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Mapping the fluid flow and shear near the core surface using the radial and horizontal components of the magnetic field

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SUMMARY

We examine the problem of determining the fluid flow and the shear (the radial derivative of the flow) at the core surface given a model of the temporal variation of the magnetic field. Whereas most previous work has focused on determining only the flow, which requires only the use of the radial component of the magnetic field, here, in addition, we determine the shear for which we must use the horizontal component of the magnetic field. Estimates of the jump in the value of the horizontal magnetic field $\mathbf{B}_h$ across the boundary layer between the top of the free stream and the base of the mantle are small, and suggest that to a high level of accuracy the mantle valuês of $\mathbf{B}_h$ can be used at the top of the core. Except in the special case of an insulating mantle, only the horizontal poloidal field is known at the core–mantle boundary and supplies one extra equation for the determination of velocity and shear. We show how the matrix elements relating the coefficients of the spectral expansion of the flow and shear are related to the geomagnetic secular variation coefficients in closed form. We examine the uniqueness of the resulting inverse problem,and show that one part of the non-uniqueness from which the shear suffers is particularly easy to describe: it takes the same form as that which affects the flow, namely a toroidal ambiguity in the field $\mathbf{u}'B_r$. However, certain uniqueness theorems can be derived: we extend the steady motions theorem of Voorhies & Backus (1985) and the geostrophic motions theorem of Hills (1979) and Backus & LeMouël (1986) to the determination of the flow and shear, and derive closely analogous results. Uniqueness in the steady case depends on the value of the same discriminant as the velocity, and in the geostrophic case the shear can be determined uniquely in the same areas as can the velocity (i.e. outside certain ambiguous patches). For the geostrophic regime, the lateral density (or temperature) variations at the top of the core can be found in a self-consistent manner. We apply our method to the temporal evolution of the field over the period 1960–1980, and produce solutions for each of the assumptions of unconstrained steady motions, geostrophic motions, and purely toroidal motions. We find that the form of the flow changes very little from solutions based only on the radial induction equation, and that the shear is weak and aligned with the flow, with a sense such that the strength of the flow decreases with depth with a length-scale for linear decay of half the core radius. This suggests that the flow near the core surface is indicative of whole core flow, rather than a flow confined to a layer near the core surface.

Key words: core–mantle boundary, Earth's magnetic field, fluid flow in core, secular variation.

1 INTRODUCTION

In recent years a number of attempts have been made to use our knowledge of the time-varying magnetic field observed at the Earth's surface to infer the pattern of fluid flow at the surface of the outer core; for a recent review see Bloxham & Jackson (1991). A major motivation for these studies has been the hope that maps of the fluid flow at the core surface might provide some insight into the dynamo process operating in the core, by which the Earth's magnetic field is maintained against Ohmic dissipation. A question which remains open is whether the fluid flow observed near the core surface bears any sensible resemblance to the pattern of fluid flow deeper within the core.

To answer this question we need to determine how the flow changes with depth. The best that we can do, unfortunately, is to estimate the radial derivative of the flow (the shear) at the core surface. Obviously, if the shear at the core surface is very vigorous then the flow at depth is likely to be very different from that at the core surface; on the other hand, if it is weak, then it is possible, although not necessarily required, that the flow at depth is similar to the pattern at the core surface.

Determinations of the flow can be based on just the radial field at the CMB. However, to determine the shear we need to use the horizontal field at the core–mantle boundary (CMB). An important step has recently been taken in this direction by Lloyd & Gubbins (1990) who derive purely toroidal motions near the core surface; they infer a radial length-scale of the flow of about 600 km. In this paper, we seek to extend their work and consider general (i.e. not purely toroidal) motions. Lloyd & Gubbins give two reasons for considering purely toroidal motions. Their first reason is based on the possibility that core fluid may be stably stratified, in which case they argue, as many others have done in the past (e.g. Whaler 1980; Bloxham 1988), that the motions must be purely toroidal. Their second reason is based on the possible corruption of poloidal ingredients of the flow by diffusion. Bloxham (1989) has argued that the toroidal part of the flow can be determined quite reliably, while the poloidal part largely depends on the assumptions that are made about it: if the flow is assumed purely toroidal, of course, the poloidal part is identically zero; if the flow is assumed geostrophic, the poloidal part is determined so as to force the total flow to be geostrophic; and if nothing is specified, the poloidal part is left poorly determined. Given this state of affairs we believe that any conclusions should be drawn from an examination of the flow using as many different assumptions as possible; those aspects which are robust to changes in the assumptions are perhaps more worthy of notice than those which seem highly dependent on the assumptions.

Throughout this paper we restrict our attention to methods of determining global maps of the large-scale flow near the core surface; we do not consider methods of obtaining estimates of the flow at points or along particular curves on the CMB. Although the flow, or particular aspects of the flow, can be determined uniquely in this way (Backus 1968) any estimates will be severely limited by the enormous uncertainties associated with point estimates of the secular variation at the CMB.

As usual, our starting point is the magnetic induction equation which governs the secular variation of the magnetic field in the core:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{1}$$

where $\mathbf{B}$, $\partial_t \mathbf{B}$ and $\mathbf{u}$ are the magnetic field, its rate-of-change, and the velocity respectively, and η is the core diffusivity ($\eta = 1/\mu_0 \sigma$ where μ_0 is the permeability of free space and σ is the core conductivity, approximately $5 \times 10^5\,\mathrm{S\,m^{-1}}$). This equation describes how the secular variation (SV) results from the effects of both advection of field by the flow (the first term on the right-hand side) and the effects of diffusion (the second term).

The well-known frozen flux approximation, put forward 25 years ago by Roberts & Scott (1965), is based on the supposition that, on the time-scale of a few decades, the core acts as a perfect conductor, so that the effects of diffusion may be neglected in the induction equation, with the consequence that field lines are frozen in the core fluid. Backus (1968) showed that for this to hold certain conditions must be satisfied by the field and its secular variation. Although several attempts have been made to test these conditions, the results are, in most cases, somewhat equivocal (again, for a review, see Bloxham & Jackson 1991). However, despite these concerns, the frozen flux approximation remains our best tool for mapping the zeroth-order non-diffusive part of the flow.

We adopt then the frozen flux version of the magnetic induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{2}$$

Unfortunately only part of the magnetic field is observable at the Earth's surface. To see this, we decompose the magnetic field $\mathbf{B}$ into toroidal $\mathbf{B}_\mathrm{T}$ and poloidal $\mathbf{B}_\mathrm{P}$ ingredients given by

$$\mathbf{B} = \mathbf{B}_\mathrm{T} + \mathbf{B}_\mathrm{P} = \nabla \times (T\hat{\mathbf{r}}) + \nabla \times \nabla \times (P\hat{\mathbf{r}}) \tag{3}$$

where $T(r, \theta, \phi)$ and $P(r, \theta, \phi)$ are the toroidal and poloidal scalars respectively (in spherical polar coordinates). Applying Ampére's law we have

$$\begin{aligned} \mu_0 \mathbf{J} = \nabla \times \mathbf{B} &= \nabla \times \mathbf{B}_\mathrm{T} + \nabla \times \mathbf{B}_\mathrm{P} \\ &\nabla \times \nabla \times (T\hat{\mathbf{r}}) + \nabla \times \nabla \times \nabla \times (P\hat{\mathbf{r}}) \\ &\nabla \times \nabla \times (T\hat{\mathbf{r}}) + \nabla \times [(-\nabla^2 P)\hat{\mathbf{r}}], \end{aligned} \tag{4}$$

from which we can see that poloidal fields result from toroidal currents, and toroidal fields from poloidal currents. In particular, we can also see that if $\mathbf{J} = 0$, for example in an insulator, then $\mathbf{B}_\mathrm{T} = 0$, while $\mathbf{B}_\mathrm{P}$ does not necessarily vanish, although P must satisfy $\nabla^2 P = 0$ (in which case we call $\mathbf{B}_\mathrm{P}$ a potential field).

At the Earth's surface, $\mathbf{B}_\mathrm{T} \simeq 0$ since $\mathbf{J} \simeq 0$ there, so we observe only the poloidal field. However, following Bullard & Gellman (1954) we can separate equation (2) into toroidal and poloidal ingredients

$$\partial_t \mathbf{B}_\mathrm{T} = \{\nabla \times [\mathbf{u} \times (\mathbf{B}_\mathrm{T} + \mathbf{B}_\mathrm{P})]\}_\mathrm{T} \tag{5}$$

and

$$\partial_t \mathbf{B}_\mathrm{P} = [\nabla \times (\mathbf{u} \times \mathbf{B}_\mathrm{P})]_\mathrm{P}. \tag{6}$$

This decomposition is only valid at the CMB where we impose the condition $u_r = 0$. We note that the toroidal field is irrelevant to the temporal evolution of the poloidal field at the CMB; in other words, we do not need to know $\mathbf{B}_\mathrm{T}$ in order to determine $\mathbf{u} = \mathbf{u}_\mathrm{h}$ at the CMB from $\partial_t \mathbf{B}_\mathrm{P}$. The effect of Ekman suction, which allows $u_r \neq 0$, is negligible (see Bloxham & Jackson 1991).

The radial component of the frozen flux induction equation can be written in the form

$$\partial_t B_r = -\nabla_\mathrm{h} \cdot (\mathbf{u} B_r) \tag{7}$$

which only involves the flow $\mathbf{u}$, whereas the other components involve both the velocity and its radial derivative, the shear $\mathbf{u}' = \partial\mathbf{u}/\partial r$. With the exception of Lloyd & Gubbins (1990), all previous studies aimed at determining the large-scale flow have used just the radial component of the induction equation.

This description of frozen flux induction is incomplete, since at the CMB the flow must obey non-slip boundary conditions and so, trivially, $\mathbf{u} = 0$ at the CMB. A viscous boundary layer (VBL) is set up, however, through which the velocity changes from its value at the top of the free stream to zero at the base of the mantle. Our interest, of course, is in the velocity at the base of this VBL (i.e. at the top of the free stream) rather than the trivially zero flow at the rigid CMB. How much does the magnetic field change across this VBL?

The depth of this boundary layer depends on the core viscosity which, unfortunately, is one of the most poorly known parameters of the Earth, but most estimates suggest that the VBL has a depth on the order of only of a few centimetres. Across this layer the radial component of the magnetic field is continuous, but the horizontal components will be discontinuous. Various authors have estimated the size of this discontinuity (Roberts & Scott 1965; Backus 1968; Hide & Stewartson 1972; Benton 1981); Hide & Stewartson argue that the discontinuity in the horizontal component of the field, labelled $[\mathbf{B}_h]$, will be of order

$$[\mathbf{B}_h] \sim B_r u/\eta\sqrt{\nu/2\Omega} \tag{8}$$

where B_r, u, η, ν and Ω are the radial field strength, velocity, core diffusivity, kinematic viscosity and diurnal frequency respectively. For typical values for the core ($B_r \sim 5\times10^{-4}\,\mathrm{T}$, $u \sim 5\times10^{-4}\,\mathrm{m\,s^{-1}}$, $\eta \sim 1\,\mathrm{m^2\,s^{-1}}$, $\nu \sim 10^{-6}\,\mathrm{m^2\,s^{-1}}$), we find that the size of the jump is $[\mathbf{B}_h] \sim 20\,\mathrm{nT}$, which gives a relative jump of $\sim 5\times10^{-5}$ when compared to the size of the fields themselves, which is certainly negligible.

Backus (1968) derived conditions for the horizontal field to be continuous across the boundary layer. Barraclough, Gubbins & Kerridge (1989) have recently examined these conditions, finding that they are reasonably well satisfied, in fact suggesting that continuity of the horizontal field across the boundary layer is as good an approximation as the frozen flux approximation.

We have discussed the fact that finite conductivity within the mantle permits the existence of a toroidal field at the CMB of which we are completely ignorant, and as (6) shows it does not contribute to the poloidal secular variation. The presence of finite mantle conductivity means that to reconstruct the poloidal field at the CMB we must solve the poloidal diffusion equation

$$\frac{\partial \mathbf{B}_P}{\partial t} = -\nabla\times[\eta_m(r)\nabla\times\mathbf{B}_P] \tag{9}$$

in the mantle, where $\eta_m = (\mu_0\sigma_m)^{-1}$ and the conductivity σ_m is assumed to be radially symmetric. Two reasons influence our choice not to follow this procedure. First, the question of the conductivity of the deep mantle is currently not resolved, especially below depths of about 2000 km where induction studies lose resolution. However, the recent experimental evidence of Peyronneau & Poirier (1989) on the conductivity of perovskite and magnesiowüstite, when extrapolated to deep mantle temperatures and pressures, is in good agreement with the mantle conductivity profile proposed by Achache, LeMouël & Courtillot (1981) based on induction studies, although other experimental evidence is still in conflict (Li & Jeanloz 1987, 1990). Second, for a range of profiles which have been proposed, Benton & Whaler (1983) find that the correction to the magnetic field in the insulating limit is only a few per cent. In particular, for the profile of Achache *et al.* (1981) the effective spectral diffusion time for degree l, $\tau(l)$, of Benton & Whaler (1983) [or equivalently the first-order delay time of Backus (1983)] is less than 1 yr for all harmonic degrees, and thus is unimportant on the time-scales of SV. Given this result, and the great uncertainty attending any mantle conductivity profile, we choose not to apply any correction for the effects of mantle conductivity; this approximation does not imply that we believe the mantle to be insulating. The scenario of a perfectly insulating mantle has been studied by Lloyd & Gubbins (1990), who show that if all three components of field are known at the CMB then a purely toroidal velocity field can be found uniquely. Their proof relies on the fact that $\partial_r\mathbf{B}_T = \mathbf{B}_T = 0$ in the mantle, which we find to be implausible.

In Section 2 we consider the forward problem of calculating the temporal evolution of both the radial and horizontal poloidal components of the field, given an initial field and the flow and shear. In Section 3 we consider the inverse problem of determining the flow and shear, given an initial field and its temporal evolution. We discuss the non-uniqueness inherent in such inversions and show that part of the non-uniqueness in the shear is closely related to that in the flow derived from just the radial induction equation. Also, we extend the steady flow uniqueness theorem of Voorhies & Backus (1985) and the tangentially geostrophic flow uniqueness theorem of Hills (1979) and Backus & LeMouël (1986) to the cases of steady flow and shear, and geostrophic flow and shear respectively. We present our results in Section 4.

2 THE FORWARD PROBLEM

This section describes how the spectral expansion of the poloidal secular variation is related to the spectral expansions of the velocity and the magnetic field. We begin by decomposing the velocity $\mathbf{u}$ into toroidal and poloidal ingredients

$$\mathbf{u} = \mathbf{u}_T + \mathbf{u}_P = \nabla\times(\mathcal{T}\hat{\mathbf{r}}) + \nabla\times\nabla\times(\mathcal{P}\hat{\mathbf{r}}). \tag{10}$$

The condition $u_r = 0$ at $r = c$ (c is the core radius) imposes the boundary condition $\mathcal{P} = 0$, so (10) can be written in the form

$$\mathbf{u} = \mathbf{u}_h = \nabla\times(\mathcal{T}\hat{\mathbf{r}}) + \nabla_h\mathcal{S} \tag{11}$$

where

$$\mathcal{S} = \frac{\partial\mathcal{P}}{\partial r}. \tag{12}$$

The representation of the shear $\mathbf{u}_h'$ involves two extra scalars; we find

$$\mathbf{u}_h' = \mathbf{u}_h'(\mathcal{T}, \mathcal{S}, \mathcal{T}', \mathcal{S}') = \nabla \times (\mathcal{T}'\hat{\mathbf{r}}) + \nabla \times \nabla \times (\mathcal{P}'\hat{\mathbf{r}}) - \mathbf{u}_h/r, \tag{13}$$

where $'$ represents radial derivatives. We expand the potentials $\mathcal{T}$, $\mathcal{S}$, $\mathcal{T}'$, and $\mathcal{S}'$ in surface spherical harmonics

$$\mathcal{T} = \sum_{l=1}^{\infty} \sum_{m=0}^{l} ({}_ct_l^m \cos m\phi + {}_st_l^m \sin m\phi) P_l^m(\cos\theta), \tag{14}$$

$$\mathcal{S} = \sum_{l=1}^{\infty} \sum_{m=0}^{l} ({}_cs_l^m \cos m\phi + {}_ss_l^m \sin m\phi) P_l^m(\cos\theta), \tag{15}$$

$$\mathcal{T}' = \sum_{l=1}^{\infty} \sum_{m=0}^{l} ({}_ct_l'^m \cos m\phi + {}_st_l'^m \sin m\phi) P_l^m(\cos\theta), \tag{16}$$

$$\mathcal{S}' = \sum_{l=1}^{\infty} \sum_{m=0}^{l} ({}_ss_l'^m \cos m\phi + {}_ss_l'^m \sin m\phi) P_l^m(\cos\theta), \tag{17}$$

where the $P_l^m(\cos\theta)$ are associated Legendre polynomials, normalized so that

$$\oint (P_l^m)^2 \sin^2 m\phi \, d\Omega = \oint (P_l^m)^2 \cos^2 m\phi \, d\Omega = \oint (P_l^0)^2 \, d\Omega = \frac{4\pi}{2l+1}. \tag{18}$$

The poloidal magnetic field is represented as in (3), with P and P' expanded as above with coefficients $\{{}_c\Pi_l^m; {}_s\Pi_l^m\}$, $\{{}_c\Pi_l'^m; {}_s\Pi_l'^m\}$. Using $\mathbf{t}$ to represent the vector with elements t_l^m, $\mathbf{s}$ to represent the vector with elements s_l^m, $\mathbf{t}'$ to represent the vector with elements $t_l'^m$, $\mathbf{s}'$ to represent the vector with elements $s_l'^m$, and $\mathbf{\Pi}$ and $\mathbf{\Pi}'$ to represent the vectors of coefficients for P and P', and substituting into (6), we can obtain, after some manipulation, the closed-form matrix equations

$$\begin{aligned}\partial_t \mathbf{\Pi} &= \mathbf{E}_r \mathbf{t} + \mathbf{G}_r \mathbf{s} \\ &= (\mathbf{E}_r : \mathbf{G}_r) \begin{pmatrix} \mathbf{t} \\ \cdots \\ \mathbf{s} \end{pmatrix} \\ &= \mathbf{H}_r \mathbf{q}\end{aligned} \tag{19}$$

and

$$\begin{aligned}\partial_t \mathbf{\Pi}' &= \mathbf{E}_h \mathbf{t} + \mathbf{G}_h \mathbf{s} + \mathbf{E}_h' \mathbf{t}' + \mathbf{G}_h' \mathbf{s}' \\ &= (\mathbf{E}_h : \mathbf{G}_h) \begin{pmatrix} \mathbf{t} \\ \cdots \\ \mathbf{s} \end{pmatrix} + (\mathbf{E}_h' : \mathbf{G}_h') \begin{pmatrix} \mathbf{t}' \\ \cdots \\ \mathbf{s}' \end{pmatrix} \\ &= \mathbf{H}_h \begin{pmatrix} \mathbf{q} \\ \mathbf{q}' \end{pmatrix}\end{aligned} \tag{20}$$

for the radial and horizontal components of (6) respectively, where $\mathbf{q} = (\mathbf{t} : \mathbf{s})$ and $\mathbf{q}' = (\mathbf{t}' : \mathbf{s}')$.

The general matrix elements of $\mathbf{H}_r$ are given in the Appendix in equations (A16) and (A18), and of $\mathbf{H}_h$ in (A25) and (A26). It is especially noteworthy that the horizontal component of (6) (a vector equation in two scalar components) yields just one additional equation, namely (20). As we have discussed, we approximate the poloidal magnetic field in our calculations as a potential field which, naturally, we represent as the gradient of a scalar magnetic potential

$$\mathbf{B}(r, \theta, \phi, t) = -\nabla\Phi(r, \theta, \phi, t) \tag{21}$$

expanded in surface spherical harmonics

$$\Phi(r, \theta, \phi, t) = a \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l+1} \times [g_l^m(t) \cos m\phi + h_l^m(t) \sin m\phi] \times P_l^m(\cos\theta), \tag{22}$$

where a is the radius of the Earth's surface (nominally 6371.2 km) and the $\{g_l^m(t), h_l^m(t)\}$ are the Gauss geomagnetic coefficients. Under this approximation the matrix elements of $\mathbf{H}_r$ are given in the Appendix in equations (A21) and (A22), and the elements of $\mathbf{H}_h$ in (A27) and (A28).

We contrast this formalism with that of Lloyd & Gubbins (1990) who consider the special case $\mathbf{u}_h = \mathbf{u}_T$. They calculate the matrix elements numerically using a transform method which can offer substantial computational advantages over our method; our method, though, has the advantage that it requires only relatively minor adaptation to extend pre-existing codes for solving (19).

3 THE INVERSE PROBLEM

3.1 Uniqueness

Despite the apparent simplicity of equation (7), Backus (1968) showed that solutions for the velocity $\mathbf{u}$ were necessarily non-unique (i.e. if one eligible velocity field existed then so did an infinity of solutions, all of which fit the data equally well). The ambiguity results from the use of one equation to derive two orthogonal components of flow. We have already described how only the radial and horizontal poloidal fields can be estimated at the CMB,and in this paper we use these two components of field to estimate the velocity and shear at the CMB.

When the radial and horizontal poloidal fields are used at the CMB there are two equations with four unknowns, and the ambiguity in solving (1) is not diminished. In this section we examine the uniqueness of flows which can be obtained from using the two poloidal components of the induction equation. We begin by applying the Helmholtz representation of an arbitrary vector field (e.g. Backus 1986) to the field $\boldsymbol{\omega} = \mathbf{u} \times \mathbf{B}$. The Helmholtz representation is more general than the poloidal–toroidal (or Mie) representation and does not require $\boldsymbol{\omega}$ to be solenoidal, which would require that both $\mathbf{u}$ and $\mathbf{B}$ are conservative (i.e. curl-free). We write

$$\boldsymbol{\omega} = f\hat{\mathbf{r}} + \nabla_1 g + \mathbf{\Lambda}_1 h \tag{23}$$

where ∇_1 and $\mathbf{\Lambda}_1$ are the surface gradient and surface curl operators defined by

$$\nabla_1 = r(\nabla - \hat{\mathbf{r}}\nabla) = r\nabla_h; \qquad \mathbf{\Lambda}_1 = \hat{\mathbf{r}} \times \nabla_1 \tag{24}$$

and h and g are determined uniquely if they are required to satisfy

$$\langle g \rangle_r = \langle h \rangle_r = 0 \tag{25}$$

where $\langle g \rangle_r$ is the mean value of g over the sphere of radius r.

The secular variation induced is given by (Backus 1986)

$$\nabla \times \boldsymbol{\omega} = r^{-1}\{\hat{\mathbf{r}}(\nabla_1^2 h) + \nabla_1[-\nabla_r(rh)] + \mathbf{\Lambda}_1[\nabla_r(rg) + \langle f \rangle_r - f]\} \tag{26}$$

where ∇_1^2 is the dimensional surface Laplacian. Denoting $\nabla_r h$ by h' we see that if the poloidal secular variation is known (26) gives the values of h and h', but nothing can be said regarding g, g' and f, apart from the fact that for (23) to hold we require that f and g satisfy

$$-(\mathbf{\Lambda}_1 h + \nabla_1 g)\cdot\mathbf{B}_h = fB_r \tag{27}$$

because of the requirement $\boldsymbol{\omega}\cdot\mathbf{B}=0$. Applying the operator $\hat{\mathbf{r}}\times$ to (23), we see that we can write the horizontal components of $\boldsymbol{\omega}$ in the alternative form

$$\begin{aligned}-\nabla_1 h + \mathbf{\Lambda}_1 g &= \mathbf{u}_h B_r - \mathbf{B}_h u_r\\ &= \mathbf{u}_h B_r\end{aligned} \tag{28}$$

because $u_r=0$ at the CMB. Since g is unconstrained, there is an ambiguity which is exactly of the form for a velocity lying in the null-space of the radial secular variation which was pointed out by Backus (1968); and g can be chosen arbitrarily subject to some mild constraints on the null-flux curves where $B_r=0$. Clearly the horizontal poloidal component of SV does not reduce the size of the null-space in $\mathbf{u}$ from that which occurs when only the radial component of secular variation is used.

An added ambiguity is contained in the toroidal part of (26). Differentiating (28) with respect to r, we write the shear in the form

$$\begin{aligned}\mathbf{u}'B_r &= -\nabla_1 h' + \mathbf{\Lambda}_1 g' - \mathbf{u}B_r' + u_r'\mathbf{B}_h\\ &= -\nabla_1 h' + \mathbf{\Lambda}_1 g' - \frac{B_r'}{B_r}(-\nabla_1 h + \mathbf{\Lambda}_1 g)\\ &\quad - \frac{\mathbf{B}_h}{r}\nabla_1\cdot[B_r^{-1}(-\nabla_1 h + \mathbf{\Lambda}_1 g)].\end{aligned} \tag{29}$$

Now h and h' are known, and the shear is subject to two ambiguities, one of which is a complicated function of g with no simple interpretation, and a simpler one containing g'. (On null-flux curves, certain conditions must hold which are not investigated here.) Even if the flow $\mathbf{u}$ is known (g is known), then because g' is unconstrained, if $\mathbf{u}_0'B_r$ is a valid solution then so is

$$\mathbf{u}'B_r = \mathbf{u}_0'B_r + \mathbf{\Lambda}_1 g' \tag{30}$$

A heuristic demonstration of the toroidal part of the ambiguity in the shear $\mathbf{u}'$ is easily given by assuming the flow to be known and by writing the induction equation in the form (Lloyd & Gubbins 1990)

$$\partial_t\mathbf{B} = (\mathbf{B}\cdot\nabla_h)\mathbf{u} - (\mathbf{u}\cdot\nabla_h)\mathbf{B} + B_r\mathbf{u}'. \tag{31}$$

Now by applying the Helmholtz decomposition to $\mathbf{u}'B_r$ we find that even if $\mathbf{u}$ is found uniquely, there is an ambiguity in the shear of the form

$$\mathbf{u}'B_r = \mathbf{\Lambda}_1 g' \tag{32}$$

where g' is unknown.

3.2 Resolving the non-uniqueness

3.2.1 Steady motions

A trivial extension of the steady motions theorem for velocity shows that if the shear is steady over a certain interval of time, then measurement of $\mathbf{B}$ and $\partial_t\mathbf{B}$ at three different times will serve to resolve the shear uniquely, except in certain cases. To illustrate this, we rewrite equation (31) in the form

$$\partial_t\mathbf{B} - (\mathbf{B}\cdot\nabla_h)\mathbf{u} + (\mathbf{u}\cdot\nabla_h)\mathbf{B} = B_r\mathbf{u}'. \tag{33}$$

We assume that the flow has been found uniquely from the radial induction equation, and let $\partial_t\tilde{\mathbf{B}}$ represent the left-hand side of (33) which is the secular variation which is not produced by the flow $\mathbf{u}$ acting on $\mathbf{B}$. Then equation (33) becomes

$$\partial_t\tilde{\mathbf{B}} = B_r\mathbf{u}' \tag{34}$$

If only the poloidal part of (34) is known, this is equivalent to knowledge of $\nabla_h\cdot(\partial_t\tilde{\mathbf{B}})$; in contrast the quantity $\hat{\mathbf{r}}\times\nabla_h\cdot(\partial_t\tilde{\mathbf{B}})$ is unknown. Thus (34) becomes

$$\nabla_h\cdot(\partial_t\tilde{\mathbf{B}}) = \nabla_h\cdot(B_r\mathbf{u}'), \tag{35}$$

in striking similarity to the radial induction equation, although note that the data on the left-hand side are now spatial gradients of secular variation, which are likely to be much more noisy than the data (radial secular variation) used in the determination of velocity. The condition for a unique solution for $\mathbf{u}'$ and $\nabla_h\cdot\mathbf{u}'$ is $\Delta\neq 0$ where Δ is the discriminant described by Voorhies & Backus (1985). The solution is found by substituting $-\nabla_h\cdot[\partial_t\tilde{\mathbf{B}}(t_i)]$ at time t_i for $B_r(t_i)$ in equations (5) and (6) of Voorhies & Backus (1985).

3.2.2 Geostrophic motions

The dynamics of the core are determined by the Navier–Stokes equation, which, under the Boussinesq approximation, is given by

$$\begin{aligned}D_t\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} &= -1/\rho_0\nabla_p + (\rho'/\rho_0)\mathbf{g} + \nu\nabla^2\mathbf{u}\\ &\quad + 1/(\rho_0\mu_0)\nabla\times\mathbf{B}\times\mathbf{B}\end{aligned} \tag{36}$$

where D_t, $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$, ρ_0, ρ', p, $\mathbf{g}$ are the Stokes derivative, rotation vector, unperturbed and perturbed densities, pressure and gravitational vector respectively. Comparing sizes of the terms in (36) shows that in the core only a few of the forces are important. The Ekman number $\nu/\Omega c^2\sim 10^{-15}$ (c is the core radius) is tiny, showing that viscosity is unimportant, regardless of the badly known value of ν. The Rossby number (measuring the strength of inertial terms relative to the Coriolis term) is similarly small ($\sim 10^{-7}$). The neglect of the viscous and inertial terms leaves a balance between Coriolis, pressure, buoyancy and Lorentz terms. The Lorentz terms contribute the largest unknown to the equations, mainly because the radial dependencies of the magnetic fields and the strength of the toroidal field within the core are almost completely unknown. The Elsasser number (measuring the ratio of the Lorentz and Coriolis terms) is small if only the poloidal field is considered; the primary contributions comes from the coupling of the toroidal field with itself or with the poloidal field.

We write the Lorentz force $\mathfrak{L}$ in the form

$$\mathfrak{L} = \frac{1}{\mu_0}\left[(\mathbf{B}_h\cdot\nabla)\mathbf{B} + B_r\frac{\partial\mathbf{B}}{\partial r} - \nabla\tfrac{1}{2}B^2\right]. \tag{37}$$

The final term is an equivalent magnetic pressure which can be absorbed into the fluid pressure term in (36), but it is not clear which of the first two terms will give the largest contribution. Almost all discussion of the Lorentz force at the CMB has centred on assuming the mantle to be insulating, which gives $\mathbf{B}_T=0$ at the CMB. Then provided

$\partial_r \mathbf{B}_T < 10^3$ nT m^{-1} the second term is sufficiently small to be unimportant, though this last condition seems unlikely to be satisfied unless the toroidal field is everywhere small, since large radial gradients in the toroidal field are otherwise to be expected at the CMB.

Some models of the geodynamo, though, do have toroidal fields which are comparable in size to the poloidal field, and if the geodynamo operated in such a fashion the Lorentz terms would be small compared to the Coriolis terms in (36). Other models, however, operate in a 'strong field' regime suggesting that the toroidal field may be many times larger (say 10^2) than the poloidal field, in which case the Lorentz term is relevant.

To pursue the geostrophic approximation, we adopt the 'weak field' hypothesis. Let us define

$$q = p/(2\rho_0 \Omega r). \tag{38}$$

Then the flow must satisfy

$$\mathbf{u}\cos\theta = \mathbf{\Lambda}_1 q + u_r\hat{\mathbf{z}} - (2\rho_0\Omega)^{-1}\hat{\mathbf{r}} \times \mathfrak{L} \tag{39}$$

where $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$ and $\mathbf{\Lambda}_1 = \hat{\mathbf{r}} \times \nabla_1$ as in (24). At the CMB $u_r = 0$ and geostrophy requires that the term in $\mathfrak{L}$ be small, so that (39) becomes (LeMouël 1984)

$$\mathbf{u}\cos\theta = \mathbf{\Lambda}_1 q. \tag{40}$$

Now what is the constraint on $\mathbf{u}'$? The derivative of (39) shows that $\mathbf{u}'$ must obey

$$\mathbf{u}'\cos\theta = \mathbf{\Lambda}_1 q^\dagger + u_r'\hat{\mathbf{z}} + (2\rho_0\Omega)^{-1}\hat{\mathbf{r}} \times (\alpha\mathfrak{L} - \partial_r\mathfrak{L}) \tag{41}$$

where $'$ represents differentiation with respect to r, $\alpha = \rho_0'/\rho_0$ and $q^\dagger = (p' - p/r - \alpha p)/(2\rho_0\Omega r)$. The value of α measures the variation in density in the adiabatic reference state, and can be found from seismology: $\alpha = g/V_p^2$ where g is the size of gravity and V_p is the compressional wavespeed, which gives $\alpha d \approx 0.2$ over a length-scale d of the size of the core radius. To compare sizes of terms in (41) we need to estimate the size of $|\mathbf{u}'|$. Nothing concrete can be said, because the only estimate available to us is that obtained from prior core velocity studies using the radial component of the induction equation, which supply u_r', and all we offer is a plausibility argument. The radial shear u_r' is known to be badly determined, and can vary between zero for toroidal flows to $u_r' \sim 10^{-2}$ yr^{-1} for other flows. Taking the latter to be a representative value for the shear, and with $|\mathbf{u}| \sim 10$ km yr^{-1} we find that the length-scale for shear L is roughly 1000 km. In this case $Lq^\dagger \sim q$ and the term $\alpha\mathfrak{L}$ can be neglected. The size of the term $\partial_r\mathfrak{L}$ is particularly difficult to estimate, since even in the case of an insulating mantle ($\mathbf{B}_T = 0$ at $r = c$) it involves terms of the form

$$\frac{\partial\mathfrak{L}}{\partial r} \sim \frac{1}{\mu_0}\left(\frac{\partial B_r}{\partial r}\frac{\partial \mathbf{B}_h}{\partial r} + B_r\frac{\partial^2 \mathbf{B}_h}{\partial r^2}\right). \tag{42}$$

Provided these terms are sufficiently small, equation (41) becomes

$$\mathbf{u}'\cos\theta = \mathbf{\Lambda}_1 q^\dagger + u_r'\hat{\mathbf{z}} \tag{43}$$

and we find that the shear is coupled to the flow and that the shear and flow must satisfy the two equations

$$\nabla_h \cdot (\mathbf{u}\cos\theta) = 0 \tag{44}$$

$$\nabla_h \cdot (\mathbf{u}'\cos\theta) = \nabla_h \cdot (\mathcal{U}\sin\theta\hat{\boldsymbol{\theta}}) \tag{45}$$

where $\mathcal{U}$ is the upwelling $= \nabla_h \cdot \mathbf{u}$. A similar discussion of the extension of tangential geostrophy away from the boundary into a region where the Lorentz force remains small in comparison to the Coriolis force can be found in LeMouël, Jault & Gire (1987).

To prove that the shear can be found uniquely, we must briefly review the geostrophic velocity uniqueness analysis of Backus & LeMouël (1986). Recall $q = p/(2\rho_0\Omega r)$ from (38) and let us define

$$\psi = B_r/\cos\theta. \tag{46}$$

Then we can write the radial component of the induction equation at the CMB as

$$\nabla_h q \cdot \mathbf{\Lambda}_1 \psi = \partial_t B_r. \tag{47}$$

Given that ψ and $\partial_t B_r$ are known, we should expect to be able to solve equation (47) for q, and hence use (40) to obtain $\mathbf{u}_h$. However, as discussed by Backus & LeMouël, (47) merely gives the derivative of q along the arc length of the contours of ψ, so that on each contour of ψ, q may be determined only up to an arbitrary constant which may be different on each level line of ψ. Backus & LeMouël show that this problem is aleviated somewhat by noting that first, since $(\mathbf{\Omega} \times \mathbf{u})_h = \mathbf{0}$ at the geographic equator, q must be constant along the equator; and second ψ is singular at the geographic equator. Adopting the terminology of Backus & LeMouël, consider two sets of points: the 'visible belt' defined as the set of points connected to the geographic equator by contours of ψ; and the 'ambiguous patches' as the sets of points consisting of closed contours of ψ which do not intersect the geographic equator. Then, since the visible belt includes the magnetic equator where q is constant, q is determined (up to a single additive constant) in the visible belt and $\mathbf{u}_h$ is determined uniquely; in the ambiguous patches the non-uniqueness remains. In the ambiguous patches the component of $\mathbf{u}_h$ perpendicular to contours of ψ can always be found, because the induction equation takes the form

$$\partial_t\psi + \mathbf{u}_h \cdot \nabla_h\psi = 0. \tag{48}$$

In Fig. 1 we plot the ambiguous patches for the 1980 main field model of Gubbins & Bloxham (1985); they account for 41 per cent of the area of the CMB, so one component of the flow is known everywhere, and the second component over almost 60 per cent of the core surface.

Now to prove an analogous result for the uniqueness of the shear, we restrict attention to the areas outside the ambiguous patches where the velocity $\mathbf{u}_h$ is known uniquely. Using (43) in (31) gives

$$\begin{aligned}\partial_t\mathbf{B} &= (\mathbf{B}\cdot\nabla_h)\mathbf{u} - (\mathbf{u}\cdot\nabla_h)\mathbf{B} + B_r\mathbf{u}' \\ &= (\mathbf{B}\cdot\nabla_h)\mathbf{u} - (\mathbf{u}\cdot\nabla_h)\mathbf{B} + \psi\mathbf{\Lambda}_1 q^\dagger + \psi u_r'\hat{\mathbf{z}}.\end{aligned} \tag{49}$$

As we have asserted throughout, only the poloidal SV is known, which enables us to calculate only the horizontal divergence of the left-hand side. Then, applying the operator $\nabla_1\cdot$ to (49), we find

$$\begin{aligned}&\nabla_1\cdot(\partial_t\mathbf{B}) - \nabla_1\cdot[(\mathbf{B}\cdot\nabla_h)\mathbf{u} - (\mathbf{u}\cdot\nabla_h)\mathbf{B} + \psi u_r'\hat{\mathbf{z}}] \\ &\quad = \nabla_1\cdot(\psi\mathbf{\Lambda}_1 q^\dagger) \\ &\quad = -\mathbf{\Lambda}_1\psi\cdot\nabla_1 q^\dagger.\end{aligned} \tag{50}$$

Figure 1. Ambiguous patches for 1980. The shaded patches depict those areas within which the tangentially geostrophic flow hypothesis fails to resolve the non-uniqueness in the determination of the flow. The patches in the figure were calculated using the 1980 field model (D80111) of Gubbins & Bloxham (1985). The map is plotted using an Aitoff equal area projection.

Outside the ambiguous patches the left-hand side of (50) is known completely (recall $u'_r = -\nabla_h \cdot \mathbf{u}$), and the right-hand side gives the derivative of $q^\dagger$ along level lines of ψ in the same manner as did (47) for the velocity case; there is a similar arbitrary constant on each level line of ψ. On the equator we find

$$0 = \Lambda_1 q^\dagger + u'_r \hat{\boldsymbol{\theta}} \tag{51}$$

and $u'_r = -\mathcal{U}$ is known on the equator from the radial equation. Thus

$$q^\dagger(\pi/2, \phi) = q_0^\dagger - \int_0^\phi \mathcal{U}(\pi/2, \phi')\, d\phi' \tag{52}$$

so that $q^\dagger(\pi/2, \phi)$ can be found up to an additive constant $q_0^\dagger$. To proceed requires a little care: $q^\dagger$ is no longer constant on the equator as before. Because each of the level lines of ψ within the visible belt meets the equator at some point where $q^\dagger$ is known, the additive constant for $q^\dagger$ can be fixed to be the same on each level line (i.e. there is one common arbitrary constant $q_0^\dagger$). Then $\mathbf{u}'$ is found uniquely eveywhere within the visible belt.

In this scenario flows are driven entirely by lateral density variations. Curling the Navier–Stokes equation gives the vorticity equation

$$2\rho_0 \nabla \times (\boldsymbol{\Omega} \times \mathbf{u}) = \nabla \times (\rho' \mathbf{g}) \tag{53}$$

or, if density variations are interpreted in terms of variations in temperature Θ from the adiabatic background state where

$$\rho = \rho_0(1 - \alpha\Theta) \tag{54}$$

the vorticity equation becomes

$$\frac{2}{g\alpha}(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u}_h = -\nabla_h \times (\Theta \hat{\mathbf{r}}). \tag{55}$$

The temperature variations Θ can be found uniquely from (55) outside the ambiguous patches. The two geostrophic constraints introduced in (44) and (45) are sufficient for (55) to be satisfied exactly by a flow $\{\mathbf{u}, \mathbf{u}'\}$. The requirement of vanishing radial vorticity leads directly to (44), whilst requiring the horizontal component of (55) to be purely toroidal leads to (45).

4 RESULTS AND DISCUSSION

We have constructed solutions using a smoothly varying time-dependent model of the magnetic field (unpublished) in which the field is represented spatially by spherical harmonics to degree and order 14 and temporally by cubic B-splines. We calculate the magnetic field at the CMB as a potential field; as discussed we could correct for mantle conductivity, but chose not to. This does not imply that we believe the mantle to be an insulator, and thus we place no constraints on the toroidal SV. We seek the smoothest possible flow and shear compatible with the magnetic field; the same regularizing condition as was used on the flow by Bloxham (1988, 1989) is applied to the fields $\mathbf{u}$ and $c\mathbf{u}'$, using the same damping parameter for both.

Figures 2–4 show steady solutions for the flow and shear constructed over the time-span 1960–1980. The shear has been scaled by the core radius $c = 3485$ km to give the same units as the velocity. We will not dwell on describing the flow, suffice to mention that the pattern of flow is very similar to that described by several authors (e.g. Voorhies 1986; Whaler & Clarke 1988; Bloxham 1988, 1989), the differences being much more pronounced in the poloidal component rather than the toroidal component of the flow. Poloidal motions are not particularly important in fitting the secular variation, because the flows in Fig. 2 and 3 both explain over 90 per cent of the variance in the data, whereas there is no poloidal ingredient to Fig. 3.

Since the poloidal ingredient of the flow is poorly determined, two options avail themselves. First it can be set to zero (Fig. 3), the toroidal motions hypothesis, or the flow can be assumed to be geostrophic (Fig. 4); the poloidal motions are then fixed uniquely because they are determined by the toroidal members of the basis (Backus & LeMouël 1986).

In each of the solutions we find the shear (scaled as

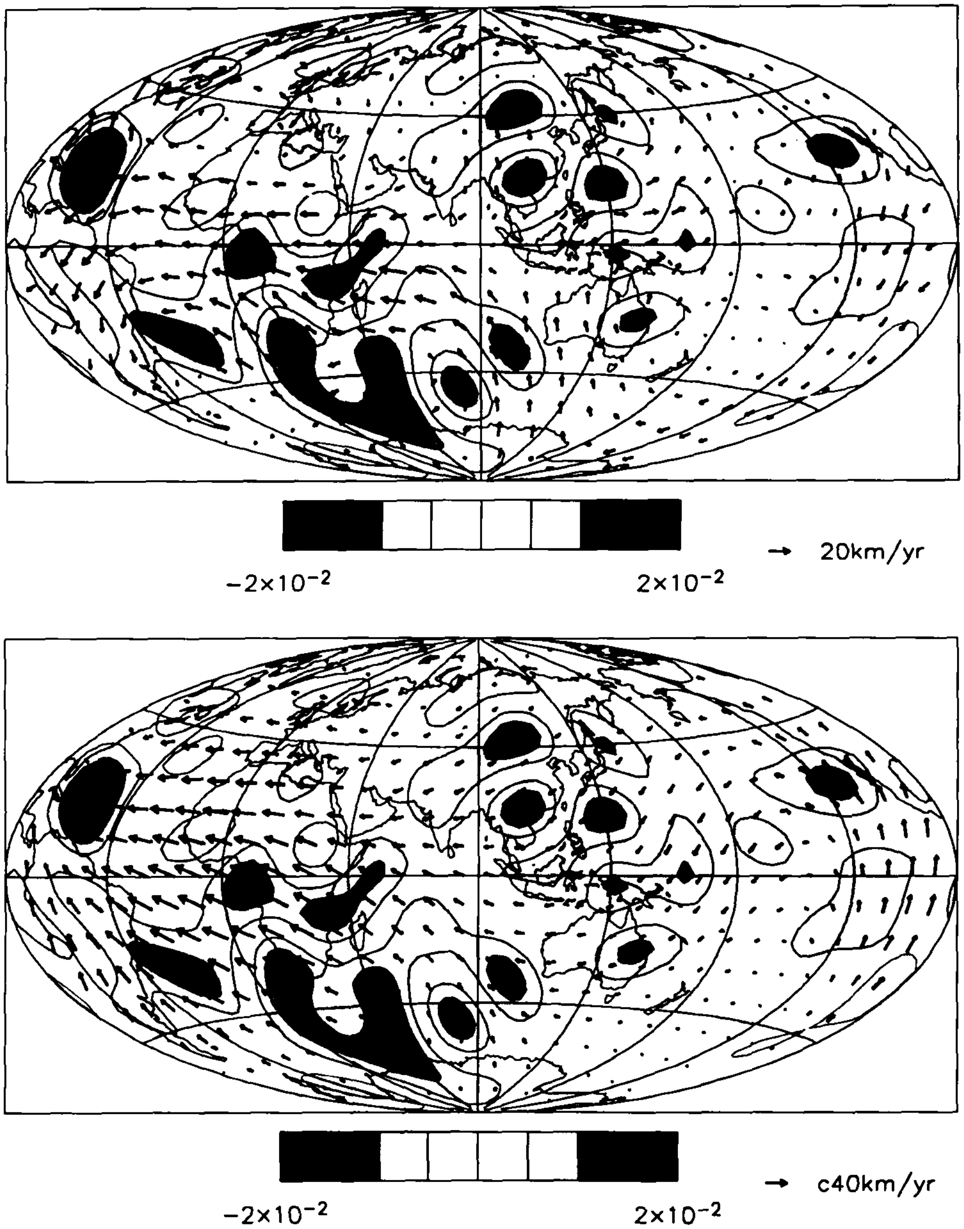

Figure 2. Steady unconstrained velocity (a) and shear (b) for the period 1960–1980. The vectors show the speed and direction of the flow at the core surface and the grey scale the intensity of the horizontal divergence (upwelling and downwelling) of the flow (in units of yr^{-1}). The sign of the horizontal divergence can be determined from the flow vectors using the fact that the flow is incompressible. To relate the shear to flow, the shear has been multiplied by the radius of the CMB and the scale of the arrows is changed. The projection is Aitoff equal area.

described above) to be approximately twice as strong as the flow. In most areas of the maps, especially where the flow is strongest, we find the flow and shear to be almost parallel. This correspondence is significant because it means that the flow decays with depth with very little change in orientation; the length-scale of this decay is of the order of half the core radius. This suggests that the flow at the core surface is highly correlated with the flow deep within the core, rather than being a localized effect; this result appears robust to whether we assume the flow to be geostrophic or toroidal. The root-mean-square values for the flow and shear for the three models are given in Table 1.

This result for the shear is in some disagreement with that reported by Lloyd & Gubbins (1990) who, as we have mentioned, report a vertical length-scale of about 600 km for the flow; our estimate is almost three times their figure. This difference is important: 600 km could be reasonably considered consistent with flow in a weakly stratified upper layer of the core (by weakly stratified we mean Brunt–Väisälä period greater than 12 hr) and Lloyd & Gubbins argue that this flow could be largely driven by thermal anomalies in the mantle near the CMB (as envisaged by Bloxham & Gubbins 1987), whilst our estimate implies whole core convection. Whole core convection could be driven both from above (by thermal anomalies in the mantle) and internally (by growth of the inner core or

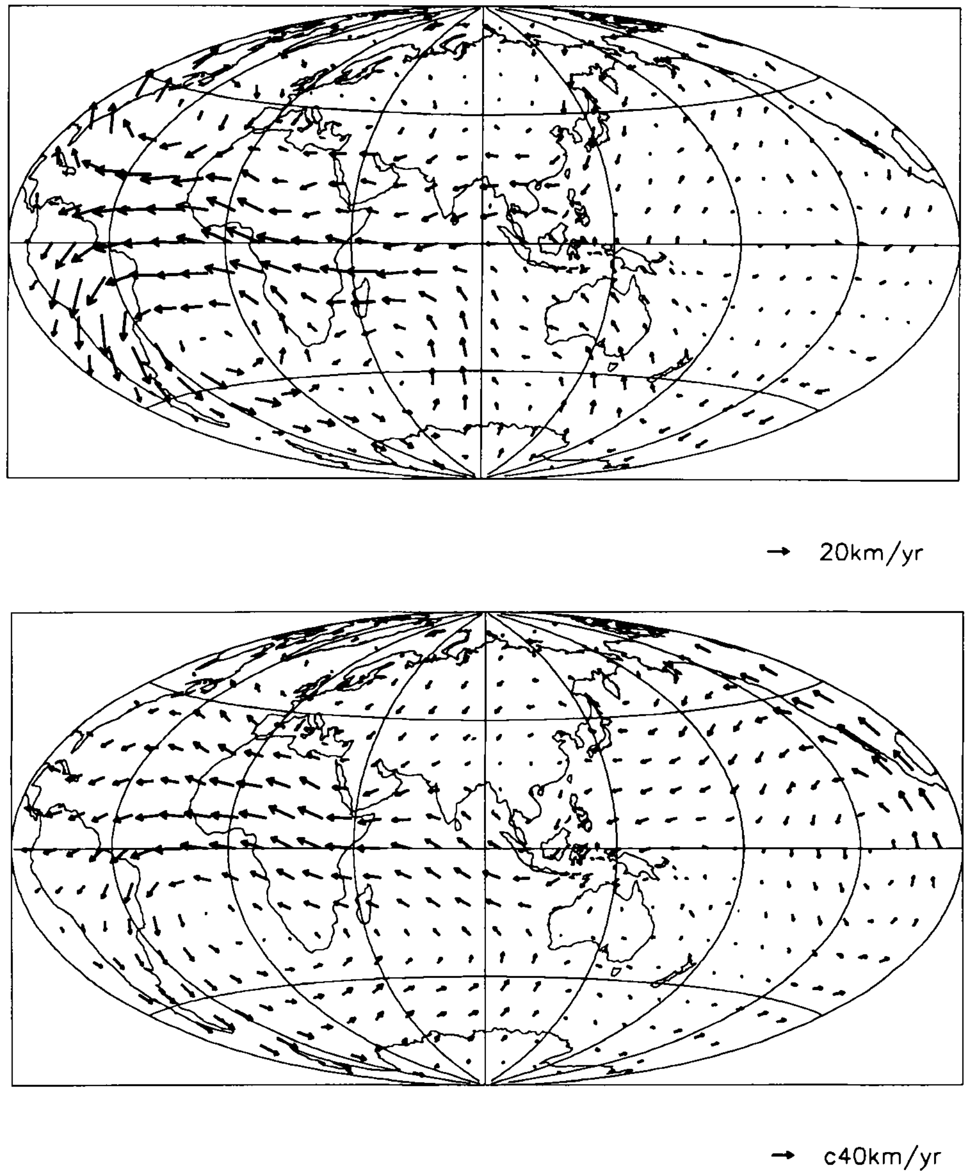

Figure 3. Steady toroidal velocity (a) and shear (b) for the interval 1960–1980.

radiogenic heating in the core). An investigation of the relative importance of these two contributions can be found in Bloxham & Jackson (1990).

5 CONCLUSIONS

We have adopted the frozen flux hypothesis which ascribes the observed SV entirely to the effects of advection. All three components of magnetic field are continuous at the CMB ($r = c$), the boundary between the mantle and core with possibly different conductivities of $\sigma_m(c)$ and $\sigma_c(c)$ respectively. At $r = c$, stress-free boundary conditions require the velocity $\mathbf{u}$ to vanish, and a thin viscous boundary layer is set up in which sheet currents can flow, causing a jump (denoted $[\mathbf{B}]$) in the value of $\mathbf{B}$ between the value at the top of the free stream and the value at the CMB. Certainly $[B_r]$ is negligible, and theoretical and observational evidence suggest $[\mathbf{B}_h]$ is small enough to be consistent with other approximations, although this is the most equivocal point of our argument. We note that, even if $\sigma_m(c) \neq 0$ at the CMB, with the consequence that a large toroidal field is permitted at $r = c$, because $u_r = 0$ at $r = c$ the induction equation separates and the poloidal SV depends solely on the poloidal field. In considering the problem of reconstructing the flow $\mathbf{u}$ and shear $\mathbf{u}'$ at the CMB given only the knowledge of poloidal SV, we show that determinations of $\mathbf{u}$ and $\mathbf{u}'$ suffer from toroidal ambiguities in the fields $\mathbf{u}B_r$ and $\mathbf{u}'B_r$ respectively. Further assumptions are necessary to resolve the non-uniqueness, and indeed assuming the flow and shear to be steady resolves the non-uniqueness everywhere, whilst assuming the flow to be geostrophic resolves the flow and shear everywhere on the core surface except within certain ambiguous patches, which cover 40 per cent of the core surface in 1980.

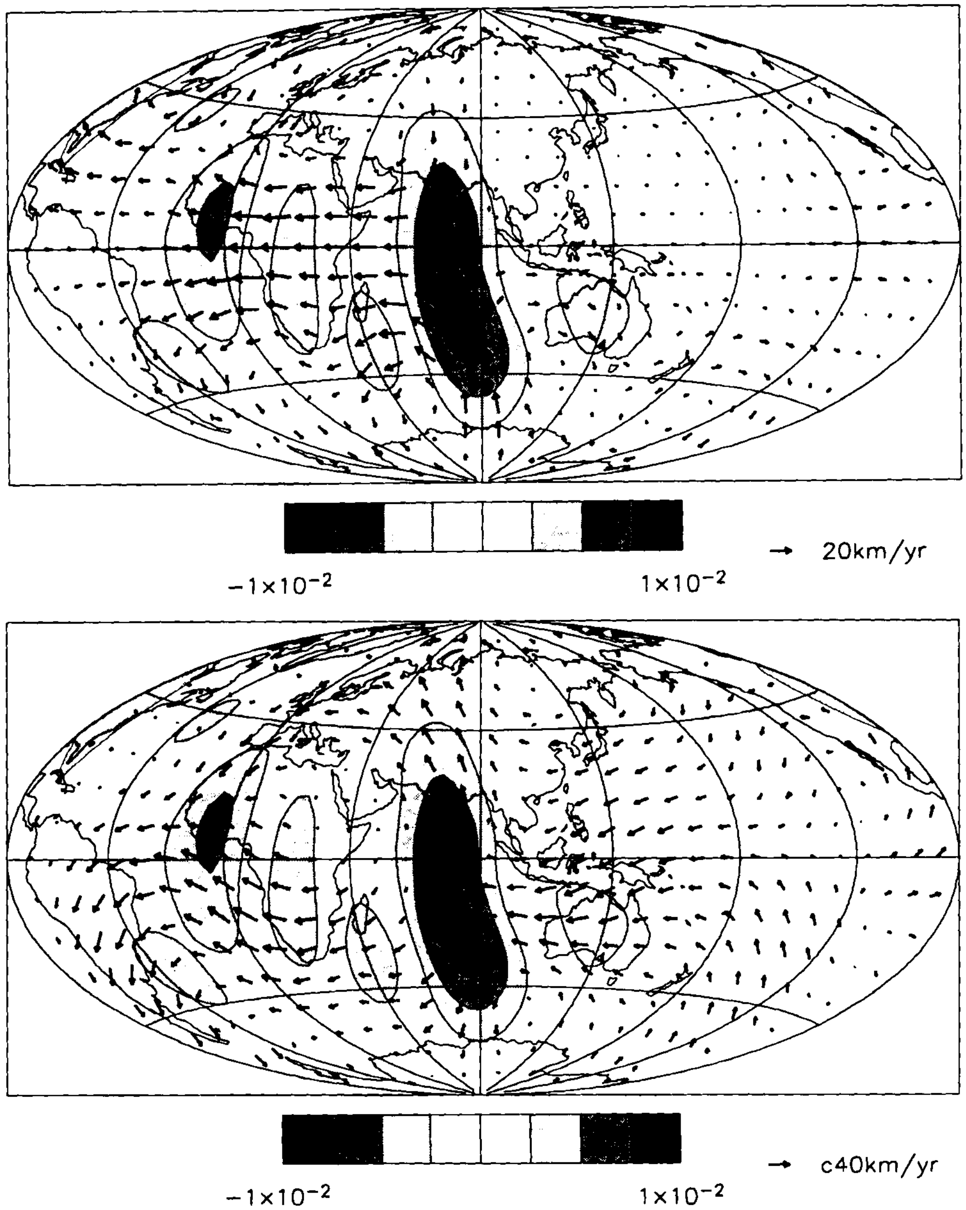

Figure 4. Steady fully geostrophic velocity (a) and shear (b) for the interval 1960–1980. Note the change of grey-scale from that in Fig. 2.

Reconstructing the poloidal magnetic field at the CMB requires a model of mantle conductivity, which is poorly known. If we adopt the results of induction studies and recent high-pressure experiments, then probably $\sigma_m(c) \sim 200\,\mathrm{S\,m^{-1}}$, and on the time-scale appropriate to the secular variation we commit a *numerical* error of less than 10 per cent in approximating the poloidal field with its insulating value. Calculations for steady flows and shears indicate that the two are highly correlated at the core surface and the flow decays with depth on a length-scale approximately equal to half the core radius; these results are fairly robust to the other restrictions which we place on the flow.

Table 1. Root mean square values of the velocity $\langle v \rangle$ and scaled shear $\langle cv' \rangle$ for the models of Figs 2–4.

	Unconstrained	Toroidal	Geostrophic
$\langle v \rangle$/km yr^{-1}	10.1	13.9	10.2
$\langle cv' \rangle$/km yr^{-1}	24.5	23.5	21.6

Responding to our initial question, we believe on the basis of this study that it is at least possible that the pattern of flow deeper within the core may not be vastly different from that imaged near the core surface. This result is not entirely surprising since studies of the main magnetic field suggest that the pattern of the steady part of the field at the CMB reveals the effect of the inner core (Gubbins & Bloxham 1987), which would be rather unexpected if the flow at the core surface were vastly different from that deeper within the core. This is encouraging for it implies that both the steady part of the field and the flow may be the result of core-wide processes, and not necessarily far-removed from the true dynamo-generated fields, although they are probably somewhat perturbed by processes near the CMB.

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APPENDIX: MATRIX ELEMENTS

In this section we derive the elements of the matrices H_r and H_h in equations (19) and (20). The derivation of the elements of H_r has been given previously by a number of authors; see for example Whaler (1986). The derivation of the elements of H_h has not been previously given. We derive explicit closed form relations for the elements. We find that this is most readily accomplished by using the complex canonical basis of Burridge (1969) and Phinney & Burridge (1973); for an application to the magnetic induction equation see James (1974).

We begin from the usual spherical polar coordinates ordered (θ, ϕ, r) defined by basis vectors $\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r$. Following Burridge

(1969) we define the complex canonical basis

$$\mathbf{e}_- = \frac{1}{\sqrt{2}}(\mathbf{e}_\theta - i\mathbf{e}_\phi) \quad \text{(A1)}$$

$$\mathbf{e}_0 = \mathbf{e}_r \quad \text{(A2)}$$

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}(-\mathbf{e}_\theta - i\mathbf{e}_\phi) \quad \text{(A3)}$$

where $i = \sqrt{-1}$.
The contravariant components x^α of $\mathbf{x}$ may then be expanded

$$x^\alpha = \sum_{lm} X_l^{\alpha m}(r) Y_l^{\alpha m}(\theta, \phi) \quad \text{(A4)}$$

where the generalized spherical harmonics $Y_l^{\alpha m}(\theta,\phi)$ are as defined by Phinney & Burridge, with

$$\int_0^{2\pi}\int_0^{\pi} Y_l^{Nm}(\theta,\phi)\left(Y_{l'}^{Nm'}\right)^*(\theta,\phi)\sin\theta\, d\theta\, d\phi = \left(\frac{4\pi}{2l+1}\right)\delta_{ll'}\delta_{mm'} \quad \text{(A5)}$$

and

$$\int_0^{2\pi}\int_0^{\pi} \left(Y_l^{Nm}\right)^* Y_{l'}^{N'm'} Y_{l''}^{N''m''} \sin\theta\, d\theta\, d\phi = 4\pi(-1)^{N-m}\begin{pmatrix} l & l' & l'' \\ -N & N' & N'' \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ -m & m' & m'' \end{pmatrix} \quad \text{(A6)}$$

and the Wigner 3-j symbol is as defined by Edmonds (1960).
Rules for differentiation are given by Phinney & Burridge and Mochizuki (1988). We have

$$x^{\alpha,\beta} = \sum_{lm} X_l^{\alpha|\beta m}(r) Y_l^{(\alpha+\beta)m}(\theta,\phi) \quad \text{(A7)}$$

where

$$\begin{aligned} X_l^{\alpha|-m} &= \frac{1}{r}\left(\Omega_\alpha^l X_l^{\alpha m} + e_{\alpha -}X_l^{0m} - e_{\alpha 0}X_l^{-m}\right) \\ X_l^{\alpha|0m} &= \frac{d}{dr}X_l^{\alpha m} \\ X_l^{\alpha|+m} &= \frac{1}{r}\left(\Omega_{\alpha+1}^l X_l^{\alpha m} + e_{\alpha +}X_l^{0m} - e_{\alpha 0}X_l^{+m}\right) \end{aligned} \quad \text{(A8)}$$

where $\Omega_\alpha^l = \sqrt{[(l+\alpha)(l-\alpha+1)/2]}$, and $e_{\alpha\beta}$ are the covariant components of the isotropic tensor δ_{ij}.

Specializing to the case of solenoidal fields, $\mathbf{x}$ may be decomposed into toroidal and poloidal ingredients, which may be expanded in vector spherical harmonics (see §2). We depart slightly from the notation of §2 by using complex spherical harmonic expansions. If $\mathbf{x}$ has defining scalars $\mathcal{T}$ and $\mathcal{P}$, we write

$$\mathcal{T} = \sum_{lm} t_l^m(r) Y_l^m(\theta,\phi) \quad \text{(A9)}$$

and likewise for $\mathcal{P}$, where $Y_l^m = Y_l^{0m}$, and ${}_ct_1^0 = \mathsf{Re}(t_1^0)$, ${}_ct_l^m = 2\mathsf{Re}(t_l^m)$, and ${}_st_l^m = -2\mathsf{Im}(t_l^m)$. Here, we use t_l^m and $s_l^m = dp_l^m/dr$ for the expansion of the velocity potentials, and P_l^m for the poloidal magnetic field.
Mochizuki shows how the coefficients of the expansions of the toroidal and poloidal scalars are related to the generalized spherical harmonic exapnsion coefficients. For the toroidal part, we have

$$\begin{aligned} X_l^{0m} &= 0 \\ X_l^{\pm m} &= \pm i\Omega_0^l \frac{1}{r} t_l^m \end{aligned} \quad \text{(A10)}$$

and for the poloidal part

$$\begin{aligned} X_l^{0m} &= \frac{l(l+1)}{r^2} p_l^m \\ X_l^{\pm m} &= \Omega_0^l \frac{1}{r}\frac{dp_l^m}{dr} \end{aligned} \quad \text{(A11)}$$

Note in particular that the two horizontal canonical components of a poloidal vector are identical; hence we obtain only one equation for the horizontal component of the poloidal magnetic induction equation.

Writing the frozen flux induction equation in the form

$$\partial_t \mathbf{B} = \mathbf{B}.\nabla\mathbf{u} - \mathbf{u}.\nabla\mathbf{B} \quad \text{(A12)}$$

we have

$$\partial_t B^\alpha = \left(B^\beta U^{\alpha,\gamma} - U^\beta B^{\alpha,\gamma}\right) e_{\beta\gamma} \quad \text{(A13)}$$

We then expand the contravariant components B^α of $\mathbf{B}$ and U^α of $\mathbf{u}$.
Considering the 0-component of $\partial_t B^\alpha$, and applying the condition $u_r = 0$, we obtain

$$\sum_1 \partial_t B_{l_1}^{0m_1} Y_{l_1}^{0m_1} = \sum_{2,3} \frac{dU_{l_2}^{0m_2}}{dr} B_{l_3}^{0m_3} Y_{l_2}^{0m_2} Y_{l_3}^{0m_3} + \sum_{2,3} \frac{1}{r} \Omega_0^{l_3} U_{l_2}^{+m_2} B_{l_3}^{0m_3} Y_{l_2}^{+m_2} Y_{l_3}^{-m_3} + \sum_{2,3} \frac{1}{r} \Omega_1^{l_3} U_{l_2}^{-m_2} B_{l_3}^{0m_3} Y_{l_2}^{-m_2} Y_{l_3}^{+m_3} \quad \text{(A14)}$$

Multiplying through by $\left(Y_{l_1}^{0m_1}\right)^*$, and integrating over the unit sphere, we obtain

$$\frac{l_1(l_1+1)}{2l_1+1} \partial_t P_{l_1}^{m_1} = (-1)^{m_1} \sum_{2,3} l_3(l_3+1) P_{l_3}^{m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \times$$
$$\left\{ \frac{dU_{l_2}^{0m_2}}{dr} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{r}\sqrt{\frac{l_3(l_3+1)}{2}} \left[U_{l_2}^{+m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 1 & -1 \end{pmatrix} + U_{l_2}^{-m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & -1 & 1 \end{pmatrix} \right] \right\} \quad \text{(A15)}$$

Then for a toroidal velocity $\mathbf{u}$ we obtain

$$\frac{l_1(l_1+1)}{2l_1+1} \partial_t P_{l_1}^{m_1} = i\frac{(-1)^{m_1}}{2r^2} \sum_{2,3} l_3(l_3+1) t_{l_2}^{m_2} P_{l_3}^{m_3} E(l_1, l_2, l_3, -m_1, m_2, m_3) \quad \text{(A16)}$$

where

$$E(l_1, l_2, l_3, m_1, m_2, m_3) =$$
$$[(l_1+l_2+l_3-1)(l_1+l_2+l_3+1)]^{1/2} \left[\frac{(-l_1+l_2+l_3)(l_1-l_2+l_3)(l_1+l_2-l_3)}{l_1+l_2+l_3} \right]^{1/2}$$
$$\times \begin{pmatrix} l_1-1 & l_2-1 & l_3-1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \text{(A17)}$$

and for a poloidal velocity $\mathbf{u}$ we obtain

$$\frac{l_1(l_1+1)}{2l_1+1} \partial_t P_{l_1}^{m_1} = \frac{(-1)^{m_1}}{2r^2} \sum_{2,3} l_3(l_3+1) s_{l_2}^{m_2} P_{l_3}^{m_3} \left[l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1) \right] G(l_1, l_2, l_3, -m_1, m_2, m_3) \quad \text{(A18)}$$

where

$$G(l_1, l_2, l_3, m_1, m_2, m_3) = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \text{(A19)}$$

For the special case where $\mathbf{B}$ is potential, with $\mathbf{B} = -\nabla\Phi$ with

$$\Phi = a \sum_{lm} \left(\frac{a}{r}\right)^{l+1} \phi_l^m Y_l^m(\theta, \phi) \quad \text{(A20)}$$

we have for a toroidal velocity

$$\frac{l_1+1}{2l_1+1} \left(\frac{a}{r}\right)^{l_1+2} \partial_t \phi_{l_1}^{m_1} = i\frac{(-1)^{m_1}}{2r^2} \sum_{2,3} (l_3+1) t_{l_2}^{m_2} \left(\frac{a}{r}\right)^{l_3+2} \phi_{l_3}^{m_3} E(l_1, l_2, l_3, -m_1, m_2, m_3) \quad \text{(A21)}$$

and for a poloidal velocity

$$\frac{l_1+1}{2l_1+1} \left(\frac{a}{r}\right)^{l_1+2} \partial_t \phi_{l_1}^{m_1} = \frac{(-1)^{m_1}}{2r^2} \sum_{2,3} (l_3+1) s_{l_2}^{m_2} \left(\frac{a}{r}\right)^{l_3+2} \phi_{l_3}^{m_3} \left[l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1) \right] G(l_1, l_2, l_3, -m_1, m_2, m_3) \quad \text{(A22)}$$

Next we consider the +-component $\partial_t B^\alpha$, apply the condition $u_r = 0$, and obtain

$$\sum_1 \partial_t B_{l_1}^{+m_1} Y_{l_1}^{+m_1} =$$

$$\sum_{2,3} \frac{dU_{l_2}^{+m_2}}{dr} B_{l_3}^{0m_3} Y_{l_2}^{+m_2} Y_{l_3}^{0m_3} - \sum_{2,3} \frac{1}{r} \Omega_1^{l_2} U_{l_2}^{+m_2} B_{l_3}^{+m_3} Y_{l_2}^{0m_2} Y_{l_3}^{+m_3} + \sum_{2,3} \frac{1}{r} \Omega_1^{l_3} U_{l_2}^{+m_2} B_{l_3}^{+m_3} Y_{l_2}^{+m_2} Y_{l_3}^{0m_3} -$$
$$\sum_{2,3} \frac{1}{r} U_{l_2}^{+m_2} B_{l_3}^{0m_3} Y_{l_2}^{+m_2} Y_{l_3}^{0m_3} - \sum_{2,3} \frac{1}{r} \Omega_2^{l_2} U_{l_2}^{+m_2} B_{l_3}^{-m_3} Y_{l_2}^{+2m_2} Y_{l_3}^{-m_3} + \sum_{2,3} \frac{1}{r} \Omega_2^{l_3} U_{l_2}^{-m_2} B_{l_3}^{+m_3} Y_{l_2}^{-m_2} Y_{l_3}^{+2m_3} \tag{A23}$$

Multiplying through by $\left(Y_{l_1}^{0m_1}\right)^*$, integrating over the unit sphere, and considering only the poloidal part of **B**, we obtain

$$\frac{1}{2l_1+1}\sqrt{\frac{l_1(l_1+1)}{2}} \frac{d}{dr} \partial_t P_{l_1}^{m_1} = \frac{(-1)^{1-m_1}}{2r} \sum_{2,3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \times \Bigg\{$$
$$2l_3(l_3+1) \frac{dU_{l_2}^{+m_2}}{dr} P_{l_3}^{m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & 1 & 0 \end{pmatrix} - \sqrt{l_2(l_2+1)}\sqrt{l_3(l_3+1)} U_{l_2}^{+m_2} \frac{dP_{l_3}^{m_3}}{dr} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & 0 & 1 \end{pmatrix} +$$
$$l_3(l_3+1) U_{l_2}^{+m_2} \frac{dP_{l_3}^{m_3}}{dr} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & 1 & 0 \end{pmatrix} - \frac{2}{r} l_3(l_3+1) U_{l_2}^{+m_2} P_{l_3}^{m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & 1 & 0 \end{pmatrix} -$$
$$\sqrt{(l_2-1)(l_2+2)}\sqrt{l_3(l_3+1)} U_{l_2}^{+m_2} \frac{dP_{l_3}^{m_3}}{dr} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & 2 & -1 \end{pmatrix} +$$
$$\sqrt{l_3(l_3+1)}\sqrt{(l_3-1)(l_3+2)} U_{l_2}^{-m_2} \frac{dP_{l_3}^{m_3}}{dr} \begin{pmatrix} l_1 & l_2 & l_3 \\ -1 & -1 & 2 \end{pmatrix} \Bigg\} \tag{A24}$$

Then for a toroidal velocity **u** we obtain

$$\frac{l_1(l_1+1)}{2l_1+1} \frac{d}{dr} \partial_t P_{l_1}^{m_1} = i \frac{(-1)^{m_1}}{2r^2} \sum_{2,3} l_3(l_3+1) \left(\frac{dt_{l_2}^{m_2}}{dr} P_{l_3}^{m_3} - \frac{2}{r} t_{l_2}^{m_2} P_{l_3}^{m_3} + t_{l_2}^{m_2} \frac{dP_{l_3}^{m_3}}{dr} \right) E(l_1, l_2, l_3, -m_1, m_2, m_3) \tag{A25}$$

and for a poloidal velocity **u** we obtain

$$\frac{l_1(l_1+1)}{2l_1+1} \frac{d}{dr} \partial_t P_{l_1}^{m_1} =$$
$$\frac{(-1)^{m_1}}{2r^2} \sum_{2,3} \Bigg\{ l_3(l_3+1) \left(l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1)\right) \left(\frac{ds_{l_2}^{m_2}}{dr} P_{l_3}^{m_3} - \frac{2}{r} s_{l_2}^{m_2} P_{l_3}^{m_3} + s_{l_2}^{m_2} \frac{dP_{l_3}^{m_3}}{dr} \right) -$$
$$l_2(l_2+1) \left(l_1(l_1+1) + l_3(l_3+1) - l_2(l_2+1)\right) s_{l_2}^{m_2} \frac{dP_{l_3}^{m_3}}{dr} \Bigg\} G(l_1, l_2, l_3, -m_1, m_2, m_3) \tag{A26}$$

For the special case where **B** is potential, we have for a toroidal velocity

$$\frac{l_1+1}{2l_1+1} \left(\frac{a}{r}\right)^{l_1+2} \partial_t \phi_{l_1}^{m_1} = i \frac{(-1)^{m_1}}{2r^2} \sum_{2,3} (l_3+1) \left((l_3+2) t_{l_2}^{m_2} - r \frac{dt_{l_2}^{m_2}}{dr} \right) \left(\frac{a}{r}\right)^{l_3+2} \phi_{l_3}^{m_3} E(l_1, l_2, l_3, -m_1, m_2, m_3) \tag{A27}$$

and for a poloidal velocity

$$\frac{l_1+1}{2l_1+1} \left(\frac{a}{r}\right)^{l_1+2} \partial_t \phi_{l_1}^{m_1} =$$
$$\frac{(-1)^{m_1}}{2r^2} \sum_{2,3} \Bigg\{ s_{l_2}^{m_2} \left[(l_3+1)(l_3+2)\left(l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1)\right) - l_2(l_2+1)\left(l_1(l_1+1) + l_3(l_3+1) - l_2(l_2+1)\right) \right]$$
$$- r \frac{ds_{l_2}^{m_2}}{dr} (l_3+1) \left(l_1(l_1+1) + l_2(l_2+1) - l_3(l_3+1)\right) \Bigg\} \left(\frac{a}{r}\right)^{l_3+2} \phi_{l_3}^{m_3} G(l_1, l_2, l_3, -m_1, m_2, m_3) \tag{A28}$$