J. Math. Soc. Japan
Vol. 14, No. 1, 1962

Mappings defined on 0-dimensional spaces and dimension theory

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(Received July 24, 1961)

§1. Introduction.

The following is a well-known Hurewicz-Kuratowski's theorem for separable metric spaces R and A (W. Hurewicz [**4**], C. Kuratowski [**6**; **7**]):

In order that a non-empty space R has the covering dimension $\leqq n$, it is necessary and sufficient that there exist a space A with $\dim A=0$ and a closed continuous mapping f of A onto R such that the order of f is at most $n+1$.

In the above $\dim A$ denotes the covering dimension of A, and the order of f is the supremum of $\{|f^{-1}(x)| ; x\in R\}$, where $|f^{-1}(x)|$ are the cardinal numbers of the sets $f^{-1}(x)$. This theorem has been extended by K. Morita [**14**] to the case when R and A are metric spaces. The classical Hurewicz-Kuratowski's theorem had been rather isolated from the general trends of dimension theory for separable metric spaces. In the framework of dimension theory for general metric spaces which has been constructed by the author this theorem occupies an important position [**17**, §3]. It seems to the author that closed mappings defined on 0-dimensional spaces will be one of powerful instruments to clear up the relation between the covering dimension and the inductive one of non-separable spaces.

In §§2 and 3 we shall characterize a non-metrizable space R which has the following property:

(*) *R is the image of a 0-dimensional space under a closed continuous mapping of order $\leqq n+1$.*

It will be shown that a space has this property if and only if there exists a directed family of closed coverings of order $\leqq n+1$, which follows out the topology of a space (cf. Definitions 2.1 and 2.2 below). We shall notice in §4 that the inductive dimension of a space which admits a directed family with the property stated above cannot be greater than n. It is to be noted that Theorem 4.1 below has been obtained independently by Soviet mathematicians, I. Proskuryakov—B. Ponomarev—B. Pasynkov, under a more restrictive assumption (P. Alexandroff [**1**, p. 80], B. Pasynkov [**21**]). It is also to be noted that Corollaries 4.2 and 4.4 had been essentially proved by K. Morita (cf. Remark 4.7). As an immediate consequence of our results it will be shown, with the aid of examples constructed by Lunz and others, that we cannot expect that Hurewicz-Kuratowski's theorem may be valid even for the case when R is a compact Hausdorff space (Remark 4.11 below). In §§5 and 6 we shall give analogous theorems to Hurewicz-Kuratowski's one for the case when R is a non-metrizable space, by introducing the notion 'vague order', instead of 'order', of mappings. In §7 we shall prove that any CW-complex R whose combinatorial dimension is n has the property (*).

This paper includes a development in detail of our brief note [**16**]. The author wishes to thank very much Professor K. Morita for his advice and encouragement. He also expresses here his hearty thanks to Mr. Y. Sasaki who was kind enough to translate voluminous Russian literature.

§2. Construction of mappings defined on 0-dimensional spaces, from directed families.

Let R be a topological space. The small and the large inductive dimension, $\operatorname{ind} R$ and $\operatorname{Ind} R$, are defined inductively as follows. For the empty set ϕ let $\operatorname{ind}\phi=\operatorname{Ind}\phi=-1$. We call $\operatorname{ind} R\leqq n$, if for any point x of R and any neighborhood G of x there exists an open neighborhood H of x with $H\subset G$ such that $\operatorname{ind}(\bar{H}-H)\leqq n-1$. We call $\operatorname{Ind} R\leqq n$, if for any pair $F\subset G$ of a closed set F and an open set G there exists an open set H with $F\subset H\subset G$ such that $\operatorname{Ind}(\bar{H}-H)\leqq n-1$.

Let $\mathfrak{F}=\{F_\alpha;\alpha\in A\}$ be a collection of subsets of R and x a point of R. Then the order of $\mathfrak{F}$ at x, order $(x,\mathfrak{F})$, is the number of elements of $\mathfrak{F}$ which contain x. The order of $\mathfrak{F}$, order $\mathfrak{F}$, is the supremum of $\{\text{order}(x,\mathfrak{F});x\in R\}$. Let H be a subset of R. Then the star of H with respect to $\mathfrak{F}$, $S(H,\mathfrak{F})$, is the sum of $F_\alpha\in\mathfrak{F}$ with $H\frown F_\alpha\neq\phi$. The restriction of $\mathfrak{F}$ to H, $\mathfrak{F}\wedge H$, is the collection $\{F_\alpha\frown H;\alpha\in A\}$. $\bar{\mathfrak{F}}$ denotes a closed collection $\{\bar{F}_\alpha;\alpha\in A\}$. Let $\mathfrak{H}=\{H_\beta;\beta\in B\}$ be another collection of subsets of R. A mapping φ of A into B is called a refine-mapping if for any $\alpha\in A$, $F_\alpha\subset H_{\varphi(\alpha)}$ is valid. When there is a refine-mapping $\varphi:A\to B$, we say that $\mathfrak{H}$ is refined by $\mathfrak{F}$ or abbreviatedly $\mathfrak{H}>\mathfrak{F}$. Let $\boldsymbol{F}=\{\mathfrak{F}_\lambda;\lambda\in\Lambda\}$ be a system of collections of subsets of a space R. Then the order of $\boldsymbol{F}$, order $\boldsymbol{F}$, is the supremum of $\{\text{order}\,\mathfrak{F}_\lambda;\lambda\in\Lambda\}$.

DEFINITION 2.1. Let $\boldsymbol{F}=\{\mathfrak{F}_\lambda;\lambda\in\Lambda\}$ be a system of collections of subsets of a topological space R. $\boldsymbol{F}$ is called to *follow out* (the topology of) R *locally, globally* and *fully* if the following conditions are respectively satisfied.

(1) For any point x of R and any open set G with $x\in G$ there exists a $\lambda\in\Lambda$ with $S(x,\mathfrak{F}_\lambda)\subset G$.

(2) For any pair $F\subset G$ of a closed set F and an open set G of R there exists a $\lambda\in\Lambda$ with $S(F,\mathfrak{F}_\lambda)\subset G$.

(3) For any open covering $\mathfrak{G}$ of R there exists a $\lambda \in \Lambda$ with $\mathfrak{G} > \mathfrak{F}_\lambda$.

DEFINITION 2.2. Let $\boldsymbol{F} = \{\mathfrak{F}_\lambda = \{F_\alpha ; \alpha \in A_\lambda\} ; \lambda \in \Lambda\}$ be a system of collections of subsets of a topological space R. $\boldsymbol{F}$ is called a *directed family with* $\{A_\lambda, \varphi_{\lambda\mu}\}$ if the following three conditions are satisfied.

(4) Λ is a directed set.

(5) For any ordered pair $\mu < \lambda$ there exists a mapping $\varphi_{\lambda\mu} : A_\lambda \to A_\mu$ such that $\{A_\lambda, \varphi_{\lambda\mu} ; \lambda \in \Lambda\}$ forms an inverse limiting system of A_λ.

(6) For any ordered pair $\mu < \lambda$ and any $\alpha \in A_\mu$ it holds that

$$F_\alpha = \bigcup \{F_\beta ; \varphi_{\lambda\mu}(\beta) = \alpha\} .$$

THEOREM 2.3. *If a non-empty topological space R has a directed family $\boldsymbol{F} = \{\mathfrak{F}_\lambda = \{F_\alpha ; \alpha \in A_\lambda\} ; \lambda \in \Lambda\}$, with an inverse limiting system $\{A_\lambda, \varphi_{\lambda\mu}\}$, of locally finite closed coverings of order $\leqq n+1$ which follows out R locally, then there exist a completely regular space A with* $\operatorname{ind} A = 0$ *and a closed continuous mapping f of A onto R with* $\operatorname{order} f \leqq n+1$.

PROOF. Consider A_λ, $\lambda \in \Lambda$, as topological spaces with the discrete topology. Let B be the limit space of $\{A_\lambda, \varphi_{\lambda\mu}\}$. Let x be an arbitrary point of R and $B_\lambda = \{\alpha ; x \in F_\alpha \in \mathfrak{F}_\lambda\}$, $\lambda \in \Lambda$. Then for any $\lambda \in \Lambda$, B_λ is a non-empty finite subset of A_λ. Moreover for any $\mu < \lambda$ we have $\varphi_{\lambda\mu}(B_\lambda) \subset B_\mu$. Hence $\{B_\lambda, \varphi_{\lambda\mu} | B_\lambda\}$ forms an inverse limiting system and we have $\lim \{B_\lambda, \varphi_{\lambda\mu} | B_\lambda\} \neq \phi$. Let A be the aggregate of points $a = (\alpha_\lambda ; \lambda \in \Lambda) \in B$ such that $\bigcap \{F_{\alpha_\lambda} ; \lambda \in \Lambda\} \neq \phi$. Then A is a completely regular space with $\operatorname{ind} A = 0$.

Define $f : A \to R$ such as $f(a) = \bigcap \{F_{\pi_\lambda(a)} ; \lambda \in \Lambda\}$, where π_λ, $\lambda \in \Lambda$, are the projections of A into A_λ. Then f is onto from the above observation. Moreover f is continuous, since $\boldsymbol{F}$ follows out R locally. To prove $\operatorname{order} f \leqq n+1$, assume that there exists a point x of R such that $|f^{-1}(x)| > n+1$. Let $\{a_i ; i = 1, \cdots, n+2\}$ be a system of mutually different points of A with $f(a_i) = x$, $i = 1, 2, \cdots, n+2$. Let λ be an index of Λ such that $\{\pi_\lambda(a_i) ; i = 1, \cdots, n+2\}$ forms a system of mutually different indices of A_λ. Then the order of $\mathfrak{F}_\lambda$ at x is not less than $n+2$, which is a contradiction. Hence we have $\operatorname{order} f \leqq n+1$.

To prove the closedness of f, let C be an arbitrary non-empty closed subset of A and x an arbitrary point of $\overline{f(C)}$. Let $D_\lambda = \{\alpha ; x \in F_\alpha \in \mathfrak{F}_\lambda, f(C) \cap F_\alpha \neq \phi\}$, $\lambda \in \Lambda$; then $|D_\lambda| < \infty$. Since $\mathrm{S}(x, \mathfrak{F}_\lambda)$ contains $R - \bigcap \{F_\alpha ; x \notin F_\alpha \in \mathfrak{F}_\lambda\} = U_\lambda$ and the latter is an open neighborhood of x by the local finiteness of $\mathfrak{F}_\lambda$, we have $D_\lambda \neq \phi$ for any $\lambda \in \Lambda$. Let $\lambda < \mu$ be an arbitrary ordered pair and β an arbitrary index of D_μ; then $f(C) \cap F_\beta \neq \phi$ and $x \in F_\beta$. Let $\alpha = \varphi_{\mu\lambda}(\beta)$; then $f(C) \cap F_\alpha \neq \phi$ and $x \in F_\alpha$ by the inequality $F_\beta \subset F_\alpha$. Therefore $\varphi_{\mu\lambda}(D_\mu) \subset D_\lambda$. Since $f(C) \cap U_\lambda \neq \phi$, we have $C \cap f^{-1}(U_\lambda) \neq \phi$. Since $f^{-1}(U_\lambda) \subset \bigcup \{\pi_\lambda^{-1}(\alpha) ; \alpha \in D_\lambda\}$, we have $E_\lambda = \{\alpha ; \alpha \in D_\lambda, C \cap \pi_\lambda^{-1}(\alpha) \neq \phi\} \neq \phi$ for every $\lambda \in \Lambda$.

Since for any pair $\mu < \lambda$ it holds that $\varphi_{\lambda\mu}(E_\lambda) \subset E_\mu$, $\{E_\lambda, \varphi_{\lambda\mu} | E_\lambda ; \lambda \in \Lambda\}$ forms an inverse limiting system consisting of non-empty compact spaces E_λ.

Hence $E=\lim\{E_\lambda, \varphi_{\lambda\mu}|E_\lambda\}$ is not empty. Let $(\alpha_\lambda; \lambda\in\Lambda)$ be an arbitrary point of E; then $\cap\{F_{\alpha_\lambda}; \lambda\in\Lambda\}=x$. Hence we have $E\subset A$. Let a be an arbitrary point of E; then $\pi_\lambda^{-1}(\pi_\lambda(a))\cap C\neq\phi$ for any $\lambda\in\Lambda$. Hence we have $a\in\bar{C}=C$. On the other hand we have already known that $f(a)\in f(E)=x$. Hence $x\in f(C)$ and the closedness of f is proved. Thus the proof is completed.

THEOREM 2.4. *If a non-empty topological space R has a directed family $\boldsymbol{F}=\{\mathfrak{F}_\lambda=\{F_\alpha; \alpha\in A_\lambda\}; \lambda\in\Lambda\}$, with an inverse limiting system $\{A_\lambda, \varphi_{\lambda\mu}\}$, of locally finite closed coverings of order $\leqq n+1$ which follows out R fully, then there exist a paracompact Haudorff space A with $\operatorname{Ind} A=0$ and a closed continuous mapping f of A onto R with $\operatorname{order} f\leqq n+1$.*

PROOF. Let A, $f: A\to R$ and $\pi_\lambda: A\to A_\lambda$ be the same as constructed in the proof of the above theorem. Since $\boldsymbol{F}$ follows out R fully, it does so locally. Hence f is a closed continuous onto mapping of order $\leqq n+1$. Thus what we have to do is to prove that A is a paracompact space with $\operatorname{Ind} A=0$.

Let $\mathfrak{G}$ be an arbitrary open covering of A. For every point x of R let $f^{-1}(x)=\{a(x,1),\cdots,a(x,m(x))\}$, where $m(x)=|f^{-1}(x)|$. Let $V(x,i)$ be an open neighborhood of $a(x,i)$ such that $\{V(x,i); i=1,\cdots,m(x)\}$ is a mutually disjoint collection which refines $\mathfrak{G}$. Let $W_x=\cup\{V(x,i); i=1,\cdots,m(x)\}$ and $V(x)=R-f(A-W_x)$; then $V(x)$ is an open neighborhood of x.

Since $\boldsymbol{F}$ follows out R fully, there exists an index $\lambda\in\Lambda$ such that $\mathfrak{F}_\lambda$ refines $\{V(x); x\in R\}$. Then the following inequalities hold:

$$\begin{aligned}\{W_x; x\in R\} &> \{f^{-1}(V(x)); x\in R\}\\ &> \{f^{-1}(F_\alpha); \alpha\in A_\lambda\} > \{\pi_\lambda^{-1}(\alpha); \alpha\in A_\lambda\}.\end{aligned}$$

Since $\{\pi_\lambda^{-1}(\alpha); \alpha\in A_\lambda\}$ is mutually disjoint, we can get a mutually disjoint open covering $\{U_x; x\in R\}$ of A such that $U_x\subset W_x$ for every $x\in R$ by an easy transfinite induction on x with an arbitrary well-ordering. Since $U_x\cap(\cup\{V(x,i); i=1,\cdots,m(x)\})=U_x\cap W_x=U_x$,

$$\{U_x\cap V(x,i); i=1,\cdots,m(x), x\in R\}$$

is a mutually disjoint open covering of A which refines $\mathfrak{G}$. Thus we can conclude that A is a paracompact space with $\operatorname{Ind} A=0$ and the theorem is proved.

Let us state here a sufficient condition for the existence of a directed family of closed coverings of order $\leqq n+1$.

THEOREM 2.5. *Let $\boldsymbol{U}=\{\mathfrak{U}_\lambda=\{U_\alpha; \alpha\in A_\lambda\}; \lambda\in\Lambda\}$ be a family of locally finite coverings of a topological space R with $\operatorname{order}\boldsymbol{U}\leqq n+1$ and $\{A_\lambda, \varphi_{\lambda\mu}; \lambda\in\Lambda\}$ an inverse limiting system, which sayisfy the following condition:*

(7) *For any ordered pair $\lambda<\mu$ and any $\beta\in A_\mu$, $\bar{U}_\beta$ is contained in U_α, where $\alpha=\varphi_{\mu\lambda}(\beta)$.*

Setting, for any λ and any $\alpha\in A_\lambda$,

$$F_\alpha=\bigcap_{\mu>\lambda}(\cup\{\bar{U}_\beta; \beta\in A_\mu, \varphi_{\mu\lambda}(\beta)=\alpha\}),$$

$\boldsymbol{F}=\{\mathfrak{F}_\lambda=\{F_\alpha\,;\,\alpha\in A_\lambda\}\,;\,\lambda\in\Lambda\}$ *is a directed family of locally finite closed coverings of order* $\leqq n+1$ *with an inverse limiting system* $\{A_\lambda,\varphi_{\lambda\mu}\}$.

PROOF. Let $\lambda<\mu$ be an arbitrary ordered pair of Λ and α an arbitrary element of A_λ. First we prove $F_\alpha\supset\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$. Let $\xi>\lambda$; then there exists a ν with $\nu>\xi$ as well as $\nu>\mu$, and it holds that

$$\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\lambda}(\gamma)=\alpha\}=\bigcup\{\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\xi}(\gamma)=\delta\}\,;\,\varphi_{\xi\lambda}(\delta)=\alpha\}$$
$$\subset\bigcup\{U_\delta\,;\,\varphi_{\xi\lambda}(\delta)=\alpha\}\,.$$

Hence we have

$$\begin{aligned}F_\alpha&=(\bigcap\{\bigcup\{\bar{U}_\delta\,;\,\varphi_{\nu\lambda}(\gamma)=\alpha\}\,;\,\nu>\mu\})\\&\quad\cap(\bigcap\{\bigcup\{\bar{U}_\delta\,;\,\varphi_{\xi\lambda}(\delta)=\alpha\}\,;\,\xi\not>\mu,\,\xi>\lambda\})\\&=\bigcap\{\bigcup\{\bar{U}_\delta\,;\,\varphi_{\nu\lambda}(\gamma)=\alpha\}\,;\,\nu>\mu\}\\&=\bigcap\{\bigcup\{\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\mu}(\gamma)=\beta\}\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}\,;\,\nu>\mu\}\,.\end{aligned}$$

On the other hand it is evident that

$$\begin{aligned}&\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}\\&\quad=\bigcup\{\bigcap\{\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\mu}(\gamma)=\beta\}\,;\,\nu>\mu\}\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}\,.\end{aligned}$$

Therefore we have $F_\alpha\supset\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$.

To prove $F_\alpha\subset\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$, let x be an arbitrary point of F_α. For each $\nu>\mu$, x is a point of $\bigcup\{\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\mu}(\gamma)=\beta\}\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$. Hence

$$B_\mu(\nu)=\{\beta\,;\,x\in\bigcup\{\bar{U}_\delta\,;\,\varphi_{\nu\mu}(\gamma)=\beta\},\,\varphi_{\mu\lambda}(\beta)=\alpha\}$$

is a finite and non-empty subset of A_μ. When $\nu>\nu'>\mu$, it is evident that $B_\mu(\nu)\subset B_\mu(\nu')$. To prove $\bigcap\{B_\mu(\nu)\,;\,\nu>\mu\}\neq\phi$, assume the contrary. Let ν_0 be a fixed index with $\nu_0>\mu$ and $B_\mu(\nu_0)=\{\beta_1,\cdots,\beta_m\}$. Then for every i with $1\leqq i\leqq m$ there exists an index ν_i with $\nu_i>\mu$ such that $\beta_i\notin B_\mu(\nu_i)$. Let ν_{m+1} be an index such that $\nu_{m+1}>\nu_i$ for $0\leqq i\leqq m$; then $B_\mu(\nu_{m+1})\subset B_\mu(\nu_0)$ and $B_\mu(\nu_i)\supset B_\mu(\nu_{m+1})\not\ni\beta_i$ for $1\leqq i\leqq m$. Hence we have $B_\mu(\nu_{m+1})=\phi$, which is a contradiction. Therefore $\bigcap\{B_\mu(\nu)\,;\,\nu>\mu\}$ is not empty and contains an element, say β_0. Then $\varphi_{\mu\lambda}(\beta_0)=\alpha$ and for any ν with $\nu>\mu$, x is contained in $\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\mu}(\gamma)=\beta_0\}$. Thus x is contained in $\bigcap\{\bigcup\{\bar{U}_\gamma\,;\,\varphi_{\nu\mu}(\gamma)=\beta_0\}\,;\,\nu>\mu\}$ and hence so in $\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$. Therefore the inequality $F_\alpha\subset\bigcup\{F_\beta\,;\,\varphi_{\mu\lambda}(\beta)=\alpha\}$ is proved.

Finally we show that each element of the family $\boldsymbol{F}$ is a closed covering of order $\leqq n+1$. For any λ and any $\alpha\in A_\lambda$, it is almost evident that F_α is a closed subset of R which is contained in U_α. Thus the order of $\mathfrak{F}_\lambda$ is at most $n+1$. To prove that $\mathfrak{F}_\lambda$ covers R, let x be an arbitrary point of R. Setting

$$C_\mu=\{\alpha\,;\,x\in U_\alpha\in\mathfrak{U}_\mu\}\,,\qquad\qquad\mu>\lambda\,,$$

it is evident that $\{C_\mu,\varphi_{\mu\nu}|C_\mu\,;\,\mu>\nu>\lambda\}$ forms an inverse limiting system.

Since C_μ is a finite and non-empty subset of A_μ for every $\mu > \lambda$, $\lim\{C_\mu, \varphi_{\mu\nu} | C_\mu\}$ is not empty and hence contains an element $(\alpha^0(\mu); \alpha^0(\mu) \in C_\mu, \mu > \lambda)$. For any μ, ν with $\mu > \lambda$, $\nu > \lambda$, $\varphi_{\mu\lambda}(\alpha^0(\mu))$ coincides with $\varphi_{\nu\lambda}(\alpha^0(\nu))$. Denote this common value by $\alpha^0(\lambda)$. Then $F_{\alpha^0(\lambda)}$ contains x, since $x \in U_{\alpha^0(\mu)}$ for every $\mu > \lambda$. Therefore $\mathfrak{F}_\lambda$ is a covering and the theorem is proved.

§ 3. Construction of directed families from mappings defined on 0-dimensional spaces.

LEMMA 3.1. *For a topological space A the following conditions are equivalent.*

(1) $\operatorname{ind} A = 0$.

(2) *A is homeomorphic to a non-empty dense subset of the limit space of an inverse limiting system of finite discrete spaces.*

This is a part of [18, Corollary 2].

THEOREM 3.2. *If a non-empty topological space R admits a closed continuous mapping f, with $\operatorname{order} f \leqq n+1$, of a completely regular space A, with $\operatorname{ind} A = 0$, onto R, then R is a regular space and has a directed family $\boldsymbol{F}$, with $\operatorname{order} \boldsymbol{F} \leqq n+1$, of finite closed coverings of R which folllows out the topology of R locally.*

PROOF. R is regular from the regularity of A and the compactness of $f^{-1}(x)$. By Lemma 3.1 we can consider A as a subset of the limit space of an inverse limiting system $\{A_\lambda, \varphi_{\lambda\mu}; \lambda \in \Lambda\}$ of finite discrete spaces A_λ. Let π_λ: $A \rightarrow A_\lambda$, $\lambda \in \Lambda$, be the projections. Then $\boldsymbol{F} = \{\mathfrak{F}_\lambda = \{F_\alpha = f(\pi_\lambda^{-1}(\alpha)); \alpha \in A_\lambda\}; \lambda \in \Lambda\}$ is a directed family of finite closed coverings of R with an inverse limiting system $\{A_\lambda, \varphi_{\lambda\mu}; \lambda \in \Lambda\}$. Moreover it is evident that $\operatorname{order} \boldsymbol{F} \leqq n+1$.

To prove that $\boldsymbol{F}$ actually follows out R locally, let x be an arbitrary point of R and U an arbitrary open neighborhood of x. Let $|f^{-1}(x)| = j$ and $f^{-1}(x) = \{a_1, \cdots, a_j\}$; then there exists, for every i with $1 \leqq i \leqq j$, an index $\lambda_i \in \Lambda$ such that $f(\pi_{\lambda_i}^{-1}(\pi_{\lambda_i}(a_i))) \subset U$, $i = 1, \cdots, j$. Let μ be an index of Λ such that i) $\mu > \lambda_i$ for $i = 1, \cdots, j$, ii) $\{\pi_\mu(a_i); i = 1, \cdots, j\}$ consists of mutually different representatives. Then we have $f(\pi_\mu^{-1}(\pi_\mu(f^{-1}(x)))) \subset U$ and $|\pi_\mu(f^{-1}(x))| = j$. Suppose that there exists an $\alpha \in A_\mu$ such that $\alpha \notin \pi_\mu(f^{-1}(x))$ and $x \in f(\pi_\mu^{-1}(\alpha)) = F_\alpha$. Then we have $|f^{-1}(x)| \geqq |\pi_\mu(f^{-1}(x))| \geqq j+1$, which is a contradiction. Therefore we have $S(x, \mathfrak{F}_\mu) \subset U$ and the theorem is proved.

THEOREM 3.3. *If a non-empty topological space R admits a closed continuous mapping f, with $\operatorname{order} f \leqq n+1$, of a normal space A, with $\operatorname{Ind} A = 0$, onto R, then R is a normal space and has a directed family $\boldsymbol{F}$, with $\operatorname{order} \boldsymbol{F} \leqq n+1$, of finite closed coverings of R which follows out R globally.*

PROOF. R is normal from the normality of A. Let βA be the Stone-Čech-compactification of A; then it is evident that $\operatorname{Ind} \beta A = 0$. Consider βA as the limit space of an inverse limiting system $\{A_\lambda, \varphi_{\lambda\mu}; \lambda \in \Lambda\}$ of finite discrete spaces A_λ. Let $\tilde{\pi}_\lambda$ be the projection of βA onto A_λ and π_λ the restriction of $\tilde{\pi}_\lambda$ to A. Then

$$\boldsymbol{F}=\{\mathfrak{F}_\lambda=\{F_\alpha=f(\pi_\lambda^{-1}(\alpha))\,;\,\alpha\in A_\lambda\}\,;\,\lambda\in\Lambda\}$$

is a directed family of finite closed coverings of R with an inverse limiting system $\{A_\lambda,\varphi_{\lambda\mu}\,;\,\lambda\in\Lambda\}$. Moreover it is evident that order $\boldsymbol{F}\leqq n+1$.

To prove that $\boldsymbol{F}$ follows out R globally, let $F\subset G$ be an arbitrary pair of a closed set F and an open set G of R. Since $f^{-1}(F)\subset f^{-1}(G)$, there exists a bounded real-valued continuous function φ of A such that $\varphi(a)=0$ if $a\in f^{-1}(F)$ and $\varphi(a)=1$ if $a\in A-f^{-1}(G)$. Let ψ be a continuous extension of φ to βA. If we set $F_1=\{a\,;\,\psi(a)=0\}$ and $F_2=\{a\,;\,\psi(a)=1\}$, then we have an open covering $\mathfrak{G}=\{\beta A-F_1,\beta A-F_2\}$ of βA. Since βA is compact, there exists an index μ of Λ such that $\{\tilde{\pi}_\mu^{-1}(\alpha)\,;\,\alpha\in A_\mu\}$ refines $\mathfrak{G}$. Let

$$G_1=\mathrm{S}(F_1,\{\tilde{\pi}_\mu^{-1}(\alpha)\,;\,\alpha\in A_\mu\})\,;$$

then $F_1\subset G_1\subset\beta A-F_2$. Hence we have $f^{-1}(F)\subset A\cap F_1\subset A\cap G_1\subset A\cap(\beta A-F_2)$ $=A\cap\beta A-A\cap F_2=A-A\cap F_2\subset A-(A-f^{-1}(G))=f^{-1}(G)$. On the other hand $\mathrm{S}(f^{-1}(F),\{\pi_\mu^{-1}(\alpha)\,;\,\alpha\in A_\mu\})\subset A\cap G_1$ holds. Hence we have $f^{-1}(F)\subset\mathrm{S}(f^{-1}(F),$ $\{\pi_\mu^{-1}(\alpha)\,;\,\alpha\in A_\mu\})\subset f^{-1}(G)$. Therefore we have $F\subset\mathrm{S}(F,\mathfrak{F}_\mu)\subset G$ and we know that $\boldsymbol{F}$ follows out R globally. Thus the theorem is proved.

LEMMA 3.4 (E. Michael [**10**, Corollary 1]). *A regular space which is a closed continuous image of a pracompact space is paracompact.*

Let $\{A_\lambda,\varphi_{\lambda\mu}\}$ be an inverse limiting system of discrete spaces A_λ. Let π_λ, $\lambda\in\Lambda$, be the projection of $A=\lim\{A_\lambda,\varphi_{\lambda\mu}\}$ into A_λ. We call the system *full* if every open covering of A can be refined by $\{\pi_\lambda^{-1}(\alpha)\,;\,\alpha\in A_\lambda\}$ for some $\lambda\in\Lambda$.

LEMMA 3.5 (K. Nagami [**18**, Theorem 2]). *In order that a topological space A be a paracompact Hausdorff space with* $\mathrm{Ind}\,A=0$ *it is necessary and sufficient that A is homeomorphic to the non-empty limit space obtained from an inverse limiting full system which consists of discrete spaces.*

THEOREM 3.6. *If a non-empty topological space R admits a closed continuous mapping f, with* order $f\leqq n+1$, *of a paracompact Haudorff space A, with* $\mathrm{Ind}\,A=0$, *onto R, then R is a paracompact Hausdorff space and has a directed family $\boldsymbol{F}$, with* order $\boldsymbol{F}\leqq n+1$, *of locally finite closed coverings of R which follows out R fully.*

PROOF. By Lemma 3.5 there exists an inverse limiting full system $\{A_\lambda,$ $\varphi_{\lambda\mu}\,;\,\lambda\in\Lambda\}$ of discrete spaces A_λ such that $A=\lim\{A_\lambda,\varphi_{\lambda\mu}\}$. Let $\pi_\lambda: A\to A_\lambda$, $\lambda\in\Lambda$, be the projections. Then

$$\boldsymbol{F}=\{\mathfrak{F}_\lambda=\{F_\alpha=f(\pi_\lambda^{-1}(\alpha))\,;\,\alpha\in A_\lambda\}\,;\,\lambda\in\Lambda\}$$

is a directed family of locally finite closed coverings of R with an inverse limiting system $\{A_\lambda,\varphi_{\lambda\mu}\}$. Moreover we have order $\boldsymbol{F}\leqq n+1$.

To prove that $\boldsymbol{F}$ follows out R fully, let $\mathfrak{G}$ be an arbitrary open covering of R. Then by the fullness of $\{A_\lambda, \varphi_{\lambda\mu}\}$ there exists an index $\lambda \in \Lambda$ such that $\{\pi_\lambda^{-1}(\alpha); \alpha \in A_\lambda\}$ refines $\{f^{-1}(G); G \in \mathfrak{G}\}$. It is evident that $\mathfrak{F}_\lambda$ refines $\mathfrak{G}$. By Lemma 3.4 R is paracompact. Moreover it is almost evident that R is regular. Thus the theorem is proved.

§ 4. Inductive dimension.

THEOREM 4.1. *If a topological space R has a directed family $\boldsymbol{F} = \{\mathfrak{F}_\lambda = \{F_\alpha; \alpha \in A_\lambda\}; \lambda \in \Lambda\}$ of locally finite closed coverings of order $\leqq n+1$ which follows out R locally, then R is a regular space with* $\operatorname{ind} R \leqq n$.

PROOF. To prove the proposition by the induction on n let (P_i) be the assertion of the proposition for the case $n = i$. Then (P_{-1}) is evidently true. Let $n > -1$ and order $\boldsymbol{F} \leqq n+1$. Make the induction assumption that (P_i) is true for $i < n$. Let x be an arbitrary point of R and G an arbitrary open set which contains x. Then there exists a $\lambda \in \Lambda$ with $\mathrm{S}(x, \mathfrak{F}_\lambda) \subset G$.

Let H be the open kernel of $\mathrm{S}(x, \mathfrak{F}_\lambda)$. Since $H_1 = R - \bigcup \{F_\alpha; x \notin F_\alpha \in \mathfrak{F}_\lambda\}$ is an open set with $x \in H_1 \subset \mathrm{S}(x, \mathfrak{F}_\lambda)$, we have $x \in H \subset \bar{H} \subset G$. Thus R is a regular space. Since $\bar{H} - H \subset R - H \subset R - H_1$, $\bar{H} - H$ is covered by $\mathfrak{F}'_\lambda = \{F_\alpha; \alpha \in B_\lambda\}$ where $B_\lambda = \{\alpha; x \notin F_\alpha, \alpha \in A_\lambda\}$. Let for every $\mu > \lambda$, $B_\mu = \{\beta; \varphi_{\mu\lambda}(\beta) \in B_\lambda\}$. Then for every $\mu \in M = \{\nu; \nu > \lambda\}$, $\mathfrak{F}'_\mu = \{F_\alpha; \alpha \in B_\mu\}$ covers $\bar{H} - H$. Let $\mathfrak{H}_\mu$ be the restriction of $\mathfrak{F}'_\mu$ to $\bar{H} - H$, $\mu \in M$. Since $\bar{H} \subset \bigcup \{F_\alpha; \alpha \in A_\lambda - B_\lambda\}$, order $\mathfrak{H}_\mu \leqq n$ for every $\mu \in M$. It can easily be seen that $\boldsymbol{H} = \{\mathfrak{H}_\mu; \mu \in M\}$ is a directed family with an inverse limiting system $\{B_\mu, \varphi_{\mu\nu} | B_\mu; \mu > \nu > \lambda\}$ of locally finite closed coverings of $\bar{H} - H$ which follows out $\bar{H} - H$ locally. Thus we have $\operatorname{ind}(\bar{H} - H) \leqq n-1$ by the induction assumption. Hence we have $\operatorname{ind} R \leqq n$ and the theorem is proved.

COROLLARY 4.2. *If there exists a closed continuous mapping f, with* order $f \leqq n+1$, *of a completely regular space A, with* $\operatorname{ind} A = 0$, *onto a topological space R, then R is a regular space with* $\operatorname{ind} R \leqq n$.

This is a direct consequence of Theorems 3.2 and 4.1.

By a similar way used in the proof of Theorem 4.1 we get the following

THEOREM 4.3. *If a topological space R has a directed family of locally finite closed coverings of order $\leqq n+1$ which follows out R globally, then R is a normal space with* $\operatorname{Ind} R \leqq n$.

COROLLARY 4.4. *If there exists a closed continuous mapping f, with* order $f \leqq n+1$, *of a normal space A, with* $\operatorname{Ind} A = 0$, *onto a topological space R, then R is a normal space with* $\operatorname{Ind} R \leqq n$.

This is a direct consequence of Theorems 3.3 and 4.3.

COROLLARY 4.5. *If there exists a closed continuous mapping f, with* order $f \leqq$

$n+1$, of a paracompact Hausdorff space A, with $\operatorname{Ind} A=0$, *onto a topological space R, then R is a paracompact Hausdorff space with* $\operatorname{Ind} R \leqq n$.

This is a direct consequence of Corollary 4.4 and Lemma 3.4.

THEOREM 4.6. *If a topological space R has a directed family of locally finite closed coverings of order $\leqq n+1$ which follows out R fully, then R is a paracompact Hausdorff space with* $\operatorname{Ind} R \leqq n$.

This is an immediate consequence of Theorem 4.3 and Michael's theorem [**10**, Theorem 1]: A regular space is paracompact if every open covering can be refined by a closure-preserving covering, where a covering $\{F_\alpha ; \alpha \in A_0\}$ is called closure-preserving if for any subset B of A_0 we have $\bigcup\{\bar{F}_\alpha ; \alpha \in B\}=\overline{\bigcup\{F_\alpha ; \alpha \in B\}}$.

REMARK 4.7. Corollaries 4.2 and 4.4 have been already essentially proved by Morita in the proof of [**13**, Theorem 1]. Therefore Propositions 4.1 and 4.3 can also be obtained, with the aid of the results in §3, as consequences of Corollaries 4.2 and 4.4. Professor Morita pointed out these remarks. Let the author take this oppotunity to correct a misprint in the paper cited now. For Morita [**13**, Remark] read i) *if f is a closed continuous mapping of a normal space X onto a totally normal space Y such that the order of f is at most $n+1$, then* $\operatorname{Ind} Y \leqq \operatorname{Ind} X+n$. According to C. H. Dowker [**3**], a topological space X is called *totally normal* if it is normal and for any open set G of X there exists a collection of open F_σ-sets of X which is locally finite in G and forms a covering of G.

On the other hand J. Nagata [**20**] proved that ii) *if f is a closed continuous mapping of a normal space X onto a perfectly normal space Y such that for any $y \in Y$ the boundary of $f^{-1}(y)$ consists of at most $n+1$ points, then* $\operatorname{Ind} Y \leqq \operatorname{Ind} X+n$.

It is to be noted that the following proposition is a generalization of both i) and ii): iii) *If f is a closed continuous mapping of a normal space X onto a totally normal space Y such that for any $y \in Y$ the boundary of $f^{-1}(x)$ consists of at most $n+1$ points, then* $\operatorname{Ind} Y \leqq \operatorname{Ind} X+n$. Since every perfectly normal space is totally normal, it is evident that iii) implies ii). Let f, X, Y be those of iii). Let Y_1 be the aggregate of $y \in Y$ such that the boundary of $f^{-1}(y)$ is empty. Let X_1 be the inverse image of Y_1 and X_2 the sum of boundaries of $f^{-1}(y)$ with $y \notin Y_1$. Then $f|X_2$ is a closed continuous mapping of a normal space X_2 onto a totally normal space $Y-Y_1$ such that the order of f is at most $n+1$. Since Y_1 is discrete and $Y-Y_1$ is closed, we have $\operatorname{Ind} Y=\max(\operatorname{Ind}(Y-Y_1), \operatorname{Ind} Y_1)$ by the hereditary normality of Y [**3**]. By these observations iii) is a direct consequence of i).

THEOREM 4.8. *Let $\boldsymbol{F}=\{\mathfrak{F}_\lambda=\{F_\alpha ; \alpha \in A_\lambda\} ; \lambda \in \Lambda\}$ be a directed family of locally finite closed coverings of a topological space R with an inverse limiting* *system* $\{A_\lambda, \varphi_{\lambda\mu}\}$. *If* $\boldsymbol{F}$ *satisfies the following two conditions:*

(1) order $\boldsymbol{F} \leqq n+1$,

(2) $\boldsymbol{F}$ *follows out* R *locally,*

then for any $\lambda \in \Lambda$ *and any mutually different indices* $\alpha_1, \cdots, \alpha_m$ *of* A_λ, $1 \leqq m \leqq n+1$, *it holds that*

$$\operatorname{ind} \bigcap_{i=1}^{m} F_{\alpha_i} \leqq n-m+1 .$$

PROOF. Let λ be an arbitrary index of Λ and $\alpha_1, \cdots, \alpha_m$ be arbitrary mutually different indices of A_λ, $1 \leqq m \leqq n+1$. Let $M = \{\mu\,;\, \mu > \lambda\}$ and

$$B_\mu = \{\beta\,;\, \varphi_{\mu\lambda}(\beta) \in A_\lambda - \{\alpha_2, \cdots, \alpha_m\}\}, \qquad \mu \in M .$$

Let $\mathfrak{H}_\mu$ be the restriction of $\{F_\alpha\,;\, \alpha \in B_\mu\}$ to $F = \bigcap_{i=1}^{m} F_{\alpha_i}$. Then it can easily be seen that $\boldsymbol{H} = \{\mathfrak{H}_\mu\,;\, \mu \in M\}$ is a directed family of locally finite closed coverings of F with an inverse limiting system $\{B_\mu, \varphi_{\mu\nu} | B_\mu\,;\, \mu > \nu > \lambda\}$ which satisfies the following two conditions:

(3) order $\boldsymbol{H} \leqq n+1-(m-1) = n-m+2$,

(4) $\boldsymbol{H}$ follows out F locally.

Therefore we can conclude that $\operatorname{ind} F \leqq n-m+1$ by Theorem 4.1 and the proof is finished.

In a similar way employed in the above proof we have the following with the aid of Theorem 4.3.

THEOREM 4.9. *Let* $\boldsymbol{F} = \{\mathfrak{F}_\lambda = \{F_\alpha\,;\, \alpha \in A_\lambda\}\,;\, \lambda \in \Lambda\}$ *be a directed family of locally finite closed coverings of a topological space* R. *If* $\boldsymbol{F}$ *satisfies the following two conditions:*

(5) order $\boldsymbol{F} \leqq n+1$,

(6) $\boldsymbol{F}$ *follows out* R *globally, then for any* $\lambda \in \Lambda$ *and any mutually different indices* $\alpha_1, \cdots, \alpha_m$ *of* A_λ, $1 \leqq m \leqq n+1$, *it holds that*

$$\operatorname{Ind} \bigcap_{i=1}^{m} F_{\alpha_i} \leqq n-m+1 .$$

PROBLEM 4.10. It is a well-known Katětov-Morita's theorem that the large inductive dimension coincides with the covering dimension for metric spaces (M. Katětov [5] and K. Morita [12]). It seems to the author an interesting problem to construct, for a paracompact and perfectly normal space R with $\dim R \leqq n$, a directed family of locally finite closed coverings of order $\leqq n+1$ which follows out R globally. It is to be noted that every metric space is paracompact and perfectly normal. If this could be done, we should have $\dim R = \operatorname{Ind} R$ for a paracompact and perfectly normal space R by Theorem 4.3.

It seems also an interesting problem to learn whether the converse of Theorem 4.1 for a metric space R is valid or not. This problem will penetrate into the essence of the small inductive dimension, one of the most important but undeveloped region in dimension theory, of metric spaces.

REMARK 4.11. We cannot expect that Hurewicz-Kuratowski's theorem cited in §1 may be valid even for the case when R is a compact Hausdorff space for the following reasons: Assume that if R is a compact Hausdorff space with $\dim R \leqq n$, there exist a normal space A with $\dim A=0$ and a closed continuous mapping f of A onto R such that the order of f is at most $n+1$. Then we know that $\operatorname{Ind} R \leqq n$ by Corollary 4.4, since $\dim A=0$ if and only if $\operatorname{Ind} A=0$. Hence we have $\operatorname{Ind} R \leqq \dim R$, which contradicts to the fact that there exists a compact Hausdorff space whose large inductive dimension is actually greater than its covering dimension (Lunz [**9**], Lokutsievski [**8**], P. Vopenka [**23**]).

§5. Closed mappings of finite vague order.

DEFINISION 5.1. Let f be a mapping of a topological space A onto another topological space R. Then the *vague order* of f is the minimum of the number n which has the following property: For an arbitrary finite open covering $\mathfrak{U}$ of R there exists an open covering $\mathfrak{V}$ of A such that i) $f(\mathfrak{V})=\{f(V);\ V\in\mathfrak{V}\}$ refines $\mathfrak{U}$, ii) for any point x of R the number of $V\in\mathfrak{V}$ with $f^{-1}(x)\cap V\neq\phi$ is at most n.

REMARK 5.2. It is almost evident that the vague order of f is the same with the minimum of the number n which has the following property: For an arbitrary finite open covering $\mathfrak{U}=\{U_1,\cdots,U_m\}$ of R there exists an open covering $\mathfrak{V}=\{V_1,\cdots,V_m\}$ of A such that i) $f(V_i)\subset U_i$ for $i=1,\cdots,m$, ii) for any point x of R the number of $V_i\in\mathfrak{V}$ with $f^{-1}(x)\cap V_i\neq\phi$ is at most n.

LEMMA 5.3. *Let f be a closed mapping of a normal space A onto a normal space R. If the vague order of f is at most $n+1$, then we have* $\dim R\leqq n$.

PROOF. Let $\mathfrak{U}=\{U_1,\cdots,U_k\}$ be an arbitrary finite open covering of R. Since the vague order of f is at most $n+1$, there exists, by Remark 5.2, a finite open covering $\mathfrak{V}=\{V_1,\cdots,V_k\}$ of A such that i) $f(V_i)\subset U_i$ for $i=1,\cdots,k$, ii) for any $x\in R$ the number of $V_i\in\mathfrak{V}$ with $f^{-1}(x)\cap V_i\neq\phi$ is at most $n+1$. Since A is normal, there exists a closed covering $\mathfrak{F}=\{F_1,\cdots,F_k\}$ of A such that $F_i\subset V_i$ for $i=1,\cdots,k$. Then $\{f(F_1),\cdots,f(F_k)\}$ is a closed covering of R of order $\leqq n+1$ such that $f(F_i)\subset U_i$ for $i=1,\cdots,k$. By [**2**, Theorem 6, p. 71] there exists an open covering $\{W_1,\cdots,W_k\}$ of R of order $\leqq n+1$ such that $F_i\subset W_i\subset U_i$ for $i=1,\cdots,k$. Thus we have $\dim R\leqq n$ and the lemma is proved.

LEMMA 5.4. *Let R be a non-empty paracompact Hausdorff space with* $\dim R \leqq n$. *Then there exist a paracompact Hausdorff space A with* $\dim A=0$ *and a closed continuous onto mapping $f: A\rightarrow R$ of the vague order* $\leqq n+1$ *such that $f^{-1}(x)$ is compact for every point x of R.*

PROOF. Let $\{\mathfrak{F}_\lambda=\{F_\alpha;\ \alpha\in A_\alpha\};\ \lambda\in\Lambda\}$ be the collection of all locally finite closed coverings of R whose orders are at most $n+1$. Let A be the aggregate of points $a=(\alpha_\lambda;\lambda\in\Lambda)$ of the product space $\prod\{A_\lambda;\lambda\in\Lambda\}$, where A_λ are topological spaces with the discrete topology, such that $\bigcap\{F_{\alpha_\lambda};\lambda\in\Lambda\}\neq\phi$. Define $f: A\to R$ as $f(a)=\bigcap\{F_{\pi_\lambda(a)};\lambda\in\Lambda\}$, where $\pi_\lambda: A\to A_\lambda$, $\lambda\in\Lambda$, are the projections. It can easily be seen that f is continuous and onto. The following argument is the same as is employed by the author in the proof of [**19**, Theorem 2] but we state it here for the sake of completeness.

To show the closedness of f, let B be an arbitrary non-empty closed subset of A and x an arbitrary point of the closure of $f(B)$. Let λ be an arbitrary element of Λ. Let

$$B_\lambda=\{\alpha\,;\,x\in F_\alpha\in\mathfrak{F}_\lambda\}\,;$$

then

$$U_\lambda=R-\bigcup\{F_\alpha\,;\,\alpha\in A_\lambda-B_\lambda\}$$

is an open neighborhood of x by the local finiteness of $\mathfrak{F}_\lambda$. Since $f(B)\cap U_\lambda\neq\phi$, it holds that $B\cap f^{-1}(U_\lambda)\neq\phi$. Since $f^{-1}(U_\lambda)\subset\bigcup\{\pi_\lambda^{-1}(\alpha);\alpha\in B_\lambda\}$, there exists an index $\alpha(\lambda)\in B_\lambda$ with $\pi_\lambda^{-1}(\alpha(\lambda))\cap B\neq\phi$.

Let $a=(\alpha(\lambda);\lambda\in\Lambda)$; then it is obvious that $f(a)=x$. Since, for any λ, $\pi_\lambda^{-1}(\pi_\lambda(a))\cap B=\pi_\lambda^{-1}(\alpha(\lambda))\cap B\neq\phi$, a is a point of $\bar{B}=B$. Therefore we get $x=f(a)\in f(B)$ and hence $\overline{f(B)}\subset f(B)$. Thus the closedness of f is proved. Moreover $f^{-1}(x)$ is compact, since $f^{-1}(x)=\prod\{B_\lambda;\lambda\in\Lambda\}$ and B_λ is finite for every $\lambda\in\Lambda$.

Next let us prove that A is a paracompact Hausdorff space with $\dim A=0$. Let $\mathfrak{U}$ be an arbitrary open covering of A; then $\mathfrak{U}$ can be refined by a covering $\mathfrak{V}$ whose elements are open and closed, by the equality $\operatorname{ind} A=0$. Since, for any $x\in R$, $f^{-1}(x)$ is compact, there exist a finite number of elements $V_{x,1},\cdots,V_{x,m(x)}$ of $\mathfrak{V}$ with $f^{-1}(x)\subset V_{x,1}\cup\cdots\cup V_{x,m(x)}=W_x$, where we can put $V_{x,1}=\phi$, $x\in R$, without loss of generality. Put $D(x)=R-f(A-W_x)$; then there exists an index $\lambda_0\in\Lambda$ such that $\mathfrak{F}_{\lambda_0}$ refines $\{D(x);x\in R\}$. Since i) $\{\pi_{\lambda_0}^{-1}(\alpha);\alpha\in A_{\lambda_0}\}$ refines $\{f^{-1}(D(x));x\in R\}$ and the latter refines $\{W_x;x\in R\}$ and ii) the order of $\{\pi_{\lambda_0}^{-1}(\alpha);\alpha\in A_{\lambda_0}\}$ is 1, we can prove by an easy transfinite induction on $x\in R$, with an arbitrary but fixed ordering, the existence of an open covering $\{U_x;x\in R\}$ of order 1 with $U_x\subset W_x$ for every $x\in R$. Let

$$\mathfrak{E}=\{U_x\cap(V_{x,i}-\bigcup_{j<i}V_{x,j});\,i=2,\cdots,m(x),x\in R\}\,;$$

then $\mathfrak{E}$ is an open covering of A of order 1 which refines $\mathfrak{U}$. Thus A is a paracompact Hausdorff space with $\dim A=0$.

To prove the vague order f of is at most $n+1$, let $\mathfrak{U}$ be an arbitrary finite open covering of R. Since $\dim R\leq n$, there exists an index $\lambda\in\Lambda$ such that $\mathfrak{F}_\lambda$ refines $\mathfrak{U}$. Let

$$\mathfrak{V}=\{\pi_\lambda^{-1}(\alpha);\,\alpha\in A_\lambda\}\,.$$

Since $f(\pi_\lambda^{-1}(\alpha))=F_\alpha$ for any $\alpha\in A_\lambda$, $f(\mathfrak{V})$ refines $\mathfrak{U}$. Let x be an arbitrary point of R. Since the order of $\mathfrak{F}_\lambda$ is at most $n+1$, the number of elements of $\mathfrak{F}_\lambda$ which contain x is at most $n+1$. Hence the number of indices α of A_λ with $x\in f(\pi_\lambda^{-1}(\alpha))$ is at most $n+1$. Thus the vague order of f is at most $n+1$ and the proof is completed.

Now the following theorem is evident from Lemmas 3.4, 5.2 and 5.3.

THEOREM 5.5. *In order that a non-empty topological space R be a paracompact Hausdorff space with* $\dim R\leqq n$ *it is necessary and sufficient that there exist a paracompact Hausdorff space A with* $\dim A=0$ *and a closed continuous onto mapping $f: A\to R$ of the vague order* $\leqq n+1$.

§ 6. Open mappings of finite vague order.

The following is to be compared with Theorem 5.5.

THEOREM 6.1. *In order that a non-empty normal space R be of the covering dimension $\leqq n$, it is necessary and sufficient that there exist a completely regular space A with* $\operatorname{ind} A=0$ *and an open continuous mapping f of A onto R such that the vague order of f is at most $n+1$.*

It is clear that the condition is sufficient. The necessity of the condition is guaranteed by the following lemma.

LEMMA 6.2. *For a normal space R with* $\dim R\leqq n$ *there exist a completely regular space A with* $\operatorname{ind} A=0$ *and an open continuous onto mapping $f: A\to R$ of the vague order* $\leqq n+1$ *such that $f^{-1}(x)$ is compact for every point x of R.*

PROOF. Let $\{\mathfrak{U}_\lambda=\{U_\alpha\,;\,\alpha\in A_\lambda\}\,;\,\lambda\in\Lambda\}$ be the family of all finite open coverings of R of order $\leqq n+1$. Let A be the aggregate of points $a=\{\alpha_\lambda\,;\,\lambda\in\Lambda\}$ of the product space $\prod\{A_\lambda\,;\,\lambda\in\Lambda\}$, where A_λ are topological spaces with the discrete topology, such that $\bigcap\{U_{\alpha_\lambda}\,;\,\lambda\in\Lambda\}\neq\phi$. Let $f(a)=\bigcap\{U_{\pi_\lambda(a)}\,;\,\lambda\in\Lambda\}$, where $\pi_\lambda: A\to A_\lambda$, $\lambda\in\Lambda$, are the projections. Then A is a completely regular space with $\operatorname{ind} A=0$ and f is a mapping of A onto R. Since for any $\lambda\in\Lambda$ and any $\alpha\in A_\lambda$ we have $f(\pi_\lambda^{-1}(\alpha))=U_\alpha$, f is an open continuous mapping. Let x be an arbitrary point of R and $B_\lambda=\{\alpha\,;\,x\in U_\alpha\in\mathfrak{U}_\lambda\}$, $\lambda\in\Lambda$. Then $f^{-1}(x)=\prod B_\lambda$ and hence it is compact.

To compute the vague order of f, let $\mathfrak{U}$ be an arbitrary finite open covering of R and x an arbitrary point of R. Then there exists a $\lambda\in\Lambda$ such that $\mathfrak{U}_\lambda$ refines $\mathfrak{U}$, since the covering dimension of R is at most n. Let $\mathfrak{V}=\{\pi_\lambda^{-1}(\alpha)\,;\,\alpha\in A_\lambda\}$; then $\mathfrak{V}$ is an open covering of A such that $f(\mathfrak{V})<\mathfrak{U}_\lambda<\mathfrak{U}$. Since $\pi_\lambda^{-1}(\alpha)\cap f^{-1}(x)\neq\phi$ implies $f^{-1}(\pi_\lambda^{-1}(\alpha))=U_\alpha\ni x$, the number of indices α with $\pi_\lambda^{-1}(\alpha)\cap f^{-1}(x)\neq\phi$ is at most the order of $\mathfrak{U}_\lambda$. Since the order of $\mathfrak{U}_\lambda$ is at most $n+1$, the vague order of f is at most $n+1$, and the lemma is proved.

REMARK 6.3. In view of Hurewicz-Kuratowski's theorem cited in § 1 it is natural to raise the question: When 'the vague order' in Theorem 6.1 is replaced with 'the order', does the theorem thus obtained remain valid? The answer for this problem, as well as for the case when f is closed (cf. Remark 4.11), is negative under some additional conditions imposed on A and R, since the following assertion [17, Theorem 4.1] is valid: A paracomact Hausdorff space R which is the image of a paracompact Hausdorff space A with $\dim A=0$, under an open continuous mapping f such that $f^{-1}(x)$ is finite for every $x\in R$, is unable to be of positive covering dimension.

§7. An example.

Let $K^n\neq\phi$ be a CW-complex given by J. H. C. Whitehead [24], where n is the maximal dimensional number of cells contained in K^n. e^i denotes an i-cell in K^n, and K^m denotes an m-section of K^n. The main purpose of this paragraph is to show the following

THEOREM 7.1. *For any* CW-*complex* K^n *there exists a directed family* $\boldsymbol{F}_n$ *of locally finite closed coverings* $\mathfrak{F}_\sigma$, $\sigma\in M$, *of* K^n *such that* $\boldsymbol{F}_i=\boldsymbol{F}_n\wedge K^i=\{\mathfrak{F}_\sigma\wedge K_i;\sigma\in M\}$ *is a directed family with* order $\boldsymbol{F}_i\leqq i+1$ *which follows out* K^i *fully for* $i=0,1,\cdots,n$.

PROOF. Let (P_m) be the assertion of the existence of spaces A_i, $i=0,\cdots,m$, and of mappings $f_i, i=0,\cdots,m$, which satisfy the following conditions:

i) A_i is a paracompact Hausdorff space with $\operatorname{Ind}A_i\leqq 0$ for $i=0,\cdots,m$.

ii) f_i is a closed continuous mapping of A_i onto K^i with order$f_i\leqq i+1$ for $i=0,\cdots,m$.

iii) $f_{i+1}|A_i=f_i$ for $i=0,\cdots,m-1$.

iv) $f_i^{-1}(\overline{e^i})$ is metrizable for any $e^i\subset K^n$ and for $i=0,\cdots,m$.

Since K^0 is discrete, (P_0) is clearly true. Make the induction assumption that (P_{m-1}) is valid for $m>0$ and let us prove that (P_m) holds.

Let $\{e_\xi^m;\xi\in X\}$ be the collection of all m-cells of K^n. Fix an arbitrary m-cell e_ξ^m. Set

$$B_\xi=f_{m-1}^{-1}(\overline{e_\xi^m}-e_\xi^m)$$

and

$$f_\xi=f_{m-1}|B_\xi\,.$$

Since B_ξ is closed in A_{m-1}, f_ξ is a closed continuous mapping of B_ξ onto a compact space $\overline{e_\xi^m}-e_\xi^m$ such that, for every point $x\in\overline{e_\xi^m}-e_\xi^m$, $f_\xi^{-1}(x)$ is compact. Hence B_ξ is compact. Let $e_j^{i_j}, j=1,\cdots,t$, be a finite number of cells of K^{m-1} such that $\overline{e_\xi^m}-e_\xi^m\subset e_1^{i_1}\cup\cdots\cup e_t^{i_t}$. Since $B_\xi\subset\bigcup\{f_{m-1}^{-1}(\overline{e_j^{i_j}});j=1,\cdots,t\}$ and each summand $f_{m-1}^{-1}(\overline{e_j^{i_j}})$ is metrizable, B_ξ is a compact metrizable space.

Since $\operatorname{Ind}B_\xi\leqq 0$, we can consider B_ξ as the limit space of an inverse limiting system $\{B_i,\varphi_{ij}\}$, where $B_i, i=1,2,\cdots$, are finite discrete spaces, by Nagami [17, §2]. Let π_i be the projection of B_ξ into B_i for $i=1,2,\cdots$. Set

$$\mathfrak{H}_i=\{H_\alpha=f_\xi(\pi_i^{-1}(\alpha))\,;\,\alpha\in B_i\},\qquad i=1,2,\cdots\,;$$

then this is a sequence of closed coverings of $\overline{e_\xi^m}-e_\xi^m$ of order $\leqq m$. Let ρ be a metric of $\overline{e_\xi^m}$ agreeing with the preassigned topology of $\overline{e_\xi^m}$. There exists, for every $\alpha\in B_1$, an open set G_α of $\overline{e_\xi^m}$ such that i) $S(H_\alpha,1/2)=\{x\,;\,\rho(x,H_\alpha)<1/2\}\supset G_\alpha$, ii) order $\{G_\alpha\,;\,\alpha\in B_1\}\leqq m$, by Alexandroff-Hopf [2, Theorem 6, p. 71]. Let, for any point x of $\overline{e_\xi^m}-\bigcup\{G_\alpha\,;\,\alpha\in B_1\}$, $V(x)$ be an open neighborhood of x such that i) $\overline{V(x)}\cap(\overline{e_\xi^m}-e_\xi^m)=\phi$, ii) $V(x)\subset S(x,1/2)$. Since $\dim\overline{e_\xi^m}\leqq m$, an open covering

$$\{G_\alpha,\,V(x)\,;\,\alpha\in B_1,\,x\in\overline{e_\xi^m}-\bigcup\{G_\alpha\,;\,\alpha\in B_1\}\}$$

of $\overline{e_\xi^m}$ can be refined by a finite open covering

$$\mathfrak{G}_1'=\{E_\alpha',\,E_{\alpha'}\,;\,\alpha\in B_1,\,\alpha'\in B_1'\}$$

such that i) $E_\alpha'\subset G_\alpha$ for any $\alpha\in B_1$, ii) $\overline{E_{\alpha'}}\cap(\overline{e_\xi^m}-e_\xi^m)=\phi$ for any $\alpha'\in B_1'$, iii) order $\mathfrak{G}_1'\leqq m+1$. Set

$$E_\alpha=E_\alpha'\cup(G_\alpha-\bigcup\{\overline{E_{\alpha'}}\,;\,\alpha'\in B_1'\})\,;$$

then it is evident that

$$\mathfrak{G}_1=\{E_\alpha,\,E_{\alpha'}\,;\,\alpha\in B_1,\,\alpha'\in B_1'\}$$

is an open covering of $\overline{e_\xi^m}$ with order $\mathfrak{G}_1\leqq m+1$.

It is easy to construct, by a successive application of the same argument as in the above, a sequence

$$\mathfrak{G}_i=\{E_\alpha,\,E_{\alpha'}\,;\,\alpha\in B_i,\,\alpha'\in B_i'\},\qquad i=1,2,\cdots,$$

of finite open coverings of $\overline{e_\xi^m}$ with order $\mathfrak{G}_i\leqq m+1$ for $i=1,2,\cdots$, which satisfies the following conditions:

i) $\overline{\mathfrak{G}_{i+1}}$ refines $\mathfrak{G}_i$ for $i=1,2,\cdots$.

ii) For any i and any $\alpha'\in B_i'$, $\overline{E_{\alpha'}}\cap(\overline{e_\xi^m}-e_\xi^m)=\phi$ and $\operatorname{dia}E_{\alpha'}$ (i.e. the diameter of $E_{\alpha'}$) $<2^{-i+1}$.

iii) For any i and any $\alpha\in B_i$, $H_\alpha\subset E_\alpha\subset S(H_\alpha,2^{-i})$.

iv) For any i and any $\alpha\in B_{i+1}$, $\overline{E_\alpha}\subset E_{\varphi_{i+1,i}(\alpha)}$.

Let C_i be a finite discrete space which is the disjoint union of B_i and B_i' for $i=1,\cdots 2,\cdots$. Define $\psi_{i+1,i}:C_{i+1}\to C_i$ for $i=1,2,\cdots$, as follows: i) $\psi_{i+1,i}(\alpha)=\varphi_{i+1,i}(\alpha)$, if $\alpha\in B_{i+1}$, ii) $\overline{E_{\alpha'}}\subset E_{\psi_{i+1,i}(\alpha')}$, if $\alpha'\in B_{i+1}'$. For any pair $i>j$ set $\psi_{ij}=\psi_{j+1,j}\cdots\psi_{i,i-1}$ and let C_ξ be the inverse limit of $\{C_i,\psi_{ij}\}$. Then C_ξ is a compact metric space with $\operatorname{Ind}C_\xi\leqq 0$ which contains B_ξ as a closed subset. Define $g_\xi:C_\xi\to\overline{e_\xi^m}$ in such a way that

$$g_\xi((\alpha_1,\alpha_2,\cdots))=\bigcap_{i=1}E_{\alpha_i}\,;$$

then g_ξ is a continuous mapping of C_ξ onto $\overline{e_\xi^m}$ with order $g_\xi\leqq m+1$ such that

$$g_\xi | B_\xi = f_\xi .$$

Let A_m be the disjoint sum of A_{m-1} and $C_\xi - B_\xi$, $\xi \in X$. Define $f_m : A_m \to K^m$ in such a way that i) $f_m | A_{m-1} = f_{m-1}$, ii) $f_m | C_\xi - B_\xi = g_\xi | C_\xi - B_\xi$, $\xi \in X$. Define the topology of A_m as follows: A subset F of A_m is closed if and only if i) $F \cap A_{m-1}$ is closed in A_{m-1}, ii) $F \cap C_\xi$ is closed in C_ξ for every $\xi \in X$. Then A_m is a topological space and f_m is a closed continuous mapping of A_m onto K^m such that $\text{order} f_m \leqq m+1$. Let $\mathfrak{U} = \{U_\delta ; \delta \in \Delta\}$ be an arbitrary open covering of A_m. Then $\mathfrak{U} \wedge A_{m-1}$ can be refined by a relatively open covering $\mathfrak{V} = \{V_\delta ; \delta \in \Delta\}$ of A_{m-1} which is locally finite in A_{m-1} such that i) $\text{order}\, \mathfrak{V} \leqq 1$, ii) $V_\delta \subset U_\delta$ for every $\delta \in \Delta$. For every $\xi \in X$, $\mathfrak{V} \wedge B_\xi$ is a finite relatively open covering of B_ξ with $\text{order}\, \mathfrak{V} \wedge B_\xi \leqq 1$. Hence we can find a relatively open covering $\mathfrak{V}_\xi = \{V_{\xi\delta} ; \delta \in \Delta\}$ of $A_{m-1} \cup C_\xi$ with $\text{order}\, \mathfrak{V}_\xi \leqq 1$ such that $V_{\xi\delta} \subset U_\delta$ and $V_{\xi\delta} \cap A_{m-1} = V_\delta$ for every $\delta \in \Delta$. Then it can easily be seen that

$$\mathfrak{W} = \{W_\delta = \bigcup \{V_{\xi\delta} ; \xi \in X\} ; \delta \in \Delta\}$$

is an open covering of A_m such that i) $\text{order}\, \mathfrak{W} \leqq 1$, ii) $W_\delta \subset U_\delta$ for every $\delta \in \Delta$. Thus A_m is a paracompact space with $\text{Ind}\, A_m \leqq 0$. To prove that A_m is a Hausdorff space, let x and y be arbitrary different points of A_m. Since x and y are closed subsets of A_m, $\{A_m - x, A_m - y\}$ is an open covering of A_m. Hence we can find, by the same way as is stated in the above, an open covering $\{W_1, W_2\}$ of A_m such that i) $W_1 \subset A_m - x$ and $W_2 \subset A_m - y$, ii) $W_1 \cap W_2 = \phi$. It is evident that $y \in W_1$ and $x \in W_2$, which shows that A_m is a Hausdorff space.

On the other hand it is evident that $f_m | A_{m-1} = f_{m-1}$ and $f_m^{-1}(\overline{e_\xi^m}) = C_\xi$ is metrizable for any $\xi \in X$. Therefore the validity of (P_m) is established and the induction is completed.

Thus we know that (P_n) is valid by the induction. By Lemma 3.5 A_n is the limit space of an inverse limiting full system $\{D_\sigma, \pi_{\sigma\tau} ; \sigma \in M\}$ of discrete spaces D_σ. Let

$$\boldsymbol{F}_n = \{\mathfrak{F}_\sigma = \{f(\pi_\sigma^{-1}(\alpha)) ; \alpha \in D_\sigma\}, \sigma \in M\} ,$$

where $\pi_\sigma : A_n \to D_\sigma$, $\sigma \in M$, are the projections. Then it can easily be seen that $\boldsymbol{F}_n$ satisfies all of the requirements of the theorem, and the proof is completed.

By an analogous argument to this proof we have the following.

COROLLARY 7.2. *Any infinite dimensional* CW-*complex* K *admits a directed family* $\boldsymbol{F}$ *of locally finite closed coverings which follows out* K *fully such that* i) $\boldsymbol{F} \wedge K^i$ *follows out* K^i *fully,* ii) $\text{order}\, \boldsymbol{F} \wedge K^i \leqq i+1$, *for* $i = 0, 1, \cdots$.

COROLLARY 7.3. *For any* CW-*complex* K^n *we have*

$$\dim K^n = \text{ind}\, K^n = \text{Ind}\, K^n = n .$$

PROOF. By Theorems 4.3 and 7.1 we have $\operatorname{Ind} K^n \leqq n$. It is well known that $\dim K^n \leqq \operatorname{Ind} K^n$ and $\operatorname{ind} K^n \leqq \operatorname{Ind} K^n$. Let e^n be an arbitrary n-cell of K^n. Then it is evident that $n = \operatorname{ind} \overline{e^n} \leqq \operatorname{ind} K^n$ and $n = \dim \overline{e^n} \leqq \dim K^n$. Thus we have the equalities $\dim K^n = \operatorname{ind} K^n = \operatorname{Ind} K^n = n$ and the proof is completed.

REMARK 7.4. It is to be noted that the equality $\dim K^n = n$ has already been proved by H. Miyazaki [11] and K. Morita [15, Theorem 2]. Recently B. Pasynkov [22] proved that, for any locally compact group G, the equalities $\dim G = \operatorname{ind} G = \operatorname{Ind} G$ hold. It seems to the author an interesting problem to study whether any n-dimensional locally compact group G admits a directed family of locally finite closed coverings of order $\leqq n+1$ which follows out G fully or not.

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