from the choice of M in its deformation class. A beautiful presentation of this argument is found in [H6, Proposition 3.7].

The second proof uses the concept of subtwistor metric d_{tw}, introduced in the Appendix, Section 9. This argument, based on the theory of intrinsic metrics and the Gleason-Palais's elaboration of the Gleason-Montgomery-Zippin solution of Hilbert's fifth problem, is much more explicit.

Proof of Proposition 6.11: the first approach. From Proposition 5.8, it follows that $\Omega_U^5(x)$ is open. Indeed, any y can be connected to x by a sequence $\mathfrak{S}(x, y)$ of 5 GHK lines. From its proof it is apparent that one can chose this sequence in such a way that it depends continuously on x, y. Let $y \in \Omega_U^5(x)$, and let $\gamma : [0, 1] \longrightarrow U$ be a path connecting x to y and sitting in the union of the GHK lines $S_i \in \mathfrak{S}(x, y)$. Given a sequence $\{y_i\}$ converging to y, and a sequence $\mathfrak{S}(x, y_i)$ converging to $\mathfrak{S}(x, y)$, we can chose a sequence of paths $\gamma_i : [0, 1] \longrightarrow \mathbb{P}\mathsf{er}$ with the following properties.

- The paths γ_i sit in $\mathfrak{S}(x, y_i)$.
- The sequence $\{\gamma_i\}$ converges to γ in the compact-open topology.
- Each γ_i connects x to y_i.

Since $[0, 1]$ is compact, U is open, and $\gamma([0, 1]) \subset U$, for any sufficiently big i, one has $\gamma_i([0, 1]) \subset U$. This implies that $y_i \in \Omega_U^5(x)$.

Proof of Proposition 6.11: the second approach. It suffices to prove Proposition 6.11 when the closure of U is compact. Indeed, each point $z \in \Omega_U^*(x)$ has a neighbourhood $V \subset U$ with its closure in U compact. If $\Omega_V^*(x)$ is open in V, then each point $z \in \Omega_U^*(x)$ has an open neighbourhood $V_1 \subset V \subset U$ contained in $\Omega_U^*(x)$.

By Theorem 9.1 (see the Appendix, Section 9), the metric d_{tw} induces the usual topology on $\mathbb{P}\mathsf{er}$. For any $x \in U$, the distance $d_{tw}(x, \partial U)$ to the boundary of U is positive, because ∂U is compact. Then, for any $r < d_{tw}(x, \partial U)$, the open ball $B_r(x, d_{tw})$ is contained in $\Omega_U^*(x)$. Indeed, let $y \in B_r(x, d_{tw})$. Then $y = s_{n+1}$ is connected to $x = s_0$ by a sequence of GHK lines $S_1, ..., S_n$, such that $\sum_{i=0}^n d_g(s_i, s_{i+1}) < r$. Consider the corresponding piecewise geodesic path $\gamma \subset \bigcup S_i$ of length $< r$. Since $d_{tw}(x, \partial U) > r$, the whole of γ belongs to U.

Therefore, y is connected to x by a union of connected segments of GHK lines which lie in U. ■

To apply Proposition 6.7 to the period map using the exceptional sets, we also need closed subsets with smooth boundary. In this situation the following lemma can be used.

Lemma 6.12: Let $K \subset \mathbb{P}\mathsf{er}$ be a compact closure of an open subset with smooth boundary, and $x \in K$ a point. Then $\Omega_K(x)$ contains an interior point of K.

Proof: Let V_x be the 2-plane in $H^2(M, \mathbb{R})$ corresponding to x via the identification $Gr(2) = \mathbb{P}\mathsf{er}$. Then the tangent space $T_x \mathsf{Per}$ is identified with $\operatorname{Hom}(V_x, V_x^\perp)$, where $V_x^\perp$ is the orthogonal complement. For a hyperkähler line C associated with a 3-dimensional space W, the corresponding 2-dimensional space $T_x C \subset T_x \mathbb{P}\mathsf{er}$ is the space $\operatorname{Hom}(V_x, (V_x^\perp \cap W))$. Since $V_x^\perp = H_x^{1,1}(M)$ and W can be chosen by adding to V_x any Kähler class in $H^{1,1}(M)$, the set of all tangent vectors $T_x C \subset T_x \mathbb{P}\mathsf{er}$ is open in the space

$$P := \{l \in \operatorname{Hom}(V_x, H_x^{1,1}(M)) \ \ | \ \ \operatorname{rk} l = 1\}$$

The condition $\operatorname{rk} l = 1$ is quadratic, and it is easy to check that an open subset $U_P \subset P$ cannot be contained inside a linear subspace of positive codimension. In particular, U_P cannot lie in the tangent space to the boundary of K,

$$U_P \not\subset T_x \partial K \subset \operatorname{Hom}(V_x, H^{1,1}(M)). \tag{6.1}$$

Take for U_P the set of all vectors tangent to GHK lines passing through x. Then (6.1) implies that for a generic GHK line C passing through x, C intersects with the interior points of K. ■

Corollary 6.13: Let $K \subset \mathbb{P}\mathsf{er}$ be a closure of an open, connected subset $U \subset \mathbb{P}\mathsf{er}$ with smooth boundary, and Ω_K the operation on subsets of K defined above. Then $\Omega_K^*(x) = K$, for any point $x \in K$.

Proof: Clearly, $\Omega_U^*(x)$ is the set of all points in U which can be connected to x within U by a finite sequence of connected segments of GHK lines. By Proposition 6.11, $\Omega_U^*(x)$ is open in U. If $y \notin \Omega_U^*(x)$, then $\Omega_U^*(y)$ does not intersect $\Omega_U^*(x)$. Then U is represented as a disconnected union of open sets $\Omega_U^*(x_i)$, for some $\{x_i\} \subset U$. This is impossible, because U is connected. We proved that $\Omega_U^*(x) = U$. Then $\Omega_K^*(x) = K$, because every point on a boundary of K is connected to some point of U by a connected segment of a GHK line (Lemma 6.12). ■

The main result of this section is the following theorem

Theorem 6.14: Let M be a Hausdorff manifold and $M \xrightarrow{\psi} \mathbb{P}\mathsf{er}$ a local diffeomorphism compatible with GHK lines. Then ψ is a covering.

Remark 6.15: It is well known that $\mathbb{P}\mathsf{er}$ is simply connected (Claim 2.9). Then Theorem 6.14 implies that ψ is a diffeomorphism.

Proof of Theorem 6.14: To prove that ψ is a covering, it suffices to show that all its exceptional sets of (ψ, K), are empty provided that K is a closure of a simply connected open subset $U \subset \mathbb{P}\mathsf{er}$ which has a smooth boundary (Proposition 6.7). Let K_α be an exceptional set, associated with a closure $K \subset \mathbb{P}\mathsf{er}$ of an open subset $U \subset \mathbb{P}\mathsf{er}$ with smooth boundary. From Lemma 6.9 and Remark 6.10 it follows that $\Omega_K(K_\alpha) = K_\alpha$, where $\Omega_K(Z)$ is a union of all connected segments of $C \cap K$ intersecting Z, for all GHK lines $C \subset \mathbb{P}\mathsf{er}$. Then $\Omega^*_K(K_\alpha) = K_\alpha$, where $\Omega^*_K(Z)$ is a union of all iterations $\Omega^i_K(Z)$. However, for any non-empty $Z \subset K$, one has $\Omega^*_K(Z) = K$ by Corollary 6.13. Therefore, any exceptional set K_α of (ψ, K) for K as above is empty, and Theorem 6.14 follows. ■

Remark 6.16: Proposition 4.30 is implied by Proposition 6.3 and Theorem 6.14 below. Indeed, by Proposition 6.3, Per is compatible with the generic hyperkähler lines (Definition 5.3), and by Theorem 6.14, any such map is necessarily a covering.

7 Monodromy group for $K3^{[n]}$.

When $M = K3^{[n]}$ is a Hilbert scheme of points on a K3 surface, fundamental results about its moduli were obtained by E. Markman ([M1], [M2]), using the Fourier-Mukai action on the derived category of coherent sheaves. In this section we relate these results with our computation of Teich_b to obtain a global Torelli theorem for $M = K3^{[p^\alpha+1]}$, p prime.

7.1 Monodromy group for hyperkähler manifolds

Definition 7.1: The **monodromy group** $\mathsf{Mon}(M)$ of a hyperkähler manifold M is a subgroup of $GL(H^*(M, \mathbb{Z}))$ generated by the monodromy of the Gauss-Manin local systems, for all holomorphic deformations of M over a connected complex analytic base.

Using the global Torelli theorem (Theorem 4.29), the monodromy group can be related to the mapping class group, as follows.

Theorem 7.2: Let (M, I) be a hyperkähler manifold, and Teich^I the corresponding connected component of a Teichmüller space. Denote by Γ^I the subgroup of the mapping class group preserving the component Teich^I, and let $\mathsf{Mon}(M)$ be the monodromy group of (M, I) defined above. Then $\mathsf{Mon}(M)$ coincides with the image of Γ^I in $GL(H^*(M, \mathbb{Z}))$.

Proof:[1] From its construction, it is obvious that the Teichmüller space is the coarse moduli space for the following functor from analytic spaces to sets. This functor associates to a complex analytic space B the set of isomorphism classes of pairs $(\mathcal{X} \longrightarrow B, \Psi)$, where $\mathcal{X} \longrightarrow B$ is a complex analytic deformation of M over B, and Ψ a smooth trivialization of the family $\mathcal{X} \longrightarrow B$, defined up to isotopy on the fibers.

Now, consider an element $\gamma \in \mathsf{Mon}(M)$ in the image of the monodromy of a holomorphic family $\mathcal{Z} \longrightarrow B$. Then there exists a point $b \in B$ and a loop $\gamma_0 : [0,1] \longrightarrow B$ such that the corresponding Gauss-Manin monodromy induces γ on $H^*(M,\mathbb{Z})$.

For a certain covering $\widetilde{B} \longrightarrow B$, the corresponding family $\widetilde{\mathcal{Z}} \longrightarrow \widetilde{B}$ admits a smooth trivialization. This gives a map $\widetilde{B} \stackrel{\psi}{\longrightarrow} \mathsf{Teich}^I$ to the corresponding coarse moduli space, such that the pullback $\widetilde{\mathcal{Z}} \longrightarrow \widetilde{B}$ of the family $\mathcal{Z} \stackrel{\pi}{\longrightarrow} B$ admits a smooth trivialization $\widetilde{\mathcal{Z}} = \widetilde{B} \times M$.

Let $\widetilde{\gamma}_0$ be a lifting of γ_0 to $\widetilde{B}$, and $x, y \in \widetilde{B}$ the ends of this path, with $\widetilde{\mathcal{Z}}_x = (M,I) = \widetilde{\mathcal{Z}}_y$ denoting the fibers of π over x, y. The trivialization of π over $\widetilde{B}$ induces a diffeomorphism $(M,I) = \widetilde{\mathcal{Z}}_x \longrightarrow \widetilde{\mathcal{Z}}_y = (M,I)$ acting on $H^*(M,\mathbb{Z})$ as γ.

In a neighbourhood of the corresponding path $\psi(\gamma_0) \subset \mathsf{Teich}^I$ the universal family of deformations of M is well defined. Replacing B by a smaller neighbourhood of γ_0 if necessary, we may assume that the family $\widetilde{\mathcal{Z}} \longrightarrow \widetilde{B}$ is a pullback of the universal family on a neighbourhood U of $\psi(\gamma_0)$. The monodromy group of this family by definition belongs to Γ_I, hence the image of γ in $GL(H^*(M,\mathbb{Z})$ lies in $i(\Gamma_I)$.

Conversely, for each $\gamma \in \Gamma^I$, consider the action of Γ^I on Teich^I, and let $x, y := \gamma(x)$ be a pair of points on Teich^I, connected by a smooth path γ_0. Denote by U a neighbourhood of γ_0, diffeomorphic to an open ball. Consider a non-normal quotient U_1, obtained from U by identifying x and y. Since U is diffeomorphic to an open ball, there exists a universal fibration $\widetilde{X} \longrightarrow U$. Gluing two fibers of this universal fibration, we obtain a holomorphic fibration $\widetilde{X}_1 \longrightarrow U_1$; its monodromy acts on $H^2(M,\mathbb{Z})$ as γ, by construction. ■

This result allows one to answer the question asked in [M2] (Conjecture 1.3).

Corollary 7.3: Let $\gamma \in \mathsf{Mon}$ be an element of the monodromy group acting trivially on the projectivization $\mathbb{P}H^2(M,\mathbb{C})$. Suppose that a general deformation of M has no automorphisms. Then γ is trivial.

Proof: Using [C, Remark 13], we may assume that there exists a universal fibration $\mathcal{Z} \stackrel{\pi}{\longrightarrow} \mathsf{Teich}_I$. This gives a local system $R\pi_*\mathbb{Z}$ over Teich^I. Let γ be an element of the mapping class group acting trivially on $H^2(M)$ preserving a connected component $\mathsf{Teich}^I \subset \mathsf{Teich}$. By Theorem 4.26, γ acts trivially on

[1] I am grateful to the referee for numerous suggestions which lead to many improvements in this proof.

Teich^I. However, Theorem 7.2 implies that an action of any $\gamma \in \mathsf{Mon}$ is induced by the parallel transport in the local system $R\pi_*\mathbb{Z}$ from x to $\gamma(x)$ along a path γ_0 connecting x to $\gamma(x)$ in Teich^I. Since the action of γ on Teich^I is trivial, we may choose γ_0 to be trivial. ■

Remark 7.4: In the above corollary, a stronger result is actually proven. Instead of defining the monodromy group as above, we could define $\widetilde{\mathsf{Mon}}$ as the image of $\pi_1(\mathcal{M})$ in the mapping class group of M. Then Corollary 7.3 implies that the natural map of $\widetilde{\mathsf{Mon}}$ to $PGL(H^2(M,\mathbb{C}))$ is injective.

Remark 7.5: The kernel of the natural projection $\Gamma_I \longrightarrow PGL(H^2(M,\mathbb{C})$ is identified with the group of holomorphic automorphisms of a generic deformation of a hyperkähler manifold M. When M is a deformation of a Hilbert scheme of a K3, this group is trivial, which can be easily seen e.g. from the results of [V5]. When M is a generalized Kummer variety, it is known to be non-trivial ([KV]).

7.2 The Hodge-theoretic Torelli theorem for $K3^{[n]}$

Definition 7.6: Let V be a vector space, g a non-degenerate quadratic form, and $v \in V$ a vector which satisfies $g(v,v) = \pm 2$. Consider the pseudo-reflection map $\rho_v: V \longrightarrow V$,

$$\rho_v(x) := \frac{-2}{g(v,v)}x + g(x,v)v.$$

Clearly, ρ_v is a reflection when $g(v,v) = 2$, and $-\rho_v$ is a reflection when $g(v,v) = -2$. Given an integer lattice in V, consider the group $\operatorname{Ref}(V)$ generated by ρ_v for all integer vectors v with $g(v,v) = \pm 2$. We call Ref **a reflection group**.

The following fundamental theorem was proven by E. Markman in [M2].

Theorem 7.7: ([M2, Theorem 1.2]) Let $M = K3^{[n]}$ be a Hilbert scheme of points on a K3 surface, and Mon^2 be the image of the monodromy group in $GL(H^2(M,\mathbb{Z}))$. Then $\mathsf{Mon}^2 = \operatorname{Ref}(H^2(M,\mathbb{Z}), q)$. ■

Comparing this with Theorem 7.2 and using the global Torelli theorem (Theorem 4.29), we immediately obtain the following result.

Theorem 7.8: Let $M = K3^{[n]}$ be a Hilbert scheme of points on K3, $\mathcal{M}_b$ its birational Teichmüller space, and $\mathcal{M}_b(I)$ a connected component of $\mathcal{M}_b$. Then $\mathcal{M}_b(I) \cong \mathbb{P}\mathsf{er} / \operatorname{Ref}$, where $\mathbb{P}\mathsf{er}$ is the period domain defined as in (1.3), and $\operatorname{Ref} = \operatorname{Ref}(H^2(M,\mathbb{Z}), q)$ the corresponding reflection group, acting on $\mathbb{P}\mathsf{er}$ in a natural way. ■

The reflection group was computed in [M2] (Lemma 4.2). When $n-1$ is a prime power, this computation is particularly effective.

Definition 7.9: Let (V, g) be a real vector space equipped with a non-degenerate quadratic form of signature (m, n), and

$$S := \{v \in V \ \ | \ \ g(v, v) > 0\}.$$

It is easy to see that S is homotopy equivalent to a sphere S^{m-1}. Define the **spinorial norm** of $\eta \in O(V)$ as ± 1, where the sign is positive if η acts as 1 on $H^{m-1}(S)$, and negative if η acts as -1. Let $O^+(V)$ denote the set of all isometries with spinorial norm 1.

Remark 7.10: It is easy to see that $\operatorname{Ref} \subset O^+(V)$, where Ref is a reflection group.

Proposition 7.11: ([M2, Lemma 4.2]). Let $M = K3^{[n]}$ be a Hilbert scheme of K3, and $\operatorname{Ref} = \operatorname{Ref}(H^2(M, \mathbb{Z}), q)$ the corresponding reflection group. Then $\operatorname{Ref} = O^+(H^2(M, \mathbb{Z}), q)$ if and only if $n - 1$ is a prime power. ■

Definition 7.12: Let V be a real vector space equipped with a non-degenerate quadratic form of signature (m, n). A choice of **spin orientation** on V is a choice of a generator of the cohomology group $H^{m-1}(S)$ (Definition 7.9). Clearly, $O^+(V)$ is a group of orthogonal maps preserving the spin orientation.

Remark 7.13: For a space V with signature (m, n), the group $O(V)$ has 4 connected components, which are given by a choice of orientation and spin orientation. Alternatively, these 4 components are distinguished by a choice of orientation on positive m-dimensional planes and negative n-dimensional planes.

Remark 7.14: Donaldson ([Don]) has shown that any diffeomorphism of a K3 surface M preserves the spin orientation, and the global Torelli theorem implies that every integer isometry of $H^2(M)$ preserving the spin orientation is induced by a diffeomorphism ([Bor]). This implies that the mapping class group Γ_M is mapped to $O^+(H^2(M, \mathbb{Z}))$ surjectively.

Remark 7.15: Let $V = H^2(M, \mathbb{R})$ be the second cohomology of a hyperkähler manifold, equipped with the Hodge structure and the BBF form, and $V^{1,1} \subset V$ the space of real (1,1)-classes. The set of vectors

$$R := \{v \in V^{1,1} \ \ | \ \ q(v, v) > 0\}$$

is disconnected, and has two connected components. Since the orthogonal complement $(V^{1,1})^\perp$ is oriented, a spin orientation on V is uniquely determined by a choice of one of two components of R. The Kähler cone of M is contained in one of two components of R. This gives a canonical spin orientation on $H^2(M, \mathbb{R})$.

Definition 7.16: Let M be a hyperkähler manifold. We say that **the Hodge-theoretic Torelli theorem holds for** M, if for any I_1, I_2 inducing isomorphic

Hodge structures on $H^2(M)$, the manifold (M, I_1) is bimeromorphically equivalent to (M, I_2), provided that this isomorphism of Hodge structures is also compatible with the spin orientation and the Bogomolov-Beauville-Fujiki form, and I_1, I_2 lie in the same connected component of the moduli space.

Remark 7.17: This is the most standard version of global Torelli theorem.

The following claim immediately follows from Theorem 7.2.

Claim 7.18: Let M be a hyperkähler manifold. Then the following statements are equivalent.

(i) The Hodge-theoretic Torelli theorem holds for M.

(ii) The monodromy group Mon of M is surjectively mapped to the group $O^+(H^2(M,\mathbb{Z}), q)$, under the natural action of Mon on $H^2(M)$.

■

Comparing this with the Markman's computation of the monodromy group (Proposition 7.11), we immediately obtain the following theorem.

Theorem 7.19: Let $M = K3^{[p^\alpha+1]}$. Then the Hodge-theoretic Torelli theorem holds. ■

Remark 7.20: For other examples of hyperkähler manifolds, the Hodge-theoretic global Torelli theorem is known to be false. For some of generalized Kummer varieties this was proven by Namikawa ([Na]), and for $M = K3^{[n]}$, $n \neq p^\alpha + 1$, this observation is due to Markman ([M2]). For O'Grady's examples of hyperkähler manifolds ([O]), it is unknown.

8 Appendix: A criterion for a covering map (by Eyal Markman)

Another version of the proof of Proposition 6.7 was proposed by E. Markman; with his kind permission, I include it here.

Proposition 8.1: (Proposition 6.7) Let $\psi : X \to Y$ be a local homeomorphism of Hausdorff topological manifolds. Assume that every open subset $U \subset Y$, whose closure $\overline{U}$ is homeomorphic to a closed ball in $\mathbb{R}^n$, and such that U is the interior of its closure, satisfies the following property. For every connected component C of $\psi^{-1}(\overline{U})$, the equality $\psi(C) = \overline{U}$ holds. Then ψ is a covering map.

Verbitsky stated the above proposition in the category of differentiable manifolds and provided a proof of the proposition, involving Riemannian-geometric constructions on the domain X. We translate his proof to an elementary point set topology language. The natural translation of the statement and its proof to the category of differentiable manifolds is valid as well. In that case ψ is a local diffeomorphism and it suffices for the assumption to hold for open subsets U, such that the boundary ∂U is smooth, and there exists a homeomorphism from $\overline{U}$ onto a closed ball in $\mathbb{R}^n$, which restricts to a diffeomorphism between the two interiors and between the two boundaries. We will need the following well known fact (see [Br], Lemma 1).

Lemma 8.2: Let $f : X \to Y$ be a local homeomorphism of topological spaces, W a connected Hausdorff topological space, $h : W \to Y$ a continuous map, x_0 a point of X, and w_0 a point of W satisfying $h(w_0) = f(x_0)$. Then there exists at most one continuous map $\widetilde{h} : W \to X$, satisfying $\widetilde{h}(w_0) = x_0$, and $f \circ \widetilde{h} = h$.

Proof of Proposition 8.1: The statement is local, so we may assume that $Y = \mathbb{R}^n$. Let x be a point of X and set $y := \psi(x)$.

Definition 8.3: An open subset $U \subset \mathbb{R}^n$ is said to be *x-star-shaped*, if it satisfies the following conditions.

1. y belongs to U.
2. For every point $u \in U$, the line segment from y to u is contained in U.
3. There exists a continuous map $\gamma : U \to X$, satisfying $\gamma(y) = x$, and $\psi \circ \gamma : U \to \mathbb{R}^n$ is the inclusion.

Claim 8.4:

1. Let $\{U_i\}_{i \in I}$ be a finite collection of x-star-shaped open subsets of $\mathbb{R}^n$. Then their intersection $\cap_{i \in I} U_i$ is x-star-shaped.
2. Let $\{U_i\}_{i \in I}$ be an arbitrary collection of x-star-shaped open subsets of $\mathbb{R}^n$. Then their union $U := \cup_{i \in I} U_i$ is x-star-shaped.
3. Let $U \subset \mathbb{R}^n$ be an x-star-shaped open subset, $W \subset \mathbb{R}^n$ a connected open subset satisfying the following conditions. a) $W \cap U$ is connected. b) For every point $t \in W \cup U$, the line segment from t to y is contained in $W \cup U$. c) There exists a continuous map $\eta : W \to X$, such that $\psi \circ \eta : W \to \mathbb{R}^n$ is the inclusion. d) There exists a point $t \in W \cap U$, such that $\eta(t) = \gamma(t)$, where $\gamma : U \to X$ is the lift of the inclusion satisfying $\gamma(y) = x$. Then $W \cup U$ is x-star-shaped.

Proof: Part 1 is clear. Proof of part 2: Let $\gamma_i : U_i \to X$ be the unique lift of the inclusion, satisfying $\gamma_i(y) = x$. Define $\gamma : U \to X$ by $\gamma(t) = \gamma_i(t)$, if t belongs to U_i. It sufficed to prove that γ is well defined. If t belongs to $U_i \cap U_j$, then $U_i \cap U_j$ is connected, being x-star-shaped, and $\gamma_i(t) = \gamma_j(t)$, by Lemma 8.2.

The proof of part 3 is similar to that of part 2. ■

Given a positive real number ε, set $B_\varepsilon(y) := \{y' \in \mathbb{R}^n \ : \ d(y,y') < \varepsilon\}$, where $d(y',y)$ is the Eucleadian distance from y' to y. Let $\overline{B}_\varepsilon(y)$ be the closure of $B_\varepsilon(y)$.

Claim 8.5: Assume that $B_\varepsilon(y)$ is x-star-shaped and let $\gamma : B_\varepsilon(y) \to X$ be the lift of the inclusion satisfying $\gamma(y) = x$, as in Definition 8.3. Then there exists an open connected subset $V \subset X$, such that V contains the closure $\overline{\gamma[B_\varepsilon(y)]}$, $\psi : V \to \psi(V)$ is injective, and $\psi(V)$ is x-star-shaped.

Proof: Let z be a point on the boundary $\partial\gamma[B_\varepsilon(y)]$. Then $\psi(z)$ belongs to the boundary of $B_\varepsilon(y)$. Now $\psi(z)$ has a basis of open neighborhoods W with the property that $U_z := W \cup B_\varepsilon(y)$ is x-star-shaped (use Claim 8.4 part 3). Let $\mathcal{U}_z$ denote the collection of all such U_z. The collection $\{B_\varepsilon(y)\} \cup \left[\cup_{z\in\partial\gamma[B_\varepsilon(y)]}\mathcal{U}_z\right]$ is thus a collection of x-star-shaped open subsets. Their union U is x-star-shaped, by Claim 8.4, so the inclusion $U \subset \mathbb{R}^n$ admits a lift $\gamma : U \to X$ satisfying $\gamma(y) = x$. Set $V := \gamma[U]$. Then V is open, since γ is a local-homeomorphism, and V contains the closure of $\gamma[B_\varepsilon(y)]$, by construction. ■

Let $D_x \subset \mathbb{R}^{>0}$ be the set of all $\varepsilon \in \mathbb{R}^{>0}$, such that there exists a continuous map $\gamma : \overline{B}_\varepsilon(y) \to X$, satisfying $\gamma(y) = x$, and such that $\psi \circ \gamma : \overline{B}_\varepsilon(y) \to \mathbb{R}^n$ is the inclusion. Clearly, D_x is a non-empty connected interval having 0 as its left boundary point. We need to show that $D_x = \mathbb{R}^{>0}$. It suffices to show that D_x is both open and closed.

Claim 8.6: D_x is open.

Proof: Let ε be a point of D_x. The image $\gamma[\overline{B}_\varepsilon(y)]$ is compact and X is Hausdorff. Hence, $\gamma[\overline{B}_\varepsilon(y)]$ is closed and is thus equal to the closure of $\gamma[B_\varepsilon(y)]$. Then $\psi\left(\overline{\gamma[B_\varepsilon(y)]}\right) = \overline{B}_\varepsilon(y)$. Hence, there exists an open x-star-shaped subset $U \subset \mathbb{R}^n$, containing $\overline{B}_\varepsilon(y)$, by Claim 8.5. Compactness of $\overline{B}_\varepsilon(y)$ implies that U contains $\overline{B}_{\varepsilon_1}(y)$, for some $\varepsilon_1 > \varepsilon$. Now ε_1 belongs to D_x, since U is x-star-shaped. Hence, D_x is open. ■

Set $s := \sup(D_x)$. If s is infinite, we are done. Assume that s is finite. $B_s(y)$ is x-star-shaped, by Claim 8.4. Let $\gamma : B_s(y) \to X$ be the lift of the inclusion satisfying $\gamma(y) = x$.

Claim 8.7: The closure $C := \overline{\gamma[B_s(y)]}$ is a connected component of the preimage $\psi^{-1}[\overline{B}_s(y)]$. Furthermore, $\psi : C \to \overline{B}_s(y)$ is injective.

Proof: There exists an open subset V of X, containing C, such that $\psi : V \to \psi(V)$ is a homeomorphism, by Claim 8.5. Hence, $V \cap \psi^{-1}[\overline{B}_s(y)] = C$, and C is both open and closed in $\psi^{-1}[\overline{B}_s(y)]$. ■

Up to now we used only the assumption that ψ is a local homeomorphism. We now use the assumption that $\psi : C \to \overline{B}_s(y)$ is surjective, for every connected component of $\psi^{-1}[\overline{B}_s(y)]$, and in particular for $C := \overline{\gamma[B_s(y)]}$. We conclude that s belongs to D_x. A contradiction, since D_x is open. This completes the proof of Proposition 8.1. ■

9 Appendix: Subtwistor metric on the period space

9.1 GHK lines and subtwistor metrics

Let $\mathbb{P}\mathsf{er}$ be a period space of a hyperkähler manifold M, identified with a Grassmannian $SO(b_2-3,3)/SO(2)\times SO(b_2-1,3)$ of oriented, positive 2-planes in $H^2(M,\mathbb{R})$. We shall consider $\mathbb{P}\mathsf{er}$ as a complex manifold, with the complex structure obtained as in (1.3). Fix an auxiliary Euclidean metric g on $H^2(M)$. Given a positive 3-dimensional plane $W\subset H^2(M)$, denote by $S_W\subset \mathbb{P}\mathsf{er}$ the set of all 2-dimensional oriented planes contained in W. Clearly, S_W is a complex curve in $\mathbb{P}\mathsf{er}$. The metric g induces a Fubini-Study metric on S_W.

Consider a sequence $S_1, ..., S_n$ of intersecting hyperkähler lines connecting $x\in\mathbb{P}\mathsf{er}$ to $y\in\mathbb{P}\mathsf{er}$, with $s_i\in S_i\cap S_{i+1}$, $i=1,...,n-1$ the intersection points, and $s_0 := x$, $s_{n+1} := y$. Denote by $l_{S_1,...,S_n}(x,y)$ the sum $\sum_{i=0}^n d_g(s_i,s_{i+1})$, where the distance $d_g(s_i,s_{i+1})$ is computed on the hyperkähler line S_{i+1} using the metric induced by g as above. Let

$$d_{tw}(x,y) := \inf_{S_1,...,S_n} l_{S_1,...,S_n}(x,y)$$

where the infimum is taken over all appropriate sequences of GHK lines, connecting x to y.

The following theorem is stated for periods of hyperkähler manifolds, but in fact it could be stated abstractly for $\mathbb{P}\mathsf{er} = SO(m-3,3)/SO(2)\times SO(m-1,3)$, for any $m>0$. No results of geometry or topology of hyperkähler manifolds are used in its proof.

Theorem 9.1: Let $\mathbb{P}\mathsf{er}$ be a period space of a hyperkähler manifold, and $d_{tw} : \mathbb{P}\mathsf{er}\times\mathbb{P}\mathsf{er} \longrightarrow \mathbb{R}^{\geqslant 0}\cup\infty$ the function defined above. Then

(i) d_{tw} is a metric on $\mathbb{P}\mathsf{er}$

(ii) The metric d_{tw} induces the usual topology on $\mathbb{P}\mathsf{er}$.

The rest of this section is taken by the proof of Theorem 9.1.

Definition 9.2: The metric d_{tw} is called **the subtwistor metric** on the period space, and a piecewise geodesic connecting x to y and going over S_i is called **a subtwistor path**.

Remark 9.3: It is in many ways similar to the sub-Riemannian metrics known in metric geometry (see e.g. [BBI]).

The triangle inequality for d_{tw} is clear from its definition. To prove that d_{tw} is a metric, we need only to show that $d_{tw} < \infty$ and $d_{tw}(x, y) > 0$ for $x \neq y$. The inequality $d_{tw} < \infty$ follows from Proposition 5.8, which claims that all points of $\mathbb{P}\mathsf{er}$ are connected by a finite sequence of GHK lines. The latter condition is clear, because $d_{tw} \geqslant d_g$, where d_g is a geodesic distance function on $\mathbb{P}\mathsf{er}$ associated with the Riemannian metric g.

9.2 Subtwistor metric on a Lie group

Let $(\mathbb{P}\mathsf{er}, d_g)$ be the space $\mathbb{P}\mathsf{er}$ equipped with a Riemannian metric associated with a scalar product g on $V := H^2(M, \mathbb{R})$, and d_{tw} its subtwistor metric. To finish the proof of Theorem 9.1, we have to show that the identity map $(\mathbb{P}\mathsf{er}, d_{tw}) \xrightarrow{\tau} (\mathbb{P}\mathsf{er}, d_g)$ is a homeomorphism.

Notice that τ is continuous, because $d_{tw} \geqslant d_g$. Brouwer's invariance of domain theorem implies that it suffices to show that $(\mathbb{P}\mathsf{er}, d_{tw})$ is homeomorphic to a manifold.

Claim 9.4: (Brouwer's invariance of domain theorem) Let $X \xrightarrow{f} Y$ be a continuous, bijective map of Hausdorff manifolds. Then f is a homeomorphism.

Proof: This is a corollary of L. E. J. Brouwer's Theorem on invariance of domain, proven in Beweis der Invarianz des n-dimensionale Gebiets, Math. Annalen 71 (1911), pages 305-315. See also Terence Tao's blog: http://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/[1] ■

We prove that $(\mathbb{P}\mathsf{er}, d_{tw})$ is a manifold by an application of the Gleason-Palais theorem on transformation groups, obtained in 1950-ies as a byproduct of the solution of Hilbert's 5th problem by Gleason, Montgomery and Zippin. This would require us to switch from $\mathbb{P}\mathsf{er}$ to a Lie group $G \subset SO(H^2(M, \mathbb{R}), q)$ transitively acting on $\mathbb{P}\mathsf{er}$. We introduce a metric d_{tw} on G, in such a way that $(\mathbb{P}\mathsf{er}, d_{tw})$ is obtained as a quotient of (G, d_{tw}), and prove that (G, d_{tw}) is a manifold, using the Gleason-Palais theorem.

Let $V := H^2(M, \mathbb{R})$, considered as a vector space with the scalar product q of signature $(3, n-3)$, and let G be the connected component of identity of $SO(V)$.

Definition 9.5: An **elementary transform** is an element $h \in G$ fixing a codimension 2 subspace $V_1 \subset V$ of signature $(1, n-3)$. **An elementary decomposition** of $h \in G$ is a decomposition $h = h_1 h_2 ... h_n$, where h_i are elementary transforms.

[1] I am grateful to the referee for this observation and the reference.

Remark 9.6: Any element of G admits an elementary decomposition, obviously non-unique. This is proven by the same argument as used to show that $SO(n)$ is generated by rotations fixing a codimension 2 subspace.

For any elementary transform $h \in G$, choose orthogonal coordinates in which h is a turn, with an angle $0 \leqslant \alpha \leqslant \pi$, and denote by $\|h\|$ the number $|\alpha|$. Consider the Lie algebra element $\log h$, corresponding to h, and let $\mathfrak{h} \in T\,\mathbb{P}\mathsf{er}$ be the corresponding tangent vector field. For any $x \in \mathbb{P}\mathsf{er}$, one has $T_x\,\mathbb{P}\mathsf{er} = \operatorname{Hom}(x, x^\perp)$, where $x^\perp$ is an orthogonal complement to x which is considered as a codimension 2 subspace in V. Fix an auxiliary positive definite Euclidean metric g on V. Clearly, at $x \in \mathbb{P}\mathsf{er}$, one has

$$|\mathfrak{h}|_g\Big|_x = \Big|\pi_{x^\perp}\left(\log h\Big|_x\right)\Big|_g \tag{9.1}$$

where $\pi_{x^\perp}$ is an orthogonal (with respect to g) projection to $x^\perp$. Since $x^\perp$ is 2-dimensional, the quantity (9.1) is bounded by $|a_1|+|a_2|$, where a_i are eigenvalues of $\log h$. This gives

$$|\mathfrak{h}|_g\Big|_x \leqslant 2\|h\|. \tag{9.2}$$

Consider the path $\gamma: [0,1] \longrightarrow \mathbb{P}\mathsf{er}$, $\gamma(t) = e^{t\log h}(x)$, connecting x to $y := h(x)$. Then $d_g(x,y) \leqslant \int_0^1 |\mathfrak{h}(e^{t\log h}(x))|dt$. Therefore, (9.2) gives $d_g(x, h(x)) \leqslant 2\|h\|$.

Definition 9.7: Define the **subtwistor norm** on G as $\|h\|_{tw} := \inf(\|h_1\| + \|h_2\| + ... + \|h_n\|)$, where the infinum is taken over all elementary decompositions $h = h_1 h_2 ... h_n$.

Remark 9.8: It is easy to check that this norm satisfies the usual axioms, that is, defines a right-invariant metric on the Lie group, using the formula $d_{tw}(x,y) := \|xy^{-1}\|_{tw}$.

Claim 9.9: Let $x, y \in \mathbb{P}\mathsf{er}$. Then $\mu \leqslant d_{tw}(x,y) \leqslant 2\mu$, where

$$\mu := \inf_{h\in G, h(x)=y} \|h\|_{tw},$$

and infimum is taken over all $h \in G$ such that $h(x) = y$.

Proof: Let $S_1, ..., S_n$ be a sequence of GHK lines connecting x to y, and $x_0, ..., x_n$ the intersection points with $x_0 = x, x_i \in S_i \cap S_{i+1}$ and $x_n = y$. By definition, $d_{tw}(x,y)$ is an infimum of $\sum_i d_g(x_i, x_{i+1})$ for all such sequences. Let $W_i \subset V$ be a 3-dimensional positive subspace corresponding to S_i, and h_i an elementary transform acting trivially on $W_i^\perp$ and mapping x_{i-1} to x_i (such h_i exists, because $SO(3)$ acts transitively on 2-planes).[2]

[2]The corresponding rotation can be chosen in such a way that its angle is equal to the distance between S_{i-1} and S_i in the twistor line; then $\|h_i\| \geqslant d_g(x_{i-1}, x_i)$.

Then $h := h_1 ... h_n$ maps x to y. The number $d_g(x_{i-1}, x_i)$ is equal to $d_{tw}(x_{i-1}, x_i)$, because these two points lie on the same twistor line. Moreover, this number is equal to a length of the smallest circle segment in S_i connecting x_{i-1} to x_i, which is equal to $\|h_i\|$. We obtained

$$d_{tw}(x, y) = \inf_{\{S_i\}} \sum_i d_g(x_{i-1}, x_i) = \sum_i \|h_i\| \geqslant \|h\|_{tw} \geqslant \mu.$$

Using (9.2), we also obtain

$$d_{tw}(x, y) \leqslant \sum_i d_g(x_{i-1}, x_i) \leqslant \sum_i 2\|h_i\| = 2\mu.$$

■

From Claim 9.9, the following observation is apparent.

Corollary 9.10: Let $G \subset SO(V)$ be a connected component, $\mathbb{P}\mathrm{er} = Gr_{++}(V)$ the period space, $x \in \mathbb{P}\mathrm{er}$ a point, and $G \xrightarrow{\tau_x} \mathbb{P}\mathrm{er}$ the map mapping g to $g(x)$. Let $y \in \mathbb{P}\mathrm{er}$, and let A be the distance between $\tau_x^{-1}(x)$ and $\tau_x^{-1}(y)$, computed with respect to the subtwistor norm on G. Then $A \leqslant d_{tw}(x, y) \leqslant 2A$.

Proof: Clearly, A is an infimum of $\|vu^{-1}\|_{tw}$ for all $u, v \in G$ satisfying $u(x) = x, v(x) = y$. In terms of Claim 9.9, this number is equal to μ, and then the inequality $A \leqslant d_{tw}(x, y) \leqslant 2A$ follows. ■

Given a norm (or a right-invariant metric) on a group, one can construct a metric on its right quotients, as follows. One defines the metric on the space of right classes G/G_x, using the Hausdorff distance between the right classes yG_x and zG_x computed with respect to the given metric on G.

Corollary 9.11: Let $G \subset SO(V)$ be a subgroup defined above, $\|\cdot\|$ the subtwistor norm, $\mathbb{P}\mathrm{er} = Gr_{++}(V)$ the corresponding period space, $x, y, z \in \mathbb{P}\mathrm{er}$, and $G \xrightarrow{\tau_x} \mathbb{P}\mathrm{er}$ the map mapping g to $g(x)$. Denote by G_x the stabilizer of x in G, $G_x := \tau_x^{-1}(x)$. We equip G/G_x with a metric, using the subtwistor metric on G as above. Then the natural map $\tau_x : G \longrightarrow \mathbb{P}\mathrm{er}$ induces a homeomorphism from G/G_x to $\mathbb{P}\mathrm{er}$.

Proof: Let $A_{x,y}$ be the Hausdorff distance between $\tau_x^{-1}(y)$ and $\tau_x^{-1}(z)$, computed in the subtwistor metric. To prove Corollary 9.11, it would suffice to show that for some number $\mu(z) > 1$ depending on z, one has

$$\mu(z)^{-1} A_{x,y} \leqslant d_{tw}(y, z) \leqslant \mu(z) A_{x,y} \tag{9.3}$$

When $z = x$, this inequality is implied directly by Corollary 9.10:

$$A_{x,y} \leqslant d_{tw}(y, x) \leqslant 2A_{x,y}.$$

For general z, we use the action of G on itself and $\mathbb{P}\mathsf{er}$.

The group $G := SO(H^2(M,\mathbb{R}),q)$ acts on $\mathbb{P}\mathsf{er}$ transitively, and each $\gamma \in G$ induces a bi-Lipschitz map on $\mathbb{P}\mathsf{er}$, distorting the metric d_g in a way which is bounded by C_γ, where C_γ is a constant depending on the largest eigenvalue of $\gamma(g)g^{-1}$. Indeed, d_g is the metric on the Grassmannian of 2-planes in $H^2(M,\mathbb{R})$ associated with the metric g on $H^2(M,\mathbb{R})$, and γ distorts this metric in a controlled way, with the Lipschitz constant bounded by the eigenvalues of $\gamma(g^{-1})g$.

The same argument shows that γ distorts $(\mathbb{P}\mathsf{er}, d_{tw})$ by the same number:

$$C_\gamma^{-1} d_{tw}(x,y) \leqslant d_{tw}(\gamma(x),\gamma(y)) \leqslant C_\gamma d_{tw}(x,y).$$

Now, let $\gamma \in G$ be an element mapping z to x. Then the right action of γ maps τ_x to τ_z and the pair (y,z) to $(\gamma(y),x)$, giving

$$A_{x,y} \leqslant d_{tw}(\gamma(y),x) \leqslant 2A_{x,y} \tag{9.4}$$

by the above argument. Since the action of γ on $\mathbb{P}\mathsf{er}$ is bi-Lipschitz,

$$C_\gamma^{-1} d_{tw}(y,z) \leqslant d_{tw}(\gamma(y),x) \leqslant C_\gamma d_{tw}(y,z). \tag{9.5}$$

Comparing this inequality with (9.4) we obtain (9.3):

$$\frac{A_{x,y}}{2C_\gamma} \leqslant \frac{A_{x,y}}{C_\gamma} \overset{(9.4)}{\leqslant} \frac{d_{tw}(\gamma(y),x)}{C_\gamma} \overset{(9.5)}{\leqslant} d_{tw}(y,z) \overset{(9.5)}{\leqslant} C_\gamma d_{tw}(\gamma(y),x) \overset{(9.4)}{\leqslant} 2C_\gamma A_{x,y}.$$

■

Corollary 9.12: Let $G \subset SO(H^2(M,\mathbb{R}),q)$ be the connected component of unit in $SO(H^2(M,\mathbb{R}),q)$ acting on the period space $\mathbb{P}\mathsf{er}$, and d_{tw} the metric defined from the subtwistor norm. Then $(\mathbb{P}\mathsf{er}, d_{tw})$ is homeomorphic to the quotient G/G_x, equipped with the quotient topology induced from d_{tw}.

Proof: Follows from Corollary 9.11. ■

To prove that $(\mathbb{P}\mathsf{er}, d_{tw}) = (G, d_{tw})/G_x$ is a manifold, it would suffice to show that G with the metric induced from a subtwistor norm is a manifold. Indeed, in this case, the subtwistor norm induces the standard topology on G by Claim 9.4. Then $(\mathbb{P}\mathsf{er}, d_{tw})$ is a manifold by Corollary 9.12.

Theorem 9.13: Let $G \subset SO(H^2(M,\mathbb{R}),q)$ be the Lie group defined as above, and d_{tw} the metric defined from the subtwistor norm. Then (G, d_{tw}) is a topological manifold.

We prove Theorem 9.13 in Subsection 9.3.

9.3 Gleason-Palais theorem and its applications

Theorem 9.13 follows directly from a theorem of Gleason and Palais about transformation groups ([GP]; for a more recent treatment and reference, see [BZ]).

Definition 9.14: Let M be a topological space. We say that M **has Lebesgue covering dimension** $\leqslant n$ if every open covering of M has a refinement $\{U_i\}$ such that each point of M belongs to at most $n+1$ element of $\{U_i\}$. A **Lebesgue covering dimension** of M (denoted by $\dim M$) is an infimum of all such n.

The following two well-known claims are easy to prove.

Claim 9.15: If M is an n-manifold, $\dim M = n$. ■

Claim 9.16: If $X \subset M$ is a subset of a topological space, with induced topology, one has $\dim X \leqslant \dim M$. ■

The following theorem is a deep and important result, obtained in 1950-ies in the course of studying transformation groups along with the solution of Hilbert's 5-th problem.

Theorem 9.17: (Gleason-Palais) Let G be a topological group, which is locally path connected, and has $\dim K < \infty$ for each compact, metrizable subset $K \subset G$. Then G is homeomorphic to a Lie group.

Proof: [GP, Theorem 7.2]. ■

Now we can finish the proof of Theorem 9.13, and Theorem 9.1. The group (G, d_{tw}) is by construction locally path connected. Moreover, one has $d_{tw} \geqslant d_g$, where d_g is a metric obtained from a positive definite metric on V; this is proven in the same way as the inequality $d_{tw} \geqslant d_g$ for $\mathbb{P}\mathsf{er}$. Therefore, the map $(G, d_{tw}) \longrightarrow (G, d_g)$ is continuous. This implies that any compact $K \subset (G, d_{tw})$ is homeomorphic to its image in (G, d_g), hence it is finitely-dimensional. Applying Gleason-Palais, we obtain that (G, d_{tw}) is a manifold. ■

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References

[BZ] Banakh, T., Zdomskyy, L., *Closed locally path-connected subspaces of finite-dimensional groups are locally compact,* Topology Proceedings 36 (2010), 399-405.

[Bea1] Beauville, A. *Varietes Kähleriennes dont la première classe de Chern est nulle.* J. Diff. Geom. **18**, pp. 755-782 (1983).

[Bea2] A. Beauville et al., *Géométrie des surfaces K3 : modules et périodes,* Papers from the seminar held in Palaiseau, Astérisque 126 (1985), 1-193.

[Bes] Besse, A., *Einstein Manifolds*, Springer-Verlag, New York (1987)

[Bo1] Bogomolov, F. A., *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR-Sb. **22** (1974), 580-583.

[Bo2] Bogomolov, F. A., *Hamiltonian Kähler manifolds*, Sov. Math. Dokl. **19** (1978), 1462–1465.

[Bor] Borcea, C., *Diffeomorphisms of a K3 surface,* Math. Ann. 275 (1986), 1-4.

[BKPS] Borcherds, R., Katzarkov, L., Pantev, T., Shepherd-Barron, N., *Families of K3 surfaces,* J. Algebraic Geom. 7 (1998), no. 1, 183–193, arXiv:alg-geom/9701013.

[Bou] Boucksom, S., *Higher dimensional Zariski decompositions*, Ann. Sci. Ecole Norm. Sup. (4) 37 (2004), no. 1, 45–76, arXiv:math/0204336

[Br] Browder, Felix E., *Covering spaces, fibre spaces, and local homeomorphisms*, Duke Math. J. Volume 21, Numb er 2 (1954), 329-336.

[BR] Burns, Dan; Rapoport, Michael, *On the Torelli problem for kählerian K-3 surfaces*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série 8 (2), 235-273

[BBI] Burago, D., Burago, Yu., Ivanov, S., *A course in metric geometry*, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001.

[C] F. Catanese, *A Superficial Working Guide to Deformations and Moduli*, arXiv:1106.1368, 56 pages.

[De] Debarre, O., *Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles,* C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 14, 681–684.

[DV] Debarre, O., Voisin, C., *Hyper-Kähler fourfolds and Grassmann geometry*, 28 pages, arXiv:0904.3974.

[DP] Demailly, J.-P., Paun, M., *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Annals of Mathematics, **159** (2004), 1247-1274, math.AG/0105176.

[Don] Donaldson, S. K., *Polynomial invariants for smooth four-manifolds,* Topology 29 (1990), 257-315.

[Dou] Douady, A., *Le probleme des modules pour les varietes analytiques complexes,* Seminaire Bourbaki, 1964/1965, No 277.

[F] Fujiki, A. *On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold*, Adv. Stud. Pure Math. 10 (1987), 105-165.

[GP] Gleason, A., Palais, R., *On a class of transformation groups,* Amer. J. Math. 79 (1957), 631-648.

[GHS1] Gritsenko, V. A.; Hulek, K.; Sankaran, G. K., *The Kodaira dimension of the moduli of $K3$ surfaces,* Invent. Math. 169 (2007), no. 3, 519–567, arXiv:math/0607339.

[GHS2] V. Gritsenko, K. Hulek, G. K. Sankaran *Moduli spaces of irreducible symplectic manifolds*, 40 pages, arXiv:0802.2078.

[Ham] Richard S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65-222

[H1] Huybrechts, D., *Compact hyperkähler manifolds: Basic results*, Invent. Math. 135 (1999), 63-113, alg-geom/9705025

[H2] Huybrechts, D., *Erratum to the paper: Compact hyperkähler manifolds: basic results*, Invent. math. 152 (2003), 209-212, math.AG/0106014.

[H3] Huybrechts, D. *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. 326 (2003), no. 3, 499–513, arXiv:math/9909109.

[H4] Huybrechts, D., *Compact hyperkähler manifolds, Calabi-Yau manifolds and related geometries,* Universitext, Springer-Verlag, Berlin, 2003, Lectures from the Summer School held in Nordfjordeid, June 2001, pp. 161-225.

[H5] Huybrechts, D., *Finiteness results for hyperkähler manifolds*, J. Reine Angew. Math. 558 (2003), 15–22, arXiv:math/0109024.

[H6] Daniel Huybrechts, *A global Torelli theorem for hyperkähler manifolds (after Verbitsky)*, Séminaire Bourbaki 63ème année, 2010-2011, no 1040, arXiv:1106.5573.

[JT] Jorgenson, J., Todorov, A., *Enriques surfaces, analytic discriminants, and Borcherds's Φ function,* Comm. Math. Phys. 191 (1998), no. 2, 249–264.

[KV] Kaledin, D., Verbitsky, M., *Partial resolutions of Hilbert type, Dynkin diagrams, and generalized Kummer varieties,* 33 pages, arXiv:math/9812078

[KS1] K. Kodaira and D. C. Spencer *On Deformations of Complex Analytic Structures, I*, Annals of Math., Vol. 67, No. 2 (1958), pp. 328-401.

[KS3] K. Kodaira and D. C. Spencer *On Deformations of Complex Analytic Structures, III, Stability Theorems for Complex Structures*, Annals of Math., Vol. 71, No. 1 (1960), pp. 43-76.

[KM] Kollár, J., Matsusaka, T. *Riemann-Roch type inequalities*, Am. J. Math. 105 (1983), 229-252.

[Ku] Kulikov, Vik. S., *Degenerations of K3 surfaces and Enriques surfaces,* Mathematics of the USSR-Izvestiya, 11:5 (1977), 957-989.

[Ku1] Kuranishi, M. *New proof for the existence of locally complete families of complex structures*, 1965 Proc. Conf. Complex Analysis (Minneapolis, 1964) pp. 142-154 Springer, Berlin.

[Ku2] Kuranishi, Masatake *A note on families of complex structures* 1969 Global Analysis (Papers in Honor of K. Kodaira) pp. 309-313 Univ. Tokyo Press, Tokyo.

[LTYZ] Liu, K., Todorov, A., Yau, S.-T., Zuo, K., *Shafarevich's Conjecture for CY Manifolds I*, Q. J. Pure Appl. Math. 1 (2005), no. 1, 28–67, arXiv:math/0308209.

[M1] Markman, E., *On the monodromy of moduli spaces of sheaves on $K3$ surfaces*, J. Algebraic Geom. 17 (2008), no. 1, 29–99, arXiv:math/0305042.

[M2] Markman, E. *Integral constraints on the monodromy group of the hyperkahler resolution of a symmetric product of a K3 surface,* International Journal of Mathematics Vol. 21, No. 2 (2010) 169-223, arXiv:math/0601304.

[M3] Markman, E. *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Proceedings of the conference "Complex and Differential Geometry", Springer Proceedings in Mathematics, 2011, Volume 8, 257–322, arXiv:math/0601304.

[Na] Namikawa, Y., *Counter-example to global Torelli problem for irreducible symplectic manifolds*, Math. Ann. 324 (2002), no. 4, 841–845, arXiv:math/0110114.

[Ni] Nieper, M., *Hirzebruch-Riemann-Roch formula on irreducible symplectic Kähler manifolds*, J. Algebraic Geom. 12 (2003), no. 4, 715–739.

[O] O'Grady, Kieran G., *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. 12 (2003), no. 3, 435–505, arXiv:math/0010187, 68 pages.

[SGA1] Grothendieck, A., *Revêments étales et groupe fondamental,* Lecture Notes in Mathematics, no. 224, Springer-Verlag, Berlin-New York, 1971.

[Si] Siu, Y.-T., *Every K3 surface is Kähler*, Invent. Math. 73 (1983), 139-150.

[Su] Sullivan, D., *Infinitesimal computations in topology*, Publications Mathématiques de l'IHÉS, 47 (1977), p. 269-331

[T1] Todorov, A., *Ray Singer Analytic Torsion of CY Manifolds II*, arXiv:math/0004046, 18 pages.

[T2] Todorov, A., *Weil-Petersson volumes of the moduli spaces of CY manifolds,* Comm. Anal. Geom. 15 (2007), no. 2, 407–434.

[V1] Verbitsky M., *Hyperkähler embeddings and holomorphic symplectic geometry II,* GAFA **5** no. 1 (1995), 92-104, alg-geom/9403006.

[V2] Verbitsky, M., *Cohomology of compact hyperkähler manifolds,* alg-geom electronic preprint 9501001, 89 pages, LaTeX.

[V3] Verbitsky, M., *Cohomology of compact hyperkähler manifolds and its applications,* alg-geom electronic preprint 9511009, 12 pages, LaTeX, also published in: GAFA vol. 6 (4) pp. 601-612 (1996).

[V4] Verbitsky, M., *Mirror Symmetry for hyperkähler manifolds,* Mirror symmetry, III (Montreal, PQ, 1995), 115–156, AMS/IP Stud. Adv. Math., 10, Amer. Math. Soc., Providence, RI, 1999, alg-geom/9512195.

[V5] Verbitsky, M., *Trianalytic subvarieties of the Hilbert scheme of points on a K3 surface*, Geom. Funct. Anal. 8 (1998), no. 4, 732–782, arXiv:alg-geom/9705004.

[V6] Verbitsky, M., *Parabolic nef currents on hyperkähler manifolds*, 19 pages, arXiv:0907.4217.

[Vi] Viehweg, E., *Quasi-projective Moduli for Polarized Manifolds*, Springer-Verlag, Berlin, Heidelberg, New York, 1995, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 30, also available at `http://www.uni-due.de/ mat903/books.html`.

[VGO] Vinberg, E. B., Gorbatsevich, V. V., Shvartsman, O. V., *Discrete Subgroups of Lie Groups*, in "Lie Groups and Lie Algebras II", Springer-Verlag, 2000.

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