Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 137-140

MAPS OF MANIFOLDS WITH INDEFINITE METRICS PRESERVING CERTAIN GEOMETRICAL ENTITIES

R. S. KULKARNI

Department of Mathematics
Indiana University
Bloomington, Indiana 47401

and

(Mathematics Department, Rutgers University)

(Received June 1, 1977)

ABSTRACT. It is shown that (i) a diffeomorphism of manifolds with indefinite metrics preserving degenerate r-plane sections is conformal, (ii) a sectional curvature-preserving diffeomorphism of manifolds with indefinite metrics of dimension ≥ 4 is generically an isometry.

1. INTRODUCTION.

Let (M^n, g), $(\overline{M}^n, \overline{g})$ be pseudo-Riemannian manifolds. A diffeomorphism $f: M \to \overline{M}$ is said to be curvature-preserving if given $p \in M$ and a 2-dimensional plane section σ at p such that the sectional curvature $K(\sigma)$ is defined then at f(p) the sectional curvature $\overline{K}(f_* \sigma)$ is defined and $K(\sigma) = \overline{K}(f_* \sigma)$. A point $p \in M$ is called isotropic if there exists a constant c(p) such that $K(\sigma) = c(p)$ for any 2-plane section σ at p for which K is defined. I studied the notion of a curvature preserving map in the Riemannian case and showed

THEOREM 1. If $n \geq 4$ (M^n, g), $(\overline{M}^n, \overline{g})$ *Riemannian manifolds and non-isotropic* *points are dense in* M *then a curvature-preserving map* $f:M \to \overline{M}$ *is an isometry.*

cf. [1] and for this and other types of "Riemannian" analogues cf. [5], [6] [2], [3], [4]. The purpose of this note is to point out Theorem 2.

THEOREM 2. *Theorem 1 is valid for pseudo-Riemannian manifolds.*

Unlike certain local results in pseudo-Riemannian geometry Theorem 2 is not obtained from Theorem 1 by formal changes of signs. Its proof is actually simpler but for an entirely different reason which seems to be well worth pointing out. One of the main steps in Theorem 1 and its other analogues mentioned above is that a curvature-preserving map is necessarily conformal on the set of nonisotropic points. This step is automatic in the case of indefinite metrics due for the next result. Let us call a subspace A of a tangent space at a point in M *degenerate* (resp. nondegerate) if $g|_A$ is degenerate (resp. nondegenerate). Sectional curvature is defined only for nondegenerate 2-plane sections. So by definition a curvature-preserving map carries degenerate 2-plane sections into degenerate 2-plane sections.

THEOREM 3. Let (M^n,g), $(\overline{M}^n,\overline{g})$ *be indefinite pseudo-Riemannian manifolds,* $n \geq 3$. *Let* $r \geq 1$. *Let* $f:M \to \overline{M}$ *be a diffeomorphism which carries degenerate r-dimensional plane sections of* M *into those of* $\overline{M}$. *Then* f *is conformal*. (i.e. *there exists a nowhere vanishing smooth function* $\Phi:M \to \mathbb{R}$ *such that* $f^*\overline{g} = \Phi\cdot g$.)

Recall that a geodesic on (M,g) whose tangent vector field X satisfies $g(X,X) = 0$ is called a *light like* geodesic.

COROLLARY 1. Let (M^n,g), $(\overline{M}^n,\overline{g})$ *be indefinite pseudo-Riemannian manifolds. Then a diffeomorphism* $f:M \to \overline{M}$ *which preserves light-like geodesics is conformal.*

This is the case $r = 1$ of Theorem 3. Note that this corollary is an extension and "Geometrization" of H. Weyl's famous observation about the conformal invariance of Maxwell's equations.

2. PROOF OF THEOREMS 2 AND 3.

First we prove Theorem 3.

The case $r = 2$ contains the essential ideas so we prove the theorem only in this case leaving the general case to the reader. Let $T_p(M)$ denote the tangent space to M at p etc. It clearly suffices to show that for each p in M $f_{*_p}: T_p(M) \to T_{f(p)}(\bar{M})$ is a homothety. Let $\{e_i, e_j, e_\alpha\}$ be an orthonormal set of vectors so that

$$\langle e_i, e_i\rangle = \langle e_j, e_j\rangle = -\langle e_\alpha, e_\alpha\rangle$$

Let $f_* e_i = \overline{e_i}$ and g or $\langle , \rangle$ also denote the canonically induced metric in all tensor powers and similarly for $\bar{g}$. Let $x^2 + y^2 = 1$. Then the 2-dimensional plane $\sigma = \mathrm{span}\,\{xe_i + ye_j + e_\alpha,\ - ye_i + xe_j\}$ is degenerate. Hence by hypothesis $f_*\sigma$ is degenerate i.e.

$$o = \bar{g}\ ((x\ \overline{e_i} + y\ \overline{e_j} + \overline{e_\alpha})\wedge(-y\ \overline{e_i} + x\ \overline{e_j}),\ (x\ \overline{e_i} + y\ \overline{e_j} + \overline{e_\alpha})\wedge(-y\ \overline{e_i} + x\ \overline{e_j}))$$

$$= \bar{g}\ (\overline{e_i} \wedge \overline{e_j} + x\ \overline{e_\alpha} \wedge \overline{e_j} - y\ \overline{e_\alpha} \wedge \overline{e_i},\ \overline{e_i} \wedge \overline{e_j} + x\ \overline{e_\alpha} \wedge \overline{e_j} - y\ \overline{e_\alpha} \wedge \overline{e_i})$$

$$= \{\bar{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_i} \wedge \overline{e_j}) + x^2\bar{g}(\overline{e_\alpha} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) + y^2\bar{g}(\overline{e_\alpha} \wedge \overline{e_i}, \overline{e_\alpha} \wedge \overline{e_i}) -$$

$$- 2xy\ \bar{g}(\overline{e_\alpha} \wedge \overline{e_i}, \overline{e_\alpha} \wedge \overline{e_j})\} + \{2x\ \bar{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) - 2y\ g(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_i})\}.$$

A similar expression with (x,y) replaced by $(-x,-y)$ is also true. Hence each $\{,\}$ is separately zero and since (x,y) are subject to the only relation $x^2 + y^2 = 1$ it follows that

$$o = \bar{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_i}) = \bar{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) = \bar{g}(\overline{e_i} \wedge \overline{e_\alpha}, \overline{e_j} \wedge \overline{e_\alpha})$$

and

$$\bar{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_i} \wedge \overline{e_j}) = -\bar{g}(\overline{e_i} \wedge \overline{e_\alpha}, \overline{e_i} \wedge \overline{e_\alpha}) = -\bar{g}(\overline{e_j} \wedge \overline{e_\alpha}, \overline{e_j} \wedge \overline{e_\alpha})$$

i.e. $\{\overline{e_i} \wedge \overline{e_j}, \overline{e_i} \wedge \overline{e_\alpha}, \overline{e_j} \wedge \overline{e_\alpha}\}$ is an orthogonal basis of the second exterior power $\Lambda^2(\text{span}\ \{\overline{e_i}, \overline{e_j}, \overline{e_\alpha}\})$. This means that f induces a homothetic map of $\Lambda^2(\text{span}\ \{e_i, e_j, e_\alpha\})$ onto $\Lambda^2(\text{span}\ \{\overline{e_i}, \overline{e_j}, \overline{e_\alpha}\})$. It is then easy to see that f induces a homothety of span $\{e_i, e_j, e_\alpha\}$ onto span $\{e_i, e_j, e_\alpha\}$. By varying the set $\{e_i, e_j, e_\alpha\}$ it is clear that f_{*_p} is a homothety. This finishes the proof. QED

PROOF OF THEOREM 2. By Theorem 3 we have $f^*\bar{g} = \Phi\cdot g$ where Φ is a nowhere vanishing function on M. Now the proof that f is an isometry i.e. $\Phi = 1$ is exactly as in [1] or [4] §7. QED

ACKNOWLEDGMENT. This work was partially supported by NSF Grant MPS - 71 - 03442.

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KEY WORDS AND PHRASES. *Riemannian and pseudo-Riemannian manifolds, diffeomorphism of manifolds.*

AMS(MOS) SUBJECT CLASSIFICATIONS (1970). 53C20, 53C25.