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MAPPING PROPERTIES OF $\log g'(z)$

BY

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1. Introduction. In recent investigations of Duren and McLaughlin (see [2] and [3]) on the Marx conjecture for starlike functions, the univalence of $\log k'(z)$ and $\sqrt{k'(z)}$, where $k(z) = z/(1-z)^2$, in the open unit disk $D = \{z: |z| < 1\}$ plays an important role. In this paper* we investigate the problem of determining the mapping properties of $\log k_c'(z)$ for the generalized Koebe function

$$k_c(z) = \frac{1}{2c}\left\{\left(\frac{1+z}{1-z}\right)^c - 1\right\} \quad (c \text{ complex})$$

and, more generally, the question of univalence and close-to-convexity of $\log g'(z)$ when $g(z)$ ranges over various classes of locally univalent functions on D.

One of our results shows that $\log k_c'(z)$ for c real, $|c| \geqslant 1$, maps D univalently onto a starlike region. This result depends on an analysis of the boundary behavior of $\log k_c'(z)$ (i.e. when $|z| = 1$) and of a boundary characterization of starlike mappings that we develop in Section 2. The boundary characterizations of starlike, convex and bounded boundary rotation mappings in Section 2 are of interest in themselves apart from our application in the proof of Theorem 3.1.

2. Boundary characterizations of starlikeness and bounded boundary rotation. A function $f(z) = z + \ldots$ analytic in D is *starlike* (with respect to the origin) in D if $\operatorname{Re}[zf'(z)/f(z)] > 0$ for all z in D. For a specific function f it can be quite difficult to verify that the condition $\operatorname{Re}[zf'(z)/f(z)] > 0$ is satisfied throughout D, and the verification can involve numerous special cases with a variety of tedious calculations for various values of r and θ, where $z = re^{i\theta}$. In Theorem 2.1 and Corollary 2.1 we cast the

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characterization of starlikeness in a form that often simplifies matters by permitting one to perform simpler calculations with $\mathrm{Re}[zf'(z)/f(z)]$ for points $z = e^{i\theta}$ on the boundary. The necessity of the conditions characterizing starlikeness in Theorem 2.1 is known, but the proof of their sufficiency seems to be new.

THEOREM 2.1. *Let $f(z) = z + \dots$ be analytic in D. Then $f(z)$ is starlike in D if and only if the following three conditions are satisfied:*

(i) *$f(z)/z \neq 0$ for all $z \in D$.*

(ii) *The harmonic function $\arg[f(z)/z]$ is bounded in D.*

(iii) *$\lim\limits_{r\to 1} \arg[f(re^{i\theta})/re^{i\theta}] \equiv V(\theta) - \theta$ exists for all $\theta \in [0, 2\pi]$, $V(2\pi) - V(0) = 2\pi$, and $V(\theta)$ is a monotone non-decreasing function.*

Proof. The necessity of conditions (i)-(iii) is known (see [6], [9], Lemma 1, and [11], p. 181). To prove their sufficiency, let $V(\theta)$ be the monotone function in (iii) and define the starlike function g by

$$g(z) = z\exp\left\{-\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})\, dV(t)\right\}.$$

Then, for all $\theta \in [0, 2\pi]$,

$$\lim_{r\to 1} \arg[g(re^{i\theta})/re^{i\theta}] = V(\theta) - \theta + c,$$

where c is a constant, with the possible exception of the countable set of points of discontinuity of $V(\theta)$ (see [9], p. 210). The function

$$h(z) = \frac{f(z)/z}{g(z)/z}$$

is analytic, does not vanish in D, and satisfies the equalities

$$\arg h(z) = \arg[f(z)/z] - \arg[g(z)/z]$$

and

$$\lim_{r\to 1} \arg h(re^{i\theta}) = \big(V(\theta) - \theta\big) - \big(V(\theta) - \theta + c\big) = -c$$

for all but a countable number of points $\theta \in (0, 2\pi]$. Furthermore, $\arg h(z)$ is harmonic and bounded in D, since $|\arg[g(z)/z]| < \pi/2$ (see [11], p. 181) and condition (ii) is assumed. Thus $\arg h(z)$ is constant in D since it is the Poisson integral of its radial limit function. It follows that $h(z)$ is constant in D and, therefore, $f(z)$ is a (real) constant multiple of the starlike function $g(z)$. The proof of Theorem 2.1 is complete.

A careful analysis of the radial limits in (iii) shows that $V(\theta)$ is necessarily equal to $\big(V(\theta+) + V(\theta-)\big)/2$ at any point of discontinuity (see [9], p. 210). Our next assertion gives a more flexible boundary criterion for starlikeness.

COROLLARY 2.1. *Let* $f(z) = z + \ldots$ *be analytic in* D, *have no zeros in* $0 < |z| < 1$, *and let* $\arg[f(z)/z]$ *be a bounded harmonic function in* D. *Suppose that*

(1) $\lim_{r \to 1} \arg[f(re^{i\theta})/re^{i\theta}] \equiv V(\theta) - \theta$ *exists for all* $\theta \in [0, 2\pi]$ *with* $V(2\pi) - V(0) = 2\pi$;

(2) *there is a finite set of points* $T = \{t_j: j = 1, \ldots, n\}$, $0 \leqslant t_1 < \ldots < t_n < 2\pi$, *such that* $V(t_j -) \leqslant V(t_j) \leqslant V(t_j +)$, $j = 1, \ldots, n$, *and, for all* $\theta \notin T$, $V(\theta) = \arg f(e^{i\theta})$ *and* $V(\theta)$ *is continuously differentiable.*

Then $f(z)$ *is starlike in* D *if*

$$\mathrm{Re}[e^{i\theta} f'(e^{i\theta})/f(e^{i\theta})] \geqslant 0 \quad \textit{for all } \theta \notin T.$$

Proof. By Theorem 2.1 and hypothesis (2), it is sufficient to show that $V(\theta)$ is monotone non-decreasing in each of the intervals $t_j < \theta < t_{j+1}$. This is immediate since

$$V'(\theta) = \frac{\partial}{\partial \theta} \arg f(e^{i\theta}) = \mathrm{Re}[e^{i\theta} f'(e^{i\theta})/f(e^{i\theta})] \geqslant 0 \quad \text{for } \theta \in (t_j, t_{j+1}).$$

Remark. If $f(z)$ is close-to-convex, then $|\arg[f(z)/z]|$ must be bounded in D (see [11], p. 181). Frequently it is easy to verify that a particular function is close-to-convex. We shall use this fact in Section 3.

COROLLARY 2.2. *Let* $f(z) = z + \ldots$ *be analytic with* $f'(z) \neq 0$ *and let* $\arg f'(z)$ *be bounded in* D. *Then* $f(z)$ *is convex univalent in* D *if and only if*

$$\lim_{r \to 1} \arg f'(re^{i\theta}) \equiv V(\theta) - \theta$$

exists for all $\theta \in [0, 2\pi]$, $V(2\pi) - V(0) = 2\pi$ *and* $V(\theta)$ *is a monotone non-decreasing function.*

Proof. This follows from Theorem 2.1 and the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike.

Similarly, a characterization of convexity corresponding to Corollary 2.1 can be obtained by substituting $f'(z)$ for $f(z)/z$ in that result.

In the next theorem we consider the class V_k of analytic functions with boundary rotation no greater than $k\pi$ (cf. [7]).

THEOREM 2.2. *Let* $f(z) = z + \ldots$ *be analytic with* $f'(z) \neq 0$ *and let* $\sup |\arg f'(z)| < \infty$ *in* D. *For* $f(z)$ *to be in* V_k *it is necessary and sufficient that*

$$\lim_{r \to 1} \arg f'(re^{i\theta}) \equiv V(\theta) - \theta$$

exists for all $\theta \in [0, 2\pi]$, $V(2\pi) - V(0) = 2\pi$ *and that* $V(\theta)$ *be a function of bounded variation with*

$$\int_0^{2\pi} |dV(\theta)| \leqslant k\pi.$$

Proof. This theorem follows directly from Theorem 2.1 and Brannan's observation [1] that $f \in V_k$ if and only if there exist univalent starlike functions $g(z)$ and $h(z)$ such that

$$f'(z) = \big(g(z)/z\big)^{(k+2)/4}\big(h(z)/z\big)^{(2-k)/4}, \qquad z \in D.$$

Remark. A more restrictive and geometrically less natural condition has been given by Flett [4] who considered only univalent $f(z)$ which satisfy

$$\sup_{r<1} \int_0^{2\pi} \log^+ |f'(re^{i\theta})|\, d\theta < \infty.$$

3. Mapping properties of $\log k_c'(z)$. The generalized Koebe function $k_c(z)$ is of the form

$$k_c(z) = \int_0^z \frac{(1+w)^{c-1}}{(1-w)^{c+1}}\, dw \qquad (c \text{ complex}).$$

Clearly,

$$k_0(z) = \frac{1}{2}\log\frac{1+z}{1-z} \quad \text{and} \quad k_c(z) = \frac{1}{2c}\left\{\left(\frac{1+z}{1-z}\right)^c - 1\right\}, \qquad c \neq 0.$$

In particular, for $c = 1$ and $c = 2$ we have the familiar $k_1(z) = z/(1-z)$ and $k_2(z) = z/(1-z)^2$.

THEOREM 3.1. *The function* $f_c(z) = (1/2c)\log k_c'(z)$, *where* $c \neq 0$, *is*

(1) *not locally univalent if* $|c| < 1$;

(2) *univalent and close-to-convex if* $|c| \geqslant 1$;

(3) *convex in the direction of the imaginary axis if* $|c| \geqslant 1$;

(4) *convex if and only if* $c = \pm 1$;

(5) *starlike if c is real and* $|c| \geqslant 1$.

Proof. (1) We have

$$f_c(z) = \frac{c-1}{2c}\log(1+z) - \frac{c+1}{2c}\log(1-z), \tag{3.1}$$

and $f_c'(z) = (1+z/c)/(1-z^2)$. Clearly, $f_c(z)$ is not locally univalent in D if $|c| < 1$, since $f_c'(-c) = 0$.

(2) If $|c| \geqslant 1$, then $f_c(z)$ is close-to-convex with respect to the univalent convex function $g(z) = (1/2)\log\big((1+z)/(1-z)\big)$, since

$$\operatorname{Re}[f_c'(z)/g'(z)] = \operatorname{Re}(1+z/c) > 0, \qquad z \in D.$$

(3) If $|c| \geqslant 1$, then

$$\operatorname{Re}[(1-z^2)f_c'(z)] = \operatorname{Re}(1+z/c) > 0, \qquad z \in D,$$

and $f_c(z)$ is convex in the direction of the imaginary axis (see [5]).

(4) If we let $Q_c(z) = 1+zf_c''(z)/f_c'(z)$, then

$$Q_c(z) = z/(c+z)+(1+z^2)/(1-z^2) \tag{3.2}$$

$$\operatorname{Re} Q_c(re^{i\theta}) = \frac{r^2+|c|r\cos(\theta-t)}{|c+re^{i\theta}|^2}+\frac{1-r^4}{|1-r^2e^{i2\theta}|^2}, \quad t = \arg c. \tag{3.3}$$

If $c = \pm 1$ and $z \in D$, then $Q_c(z) = 1/(1\mp z)$, $\operatorname{Re} Q_c(z) > 1/2$, and $f_c(z)$ is convex (of order 1/2). For $|c| > 1$ and $\theta \neq 0, \pi$, equation (3.3) yields that

$$\operatorname{Re} Q_c(e^{i\theta}) = \big(1+|c|\cos(\theta-t)\big)|c+e^{i\theta}|^{-2}$$

which is negative for values of θ near $t+\pi$. Hence, by continuity, $\operatorname{Re} Q_c(z) < 0$ for some $z \in D$. Finally, for $c = e^{it}$, $t \neq 0, \pi$, we have

$$\operatorname{Re} Q_c(-re^{it}) = \frac{-r}{1-r}+\frac{1-r^4}{|1-r^2e^{i2t}|^2}$$

which is negative for r near 1 ($2t \neq 0 \pmod{2\pi}$).

(5) Since $f_{-c}(z) = -f_c(-z)$ by (3.1), it will suffice to show that $f_c(z)$ is starlike for $c > 1$. We show that the conditions of Corollary 2.1 are satisfied with the two-point exceptional set $T = \{0, \pi\}$. The harmonic function $\arg[f_c(z)/z]$ is bounded in D since $f_c(z)$ is close-to-convex.

Choosing the branch of $\arg[f_c(z)/z]$ that vanishes at $z = 0$, letting

$$V(\theta) = \theta+\lim_{r\to 1}\arg[f_c(re^{i\theta})/re^{i\theta}]$$

and using (3.1), we see that

$$V(\theta) = \begin{cases} 0, & \theta = 0, \\ \pi, & \theta = \pi, \\ \arg\left[\dfrac{c-1}{2c}\log(1+e^{i\theta})-\dfrac{c+1}{2c}\log(1-e^{i\theta})\right], & \theta \neq 0, \pi. \end{cases}$$

Following Duren and McLaughlin [3], we introduce the polar representations

$$1-e^{i\theta} = \sqrt{2(1-\cos\theta)}\, e^{i(\theta-\pi)/2}, \quad 0 < \theta < 2\pi,$$

$$1+e^{i\theta} = \begin{cases} \sqrt{2(1+\cos\theta)}\, e^{i\theta/2}, & 0 \leqslant \theta < \pi, \\ \sqrt{2(1+\cos\theta)}\, e^{i(\theta-2\pi)/2}, & \pi < \theta \leqslant 2\pi. \end{cases}$$

Then

$$f_c(e^{i\theta}) = (1/2c)[(c-1)\log(1+e^{i\theta})-(c+1)\log(1-e^{i\theta})] = u(\theta)+iv(\theta),$$

where

$$u(\theta) = (1/4c)[-\log 4 + (c-1)\log(1+\cos\theta) - (c+1)\log(1-\cos\theta)],$$
$$0 < \theta < 2\pi, \ \theta \neq \pi,$$

and

$$v(\theta) = \begin{cases} (1/4c)[(c+1)\pi - 2\theta], & 0 < \theta < \pi, \\ (1/4c)[(3-c)\pi - 2\theta], & \pi < \theta < 2\pi. \end{cases}$$

It is easy to check that $V(\theta) = \arg[u(\theta) + iv(\theta)]$ is continuous in the interval $0 < \theta < 2\pi$, $V(0+) = 0 = V(0) = V(2\pi-) - 2\pi$, and $V(\theta)$ is continuously differentiable in each of the intervals $0 < \theta < \pi$ and $\pi < \theta < 2\pi$.

We complete the proof by showing that $V'(\theta) > 0$ in the interval $0 < \theta < \pi$. This is sufficient because of the symmetry $f_c(e^{-i\theta}) = \overline{f_c(e^{i\theta})}$. Clearly, $v(\theta)$ is a strictly decreasing linear function in $0 < \theta < \pi$, and $u(\theta)$ is strictly decreasing in $0 < \theta < \pi$. Moreover,

$$u'(\theta) = -(c + \cos\theta)/(2c\sin\theta) < 0, \quad 0 < \theta < \pi.$$

We choose the branch of the inverse cotangent with values in $(0, \pi)$ and write $V(\theta) = \operatorname{arccot}[u(\theta)/v(\theta)]$. Then

$$V'(\theta) = \big(u(\theta)v'(\theta) - v(\theta)u'(\theta)\big)/v^2(\theta)$$

is positive on $\pi/2 \leqslant \theta < \pi$, since $u(\theta)$ and $v(\theta)$ are decreasing and $v(\theta) > 0$ on $0 < \theta < \pi$, and $u(\theta) < 0$ on $\pi/2 \leqslant \theta < \pi$. Furthermore, $V'(\theta)$ is positive on $0 < \theta \leqslant \pi/2$, since $V'(\pi/2) > 0$, and

$$\frac{d}{d\theta}\big(u(\theta)v'(\theta) - v(\theta)u'(\theta)\big) = -v(\theta)u''(\theta)$$
$$= -v(\theta)(1 + c\cos\theta)/2c\sin^2\theta < 0, \quad 0 < \theta \leqslant \pi/2.$$

(Note that $v''(\theta) = 0$.) This completes the proof of the theorem.

Our proof of (5) is a generalization of the method used by Duren and McLaughlin for the case $c = 2$ (see [3], p. 272). Their proof is incomplete, since their observation that $u(\theta)$ and $v(\theta)$ both decrease in $0 < \theta < \pi$ is not enough to establish the starlikeness of the curve $\big(u(\theta), v(\theta)\big)$ for $0 < \theta < \pi$ in the first quadrant ($u > 0, v > 0$) of the plane (u, v). Our proof fills this gap and it is valid for any $c \geqslant 1$ or $c \leqslant -1$. We are indebted to P. L. Duren for a comment (private communication) that motivated us to simplify to the present form an earlier proof of ours.

The polar form $f_c(e^{i\theta}) = u(\theta) + iv(\theta)$ makes it easy to see that the starlike region $f_c(D)$ is an infinite symmetric horizontal striplike region lying in a horizontal strip of width $\pi(c+1)/(2c) = v(0+) - v(2\pi-)$. An illustration for $c = 2$ appears in [3], p. 273.

COROLLARY 3.1. *If c is real and $|c| \geqslant 1$, then $(k_c'(z))^\beta$ is univalent in D for all real β satisfying $0 < |\beta| \leqslant 2/(1+|c|)$.*

Proof. Because of symmetry it is enough to consider the case $c \geqslant 1$. The preceding remarks show that

$$\log(k_c'(z))^\beta = \beta \log k_c'(z) = 2c\beta f_c(z)$$

maps D onto a region lying in the interior of a horizontal strip of width $\beta\pi(c+1) \leqslant 2\pi$ when $|\beta| \leqslant 2/(1+c)$. The exponential function is univalent in any open horizontal strip of width no greater than 2π. Hence

$$(k_c'(z))^\beta = \exp\{\beta \log k_c'(z)\}$$

is univalent in D if $0 < |\beta| \leqslant 2/(1+|c|)$.

This corollary extends and generalizes the Duren and McLaughlin result that $\sqrt{k_2'(z)}$ is univalent in D (see [3], p. 269). In particular, for $c = 2$, we see that $(k_2'(z))^\beta$ is univalent in D for all real β satisfying $0 < |\beta| \leqslant 2/3$.

In our proof of Theorem 3.1 (5) the positive nature of $u''(\theta)$ on an interval (where $\cos\theta > -1/c$) and the linearity of $v(\theta)$ show that the corresponding portions of the curve $(u(\theta), v(\theta))$ are convex as well as starlike. In connection with his work on the Marx conjecture [2], Duren determined the radius of convexity of $f_2(z)$. We can show that the radius of convexity of $f_c(z)$ for c real, $|c| \geqslant 1$, is $R_c = \sqrt{s_c}$, where s_c is the unique root in $0 < s \leqslant 1$ of the polynomial equation

$$c^2 s^5 + (32 - 25c^2)s^4 + (27c^4 - 17c^2)s^3 + (9c^2 - 27c^4)s^2 - 27c^4 s + 27c^4 = 0.$$

We omit the proof which is elementary but rather tedious in some of its details. It is interesting that $R_c = 1$ for $c = \pm 1$, but if $|c| \to \infty$, then $R_c \to 1$, rather than tending to a minimum value. (Of course, $R_c \geqslant 2 - \sqrt{3}$, since f_c is univalent.)

This limiting behavior of R_c (c real, $|c| \geqslant 1$) can be explained if we develop one additional geometric property of the mapping $f_c(z)$. Since

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{1 + z f_c''(z)}{f_c'(z)} \right| d\theta \leqslant \int_0^{2\pi} \operatorname{Re}\left(\frac{1+z^2}{1-z^2}\right) d\theta + \int_0^{2\pi} \left| \operatorname{Re} \frac{z}{c+z} \right| d\theta \leqslant 2\pi + \frac{2\pi}{|c|+1},$$

we see that if $|c| \to \infty$, then $f_c(z)$ is in V_k with k as close to 2 as desired. The radius of convexity for any function in V_k is at least $(k - (k^2 - 4))/2$. Thus R_c must approach 1 as $|c| \to \infty$. Indeed, this holds true for complex as well as real c.

4. Univalence of $\log g'(z)$ for locally univalent $g(z)$. A family of functions $g(z) = z + \ldots$, analytic and locally univalent in D, is said to be *linear invariant* if, for every Möbius transformation $\varphi(z)$ of D onto D, the function

$$\Lambda_\varphi[g(z)] = \frac{g(\varphi(z)) - g(\varphi(0))}{\varphi'(0)g'(\varphi(0))} = z + \ldots$$

is again a member of the family (cf. [10]). If M is a linear invariant family, the *order* of M is defined to be

$$a = \sup\{|f''(0)/2| : f \in M\}.$$

It is always the case $a \geqslant 1$ (see [10]). Following Pommerenke [10], we let $\mathscr{U}_a$ denote the union of all linear invariant families of order at most a. The family $\mathscr{U}_1$ is precisely the class of all normalized univalent convex mappings of D. The family of normalized univalent functions in D is a proper subset of $\mathscr{U}_2$. Pommerenke has shown ([10], Satz 2.5) that the radius of convexity of $\mathscr{U}_a$ is given by the formula

$$R_a = a - \sqrt{a^2-1} = 1/\left(a + \sqrt{a^2-1}\right). \tag{4.1}$$

The mapping properties of $\log k_c'(z)$ established in Section 3 suggest the general problem of determining mapping properties of $f(z) = \log g'(z)$, where

$$g(z) = z + bz^2 + \ldots \tag{4.2}$$

runs over a whole family of locally univalent functions. Clearly, the univalence and local univalence of $f(z)$ are limited by the location of the zeros of $g''(z)$, since $f'(z) = g''(z)/g'(z)$. We shall determine the radii of univalence and close-to-convexity of $\log g'(z)$ when $g \in \mathscr{U}_1$ and lower bounds for these quantities when $g \in \mathscr{U}_a$. The determining factors in this work are the modulus of $b = g''(0)/2$ in (4.2) and the radius of convexity in formula (4.1).

THEOREM 4.1. *Let $g(z) = z + bz^2 + \ldots$ be univalent and convex in D (i.e. $g \in \mathscr{U}_1$). Then $f(z) = \log g'(z)$ is univalent and close-to-convex in the disk $|z| < |b|$. Furthermore, $g'(z)$ is univalent and close-to-convex in $|z| < |b|$. The function $\log k_c'(z)$ with $c = |b|$ shows that these results are sharp*

Proof. Our proof consists in showing that

$$\operatorname{Re} f'(z) = \operatorname{Re}[g''(z)/g'(z)] > 0$$

in the disk $|z| < |b|$. This will imply that f is close-to-convex (and hence univalent) with respect to the identity $z \to z$, and that $g'(z)$ is close-to--convex with respect to the convex function $g(z)$ in $|z| < |b|$.

Clearly, $f(z)$ is not univalent in any neighborhood of 0 if $b = g''(0)/2 = f'(0)/2 = 0$. It is sufficient to assume that $0 < b \leqslant 1$. Now, if $g \in \mathscr{U}_1$, then

$$g_t(z) = e^{it} g(ze^{-it}) = z + be^{-it}z + \ldots \in \mathscr{U}_1,$$

$$|be^{-it}| = |b| \quad \text{and} \quad f_t(z) = \log g_t'(z) = f(ze^{-it}).$$

If $b = 1$, then the uniqueness of the extremal function for the coefficient problem in $\mathscr{U}_1$ implies that $g(z) = z/(1-z)$, and $f(z) = -2\log(1-z)$ which is univalent and convex in D.

In the case $0 < b < 1$ we observe that the convexity of $g(z)$ in D implies that $1+zg''(z)/g'(z)$ is subordinate to $(1+z)/(1-z)$ in D. Hence

$$1+\frac{zg''(z)}{g'(z)} = \frac{1+zB(z)}{1-zB(z)} = 1+\frac{2zB(z)}{1-zB(z)}, \qquad |z|<1,$$

where $B(z) = b+b_1z+\ldots$ is analytic and $|B(z)|<1$ in D. Since $B(z)$ is bounded and $B(0) = b$, the values of $B(z)$ for z in the disk $|z|\leqslant r$ lie in the disk

$$\left|w-\frac{(1-r^2)b}{1-r^2b^2}\right| < \frac{r(1-b^2)}{1-r^2b^2} \quad (w \text{ complex}), \tag{4.3}$$

and $(b-r)/(1-rb) \leqslant \operatorname{Re}B(z) \leqslant (b+r)/(1+rb)$ (see [8], p. 167). The disk (4.3) is centered at a point on the positive real axis in D and does not contain the origin if $r<b$. Thus the inversion $w\to 1/w$ yields that

$$\operatorname{Re}\frac{1}{B(z)} \geqslant \frac{1+rb}{b+r}, \qquad |z|\leqslant r<b. \tag{4.4}$$

Since $f'(z) = g''(z)/g'(z) = 2B(z)/(1-zB(z))$ and $B(z)$ has no zeros in $|z|\leqslant r<b$, it follows that $\operatorname{Re}f'(z)$ will be positive in $|z|\leqslant r<b$ if $\operatorname{Re}[1/B(z)-z]>0$ in $|z|\leqslant r<b$. Inequality (4.4) implies that

$$\operatorname{Re}\left[\frac{1}{B(z)}-z\right] \geqslant \frac{1+rb}{b+r}-r = \frac{1-r^2}{b+r} > 0, \qquad |z|\leqslant r<b,$$

and establishes the result.

To show that our result is sharp note that the generalized Koebe function $k_c(z)$ is univalent and convex in D when $0<c<1$. Moreover, for $f(z) = \log k_c'(z)$ with $c = |b|$, we have $f'(z) = 2(z+|b|)/(1-z^2)$, $f'(-|b|) = 0$, and $k_c''(0)/2 = |b|$.

THEOREM 4.2. *If $g(z) = z+bz^2+\ldots$ belongs to $\mathscr{U}_a$, then the function $f(z) = \log g'(z)$ is univalent and close-to-convex in the disk $|z|<|b|R_a^2$.*

Proof. Suppose that $g(z) = z+bz^2+\ldots$ belongs to $\mathscr{U}_a$. Then

$$g_\varrho(z) = \frac{1}{\varrho}g(\varrho z) = z+b\varrho z^2+\ldots$$

is univalent and convex in $|z|<1$ provided $\varrho = R_a$. Hence, by Theorem 4.1,

$$f(\varrho z) = \log g'(\varrho z) = \log g_\varrho'(z), \qquad \varrho = R_a,$$

is univalent and close-to-convex in $|z|<|b|R_a$ and, consequently, $f(z)$ is univalent and close-to-convex in $|z|<|b|R_a^2$.

Theorem 4.2 is probably not sharp in the sense that one can find a function $g(z) \in \mathcal{U}_a$ that is univalent in $|z| < |g''(0)/2| R_a^2$ and not univalent in any larger disk. (**P 933**)

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