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MARCINKIEWICZ LAWS AND CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES[1]

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Consider a set of independent identically distributed random variables indexed by Z_+^d, the positive integer d-dimensional lattice points, $d \geqq 2$. The classical Kolmogorov-Marcinkiewicz strong law of large numbers is generalized to this case. Also, convergence rates in the law of large numbers are derived, i.e., the rate of convergence to zero of, for example, the tail probabilities of the sample sums is determined.

1. Introduction. Let $Z_+{}^d$, where $d \geqq 2$ is an integer, denote the positive integer d-dimensional lattice points and let $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+{}^d\}$ be a set of i.i.d. random variables. The notation $\mathbf{m} \prec \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \cdots, m_d)$ and $\mathbf{n} = (n_1, n_2, \cdots, n_d) \in Z_+{}^d$ means that $m_i \leqq n_i$, $i = 1, 2, \cdots, d$, $|\mathbf{n}|$ is used for $\prod_{i=1}^d n_i$ and $\mathbf{n} \to \infty$ is to be interpreted as $n_i \to \infty$, $i = 1, 2, \cdots, d$.

Let $S_{\mathbf{n}} = \sum_{\mathbf{k} \prec \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in Z_+{}^d$. It has been shown by Smythe [20] (see also [10]) that the strong law of large numbers holds if and only if $E|X_1| \cdot (\log^+ |X_1|)^{d-1} < \infty$. In the first part of this paper we generalize this result and prove a strong Marcinkiewicz law, i.e., we show that $|\mathbf{n}|^{-1/r} \cdot S_{\mathbf{n}} \to 0$ a.s. as $\mathbf{n} \to \infty$ if and only if $E|X_1|^r \cdot (\log^+ |X_1|)^{d-1} < \infty$, $0 < r < 2$, with $EX = 0$ if $r \geqq 1$. If $r = 1$ one recovers the result of [20]. In contrast to the case $d = 1$ where the strong and weak laws both hold under the assumption of a finite mean, only the weak law holds if $d \geqq 2$; see [10]. A result of Pyke and Root [19] enables us to prove a weak Marcinkiewicz law for $0 < r < 2$ requiring only $E|X|^r < \infty$.

The second part of the paper deals with convergence rates in the law of large numbers; that is, we investigate how fast quantities such as, e.g., $P(|S_{\mathbf{n}}| \geqq |\mathbf{n}| \cdot \varepsilon)$ tend to zero if moments higher than the first exist and $EX = 0$.

This can be done by studying these quantities directly, but also by studying convergence of sums of the type $\sum_{\mathbf{n}} |\mathbf{n}|^t \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}| \cdot \varepsilon)$, where t is a real number.

In the case $d = 1$ this was initiated by Hsu and Robbins [12] and Erdös [7] for $t = 0$ and Spitzer [22] for $t = -1$. Later Katz [14], Baum and Katz [1], Chow [5], Lai [15], Chow and Lai [6] have continued these investigations. Except for [1], they deal mainly with the convergence of the above sums. In [2] it is shown that $P(|S_n| \geqq n\varepsilon) = o(n^{1-r})$ as $n \to \infty$ if $E|X|^r < \infty$, $1 \leqq r < 2$.

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This has later been extended and generalized to $r \geqq 1$ in [1]. Jain [13] has generalized some of the results to Banach space valued variables. For $d \geqq 2$, $\sum_{\mathbf{n}} |\mathbf{n}|^t \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|\varepsilon)$ has been studied in [21] for $t = 0$ and in [4] for $t \geqq -1$.

The plan of the paper is as follows. In Section 2 we collect some auxiliary results. Section 3 deals with the Marcinkiewicz laws. In Section 4 two theorems comparing sums for convergence rates are stated and they are proved in Sections 5 and 6 respectively. The probabilities are studied directly in Section 7. Section 8, finally, contains some remarks.

2. Preliminaries. In this section we collect some general facts along with some lemmas that will be of use later.

Again $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ are i.i.d. random variables with mean zero whenever it is finite. Let X denote a random variable which has the same distribution as X_1 and which is independent of all other random variables.

The notation $\simeq$ between sums and/or integrals will be used to denote that the quantities on either side of the sign converge simultaneously. $I\{\cdot\}$ denotes the indicator function of the set in braces.

Define $d(x) = \text{Card}\,\{\mathbf{n} \in Z_+^d;\, |\mathbf{n}| = [x]\}$ and $M(x) = \sum_{k=1}^{[x]} d(k)$. We have (cf. [21])

$$M(x) = O(x(\log^+ x)^{d-1}) \quad \text{and} \quad d(x) = o(x^\delta) \quad \forall\, \delta > 0 \quad \text{as} \quad x \to \infty\,. \tag{2.1}$$

LEMMA 2.1. *Let $r > 0$ and $m = 0, 1, 2, \cdots$. For any random variable X the following statements are equivalent*:

$$E|X|^r \cdot (\log^+ |X|)^m \cdot d(X) < \infty\,, \tag{2.2}$$

$$E|X|^r \cdot (\log^+ |X|)^{m+d-1} < \infty\,, \tag{2.3}$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1} \cdot (\log |\mathbf{n}|)^m \cdot P(|X| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) < \infty\,, \qquad \alpha > 0,\ \varepsilon > 0\,, \tag{2.4}$$

$$\sum_{j=1}^\infty j^{\alpha r-1} \cdot (\log j)^{d+m-1} \cdot P(|X| \geqq j^\alpha \cdot \varepsilon) < \infty\,, \qquad \alpha > 0,\ \varepsilon > 0\,, \tag{2.5}$$

$$\sum_{j=1}^\infty j^{\alpha r-1} \cdot (\log j)^m \cdot d(j) \cdot P(|X| \geqq j^\alpha \cdot \varepsilon) < \infty\,, \qquad \alpha > 0,\ \varepsilon > 0\,. \tag{2.6}$$

This generalizes results from [21] and [4]. The proofs are similar and omitted.

LEMMA 2.2. *Let $Y_{\mathbf{n}} = X_{\mathbf{n}} \cdot I\{|X_{\mathbf{n}}| \leqq \varepsilon \cdot |\mathbf{n}|^{1/r}\}$. If $E|X|^r \cdot (\log^+ |X|)^{d-1+m} < \infty$, $0 < r < 2$, $m = 0, 1, 2, \cdots$, then*

$$\sum_{\mathbf{n}} (\log |\mathbf{n}|)^m \cdot \text{Var}\,(|\mathbf{n}|^{-1/r} \cdot Y_{\mathbf{n}}) < \infty\,, \tag{2.7}$$

$$\sum_{j=1}^\infty j^{-2/r} \cdot (\log j)^{d-1+m} \cdot \text{Var}\,(|X| \cdot I\{|X| \leqq \varepsilon j^{1/r}\}) < \infty\,. \tag{2.8}$$

Formula (2.7) reduces to [21], Lemma 2.2 if $r = 1$, $m = 0$.

PROOF. It is sufficient to prove the lemma with $\varepsilon = 1$.

$$\begin{aligned} &\sum_{\mathbf{n}} (\log |\mathbf{n}|)^m \cdot \text{Var}\,(|\mathbf{n}|^{-1/r} \cdot Y_{\mathbf{n}}) \\ &\quad \leqq \sum_{j=1}^\infty (\log j)^m \cdot j^{-2/r} \cdot d(j) \cdot E(|X|^2 \cdot I\{|X| \leqq j^{1/r}\}) \\ &\quad = \sum_{j=1}^\infty (\log j)^m \cdot j^{-2/r} \cdot d(j) \cdot \sum_{i=1}^j \int_{(i-1)^{1/r} < |x| \leqq i^{1/r}} x^2\, dF(x) \end{aligned}$$

$$= \sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} (\log j)^m \cdot j^{-2/r} \cdot d(j)) \int_{(i-1)^{1/r}<|x|\leqq i^{1/r}} x^2\, dF(x)$$
$$\leqq \text{const.} \sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} (\log j)^m \cdot j^{-2/r-1} \cdot M(j)) \cdot i^{2/r} \cdot P(i-1 < |X|^r \leqq i)$$
$$\simeq \sum_{i=1}^{\infty} (\sum_{j=i}^{\infty} (\log j)^{m+d-1} \cdot j^{-2/r}) \cdot i^{2/r} \cdot P(i-1 < |X|^r \leqq i)$$
$$\simeq \sum_{i=1}^{\infty} (\log i)^{m+d-1} \cdot i \cdot P(i-1 < |X|^r \leqq i) .$$

The last expression is finite if and only if $E|X|^r \cdot (\log^+ |X|^r)^{m+d-1} < \infty$, which is equivalent to $E|X|^r \cdot (\log^+ |X|)^{m+d-1} < \infty$. This proves (2.7). The proof of (2.8), being similar, is omitted.

Next we note that for any fixed $\mathbf{n} \in Z_+^d$, $S_\mathbf{n}$ is simply a sum of $|\mathbf{n}|$ (i.i.d.) random variables and therefore many results for partial sums remain valid when $d \geqq 2$. For example, this is true for the Marcinkiewicz–Zygmund inequalities, [17], page 109, and the theorem of [19].

For ease of reference we state some of these results in a separate lemma.

LEMMA 2.3. *Let* $\{X_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables, suppose that* $E|X|^r < \infty$, $r > 0$, *and that* $EX = 0$ *if* $r \geqq 1$. *Then,*

$$E|S_\mathbf{n}|^r = o(|\mathbf{n}|) \quad \text{as} \quad \mathbf{n} \to \infty \quad \text{if} \quad 0 < r < 2, \tag{2.9}$$

$$E|S_\mathbf{n}|^r \leqq B_r \cdot |\mathbf{n}| \cdot E|X|^r \qquad \text{if} \quad 1 \leqq r < 2, \tag{2.10}$$

$$E|S_\mathbf{n}|^r \leqq B_r \cdot |\mathbf{n}|^{r/2} \cdot E|X|^r \qquad \text{if} \quad r \geqq 2, \tag{2.11}$$

where B_r *are constants depending on r only.*

The last lemma is due to Hoffmann-Jørgensen, [11], page 164 (see also [13], page 159).

LEMMA 2.4. *Let* $\{X_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *symmetric random variables. Then, for* $j = 1, 2, \cdots$

$$P(|S_\mathbf{n}| \geqq 3^j t) \leqq C_j \cdot |\mathbf{n}| \cdot P(|X| \geqq t) + D_j(P(|S_\mathbf{n}| \geqq t))^{2^j}, \tag{2.12}$$

where C_j *and* D_j *are nonnegative constants depending on j only. Also,* $C_1 = 1$, $D_1 = 4$.

3. Marcinkiewicz laws. In view of [20], finite mean is not enough for the strong law of large numbers to hold for i.i.d. random variables; in fact, $E|X| \cdot (\log^+ |X|)^{d-1} < \infty$ is necessary and sufficient. Finite mean only, however, entails the weak law of large numbers, [10]. In this section we extend these results to $0 < r < 2$, thus yielding strong and weak Marcinkiewicz laws. (For $d = 1$, see, e.g., Loève [16], pages 242–243.)

THEOREM 3.1. *Let* $\{X_\mathbf{n}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables. Suppose that* $E|X|^r < \infty$, $0 < r < 2$, *and set* $EX = 0$ *if* $r \geqq 1$. *Then,*

$$|\mathbf{n}|^{-1/r} \cdot S_\mathbf{n} \to 0 \quad \text{in probability and in} \quad L^r \quad \text{as} \quad \mathbf{n} \to \infty . \tag{3.1}$$

PROOF. The L^r-convergence is simply a restatement of (2.9) and the convergence in probability follows by applying Markov's inequality.

THEOREM 3.2. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables. Suppose that* $E|X|^r \cdot (\log^+ |X|)^{d-1} < \infty$, $0 < r < 2$, *and set* $EX = 0$ *if* $r \geqq 1$. *Then*

$$(3.2) \qquad |\mathbf{n}|^{-1/r} \cdot S_{\mathbf{n}} \to 0 \quad \text{a.s.} \qquad \text{as} \quad \mathbf{n} \to \infty .$$

Conversely, (3.2) *implies that* $E|X|^r \cdot (\log^+ |X|)^{d-1} < \infty$.

REMARK 3.1. This result has, independently, been obtained by Chen [4] using a slightly different approach. Furthermore, the sufficiency is proven for i.i.d. random variables with a more general index set.

PROOF. (Sufficiency). We begin by proving the result for symmetric random variables.

Define $Y_{\mathbf{n}} = X_{\mathbf{n}} \cdot I\{|X_{\mathbf{n}}| \leqq |\mathbf{n}|^{1/r}\}$ as in Section 2. ($\varepsilon = 1$.) By (2.4) with $\alpha r = 1$ and $m = 0$, $\sum_{\mathbf{n}} P(X_{\mathbf{n}} \neq Y_{\mathbf{n}}) < \infty$ and by (2.7) with $m = 0$, $\sum_{\mathbf{n}} \operatorname{Var}(|\mathbf{n}|^{-1/r} \cdot Y_{\mathbf{n}}) < \infty$. In view of the symmetry $\sum_{\mathbf{n}} E(|\mathbf{n}|^{-1/r} \cdot Y_{\mathbf{n}})$ vanishes. Since the first two sums have only nonnegative terms they converge absolutely. Thus, by the Kolmogorov three series criterion as established in [8], [9],

$$(3.3) \qquad \textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{-1/r} \cdot X_{\mathbf{n}} \quad \text{converges unconditionally a.s.}$$

In particular,

$$(3.4) \qquad \textstyle\sum_{j=1}^{\infty} \sum_{|\mathbf{n}|=j} |\mathbf{n}|^{-1/r} \cdot X_{\mathbf{n}} < \infty \quad \text{a.s.}$$

An application of Kronecker's lemma yields the desired conclusion.

To remove the symmetry assumption we argue as follows. (Cf. [16].) Let $\{X_{\mathbf{n}}^s, \mathbf{n} \in Z_+^d\}$ be the symmetrized random variables. It is easy to see that $E|X^s|^r \cdot (\log^+ |X^s|)^{d-1} < \infty$. Therefore, by what has already been proved, $|\mathbf{n}|^{-1/r} \cdot S_{\mathbf{n}}^s \to 0$ almost surely as $\mathbf{n} \to \infty$. By the weak symmetrization inequalities ([16], page 245) it follows that $|\mathbf{n}|^{-1/r} \cdot (S_{\mathbf{n}} - \operatorname{med}(S_{\mathbf{n}})) \to 0$ in probability as $\mathbf{n} \to \infty$, where $\operatorname{med}(X)$ is a median of X. This fact together with (3.1) yields

$$(3.5) \qquad |\mathbf{n}|^{-1/r} \cdot \operatorname{med}(S_{\mathbf{n}}) \to 0 \qquad \text{as} \quad \mathbf{n} \to \infty ,$$

which, together with the symmetrization inequalities ([16], page 247), shows that (3.2) holds also in the nonsymmetric case.

PROOF. (Necessity). If (3.2) holds, then $|\mathbf{n}|^{-1/r} \cdot X_{\mathbf{n}} \to 0$ almost surely as $n \to \infty$. By the Borel-Cantelli lemmas this is equivalent to $\sum_{\mathbf{n}} P(|X_{\mathbf{n}}| \geqq |\mathbf{n}|^{1/r} \cdot \varepsilon) < \infty$, which is equivalent to $E|X|^r(\log^+ |X|)^{d-1} < \infty$, because of the equal distribution assumption and Lemma 2.1. This proves the theorem.

4. Convergence rates comparing sums. In this section we relate moment conditions to the convergence of certain sums. See [12], [7], [22], [14], [1] and [5] for the case $d = 1$.

THEOREM 4.1. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables, let* $r \geqq 1/\alpha$ *and* $\alpha > \frac{1}{2}$. *The following statements are equivalent*:

$$(4.1) \qquad E|X|^r \cdot (\log^+ |X|)^{d-1} < \infty \qquad \text{and, if} \quad r \geqq 1 , \quad EX = 0 .$$

(4.2) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

(4.3) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(\max_{\mathbf{k}\prec\mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

If $r > 1/\alpha$, $\alpha > \frac{1}{2}$, *then the above statements are also equivalent to*

(4.4) $\sum_{j=1}^{\infty} j^{\alpha r-2} \cdot P(\sup_{j\leqq|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^{\alpha}| \geqq \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

To see that (4.4) is not equivalent to (4.1)—(4.3) when $r = 1/\alpha$, $\alpha > \frac{1}{2}$, we also prove the following result.

THEOREM 4.2. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in Z_+^d\}$ *be* i.i.d. *random variables, let* $r = 1/\alpha$ *and* $\alpha > \frac{1}{2}$. *The following statements are equivalent:*

(4.5) $E|X|^r \cdot (\log^+ |X|)^d < \infty$ *and, if* $r \geqq 1$, $EX = 0$.

(4.6) $\sum_{\mathbf{n}} |\mathbf{n}|^{-1} \cdot \log(|\mathbf{n}|) \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{1/r} \cdot \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

(4.7) $\sum_{\mathbf{n}} |\mathbf{n}|^{-1} \cdot \log(|\mathbf{n}|) \cdot P(\max_{\mathbf{k}\prec\mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{1/r} \cdot \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

(4.8) $\sum_{j=1}^{\infty} j^{-1} \cdot P(\sup_{j\leqq|\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^{1/r}| \geqq \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

REMARK 4.1. Using the results from Section 2 we note that the moment condition in (4.1) may be replaced by $E|X|^{r-1} \cdot M(|X|) < \infty$. Chen [4] has shown that this condition (and $EX = 0$ if $r \geqq 1$) implies (4.2) for i.i.d. random variables with a more general index set. For $r = 2$, $\alpha = 1$ this has also been done by Smythe [21]. Both authors generalize the method of Erdös [7].

REMARK 4.2. In (4.5), $E|X|^r \cdot (\log^+ |X|)^d < \infty$ should be interpreted as $E|X|^r \cdot (\log^+ |X|) \cdot d(X) < \infty$ in order to compare with [1], Theorem 2.

REMARK 4.3. By replacing X by aX, where a is some constant, it follows that if one of the above series is convergent for one ε it converges for all $\varepsilon > 0$.

REMARK 4.4. Since $\alpha > \frac{1}{2}$ the condition $r\alpha = 1$ implies that $r < 2$.

The proofs of the theorems are given in Sections 5 and 6.

5. Proof of Theorem 4.1. The proof is divided into several steps. In steps (i)—(vi) the random variables are assumed to have a symmetric distribution.

(i) (4.1) $\Rightarrow$ (4.2), $\alpha r = 1$. Recall that, since $\alpha > \frac{1}{2}$, $r < 2$. Define $S_{\mathbf{n}}' = \sum_{\mathbf{k}\prec\mathbf{n}} X_{\mathbf{k}} \cdot I\{|X_{\mathbf{k}}| \leqq \varepsilon \cdot |\mathbf{n}|^{\alpha}\}$, $\mathbf{k} \prec \mathbf{n}$ and $S_{\mathbf{n}}'' = S_{\mathbf{n}} - S_{\mathbf{n}}'$.

$$\begin{aligned} P(|S_{\mathbf{n}}| \geqq 2\varepsilon \cdot |\mathbf{n}|^{\alpha}) &\leqq P(|S_{\mathbf{n}}'| \geqq \varepsilon \cdot |\mathbf{n}|^{\alpha}) + P(|S_{\mathbf{n}}''| \geqq \varepsilon \cdot |\mathbf{n}|^{\alpha}) \\ &\leqq \varepsilon^{-2} \cdot |\mathbf{n}|^{-2\alpha} \cdot \mathrm{Var}(S_{\mathbf{n}}') + |\mathbf{n}| \cdot P(|X| \geqq \varepsilon \cdot |\mathbf{n}|^{\alpha}) \\ &= \varepsilon^{-2} \cdot |\mathbf{n}|^{-(2/r)+1} \cdot \mathrm{Var}(|X| \cdot I\{|X| \leqq \varepsilon \cdot |\mathbf{n}|^{1/r}\}) \\ &\qquad + |\mathbf{n}| \cdot P(|X| \geqq \varepsilon \cdot |\mathbf{n}|^{1/r}) . \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq 2\varepsilon|\mathbf{n}|^{\alpha}) \\ &\quad = \sum_{\mathbf{n}} |\mathbf{n}|^{-1} \cdot P(|S_{\mathbf{n}}| \geqq 2\varepsilon|\mathbf{n}|^{1/r}) \\ &\quad \leqq \varepsilon^{-2} \cdot \sum_{\mathbf{n}} |\mathbf{n}|^{-2/r} \cdot \mathrm{Var}(|X| \cdot I\{|X| \leqq \varepsilon \cdot |\mathbf{n}|^{1/r}\}) + \sum_{\mathbf{n}} P(|X| \geqq |\mathbf{n}|^{1/r} \cdot \varepsilon) \\ &\quad = \varepsilon^{-2} \cdot \sum_{\mathbf{n}} \mathrm{Var}(|\mathbf{n}|^{-1/r} \cdot Y_{\mathbf{n}}) + \sum_{\mathbf{n}} P(|X| \geqq |\mathbf{n}|^{1/r} \cdot \varepsilon) , \end{aligned}$$

where $Y_{\mathbf{n}}$ is defined as in Lemma 2.2. The last two sums are finite by Lemmas 2.2 and 2.1 respectively.

(ii) (4.1) $\Rightarrow$ (4.2), $\alpha r > 1$, $1/\alpha < r \leqq 1$. By applying (2.12) with $j = 1$, Markov's inequality and the c_r-inequalities ([16]), we obtain

$$P(|S_{\mathbf{n}}| \geqq 3 \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq |\mathbf{n}| \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + 4(|\mathbf{n}|^{-\alpha r+1} \cdot \varepsilon^{-r} \cdot E|X|^r)^2 .$$

Therefore

$$\begin{aligned} &\textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq 3 \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) \\ &\qquad \leqq \textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1} \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + 4 \cdot (\varepsilon^{-r} E|X|^r)^2 \cdot \sum_{\mathbf{n}} |\mathbf{n}|^{-\alpha r} \\ &\qquad < \infty \qquad \text{by Lemma 2.1 and because} \quad \alpha r > 1 . \end{aligned}$$

(In fact $\sum_{\mathbf{n}} |\mathbf{n}|^{-\rho} = (\sum_{n=1}^{\infty} n^{-\rho})^d$, $\rho > 1$.)

(iii) (4.1) $\Rightarrow$ (4.2), $\alpha r > 1$, $1 < r < 2$. Here we use (2.12) with $j = 1$, Markov's inequality and (2.10) to obtain

$$P(|S_{\mathbf{n}}| \geqq 3 \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq |\mathbf{n}| \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + 4(\varepsilon^{-r} \cdot |\mathbf{n}|^{-\alpha r} \cdot B_r \cdot |\mathbf{n}| \cdot E|X|^r)^2$$

and consequently

$$\begin{aligned} &\textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq 3 \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) \\ &\qquad \leqq \textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1} \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + 4 \cdot (\varepsilon^{-r} \cdot B_r \cdot E|X|^r)^2 \cdot \sum_{\mathbf{n}} |\mathbf{n}|^{-\alpha r} \\ &\qquad < \infty \qquad \text{by Lemma 2.1 and because} \quad \alpha r > 1 . \end{aligned}$$

(iv) (4.1) $\Rightarrow$ (4.2), $\alpha r > 1$, $r \geqq 2$. This time we use (2.12) with $j \geqq 1$ (to be chosen later), Markov's inequality and (2.11).

$$\begin{aligned} P(|S_{\mathbf{n}}| \geqq 3^j \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) &\leqq C_j \cdot |\mathbf{n}| \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + D_j \cdot (P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon))^{2^j} \\ &\leqq C_j \cdot |\mathbf{n}| \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + D_j' \cdot |\mathbf{n}|^{-(\alpha r - r/2) \cdot 2^j} . \end{aligned}$$

Set $\beta = 2 - \alpha r + (\alpha r - (r/2)) \cdot 2^j = 2 - \alpha r + r(2\alpha - 1) \cdot 2^j$. Since $2\alpha > 1$, the last term is positive and we may therefore choose j such that $\beta > 1$. By doing so we obtain

$$\begin{aligned} &\textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq 3^j \cdot |\mathbf{n}|^{\alpha} \cdot \varepsilon) \\ &\qquad \leqq C_j \cdot \textstyle\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1} \cdot P(|X| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) + D_j' \cdot \sum_{\mathbf{n}} |\mathbf{n}|^{-\beta} \\ &\qquad < \infty \qquad \text{by Lemma 2.1 and because} \quad \beta > 1 . \end{aligned}$$

This proves that (4.1) $\Rightarrow$ (4.2) in the symmetric case. In view of Remark 4.3 the sums converge for all $\varepsilon > 0$.

(v) (4.2) $\Leftrightarrow$ (4.3). This follows immediately from

$$P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq P(\max_{\mathbf{k} < \mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq 2^d \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) .$$

The last inequality is Lévy's inequality for random variables indexed by $Z_+{}^d$; see [9], [18].

(vi) (4.2) $\Rightarrow$ (4.4). This part of the proof follows the ideas of [1]. Let

$\pi(i) = (i, 1, 1, \cdots, 1)$. Because of the i.i.d. assumption $S_{\mathbf{n}}$ and $S_{\pi(|\mathbf{n}|)}$ have the same distribution.

For reasons of clarity we collect some of the steps in a lemma.

LEMMA 5.1. *Under the assumptions of Theorem* 4.1 *and the symmetry*

(5.1) $$P(|S_{\mathbf{i}}| \geqq a) \leqq 2 \cdot P(|S_{\mathbf{k}}| \geqq a), \qquad \mathbf{i} \prec \mathbf{k},\ a > 0 .$$

(5.2) $$P(\sup_{2^{i-1} \leqq |\mathbf{n}| < 2^i} |S_{\mathbf{n}}/|\mathbf{n}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \leqq 2^d \sum_{|\mathbf{n}|=2^{i+d}} P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) .$$

(5.3) $$2^j \cdot \sum_{|\mathbf{n}|=2^j} P(|S_{\mathbf{n}}| \geqq 2 \cdot |\mathbf{n}|^\alpha \cdot \varepsilon) \leqq 2 \sum_{i=2^j}^{2^{j+1}-1} (\log i)^{d-1} P(|S_{\pi(i)}| \geqq i^\alpha \cdot \varepsilon) \qquad \textit{for large } j .$$

(5.4) $$\sum_{i=1}^\infty i^{\alpha r-2} (\log i)^{d-1} \cdot P(|S_{\pi(i)}| \geqq i^\alpha \cdot \varepsilon) < \infty , \qquad \alpha r \geqq 1,\ \alpha > \tfrac{1}{2} .$$

PROOF OF (5.1). Because of the symmetry $P(S_{\mathbf{k}} \geqq a) = P(S_{\pi(|\mathbf{k}|)} \geqq a) \geqq P(\{S_{\pi(|\mathbf{i}|)} \geqq a\} \cap \{S_{\pi(|\mathbf{k}|)} - S_{\pi(|\mathbf{i}|)} \geqq 0\}) \geqq \frac{1}{2} \cdot P(S_{\pi(|\mathbf{i}|)} \geqq a) = \frac{1}{2} \cdot P(S_{\mathbf{i}} \geqq a)$, $a > 0$, which together with a similar result for the other tail yields (5.1).

PROOF OF (5.2).

$$\begin{aligned}
&P(\sup_{2^{i-1} \leqq |\mathbf{n}| < 2^i} |S_{\mathbf{n}}/|\mathbf{n}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq P(\sup_{2^{i-1} \leqq |\mathbf{n}| < 2^i} |S_{\mathbf{n}}| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \leqq P(\sup_{|\mathbf{n}| \leqq 2^i} |S_{\mathbf{n}}| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq P(\sup_{|\mathbf{m}|=2^{i+d}} \sup_{\mathbf{n} \prec \mathbf{m}} |S_{\mathbf{n}}| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \leqq \sum_{|\mathbf{m}|=2^{i+d}} P(\sup_{\mathbf{n} \prec \mathbf{m}} |S_{\mathbf{n}}| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq \sum_{|\mathbf{m}|=2^{i+d}} 2^d P(|S_{\mathbf{m}}| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) = 2^d \cdot \sum_{|\mathbf{n}|=2^{i+d}} P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) .
\end{aligned}$$

PROOF OF (5.3). Note that $d(2^j) \sim j^{d-1} \sim (\log(2^j))^{d-1}$. Thus, for large j,

$$\begin{aligned}
&2^j \cdot \sum_{|\mathbf{n}|=2^j} P(|S_{\mathbf{n}}| \geqq 2|\mathbf{n}|^\alpha \cdot \varepsilon) \\
&\quad = 2^j d(2^j) P(|S_{\pi(2^j)}| \geqq 2 \cdot 2^{j\alpha} \cdot \varepsilon) \sim 2^j (\log 2^j)^{d-1} \cdot P(|S_{\pi(2^j)}| \geqq 2^{j\alpha+1} \cdot \varepsilon) \\
&\quad \leqq \sum_{i=2^j}^{2^{j+1}-1} (\log i)^{d-1} \cdot P(|S_{\pi(2^j)}| \geqq 2^{j\alpha+1} \cdot \varepsilon) \leqq (\text{by } (5.1)) \\
&\quad \leqq 2 \sum_{ii=2^j}^{2^{j+1}-1} (\log i)^{d-1} \cdot P(|S_{\pi(i)}| \geqq 2^{j\alpha+1} \cdot \varepsilon) \\
&\quad \leqq 2 \sum_{i=2^j}^{2^{j+1}-1} (\log i)^{d-1} \cdot P(|S_{\pi(i)}| \geqq i^\alpha \cdot \varepsilon) .
\end{aligned}$$

PROOF OF (5.4). The conclusion follows by estimating the tail probabilities as was done in steps (i)—(iii) together with Lemmas 2.1 and 2.2.

We now turn back to (vi). Let $C(\alpha, r) = 2^{\alpha r-2}$ if $\alpha r > 2$ and 1 otherwise.

$$\begin{aligned}
&\sum_{j=1}^\infty j^{\alpha r-2} \cdot P(\sup_{j \leqq |\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad = \sum_{i=0}^\infty \sum_{j=2^i}^{2^{i+1}-1} j^{\alpha r-2} \cdot P(\sup_{j \leqq |\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq \sum_{i=0}^\infty 2^i \cdot (2^i)^{\alpha r-2} \cdot C(\alpha, r) \cdot P(\sup_{2^i \leqq |\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq C(\alpha, r) \sum_{i=0}^\infty 2^{i(\alpha r-1)} \cdot \sum_{j=i+1}^\infty P(\sup_{2^{j-1} \leqq |\mathbf{k}| < 2^j} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq 2^{\alpha(1+d)} \cdot \varepsilon) \\
&\quad \leqq (\text{by } (5.2)) \leqq C(\alpha, r) \sum_{i=0}^\infty 2^{i(\alpha r-1)} \sum_{j=i+1}^\infty 2^d \sum_{|\mathbf{k}|=2^{j+d}} P(|S_{\mathbf{k}}| \geqq |\mathbf{k}|^\alpha \cdot \varepsilon)
\end{aligned}$$

(5.5) $$\begin{aligned}
&\leqq 2^d C(\alpha, r) \sum_{j=0}^\infty \Big(\sum_{i=0}^{j-1} 2^{i(\alpha r-1)}\Big) \sum_{|\mathbf{n}|=2^{j+d}} P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) \\
&\leqq 2^d C(\alpha, r) \sum_{j=0}^\infty 2^{j(\alpha r-1)} \sum_{|\mathbf{n}|=2^j} P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) \leqq (\text{by } (5.3)) \\
&\leqq \text{const.} + 2^{d+1} C(\alpha, r) \sum_{j=0}^\infty 2^{j(\alpha r-2)} \cdot \sum_{i=2^j}^{2^{j+1}-1} (\log i)^{d-1} \cdot P(|S_{\pi(i)}| \geqq i^\alpha \cdot \varepsilon/2) \\
&\leqq \text{const.} \sum_{i=1}^\infty i^{\alpha r-2} \cdot (\log i)^{d-1} \cdot P(|S_{\pi(i)}| \geqq i^\alpha \cdot \varepsilon/2) < \infty \quad \text{by } (5.4) .
\end{aligned}$$

Thus, (4.2) $\Rightarrow$ (4.4).

(vii) We now know that (4.1) $\Rightarrow$ (4.2) $\Leftrightarrow$ (4.3) and that (4.2) $\Rightarrow$ (4.4) (for α and r in appropriate regions) if the random variables have a symmetric distribution. In this step we remove that assumption.

Suppose that (4.1) holds and introduce the symmetrized random variables as in Section 3. As pointed out there, those variables also satisfy (4.1) and hence, by what has already been proven, (4.2) holds for the symmetrized variables, i.e.,

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}^s| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) < \infty . \tag{5.6}$$

By the weak symmetrization inequalities ([16], page 245), we have

$$\begin{aligned} \tfrac{1}{2} \cdot P(|S_{\mathbf{n}} - \operatorname{med}(S_{\mathbf{n}})| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) &\leqq P(|S_{\mathbf{n}}^s| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) \\ &\leqq 2P\left(|S_{\mathbf{n}} - \operatorname{med}(S_{\mathbf{n}})| \geqq |\mathbf{n}|^\alpha \cdot \frac{\varepsilon}{2}\right). \end{aligned} \tag{5.7}$$

Thus, (5.6) is equivalent to

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}} - \operatorname{med}(S_{\mathbf{n}})| \geqq |\mathbf{n}|^\alpha \cdot \varepsilon) < \infty . \tag{5.8}$$

If $r < 2$, then, by (3.5), $|\mathbf{n}|^{-1/r} \cdot \operatorname{med}(S_{\mathbf{n}}) \to 0$ as $\mathbf{n} \to \infty$, and, since $r\alpha \geqq 1$, it follows that $|\mathbf{n}|^{-\alpha} \cdot \operatorname{med}(S_{\mathbf{n}}) \to 0$ as $\mathbf{n} \to \infty$. Thus, (4.2), (5.6) and (5.8) are equivalent, in particular (4.1) $\Rightarrow$ (4.2).

If $r \geqq 2$, then an application of Chebyshev's inequality shows that $|\mathbf{n}|^{-\alpha} \cdot S_{\mathbf{n}} \to 0$ in probability as $\mathbf{n} \to \infty$, because $\alpha > \frac{1}{2}$. Thus, $|\mathbf{n}|^{-\alpha} \cdot \operatorname{med}(S_{\mathbf{n}}) \to 0$ as $\mathbf{n} \to \infty$ and the conclusion follows.

The proof of (4.1) $\Rightarrow$ (4.2) is now complete. To remove the assumption about symmetry for (4.3) and (4.4) a similar argument is used together with the symmetrization inequalities ([16], page 247).

(viii) (4.4) $\Rightarrow$ (4.1). This is the remaining part of the proof. Note that we do not assume symmetry.

According to [1], Theorem 3, we know that the conclusion is true when $d = 1$. Now,

$$\begin{aligned} \infty &> \sum_{j=1}^\infty j^{\alpha r-2} P(\sup_{j \leqq |\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq \varepsilon) \\ &\geqq \sum_{j=1}^\infty j^{\alpha r-2} P(\sup_{j \leqq |\mathbf{k}|} |S_{\pi(|\mathbf{k}|)}/|\mathbf{k}|^\alpha| \geqq \varepsilon) \end{aligned}$$

and consequently, by [1], Theorem 3, we have $E|X|^r < \infty$ and, if $r \geqq 1$, $EX = 0$. The former fact is equivalent to

$$\sum_{j=1}^\infty j^{\alpha r-1} \cdot P(|X| \geqq j^\alpha \cdot \varepsilon) < \infty \tag{5.9}$$

and, in particular (since $d(j) = o(j^\delta)$ for every $\delta > 0$ and $\alpha r > 1$) we have

$$\sum_{j=j_1}^\infty d(j) \cdot P(|X| \geqq j^\alpha \cdot \varepsilon) < 2\delta < 1 \quad \text{if } j_1 \text{ is large.} \tag{5.10}$$

Standard inequalities now yield

$$\begin{aligned} &P(\sup_{j \leqq |\mathbf{k}|} |S_{\mathbf{k}}/|\mathbf{k}|^\alpha| \geqq 2^{-d(1+\alpha)} \cdot \varepsilon) \\ &\quad \geqq P(\sup_{j \leqq |\mathbf{k}|} |X_{\mathbf{k}+1}/|\mathbf{k}+\mathbf{1}|^\alpha| \geqq 2^{-d\alpha} \cdot \varepsilon) \geqq P(\textstyle\bigcup_{j \leqq |\mathbf{k}|} \{|X_{\mathbf{k}+1}| \geqq |\mathbf{k}|^\alpha \cdot \varepsilon\}) \\ &\quad \geqq \textstyle\sum_{j \leqq |\mathbf{k}|} P(|X| \geqq |\mathbf{k}|^\alpha \cdot \varepsilon) - \frac{1}{2}(\sum_{j \leqq |\mathbf{k}|} P(|X| \geqq |\mathbf{k}|^\alpha \cdot \varepsilon))^2 \\ &\quad = \textstyle\sum_{i=j}^\infty d(i) P(|X| \geqq i^\alpha \cdot \varepsilon) - \frac{1}{2}(\sum_{i=j}^\infty d(i) P(|X| \geqq i^\alpha \cdot \varepsilon))^2 \\ &\quad \geqq (\text{by } (5.10)) \geqq (1-\delta) \cdot \textstyle\sum_{i=j}^\infty d(i) \cdot P(|X| \geqq i^\alpha \cdot \varepsilon), \qquad \text{for } j \geqq j_1 . \end{aligned}$$

Thus,

$$\begin{aligned}\infty &> \sum_{j=1}^{\infty} j^{\alpha r-2}\cdot P(\sup_{j\leq|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{\alpha}|\geqq 2^{-d(1+\alpha)}\cdot\varepsilon)\\ &\geqq (1-\delta)\cdot\sum_{j=j_1}^{\infty} j^{\alpha r-2}\cdot\{\sum_{i=j}^{\infty} d(i)\cdot P(|X|\geqq i^{\alpha}\cdot\varepsilon)\}\\ &\geqq \text{const.}\cdot\sum_{i=1}^{\infty}(\sum_{j=1}^{i} j^{\alpha r-2})\cdot d(i)\cdot P(|X|\geqq i^{\alpha}\cdot\varepsilon)\\ &\geqq \text{const.}\ \sum_{i=1}^{\infty} i^{\alpha r-1}\cdot d(i)\cdot P(|X|\geqq i^{\alpha}\cdot\varepsilon)\,.\end{aligned}$$

By Lemma 2.1, the finiteness of the last sum is equivalent to $E|X|^r\cdot(\log^+|X|)^{d-1}<\infty$ and hence (4.4) $\Rightarrow$ (4.1).

The proof of Theorem 4.1 is now complete.

6. Proof of Theorem 4.2. The proof is similar to the proof of Theorem 4.1 and will therefore not be given in full detail. Again we begin by assuming that the random variables have a symmetric distribution. Recall that $r<2$.

(a) (4.5) $\Rightarrow$ (4.6). By arguing as in step (i) of the proof of Theorem 4.1 we obtain

$$\begin{aligned}&\sum_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot\log|\mathbf{n}|\cdot P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{1/r}\cdot\varepsilon)\\ &\quad\leqq \varepsilon^{-2}\cdot\sum_{\mathbf{n}}\log|\mathbf{n}|\cdot\operatorname{Var}(|\mathbf{n}|^{-1/r}\cdot Y_{\mathbf{n}})+\sum_{\mathbf{n}}\log|\mathbf{n}|\cdot P(|X|\geqq|\mathbf{n}|^{1/r}\cdot\varepsilon)\\ &\quad<\infty\,,\qquad\text{by (2.7) and (2.4) with}\quad m=1\,.\end{aligned}$$

(b) (4.6) $\Leftrightarrow$ (4.7). This follows as in (v), Section 5.

(c) (4.6) $\Rightarrow$ (4.8). By arguing as in (vi), Section 5, we obtain

$$\begin{aligned}&\sum_{j=1}^{\infty} j^{-1}\cdot P(\sup_{j\leq|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{1/r}|\geqq 2^{(1+d)/r}\cdot\varepsilon)\\ &\quad\leqq 2^d\sum_{j=0}^{\infty}(\sum_{i=0}^{j-1}1)\sum_{|\mathbf{n}|=2^{j+d}} P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{1/r}\cdot\varepsilon)\\ &\quad= 2^d\sum_{j=0}^{\infty} j\sum_{|\mathbf{n}|=2^{j+d}} P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{1/r}\cdot\varepsilon)\\ &\quad\leqq 2^{d+1}\sum_{j=1}^{\infty} j\cdot 2^{-j}\cdot\sum_{i=2^j}^{2^{j+1}-1}(\log i)^{d-1}\cdot P(|S_{\pi(i)}|\geqq i^{1/r}\cdot\varepsilon/2)\\ &\quad\leqq \text{const.}\ \sum_{i=1}^{\infty} i^{-1}(\log i)^d\cdot P(|S_{\pi(i)}|\geqq i^{1/r}\cdot\varepsilon/2)<\infty\quad\text{by (5.4)}\,.\end{aligned}$$

(d) We now know that (4.5) $\Rightarrow$ (4.6) $\Leftrightarrow$ (4.7) and that (4.6) $\Rightarrow$ (4.8) in the symmetric case. The symmetry assumption is removed exactly as in step (vii) of the preceding proof.

(e) (4.8) $\Rightarrow$ (4.5). No symmetry is assumed. The proof proceeds by induction on the dimension. For $d=1$ see [1], Theorem 2. Now suppose the conclusion holds with $d-1$ dimensions. Obviously, $P(\sup_{j\leq|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{1/r}|\geqq\varepsilon)$ dominates $P(\sup_{j\leq|\mathbf{k}^*|}|S_{\mathbf{k}^*}/|\mathbf{k}^*|^{1/r}|\geqq\varepsilon)$ where $\mathbf{k}^*=(k_1,\cdots,k_{d-1},1)$, i.e., $\mathbf{k}^*$ equals $\mathbf{k}$ except that the last coordinate equals 1. Therefore,

$$\sum_{j=1}^{\infty} j^{-1}\cdot P(\sup_{j\leq|\mathbf{k}^*|}|S_{\mathbf{k}^*}/|\mathbf{k}^*|^{1/r}|\geqq\varepsilon)<\infty\,,\tag{6.1}$$

and from the induction hypothesis we conclude that $E|X|^r\cdot(\log^+|X|)^{d-1}<\infty$ and, if $r\geqq 1$, $EX=0$.

By Lemma 2.1 the former fact implies that

$$\sum_{j=j_1}^{\infty} d(j)\cdot P(|X|\geqq j^{1/r}\cdot\varepsilon)<2\delta<1\qquad\text{if}\quad j_1\quad\text{is large.}\tag{6.2}$$

By arguments like those following (5.10), together with (6.2), we obtain

$$(6.3)\qquad P(\sup_{j\le|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{1/r}|\geqq 2^{-d-d/r}\cdot\varepsilon)\geqq(1-\delta)\sum_{i=j}^{\infty}d(i)P(|X|\geqq i^{1/r}\cdot\varepsilon)\,,$$

for $j\geqq j_1$ and so

$$\begin{aligned}\infty &> \sum_{j=1}^{\infty} j^{-1}\cdot P(\sup_{j\le|\mathbf{k}|}|S_{\mathbf{k}}/|\mathbf{k}|^{1/r}|\geqq 2^{-d-d/r}\cdot\varepsilon)\\ &\geqq (1-\delta)\cdot\sum_{j=j_1}^{\infty} j^{-1}\cdot\{\sum_{i=j}^{\infty}d(i)\cdot P(|X|\geqq i^{1/r}\cdot\varepsilon)\}\\ &\geqq \text{const.}\cdot\sum_{i=1}^{\infty}\log i\cdot d(i)\cdot P(|X|\geqq i^{1/r}\cdot\varepsilon)\,.\end{aligned}$$

By Lemma 2.1 it follows that $E|X|^r\cdot(\log^+|X|)^d<\infty$ which concludes the proof.

7. Direct convergence rates. It follows from (4.4) that (4.1) in particular implies that

$$(7.1)\qquad \sum_{j=1}^{\infty} j^{\alpha r-2}\cdot P(\sup_{\pi(j)<\mathbf{k}}|S_{\mathbf{k}}/|\mathbf{k}|^{\alpha}|\geqq\varepsilon)<\infty$$

for all $\varepsilon>0$, $\alpha r>1$, $\alpha>\frac{1}{2}$.

This gives a way of deriving information about the rate of convergence to 0 of, e.g., $P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|\cdot\varepsilon)$.

More precisely, assuming (4.1) and $\alpha=1$, an application of [1] (lemma, page 113), yields

$$j^{r-1}\cdot P(\sup_{\pi(j)<\mathbf{k}}|S_{\mathbf{k}}/|\mathbf{k}||\geqq\varepsilon)\to 0\qquad\text{as}\quad j\to\infty\,,$$

and so

$$\begin{aligned}|\mathbf{n}|^{r-1}\cdot P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|\cdot\varepsilon)&=|\mathbf{n}|^{r-1}\cdot P(|S_{\pi(|\mathbf{n}|)}|\geqq|\mathbf{n}|\cdot\varepsilon)\\ &\leqq|\mathbf{n}|^{r-1}\cdot P(\sup_{\pi(|\mathbf{n}|)<\mathbf{k}}|S_{\mathbf{k}}/|\mathbf{k}||\geqq\varepsilon)\to 0\qquad\text{as}\quad\mathbf{n}\to\infty\,.\end{aligned}$$

Thus, if $E|X|^r\cdot(\log^+|X|)^{d-1}<\infty$, $r>1$, and $EX=0$ we conclude that

$$(7.2)\qquad |\mathbf{n}|^{r-1}\cdot P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|\cdot\varepsilon)\to 0\qquad\text{as}\quad\mathbf{n}\to\infty\,.$$

However, the same conclusion can be obtained without the logarithmic factor in the assumption by studying the probabilities directly.

THEOREM 7.1. *Let* $\{X_{\mathbf{n}},\ \mathbf{n}\in Z_+^d\}$ *be* i.i.d. *random variables with zero mean. If* $E|X|^r<\infty$, $r\geqq 1$, *then,* $\forall\,\varepsilon>0$, *the following are equivalent*:

$$(7.3)\qquad |\mathbf{n}|^{r}\cdot P(|X|\geqq|\mathbf{n}|)\to 0\qquad\textit{as}\quad\mathbf{n}\to\infty\,;$$

$$(7.4)\qquad |\mathbf{n}|^{r-1}\cdot P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|\cdot\varepsilon)\to 0\qquad\textit{as}\quad\mathbf{n}\to\infty\,;$$

$$(7.5)\qquad |\mathbf{n}|^{r-1}\cdot P(\max_{\mathbf{k}<\mathbf{n}}|S_{\mathbf{k}}|\geqq|\mathbf{n}|\cdot\varepsilon)\to 0\qquad\textit{as}\quad\mathbf{n}\to\infty\,.$$

If $r>1$, *then the above statements are also equivalent to*

$$(7.6)\qquad |\mathbf{n}|^{r-1}\cdot P(\sup_{\mathbf{n}<\mathbf{k}}|S_{\mathbf{k}}/|\mathbf{k}||\geqq\varepsilon)\to 0\qquad\textit{as}\quad\mathbf{n}\to\infty\,.$$

In [1], Theorem 4, it is shown that (7.3) and (7.4) are equivalent if $r\geqq 1$ and that (7.3), (7.4) and (7.6) are equivalent if $r>1$ without assuming that $E|X|^r<\infty$. Earlier Brillinger ([2]) proved that $E|X|^r<\infty$, $1\leqq r<2$ implies (7.4). These results are for $d=1$.

REMARK 7.1. The theorem asserts, for example, that (7.4) holds if $E|X|^r < \infty$, i.e., (7.2) is valid under a weaker assumption than before. However, when $d = 1$ they coincide.

PROOF OF THEOREM 7.1. We first note that (7.3) holds trivially and that (7.5) and (7.6) both imply (7.4).

Secondly, since $P(|S_{\mathbf{n}}| \geqq |\mathbf{n}| \cdot \varepsilon) = P(|S_{\pi(|\mathbf{n}|)}| \geqq |\mathbf{n}| \cdot \varepsilon)$, it follows that the proof given in [1], Theorem 4, for the equivalence of (7.3) and (7.4) when $d = 1$ remains valid without change. We include, however, a proof of (7.3) $\Rightarrow$ (7.4) because of its simplicity.

Now, let $1 \leqq r < 2$. If $r = 1$, (7.4) is nothing but the weak law of large numbers, see, e.g., [10], so let $1 < r < 2$. Since the sequence of arithmetic means constitutes a reversed martingale, [20], it follows from the Doob–Cairoli maximal inequality ([3], [10]) and (2.9) that

$$\begin{aligned}
&|\mathbf{n}|^{r-1} \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}| \cdot \varepsilon) \\
&\quad \leqq |\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_{\mathbf{k}}/|\mathbf{k}|| \geqq \varepsilon) \leqq |\mathbf{n}|^{r-1} \cdot \varepsilon^{-r} \cdot E \sup_{\mathbf{n}<\mathbf{k}} |S_{\mathbf{k}}/|\mathbf{k}||^r \\
&\quad \leqq |\mathbf{n}|^{r-1} \cdot \varepsilon^{-r} \cdot \gamma_{d,r} \cdot E|S_{\mathbf{n}}/|\mathbf{n}||^r = |\mathbf{n}|^{-1} \cdot \varepsilon^{-r} \cdot \gamma_{d,r} \cdot E|S_{\mathbf{n}}|^r \\
&\quad = \varepsilon^{-r} \cdot \gamma_{d,r} \cdot |\mathbf{n}|^{-1} \cdot o(|\mathbf{n}|) \to 0 \qquad \text{as} \quad \mathbf{n} \to \infty\,.
\end{aligned}$$

Here $\gamma_{d,r}$ is a constant depending on d and r only. This proves (7.4) and (7.6).

To deduce (7.5) we first assume symmetry. By the Lévy-inequality, [9], [18], $P(\max_{\mathbf{k}<\mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}| \cdot \varepsilon) \leqq 2^d \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}| \cdot \varepsilon)$. To desymmetrize we proceed as above. The case $1 \leqq r < 2$ is thus completely proved.

Now, let $r \geqq 2$. Since (2.11) is too crude an estimate for $E|S_{\mathbf{n}}|^r$ the above approach using reversed martingales does not work. To prove this part of the theorem we begin with the symmetric case.

To show (7.4) we use (2.12) with $j = 1$, (2.11) and (7.3) to obtain

$$\begin{aligned}
|\mathbf{n}|^{r-1} \cdot P(|S_{\mathbf{n}}| \geqq 3 \cdot |\mathbf{n}| \cdot \varepsilon) &\leqq |\mathbf{n}|^r \cdot P(|X| \geqq |\mathbf{n}| \cdot \varepsilon) + \text{const.}\ |\mathbf{n}|^{-1} \\
&\to 0 \qquad \text{as} \quad \mathbf{n} \to \infty\,.
\end{aligned}$$

(7.5) follows from the Lévy inequalities and it remains to prove (7.6).

Define $Z_{\mathbf{k}}' = X_{\mathbf{k}} \cdot I\{|X_{\mathbf{k}}| < 2^i \cdot \varepsilon\}$, $Z_{\mathbf{k}}'' = X_{\mathbf{k}} - Z_{\mathbf{k}}'$, $S_{\mathbf{k}}' = \sum_{\mathbf{m}<\mathbf{k}} Z_{\mathbf{m}}'$ and $S_{\mathbf{k}}'' = S_{\mathbf{k}} - S_{\mathbf{k}}'$ for $|\mathbf{k}| \leqq 2^i$ and let i be large. Then, from (5.2), (2.12) with $j = 1$, the boundedness of $Z_{\mathbf{k}}'$ and (7.4), we obtain

$$\begin{aligned}
&P(\sup_{2^{i-1-d} \leqq |\mathbf{k}| < 2^{i-d}} |S_{\mathbf{k}}'/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \\
&\quad \leqq 2^d \sum_{|\mathbf{k}|=2^i} P(|S_{\mathbf{k}}'| \geqq 2^i \cdot 3\varepsilon) = 2^d \cdot d(2^i) \cdot P(|S'_{\pi(2^i)}| \geqq 2^i \cdot 3\varepsilon) \\
&\quad \leqq 2^d \cdot d(2^i) \cdot (2^i \cdot P(|Z'| \geqq 2^i \cdot \varepsilon) + 4(P(|S'_{\pi(2^i)}| \geqq 2^i \cdot \varepsilon))^2) \\
&\quad = 2^{d+2} \cdot d(2^i)(P(|S'_{\pi(2^i)}| \geqq 2^i \varepsilon))^2 = o(2^{i\delta}) \cdot o(2^{-2i(r-1)}) = o(2^{-i(2r-\delta-2)})\,,
\end{aligned}$$

where δ may be chosen arbitrarily small.

Now, suppose that $2^{m-1} \leqq |\mathbf{n}| < 2^m$, where m is large.

$$\begin{aligned}
|\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_{\mathbf{k}}'/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) &\leqq 2^{m(r-1)} \sum_{i=m}^{\infty} P(\sup_{2^{i-1} \leqq |\mathbf{k}| < 2^i} |S_{\mathbf{k}}'/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \\
&\leqq 2^{m(r-1)} \cdot o(2^{-m(2r-\delta-2)}) = o(2^{-m(r-1-\delta)})
\end{aligned}$$

and hence

(7.7) $$|\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}'/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \to 0 \quad \text{as} \quad \mathbf{n} \to \infty .$$

We now turn to $S_\mathbf{k}''$. First we need a sharper estimate of the number of points in Z_+^d that are involved. Let $\mathbf{n}$ be fixed and define

$$A_1(m, j) = \{\mathbf{k} \in Z_+^d; \mathbf{n} \prec \mathbf{k}, |\mathbf{n}| = 2^{m-1}, |\mathbf{k}| \leqq 2^{m-1+j}\}$$
$$A_2(m, j) = \{\mathbf{k} \in Z_+^d; \mathbf{n} \prec \mathbf{k}, |\mathbf{n}| = 2^{m-1}, |\mathbf{k}| = 2^{m-1+j}\}$$
$$A_3(m, j) = \bigcup_\mathbf{i} \{\mathbf{k} \in Z_+^d; \mathbf{k} \prec \mathbf{i}, \mathbf{i} \in A_2(m, j)\} ,$$

where the union is taken over all $\mathbf{i} \in A_2(m, j)$. (Note that i above has been replaced by $m + j - 1$.)

Also, let $d_s(m, j)$ = the number of $\mathbf{k} \in Z_+^d$ belonging to $A_s(m, j)$, $s = 1, 2, 3$.

By routine calculations we find that $d_3(m, j) \sim d_1(m, j) \sim 2^{m-1+j}(\log 2^j)^{d-1}$ when m is large.

Now, the event $\{\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}''/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon\}$, where $2^{m-1} \leqq |\mathbf{n}| < 2^m$, is contained in the event $\{\sup_{\mathbf{n}'<\mathbf{k}} |S_\mathbf{k}''/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon\}$, where $\mathbf{n}' \prec \mathbf{n}$ and $|\mathbf{n}'| = 2^{m-1}$. If the latter occurs, then necessarily at least one Z'' in $\bigcup_{j=1}^\infty A_3(m, j)$ has to be nonzero. Thus,

(7.8) $$P(\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}''/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \leqq \sum_{j=1}^\infty d_3(m, j) \cdot P(|Z''| \neq 0) .$$

For m large we obtain

$$d_3(m, j) \cdot P(|Z''| \neq 0) = d_3(m, j) \cdot P(|X| \geqq 2^{m-1+j}\varepsilon)$$
$$\sim 2^{m-1+j}(\log 2^j)^{d-1} \cdot o(2^{-(m-1+j)r}) \leqq o(2^{-m(r-1)}) \cdot 2^{-j(r-1-\delta)} ,$$

where δ may be chosen arbitrarily small.

Thus, if $2^{m-1} \leqq |\mathbf{n}| < 2^m$ it follows that

$$|\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}''/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \leqq 2^{m(r-1)} \cdot o(2^{-m(r-1)}) \cdot \sum_{j=1}^\infty 2^{-j(r-1-\delta)} = o(1)$$

as $m \to \infty$, i.e.,

(7.9) $$|\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}''/|\mathbf{k}|| \geqq 2^d \cdot 6\varepsilon) \to 0 \quad \text{as} \quad \mathbf{n} \to \infty .$$

By combining (7.7) and (7.9) it follows that

$$|\mathbf{n}|^{r-1} \cdot P(\sup_{\mathbf{n}<\mathbf{k}} |S_\mathbf{k}/|\mathbf{k}|| \geqq 2^d \cdot 12\varepsilon) \to 0 \quad \text{as} \quad \mathbf{n} \to \infty ,$$

which proves (7.6) in the symmetric case.

The desymmetrization is carried through as before and the proof is complete.

REMARK 7.2. The proof that (7.6) holds for $r \geqq 2$ also applies to $1 < r < 2$ (if (7.4) has been proven earlier). However, the proof presented above for that case seems more attractive.

8. Concluding remarks.

(i) It is easy to see that the Marcinkiewicz laws can be generalized to independent random variables with zero mean if they are dominated by an $L^r(\log L)^{d-1}$-bounded random variable (cf. [16], page 242 when $d = 1$).

(ii) When $d = 1$, $d(j) = 1$ for all j. However, the proofs remain valid. Thus, for example, Section 5 provides another proof of Katz [14], Theorem 1 (the sufficiency).

(iii) In Section 4, $\alpha r = 1$ is one limiting case behaving differently from $\alpha r > 1$. Another limiting case is $\alpha = \frac{1}{2}$, which has been studied by Lai [15] when $d = 1$ and by the author $d \geqq 2$ in a forthcoming paper.

(iv) A closer look at the sums involved in Theorems 4.1 and 4.2 reveals that, in fact, more than their finiteness has been established. For example, for $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon)$, it follows from steps (ii) and (iii) of Section 5 that

$$(8.1) \qquad \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq C(\alpha, r, \varepsilon) \cdot (E|X|^r(\log^+ |X|)^{d-1} + (E|X|^r)^2),$$

when $\alpha r > 1$, $\alpha > \frac{1}{2}$ and $r < 2$. A similar inequality holds when $r \geqq 2$.

By using Chebyshev's inequality instead of Markov's inequality in the latter case we obtain ($j = 1$)

$$(8.2) \qquad \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} \cdot P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha} \cdot \varepsilon) \leqq C(\alpha, r, \varepsilon) \cdot (E|X|^r(\log^+ |X|)^{d-1} + (E|X|^2)^2).$$

In all cases $C(\alpha, r, \varepsilon)$ is a constant depending on α, r and ε only. As they stand, (8.1) and (8.2) are valid in the symmetric case, but they remain true also after desymmetrization, except that the constant then also depends on d. These results are related to estimates obtained by Chow and Lai [6].

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