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Mario Bunge's Philosophy of Mathematics: An Appraisal

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Abstract In this paper, I present and discuss critically the main elements of Mario Bunge's philosophy of mathematics. In particular, I explore how mathematical knowledge is accounted for in Bunge's systemic emergent materialism.

To Mario, with gratitude.

1 Philosophy of Mathematics

> The philosophy of mathematics is mostly a collection of interesting open problems and of ill-grounded opinions. Moreover, it is fragmentary rather than systematic. What is worse, it is quite isolated from the other branches of philosophy; in particular, none of the well-known philosophies of mathematics is a component of a philosophical system consonant with contemporary mathematics and factual science. There is not even consensus on the problematics of the philosophy of mathematics. … In sum, philosophy of mathematics is in poor shape. Therefore it poses one of the most interesting and pressing challenges to any philosopher who is aware that the most advanced and exact of all sciences has the most backward, woolly and dogmatic of all philosophies. (Bunge 1985a, p. 17)

This was written more than 25 years ago. I suspect that Bunge would probably make the same claim today. The traditional philosophies of mathematics, namely logicism, intuitionism and formalism, have taken new shapes under the name of neo-logicism, various schools of constructivism and, well, it is hard to say what formalism really amounted to in the first place, even today.[1] What can still be said is that even the new variants are not part of philosophical systems and neo-logicism, despite its claims, is certainly not consonant with contemporary mathematics and factual science. Constructive mathematics might be in slightly better shape, in large part because of its relevance to theoretical computer science, but it is certainly not seen as encompassing a philosophy of

[1] This is overstating it, I admit. Bunge himself gives a succinct presentation of Hilbert's formalism in Bunge (1985b, pp. 100–102). For a broader survey of formalism in general, see Detlefsen (2005). My point here is that the very expression "formalism" encompasses a large variety of different positions.

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mathematics.[2] What is still striking is to see how philosophy of mathematics is still isolated from the other branches of philosophy, even the other branches of philosophy of science.[3] One might believe that this state of affairs is attributable to the inherent technical character of mathematics or, along the same lines, to the fact that philosophy of mathematics was identified for a period of time with logic and technical foundational studies. True, mathematics is hard, contemporary mathematics has developed at an incredible pace and has become more and more abstract. But this is true of the other sciences as well and philosophers are following developments in contemporary physics, biology—from evolutionary biology to molecular biology—, neuropsychology and the neurosciences in general. Attempts have been made and are still being made to incorporate historical aspects as well as methodological and cognitive aspects of the practice of mathematics in the philosophical reflexion, as for instance in Cellucci and Gillies (2005), Corfield (2003), Ferreirós (2006), Mancosu (2008). One might have also thought that the so-called naturalistic turn in epistemology would have brought philosophy of mathematics closer to philosophy of science in general and in as much as the actual practice and development of mathematics is now taken seriously into consideration, it has. [For some early work in philosophy of mathematics from a Quinian naturalistic standpoint, see for instance, Maddy (1997).] But one could certainly argue that Piaget was already doing epistemology of mathematics in a naturalistic framework and Bunge has certainly been calling for such an approach, albeit in a specific manner, different from what Piaget was doing and also different, at least in the details, from some of the contemporary proposals.[4] It would certainly be interesting to compare Piaget, Quine and Bunge on the project of a naturalized epistemology and philosophy of mathematics. But we will leave this project to someone else. I will here focus on Bunge's own attempt.

Bunge is very clear about the desiderata any philosophy of mathematics should satisfy. Here are the "neutral" desiderata, directly taken from Bunge (1985a, p. 121). According to Bunge, a philosophy of mathematics has to account for the following features of mathematical research and its result, mathematics:

1. Even though some mathematical concepts have an empirical or an intuitive origin, mathematics, as a product of mathematical research, is purely conceptual, both in its content and methods;
2. There are basic differences between mathematical propositions and propositions in other research fields, between mathematical proofs and other methods of justification and verification;
3. There are differences between an abstract theory and its models and between logical models and models in science and technology;
4. Mathematics is applied as well as pure and a philosophy of mathematics has to explain how mathematics is applied so successfully to the world;
5. Mathematics is logically stratified, that is, some theory logically precedes others.

[2] Martin-Löf's heroic work might be considered to be an exception here, since very few philosophers of mathematics are defending variants of constructivism nowadays. See Martin-Löf (1998). Needless to say, constructive mathematics has nothing to do with the variants of "constructivism" in education, be it social constructivism or educational constructivism. For more about mathematical constructivism, see for instance, Beeson (1985), Bishop and Bridges (1985), Troelstra and van Dalen (1988a, b).

[3] It is even no longer included in the programs of the meetings of the PSA! In other words, it is no longer seen as an integral part of philosophy of science by some!

[4] See for instance, Beth and Piaget (1966) for Piaget's approach.

Each and every one of these items could be challenged, e.g. the second one would certainly be rejected by some contemporary philosophers of mathematics.

In his list, Bunge has included elements that are consequences of his own approach and are, therefore, more problematic from a different standpoint. These are:[5]

1. Mathematical concepts are constructions of the brain; moreover, these constructions are more likely to appear in fairly advanced societies;
2. Even though mathematical concepts are brain-children, they are universal and impersonal;
3. Some mathematical constructs are invented, while logical relations between constructs are discovered; algorithms are designed, improved and applied;
4. Mathematics develops freely within the constraints of consistency and systematicity;
5. Mathematical objects are not self-existing Platonic ideas nor does mathematics rests upon some non-rational faculties, such as intuition (except as a heuristic aid);
6. Mathematical knowledge has to be related systematically to scientific and technological knowledge; any philosophy of mathematics has to show how mathematics is an inherent component of our knowledge in general.

I will here focus on many of the elements of this second list, more specifically (1), (2) and (5), for they constitute more clearly the original components of Bunge's philosophy of mathematics. What is interesting in Bunge's case is that these components are developed in a systemic emergent materialist framework. However they are not *necessary* consequences of a systemic emergent materialist framework. This is the first point I will try to sketch. Second, I submit that the picture proposed by Bunge is not quite satisfactory. The problem, I contend, lies in the way Bunge tries to accommodate the fact that mathematical constructs are brain-children and that they exist in some sense. From my perspective, the problem is not that mathematical constructs are brain-children, but rather the way Bunge wants to account for the notion of mathematical existence in his larger framework. I should immediately admit that I do not see clearly how this spot can be fixed, even within a emergent materialist framework. I will sketch a few avenues of research, but I certainly do not pretend to have a satisfactory answer.

2 Systemic Emergent Materialism and Mathematics: Some General Remarks

The global picture of the world offered by Bunge is, to my mind at least, very appealing: the world is presented as a complex leveled system, made up at the lower level of physical systems, which organize themselves into chemical systems possessing emergent or novel properties, which, in turn, organize themselves into biological systems possessing emergent properties and so on up to sociological systems, each level introducing genuinely new features of the world. Human beings are, of course, subsystems of the whole system, with properties from all levels interacting constantly within each individual and between individuals. Humans are not *in* the world, they are intrinsic *parts* of it.

Our brain is itself an extraordinarily rich leveled system: from molecules to systems of neurons, and although we still do not have a clear view of all the levels and their interactions, we can assume that there are also in this system various emergent properties and systems which are presumably related to more familiar properties of our cognitive make-up. Indeed, we sense, perceive, feel the world—including ourselves—, we want, desire,

[5] Note that the first two constitute one single item in Bunge's original list.

fear, will, imagine, plan, speak, listen, think, learn, remember and forget, to mention but a few of the typical activities of our selves, at least when stated in the vernacular language. Furthermore, our brain allows us to *know* the world, once again including ourselves, as well as to *create* in ways no other species has exhibited.[6] Our knowledge—no matter how it is defined or characterized—is *of* the world[7] and our creations are, in one way or another, *in* the world.[8]

Now, certain parts of our knowledge and some of our creations seem to belong to a world with properties radically different from the material world. Some of our knowledge refers to real, material things, like this computer in front of me or this piece of dark chocolate, but we know other things that are not obviously material, at least not in the way my computer is, for instance Frodo Baggins, Harry Potter, Shubert's String Quartet no. 15, Anne Hébert's novel *Les chambres de bois*, the prime numbers, $e^{i\pi}$, the Klein four-group, etc. Among the latter, *mathematical* concepts seem to have peculiar properties: for most of us, they are hard to learn, hard to understand, difficult to explore, yet if there is one area where we feel confident to claim that we actually know with certainty, clarity, rigor, precision, necessity, it is in the realm of mathematical concepts. And, what is perhaps more striking, these concepts are indispensable to our understanding of the world and to intervening efficiently in the world.

At first sight, mathematics seems to differ from other kinds of knowledge by its very subject: pure mathematics is not about the real world.[9] After all, we do not see nor touch geometric figures, numbers, sets, to mention but the most elementary cases. Mathematical knowledge is about the world of mathematical ideas. If one could characterize the latter in one way or another, then one would be able to say how mathematics differ from other kinds of knowledge. There is one obvious, appealing and simple answer that has been presented in various flavors by many different philosophers: there is a *world* of mathematical ideas and this world is autonomous, transempirical and extra-mental. According to this view, mathematical ideas would be a specific part of the World of Ideas in general. Thus, our knowledge of mathematical ideas would simply and directly be our knowledge of this World. But as with any simple solution, this one brings with it new problems. How do we get to know this extra-mental and trans-empirical World? Again, there is a simple ontological answer: humans would be part of this World too or, at the very least, they would be connected to this World by some sensory capacity, some intuition. Humans would have something, be it a 'soul' or some sort of perceptual capacity, which lives in or is connected to this World. From a methodological point of view, this is a very economical strategy: no matter what our epistemology of the material world is, no matter what our theories of sense and reference for our material discourse are, we simply have to extend them more or less as they are to this other world and everything should come out fine. One of the standard problems here has to do with our theory of knowledge (and to some extent, our theory of reference): it seems obvious that there is a *causal* component to our knowledge of the material world, since at the very least, our senses are affected causally by

[6] Sadly enough, these wonderful capacities have a flip side: we err as well as destroy just as much, if not more…

[7] Need I add, once again, including ourselves as parts of the world.

[8] And also, implicitly or explicitly, *of* the world too, since they inherently reflect who and what we are.

[9] I am using this expression in Bunge's sense.

the material world, and it is hard to maintain that mathematical ideas affect us causally.[10]

Needless to say, that answer is not available to an emergent materialist like Bunge. For him, there is only one world, the material world, and if humans are indeed parts of that world, it is nonetheless the only world they are part of. One can talk about the world of ideas and the world of our creations, but it is precisely that: a world of our creations and constructs. The world of ideas, if one can talk of a world, has a specific ontological dependence with respect to the real world, a dependence which is, according to Bunge, robust and fundamental: ideas are products of our brains. No brain, no idea, period. And this is true, of course, of *all* our ideas, be they mathematical or not. But then, we are back to our original query: what is it about mathematical ideas that makes them so special if it is not that they refer to entities of the World of Ideas?

Before we consider Bunge's answer, let us first consider the so-called traditional philosophies of mathematics, for they can be interpreted in such a way as to provide an answer to our question. Had logicism succeeded, we would have another answer: mathematics is really logical knowledge and the latter is knowledge of pure Reason. Of course, this would not be the whole answer: in a emergent materialist framework, ones would still have to show that this knowledge of pure Reason is not knowledge of entities or of facts of an extra-mental, trans-empirical world. Indeed, one could claim that it is precisely via Reason that humans can connect to the World of Ideas. Thus, assuming that logicism is correct, the hard work for an emergent materialist would be to develop a theory of Reason in such a way that our logical knowledge would be explained by resorting exclusively to either properties of the material world itself or properties of our knowledge of the material world. But logicism failed: pure mathematics, at least as we know it, cannot be reduced to pure logic, at least given our understanding of what pure logic is.[11]

The logical empiricists thought they had another, more subtle, answer that would perhaps salvage what they saw as one of the central aspects of logicism: mathematical knowledge[12] is *analytic* whereas knowledge of the empirical world—to use their expression—is *synthetic*. The analytic/synthetic distinction goes back to a methodological distinction already made by the Greeks and that evolved with technical means and tools throughout the history of western philosophy. See Timmermans (1995) for more on the historical roots and the evolution of the distinction up to Kant.) Their problem was that being analytic could no longer be simply identified with being logical, since logicism had failed, and it therefore had to be extended somehow to a property of portions of languages in general, a task which, at least according to Quine's famous objection, cannot be done. Be that as it may, I want to focus on the methodological dimension of the proposal: the strategy was to focus on specific features of *language* itself in order to make the required distinction. The hope is to find certain general features of human language, features connected to sense and reference, that would show that there is a part of our language that, although it seems to point to external entities and properties, in fact, does not point to anything at all but exhibits properties of the language itself. Thus, logic and mathematics would not be about anything but aspects of our language. That strategy, had it succeeded,

[10] This latter difficulty is, of course in a nutshell, one of the so-called "Benacerraf problems". See Benacerraf (1973).

[11] The nature of logic is a thorny issue in itself, but one that can now be considered with an extremely large body of results, both model theoretical and proof theoretical. Already Tarski in the early 1930's and the mathematician Mautner in 1946 used different methods to characterize logic. Since then, many logicians and philosophers have made interesting proposals. See for instance, Feferman (2004, 2008, 2010), Mautner (1946), McGee (1996), Sher (1991, 2003), Tarski (1986).

[12] As well as logical knowledge, I should add.

would have put to rest any need to refer to a World of Ideas and would be an obvious choice for an emergent materialist.

One could claim that the formalists were also focusing on certain aspects of language, in particular of *mathematical* language, that is the specific use of symbols or signs and algorithms associated with them. Furthermore, formalism is very often associated to a form of nominalism that rejects the existence of Universals or Ideas as extra-mental and trans-empirical entities. Mathematical entities then have to be particulars in some sense and it seems that the only thing that formalists have at their disposal to play this role are the signs themselves. It usually does not take long for the standard objection to formalism to raise its head: mathematics does deal with signs and manipulation of signs according to algorithms, but if the claim is that mathematics is *about* these signs, or rather their rules of manipulation, and nothing else, then it is simply inadequate, since mathematical terms refer to other things and formalism does not provide an account of reference.[13] This is undoubtedly true, but one has to be careful not to throw away the baby with the bath water: I, for one, believe that no matter what we say about mathematical knowledge, its symbolic aspects have to be invoked and explained. It is a key feature of mathematical knowledge, one that is too easily swept under the rug precisely because of its algorithmic nature. In fact, I believe that it is of the utmost importance to explain how our brains succeed in creating and using these *specific* symbolic systems.[14]

Bunge's strategy, however, does not rely on language and its properties. The locus of the argument shifts towards ideas and their properties. Bunge starts from the assumption that ideas are constructs of the brain. In order to avoid the pitfalls of psychologism, one introduces equivalence classes of ideas. One then shows that among all ideas, there are ideas whose referents have properties which distinguishes them from other ideas. The last step consists in claiming that these ideas are precisely the formal ones. I will now try to reconstruct Bunge's strategy carefully and explore its weaknesses and consequences.

[13] This is one of Bunge's objections to nominalism in mathematics. See Bunge (1985a, pp. 112–115). It is not quite clear that everyone would agree that this is an inherent aspect of nominalism in general. In fact, various trends of contemporary nominalism are much closer to Bunge's own metaphysical position than what these criticisms suggest. I will come back to this point in a later section.

[14] I should immediately point out, since it will come back in a short while in Bunge's framework, that this is but *one* meaning of what it is to be formal and probably not the most apt. Traditionally, a formal system is a system of symbols together with formation and transformation rules that allow one to compute or prove certain formulas in the system. The system is said to be formal precisely because it does not have a fixed interpretation and one works on the signs with the given rules. This is not how Bunge uses the expression 'formal' in what follows. Of course, there is an obvious connection: since we are only looking at the signs and not thinking about what they refer to or what they mean, we are concentrating on their forms, whence the expression. One alternative would point towards the fact that mathematical objects themselves have a form or are forms. Formalism in this latter sense would have a double aspect: it would claim that mathematical knowledge depends upon various formal aspects of symbolic systems, mostly their recursive aspects, and that these signs refer to forms, which would then be characterized within the realm of ideas. I also want to emphasize the fact that the use of symbolism has an historical component that we too easily forget when we think of it nowadays. We are so used to think in symbolic terms early on in our education that we forget how hard and tortuous the road was to get to these remarkable features of mathematical knowledge. For more on the history of symbolic revolution in mathematics and its philosophical significance, see Serfati (2005).

3 Ontological Assumptions

Pure mathematics, according to Bunge, is a research field and, as such, has much in common with any research field.[15] Mathematics, however, is a *formal* research field and not a *factual* research field. To be a formal research field means that all the "objects or referents [of the field] are constructs, and all of its truth claims must be substantiated by purely conceptual means" (Bunge 1985b, p. 12). This is the key to Bunge's strategy: a basic distinction between the formal and the factual. This is a basic *ontological* distinction in Bunge's system, but not one that entails the existence of an autonomous, extra-mental and trans-empirical World of Ideas. Recall once more: there is just one world and it is a material world. Within that world, there are beings that can think and thus create, develop and use concepts, in particular mathematical concepts and systems. These are constructs of the brain, but their properties are such that they can be treated *as if* they were real entities. They do have a kind of existence. But it is precisely that: a *kind* of existence and that kind differs qualitatively from real, material existence. Furthermore, the latter existence is a type of ontological dependence. Let us now unpack these elements and examine them critically.

Recall first that the basic postulate of Bunge's ontology is the existence of substantial individuals (Bunge 1977, p. 28). These are not defined, they are given. Bunge then develops a mereology of substantial individuals, which I will ignore since it is irrelevant to my purpose. Then comes the basic ontological distinction. Bunge postulates that the world is divided into *things* and *constructs* (Bunge 1977, p. 117). Here are Bunge's definitions:

Definition 1 Let x be a substantial individual and call $p(x)$ the collection of its (unarized) properties. Then the individual together with its properties is called the *thing* or (*concrete object*) X:

$$X =_{df} \langle x, p(x) \rangle.$$

In other words, a thing is a substantial individual with properties, what Bunge calls its *form*. Notice one possibility here: Bunge could have decided to focus his attention on properties or forms and try to characterize, among the latter, those properties or the parts of the form that can be considered to be mathematical. One would then suggest that we 'abstract' or 'extract' the mathematical properties from the substantial individual. This is very close to an empiricist approach. For instance, one of the properties of a concrete object is his geometric form, and if one can isolate it from the concrete object, then one can start talking about geometry. However, Bunge rejects this possibility forcefully:

> Another obvious consequence of the preceding considerations is that concrete objects (things) have no intrinsic conceptual properties, in particular no mathematical features. This last statement goes against the grain of objective idealism, from Plato through Hegel to Husserl, according to which all objects, in particular material things, have ideal features such as shape and number. What is true is that some of our *ideas* about the world, when detached from their factual reference, can be dealt with by mathematics. (For example, by analysis and abstraction we can extract the constructs "two" and "sphere" from the proposition "That iron sphere is composed of two halves.") In particular, mathematics helps us to study the (mathematical) form of substantial properties. In short, not the world but some of our ideas about the world are mathematical. (Bunge 1977, p. 118)

[15] I will not rehearse the notion of research field here. I assume that this claim is essentially correct for the sake of argument. For details on the notion of research field, the reader can and should consult Bunge's numerous writings on the subject, e.g. Bunge (1985b).

I must admit that this is not entirely clear to me. Needless to say, the iron sphere is not, strictly speaking, a sphere in the mathematical sense. The sensory impression of the sphere presumably gives us an approximation of what a sphere in the strict sense would look like. One could perhaps say that we treat the iron sphere *as if* it were a sphere. But in order to do this, we already need to have the mathematical concept of sphere. The *mathematical concept* of sphere is not in the iron sphere.[16] The concept of sphere is given in a certain language, be it geometric, analytic or algebraic, thus in a certain context. It is, in Bunge's terminology, a *construct*.

Definition 2 x is a *construct* if, and only if,

1. there exists (really) an animal capable of conceiving x;
2. the animal conceives x as a conceptual system or a member of such.

I have emphasized the fact that the definition has two dimensions, so to speak. I, for one, might have felt that the first part was enough. Something is a construct if there is an animal that can conceive it. Notice that a construct is not identified with a thought itself. A thought is a brain process, a sequence of events in a real brain, a construct is not. In fact, it is not even clear that a construct x has to be actually conceived at any time by an animal. All that is required by the first condition is that there exists an animal *capable* of conceiving x. Thus, there might not be constructs even when there are animals capable of conceiving them. Thus, the foregoing definition says nothing about the *existence* of constructs, it stipulates, in part, what it is to be a construct. This is the first "vertical" dimension of the definition. It relates constructs to brains.[17]

What does the second part add to the definition? It says that a construct is either a conceptual system or a member of such a system. This is the "horizontal" dimension of the definition: it says that constructs come in bundles. To borrow a metaphor: no concept is an island. This is important in general and in particular for the semantics involved. The meaning of a construct depends upon the context in which it is found. An isolated construct has no meaning and the meaning of a construct can change from one context to the next. Thus, what this dimension tries to capture is the idea that constructs are context dependent in a way that things are not.

I want to emphasize how different Bunge's approach is from many contemporary metaphysics in which the basic distinction is between concrete and abstract entities. In some of these approaches, one first defines what it is to be a concrete entity and then defines abstract entities as being those which do not possess the characterizing property of concrete entities. In Bunge's framework, abstract entities are replaced by constructs and the latter are not defined by the lack of spatio-temporal properties or causal power nor is it said that they are abstract in one way or another, although Bunge's terminology seemed to have change lately (e.g., in Bunge 2010).

Given these definitions, we can introduce Bunge's fundamental metaphysical axioms (Bunge 1977, p. 119):

[16] Of course, this is not what Plato or others said. One would say that the iron sphere participates in the Form Sphere. The general idea underlying what Bunge calls "objective idealism", I take it, is to relate in one way or other the property of the real thing with the concept that applies to it. Notice that, according to Bunge, the processes of abstraction and analysis are applied to propositions, not to substantial individuals.

[17] It is interesting to consider a few specific cases. It is clear that any concrete object, e.g., this cup of coffee on my table, cannot be a construct in this sense. Any concept, e.g., the concept of cup, is a construct. Is the concept of infinite a construct? If so, was it a construct before Cantor elaborated his theory? Does this mean that it was not before? I am not sure how to answer these questions.

Axiom 1 Every object is either a thing or a construct, no object is neither and none is both.

Notice that this axiom puts Bunge at variance with many naturalistic approaches, especially of the Quinean type.

But Bunge *does* introduce a difference between things and constructs. Let us recall a fundamental definition (Bunge 1977, p. 218):

Definition 3 Let x be a thing. Then

1. x is *unchangeable* if and only if $S_{\mathbb{L}}(x)$, the lawful state space of x, is a singleton for all choices of state function for x;
2. x is *changeable* (or *mutable*) if and only if $S_{\mathbb{L}}(x)$ has at least two distinct members for all choices of state function for x.

Axiom 2 Every (concrete) thing has at least two distinct states and the state space of any construct is empty.

Bunge thus obtains the following corollary (Bunge 1977, p. 219):

Corollary 1 *It follows that all (concrete) things are changeable, and constructs are neither unchanging nor changeable.*

This is the basic metaphysical difference between constructs and things. According to Bunge, only the latter are changeable. Notice that it is *assumed* that constructs are neither unchanging nor changeable. This might go against some views according to which constructs or concepts do change. It can be argued, for instance, that the concept of electron changed during the nineteenth and twentieth century. Bunge would simply rebut that these are different concepts, with perhaps a common core, with the same name.[18] It should be emphasized that although they are different concepts, they can be compared and contrasted (see Bunge 1974, Chap. 7 for Bunge's proposals on these issues). Perhaps one should think of the difference here as reflecting the basic ontological dependence underlying the distinction. A concrete thing has an autonomous existence (in some sense) and can be in different states. In fact to be in different states is the characteristic property of being material. Since constructs do not have an autonomous existence, they are in as much as they can be conceived, they cannot be in different states.

Recently, Bunge has given a slightly different twist to his approach. We now find that to have energy is to be changeable. [See Bunge (2010, p. 63).] The axiom then becomes:

Axiom 3 All and only material (concrete) objects are changeable (Bunge 2010, p. 63).

Along with this axiom, Bunge now talks about *ideal* objects and they are now *defined* thus:

Definition 4 An object is an ideal object if it is in no state (Bunge 2010, p. 274)

Notice that we now have a negative definition which does not refer to an animal and its cognitive capacities. In his *Treatise*, Bunge sums up the basic differences between things and constructs thus (Bunge 1985a, p. 27):

[18] Thus, in some sense, conceptual change does occur, even according to Bunge. However, the change is in the real world: there are real changes in brains and in research communities.

> We take it that the keys to real existence are *absoluteness* and *mutability*, whereas those to formal existence are *context-dependence*, *immutability*, and *conceivability*.

Bunge here speaks of real and formal existence. However, as we have already indicated, there is absolutely nothing about the notion of constructs that tells us that they exist. What we know is that, *if* they exist, then they have these properties. One should perhaps say that the existence of constructs is relative (as opposed to absolute in Bunge's sense). But we are getting ahead of ourselves since we still do not know what it means to exist for constructs.

Bunge then goes on and comments on these claims as follows:

> Concerning absoluteness: whereas real things exist absolutely, every construct exists in some context or other, e.g. by fiat or by proof in some theory. For example, the natural numbers exist (formally) in number theory but not in lattice theory. On the other hand electrons and cells, mountains and forests, brains and societies, do not exist relative to some context only to vanish in another: they exist absolutely. (Caution: although things exist absolutely, certain properties, and consequently the changes in them, are frame-dependent.) (Bunge 1985a, p. 27)

One more difference has been added by Bunge recently. Thus:

Theorem 1 *For all x: If x is a material object, then x has energy, and vice versa* (Bunge 2010, p. 63)

Thus to be material is the same thing as having energy. And now Bunge talks about abstract objects:

Corollary 2 *The abstract (ideal, imaginary, non-concrete) objects lack energy* (Bunge 2010, p. 64).

And, therefore, abstract objects are not material. Notice that now Bunge talks about abstract or ideal or imaginary objects. We will stick to the earlier terminology and talk about constructs.

Thus, one might want to make the distinction with respect to the kind of dependence involved here. One possibility would be to claim that the existence of construct is *relative* to the brain capacities of animals, whereas the existence of real things is not. Furthermore, we also know that their existence *depends* upon real thinking brains. But this is different from what Bunge is saying in the foregoing quote. All the properties mentioned by Bunge, i.e., being in no state and lacking energy, do not depend in any way, or so it seems to me, on the fact that constructs are products of the brain. Bunge emphasizes the fact that constructs are relative to contexts, i.e. conceptual contexts, whereas I want to emphasize that constructs are relative to some animals. For I believe that the property of belonging to conceptual contexts follows from our very cognitive make-up. Be that as it may, I want to underline the vertical component of the definition of constructs and connect that component to an issue of ontological dependence whereas Bunge emphasizes the horizontal component and connects that component to a semantical issue. I believe, however, that the ontological dependence is the key conundrum here: what kind of dependence is involved? It is certainly not a logical dependence nor a causal dependence, for in the first case, the relation is strictly between constructs whereas in the second case, the relation is strictly between things or real systems. Once this relation has been clarified, one has next to clarify whether this relation of dependence is compatible with the horizontal component of the definition and the other properties listed here by Bunge. But, before I look into these problems more carefully, I still have to see whether within constructs, mathematical constructs can be characterized.

4 Mathematical Constructs

But what is it to be a *mathematical* construct? There is no doubt that there are numerous constructs that have nothing to do with mathematics, e.g. concepts of philosophy, religion, ideology in general, but also more practical, down to earth concepts. Here, we have a hint that an historical component has to be taken into account to provide an adequate answer.

> Historians of mathematics have noted that, until about the mid-nineteenth century, the bulk of mathematical research was concerned with individual constructs, such as particular figures, equations, functions, or algorithms. From then on, and particularly since mid-twentieth century, mathematics has been conceived as the study of *conceptual systems*, such as groups of transformations (or even the whole category of groups in general), families of functions (or even entire functional spaces), and topological spaces (such as metric spaces in general). (Bunge 1985a, p. 19)

Thus, Bunge concentrates on conceptual systems. But no one would claim that the individual constructs that came before conceptual systems were not mathematical. They certainly were. Bunge is focusing on contemporary mathematics, which is fine. However, if we want to cover mathematical constructs in general, we will need a different strategy. Either there is an intrinsic element to all mathematical constructs, independent of any historical dimension, or there is an historical component to mathematical knowledge and it is possible to characterize mathematical concepts during specific periods and places *and* explain how these properties have come to change. For instance, for a long period of time in Occident, mathematical constructs were taken to fall under the category of *quantities*, discrete or continuous, and that was their specific property. The very object of mathematical constructs was that they were dealing with quantities. One can also explain how this view of mathematics was progressively transformed in the nineteenth century in the Western hemisphere.

According to Bunge, mathematics is the study of conceptual systems. But again, there are conceptual systems in many other disciplines, e.g., in philosophy. What distinguishes *mathematical* conceptual systems?

> What makes the mathematical study of conceptual systems unique is that (a) it is *purely conceptual* (i.e. does not make essential use of any empirical data or procedures) and it involves, at some point or other, (b) *positing or conjecturing the laws* (general patterns) satisfied by the members of those conceptual systems, as well as (c) *proving or disproving conclusively* some such conjectures (Mathematical proofs are perfectible, but they are conclusive.) (Bunge 1985a, p. 22)

Does this characterize mathematics as the study of conceptual systems? One could argue that there are large parts of philosophy, at least at it is traditionally conceived,[19] that satisfy these three properties. For instance, metaphysics is purely conceptual; philosophers posit and conjecture general patterns and some philosophers try to prove or disprove some of their conjectures. In fact, Bunge's own work in ontology seems to fall under this characterization. I, for one, am far from being satisfied by this enumeration.

There are still at least two avenues left open by Bunge. The first one is to stipulate what the conceptual mathematical systems are, period. The second is to fall back on a distinction between truth of reason and truth of facts. The idea underlying this last strategy, closely related to the path taken by the logical empiricists, is to introduce a clear-cut distinction between kinds of truth and then show that mathematical propositions are truths of reason. Let us quickly explore these avenues in turn.

Bunge claims that:

> Every mathematical system ("structure") can be characterized in either of two ways: (a) as a set equipped with a structure consisting of one or more operations or functions defined on that set [here,

[19] I am excluding what is now called experimental philosophy.

> Bunge inserts a reference to Bourbaki]; (b) as a collection of objects together with one or more morphisms relating those objects—i.e. a category [here Bunge refers to Mac Lane's standard textbook on categories]. (Actually the second concept subsumes and supersedes the first.) (Bunge 1985b, pp. 19–20)

In other words, mathematics is about sets or categories. Bunge is adopting here what I will call a *foundational stance*. To put it differently, one is not looking at mathematics with its diverse fields, disciplines, methods, connections, etc., but at a *reconstruction* of mathematics from a foundational point of view.[20] It is possible to define (almost) all mathematical objects as sets and prove theorems within ZFC or some extension thereof as it is possible to do the same (modulo some adjustments) within category theory. But this is not the whole philosophical story. It is certainly an important beginning and is worth the while mostly because it yields interesting and revealing results—completeness, incompleteness, proof-theoretical strength, complexity, etc. are all revealing results in the foundations of mathematics with implications about mathematical knowledge. These reconstructions, both the set-theoretical one and the category-theoretic one, also have weaknesses and limitations. Although they capture many aspects of mathematical knowledge, they introduce distortions and inadequacies. The distortions are mostly epistemological. Indeed, when one considers, for instance, a proof fully written in a formal system, one gains at the level of the justification, since each and every step is justified by an explicit rule, an explicit definition or an axiom, but one looses the epistemological weight, so to speak, of the steps, those that play a role in our understanding of the proof. The inadequacies are more at the level of reference. For instance, in the context of set theories, which set *is* the Klein 4-group? In the context of category theory, one has to face the dual problem so to speak, that is, the ambiguity of reference. In a categorical framework, one can only speak to entities up to isomorphism and can never talk about "the" real numbers, for instance. In both cases, these features now have to be addressed and clarified before anyone can claim to have solved the nature of mathematical knowledge. Furthermore, from my point of view, given what we are trying to answer, this is simply pushing the same issue at a different level. Indeed, if the answers to the question "what is specific to mathematical conceptual systems" is simply "mathematics is about sets (or categories)", then one has to clarify what is specific about sets (or categories) as mathematical entities. In other words, what makes *them* mathematical. We are thus back to our original problem: what does characterize a *mathematical* conceptual system? Finally, one might object that there is another problem with this answer. What about the totality of sets or the totality of categories?[21] But Bunge cuts short to this query:

> In some cases it exists (formally) by virtue of being contained in a larger object, so we can answer the question 'Where (in which set, class, or category) is x?' But if the more comprehensive object happens to be the totality of constructs, then the question makes no more sense than the question 'Where is the physical universe?'. (Bunge 1985a, p. 30)

[20] Notice that the foundational stance was motivated, when it was more or less introduced at the end of the nineteenth century and the beginning of the twentieth century, by philosophical claims. Thus, logicism, as we have said, wanted to show that mathematical knowledge rested on reason alone, i.e. that there was no need for a so-called mathematical intuition in the kantian sense of the word. Hilbert's program wanted to justify the use of transfinite methods in mathematics via metamathematics. It was thus also an epistemological program. By saying that every mathematical system is a set or a category, Bunge is certainly not making a claim of that sort.

[21] The totality of sets cannot be a set. This is clear. It can be a class, provided that we work in such a framework. The totality of categories can be a category, but this answer falls short of the requirements of mathematics itself. The totality of categories is a higher-dimensional category. This is a totally different foundational framework which is still undergoing important development as I write.

It is not quite clear to me that the analogy is sound. After all, the physical universe is a thing (or a concrete system) and there may be ways of seeing that the question is senseless. (Again, notice how Bunge is using the same strategy as the logical empiricists. Some metaphysical issues are simply pseudo-problems.) One does not ask *where* is the totality of mathematical constructs, but *what* is the totality of constructs of a certain kind. And in the case of sets and categories, there is nothing *a priori* that prevents us from asking the question. It seems to be perfectly reasonable to inquire about the totality of sets or the totality of categories. It is even more interesting to consider answers which give a reason why the given totality has to be of a different kind or why the question is ill-formed. Be that as it may, Bunge's answer that a mathematical conceptual system is a universe of sets or a category does not give us any insight into the nature of mathematical conceptual systems as such. To put it differently: the foundational stance does not yield, by itself, a satisfactory philosophical answer. The foundational stance has to be accompanied by a philosophical motivation or point of view.

Let us thus turn to the distinction between truth of reason and truth of facts. Here is how Bunge introduce the distinction (Bunge 1974, p. 90):

> A truth of reason is, of course, one that can be established by reason alone. A number of kinds of truth of reason have been distinguished, among them the following.

1. *Dictionary truth* or *veritas ex vi terminorum*—e.g. a nominal definition.
2. *Truth by request* or postulation—e.g., a postulate in a mathematical theory.
3. *Truth by proof or deduction*—e.g. a theorem in a mathematical theory.
4. *Logical truth* or *veritas ex vi formarum* or tautology—e.g., any valid formula in a given system of logic.
5. *Truth by exemplification* or satisfaction in a model.

But this enumeration is misleading. For only the last two cases are, according to Bunge, cases of truth of reason.

> The first two hardly deserve to be classed among truths: dictionary "truths" are mere conventions; and mathematical postulates are proposed because they summarize theories not because they are assumed to be true in themselves. (…) The third, truth by deduction, is another case of abuse of the word 'truth': if its peculiarity is that it is deducible from a set of assumptions then the concept of truth is redundant here. Only the last two concepts of truth are legitimate, namely those of true under all interpretations (logical truth) and true under some interpretations (mathematical truth). They are the object of model theory. (Bunge 1974, p. 90)

Thus, mathematical truth is here taken to be elucidated by model theory. This is an extremely important claim. From a philosophical point of view, the specific details of model theory, e.g. the nature of the formal systems involved, taking interpretations over sets or in categories, is irrelevant. What matters are the general norms that one can extract from the framework. For instance, the assumption that mathematics is or can be fully formalized in a certain manner, that is, one can develop a fully formal system for any mathematical theory and such that the formal system satisfies clear requirements, e.g. the syntax is fully explicit, the rules of formation and deduction are recursive.[22] This assumption is a way to capture what I consider to be the important portions of the formalist approach. But we do not stop at the formalism and its properties. Indeed, the assumption goes hand in hand with a parallel methodological assumption: existing mathematical

[22] Again, without specifying the details about this formal system, whether it is first-order, second-order, type-theoretical or otherwise. It is certainly one of the important contributions of foundational studies of the twentieth century to have provided a precise and clear notion of formal system.

theories that have been fully formalized can be interpreted in a system of mathematical entities and the interpretation is also fully explicit. The upshot, according to Bunge, is that mathematical truth is a matter of *coherence* between two conceptual systems. Notice that Bunge goes against Tarski's own interpretation of his definition of truth since the latter claimed that he was providing a correspondence theory of truth, even for mathematics.

We are back to the previous case, but with a totally different perspective. Indeed, on the one hand, we have deductive systems and, on the other hand, we have systems in which these are interpreted, i.e. sets with structures or categories with structures. However, we are no longer in a foundational stance. There is a definite claim made about mathematical knowledge: (1) it is precisely the kind of knowledge that can be *fully* formalized (according to specific norms of formalization); (2) it is precisely the kind of knowledge that can be *fully* interpreted in structures of a certain kind, e.g. sets or categories (according to specific norms of interpretation); (3) it is precisely the kind of knowledge in which the truth value of the propositions can be *fully* established by relating the two foregoing dimensions systematically (according to specific norms of satisfaction), in other words, nothing else is required to establish this truth value.[23] Notice that we need all three of these properties. Of course, some physical theories can be fully formalized and can be fully interpreted in appropriate mathematical structures. However, one would not claim, in this case, that the truth value of a proposition of a physical theory is fully determined by a systematic relationship between the formal system and the formal interpretation. In this case, something else is required: a link to the real world. The same holds, I contend, with other cases. In the case of philosophy and philosophical systems, it is unlikely that it could satisfy any of the requirements and, I claim, even if it did, it would also require, directly or indirectly, a link to the real world.[24]

Are we back to the analytic/synthetic distinction? Bunge toyed with the notion of analyticity at various moments in his work. [See for instance, Bunge (1961).] In the early 1960s, Bunge accepted the analytic/synthetic distinction: a formula is analytic if and only if it is merely a formula of a formal language that is either a definition or holds under all interpretations (in all models). But Bunge did not give it an important role to play in his philosophical system. To wit:

> We uphold then the analytic/synthetic distinction, indicted in recent times (Quine 1951). However, we do not define analyticity in terms of the information needed for understanding a sentence, this being a pragmatic not a semantic concept of analyticity; hence we are not bothered by Quine's examples. Furthermore we do not regard the distinction as a dichotomy or as being central to the whole of semantics and the philosophy of formal science. The essential distinction, as far as epistemology and the philosophy of science is concerned, is that between truths (or falsities) of reason and truths (or falsities) of fact. (Bunge 1974, p. 170)

Bunge later more or less pushed to the side the distinction between analytic and synthetic propositions. It is simply replaced by the distinction between formal and factual propositions.

> The preceding classification is more correct than either of the popular dichotomies analytic/synthetic or a priori/a posteriori. For one thing the concepts of analyticity and aprioriness have not been well defined except in extreme cases. Thus, a tautology is clearly analytic, but what about a theorem in theoretical physics, which has been derived by purely conceptual means? And an empirical datum is

[23] We are not making a clear-cut distinction between logic and mathematics. A distinction can be made, but it rests on subtle issues related to the above. We refer the reader to the literature mentioned previously. Furthermore, our claim does not rest upon a clear-cut distinction about logic and mathematics nor does it require that we fix a specific logic.

[24] Bunge would certainly agree to the claim that mathematics is a peculiar *semiotic system*. See Bunge (2003) for more on the notion of semiotic system.

> clearly a posteriori, but what about a factual hypothesis not built inductively and which happens to anticipate experience correctly? Neither of the two distinctions amounts to a dichotomy, hence neither is a sound principle of classification. (Bunge 1983, p. 181)

Be that as it may, if I am correct, we end up with a characterization of mathematics that is extremely close to the traditional strategy encoded by the analytic/synthetic distinction. After all, being analytically true can be roughly described as being true by virtue of the meaning of the terms involved *and nothing else*. We certainly keep that last bit of the conjunction. What has changed dramatically is the precision of the first part. And this is an empirical claim: I claim that the norms associated with model theory upon which the notions of satisfaction, interpretation and truth are built are here to stay and that they encapsulate what it is to be *mathematically* true.[25]

Thus, I believe that Bunge suggests a way to characterize mathematical conceptual systems. Whether it is satisfactory is not entirely clear to me at this stage. But we have other, perhaps more important, unfinished business to attend to. Indeed, so far, nothing has been explicitly said about existence as such. Let us look more carefully at the notion of mathematical existence.

5 Matters of Existence

In the same way that he has distinguished between truth of fact and truth of reason, Bunge introduces a distinction between real existence and formal existence.

Bunge makes a series of assumptions about existence.

Axiom 4 All and only material objects exist objectively (really) (Bunge 2010, p. 70).

And hence:

Corollary 3 *No ideal (or imaginary) objects exist objectively (really)* (Bunge 2010, p. 70).

Whence:

Corollary 4 *No mathematical structures exist out there* (Bunge 2010, p. 70).

Bunge could not be more clear. Mathematical objects do not really exist. Notice how radically different Bunge's approach is to Quine's approach on the issue. Indeed, in a Quinean perspective, the entities that exist are those over which our best scientific theories quantify if their statements are to be true. Thus, in a Quinean framework, existence is established by the existential quantifier in first-order logic and since we quantify over mathematical entities in our best scientific theories, we have to admit that mathematical objects exist just as much as any other object falling under the existential quantifier. One

[25] In a sense, though, I still haven't provided the answer. Two important elements have to be established. First, I still have to exhibit these norms explicitly. I believe that they are fairly clear from model theory itself. In fact, I claim that they are nowadays inherent in the practice of contemporary mathematical logic. Notice that the definitions of interpretation, satisfaction and truth found in categorical model theory *are* merely a generalization of Tarski's definition in the context of category theory. This fact is not appreciated enough by philosophers of mathematics and logic. Thus all the norms at work in the set-theoretical framework are just as present in the categorical framework. I would in fact be ready to claim that category theory helps clarify some of the norms implicit in the semantics. (For more on this, see Marquis (2009)) Second, I believe that one still has to explain why mathematical knowledge does satisfy these norms! I do believe that there is something about mathematical constructs, in Bunge's sense of the expression, that explains these facts. The key, I believe, is to capture the various ways it is *formal* and *abstract*. I distinguish the two.

might think that Bunge would therefore be a nominalist, since he denies the existence of abstract objects. However, he rejects this possibility forcefully:

> … strict nominalists have no use for any of the above [i.e. his definition of ideal objects], since they deny the existence of properties, or else assert that these can be defined as sets of individuals. But all knowledge consists of attributing properties or changes thereof to individuals. In particular, law statements relate properties. And identifying properties with set of individuals amounts to confusing predicates with their extensions. (Bunge 2010, p. 275)

As far as I know, no one has attempted to make a clear cut distinction between factual existence and formal existence the way Bunge does.[26] Indeed, Bunge rejects Quine's reliance on the existential quantifier. Bunge introduced two predicates to talk about existence. They are defined at various places in his work, although the formal definition only appeared in the 1980s and not in his earlier work on semantics where the distinction is introduced informally only. Here is the formal definition:

Definition 5 Let C be non-empty [sic] subset of some set X, and χ_C the characteristic function of C, that is, the function $\chi_C : X \to \{0, 1\}$ such that $\chi_C(x) = 1$ if and only if x is in C, and $\chi_C(x) = 0$ otherwise. The *relative (or contextual) existence predicate* is the statement-valued function

$$E_C : C \to \text{The set of statement containing } E_C$$

such that "$E_C(x)$" is interpreted as "x exists in C", and it is equivalent to $\chi_C(x) = 1$ (Bunge 2010, pp. 268–269; see also Bunge 1977, p. 156).

Why does Bunge make this definition in the first place? What does it do? It marks a clear distinction between the logical quantifier "$\exists$" and the existential *predicate*. Thus, the main point is simple: existence is not, in general, a logical matter.

> Notice that the above existence predicate is unrelated to the "existential" quantifier, which I prefer to call "particularizer". I submit that "$\exists\, xPx$" only says that *some* individuals have the property P. Their existence must be assumed or denied separately. For instance "Some postulated entities exist in the real world" can be symbolized as "$\exists\, x\, E_W\, Px$", where W stands for the collection of real things. (Bunge 2010, p. 269)

We might add that categorical logic has shed a new light on the nature of the quantifiers: they naturally arise as adjoints to the substitution operation, which is a purely formal affair. Moreover, still in categorical logic, sequents, i.e. deductions, are done over explicitly declared contexts, which allows the possibility of having empty domains in logic and using the existential quantifier over empty domains coherently. Finally, there are mathematical contexts, e.g. toposes, where existence is more subtle than in classical logic, i.e. it is easy to construct toposes in which there are non-empty sets without elements. Stated in this way, the statement sounds absurd. However, once one knows how to interpret "being non-empty" and "element" in this context, then it is very natural. And we know how to treat the existential quantifier even in these cases (see for instance, Marquis (2009) for more).

Thus, in Bunge's system, we have contextual existence predicates and therefore at least two different existence predicates: one for real existence and one for ideal existence. Here they are (Bunge 1977, p. 157):

[26] Quine's position and the variant introduced by Putnam is now known under the name of the *indispensability argument*. It has been enormously influential, even as a springboard for those who believed in a form of naturalism and that mathematical objects do not exist (really). It has led to the reintroduction of 'nominalism' in contemporary philosophy of mathematics. For a survey of the main positions in the field, see Burgess and Rosen (1997). For recent attempts defending a form or another of nominalism in mathematics, see among others, Leng (2005, 2007, 2010) or Chihara (2004, 2007, 2008, 2010).

Definition 6

1. *x exists conceptually* $=_{df}$ For some set C of constructs, $E_C\, x$;
2. *x exists really* $=_{df}$ For some set Θ of things, $E_\Theta x$.

The astute reader will have observed that there are numerous existence predicates, in fact, just as many as there are contexts of relevance. Thus Bunge introduces an existence predicate for the set M of characters in Greek mythology and an existence predicate E_M for these. One wonders how these existence predicates arise and are related. One can say that the foregoing axioms tell us that the two predicates in Definition 6 are certainly the most fundamental. Thus, every character in Greek mythology is certainly a construct. Notice that conceptual existence and real existence are mutually exclusive, i.e. an object cannot exist both conceptually and really. Thus, we have two parallel realms, so to speak. However, we know that the realm of conceptual existence depends upon the realm of real existence, via thinking brains. Thus, although concrete existence and real existence are mutually exclusive, the former depends upon the later for its being. The link between the two realms is left completely unspecified.

These are mere definitions and, as such, do not tell whether anything exists or how we can determine whether something exists or not. For this, Bunge introduces the:

Criterion 1 An object other than the entire world exists really if it is shown to be connected to some real object other than it (Bunge 1977, p. 160).

This is more delicate than it might seem. What kind of connections are allowed here? According to Bunge, constructs *are* connected to (some) real objects, e.g. brains. If I can show that an animal does think a construct c, should I conclude that c really exists? This is incompatible with the foregoing definitions: we would have an object that does exist both conceptually and really. This is certainly not what Bunge wants. It seems that we have to stipulate the kind or kinds of connections allowed. One obvious possibility would be to say that the connection involves an exchange of energy, but since I am not sure that this possibility does the work, I will simply leave it open.[27] Be that as it may, there seems to be a genuine problem with the criterion proposed.

Mathematical objects are constructs and therefore they exist conceptually. However, for mathematical constructs to exist, something more is required, they have to satisfy an additional property:

Definition 7 If x is a construct, then *x exists mathematically* $=_{df}$ For some C, C is a set, class, or category, such that (1) x is in C, and (2) C is specified by an exact and consistent theory.[28] (Bunge 1985a, p. 30)

[27] One thing that does seem to be obvious to me is that Bunge would not want the connection to be causal. He is very clear about the fact that there are determinate non-causal relations between things. If exchanging energy is equivalent to having a causal connection, then our proposal is inadequate.

[28] Compare Bunge's definition with the following passage:

> Obviously, an object is a mathematical object and is known as such, insofar as it belongs to, or is a member of, a mathematical system of objects and logical relations. This belonging-to, or membership is what constitutes the meaning of the expression 'there exists' in mathematical existential propositions; and insofar as the system to which it is referred is a different one, this expression means something different in every system. (von Freytag-Löringhoff, quoted by Thomas (2000, p. 324)

This was written by a disciple of Hans Vaihinger immediately after the second World War. We will come back to this connection later.

We are back to the foundational stance. Bunge is here considering mathematical existence in the way that mathematicians would use the expression themselves (more or less, ignoring some qualms coming from the constructivists). Hilbert, for instance, used the expression in this way at various times. See Ferreirós (2009) for a detailed analysis of Hilbert's conceptions on this issue.) This is, in some sense, a necessary condition to mathematical existence. I have no objection as such to this usage. It certainly would deserve to be explored and analyzed, but I will simply let it stand as it is for the sake of argument. For as we have seen mathematical constructs are constructs and as such, one could say that they do not *really* exist. Is this it? Are we done? Is this all that an ontological study of mathematical constructs can deliver in a emergent materialist framework. It might seem to be. To wit:

> … our postulate that ideal objects are immaterial, and conversely, precludes all talk about the ontology of mathematics. One should talk instead of the reference class(es) of mathematical predicates and structures. For example, the domain D of a function *f* of a single variable is the reference class of *f*, not its ontology. The reason is that ontologies are theories about the world, not sets. Hence we do not make any ontological commitment when positing that a certain domain is non-empty. Ontology starts when specifying the nature of the members of the domain in question—e.g., material, spiritual, or hybrid. (Bunge 2010, p. 275)

We are entering a meta-ontological debate at this point. I, for one, do not find this claim satisfactory: the foregoing quote seems to suggest that there is no *ontological* issue involved. Bunge does accept the claim that ontology makes basic distinctions, e.g., between things and their properties. Why couldn't we include a discussion about modes of being? Isn't it what is done by Bunge after all? By introducing these distinctions between kinds of existence, isn't he introducing at the same time modes of existence? In other words, no one doubts that we think about mathematical objects, no one would dispute that they do not really exist or at least not in the same way as concrete objects do. Some do in fact deny that they exist in any sense of that word.[29] But if we grant that it makes sense to talk about mathematical object—and I certainly believe it does—one has to explain *the kind of objects they are* and this, I seems to me, is an ontological issue. Bunge provides important hints and he is certainly sensitive to these issues. For instance, I find his analysis of energy as a *property* of things to be illuminating in that it teaches us what kind of being energy is. In other words, what kind of existence is conceptual existence?

6 Pretensions and Fictions

It is at this particular point that Bunge uses a surprising card.

> For example the Pythagorean theorem exists in the sense that it belongs in Euclidean geometry. Surely it did not come into existence before someone in the Pythagorean school invented it. But it has been in conceptual existence, i.e. in geometry, ever since. Not that geometry has an autonomous existence, i.e. that it subsists independently of being thought about. *It is just that we make the indispensable pretence that constructs exist provided they belong in some body of ideas*—which is a roundabout fashion of saying that constructs exist as long as there are rational beings capable of thinking them up. Surely this mode of existence is neither ideal existence (or existence in the Realm of Ideas) nor real or physical existence. To invert Plato's cave metaphor we may say that ideas are but the shadows of things—and shadows, as is well known, have no autonomous existence. (Bunge 1977, p. 157, our emphasis)

[29] I thank Robert Thomas for reminding me of this important fact.

Two passages are worth commenting. The first one is the passage I have underlined: *It is just that we make the indispensable pretence that constructs exist provided they belong in some body of ideas*. What kind of claim is that? Is it a fact, a human fact? Is it an empirical fact? Can we and should we put it to the test? If it is not a fact, what is it? A methodological rule? It seems to be too easy to say that we pretend that they exist. Many mathematicians would rather say that they exist, in some strong sense of the word, most likely in the sense that their existence is independent of our capacity of conceiving them. Others might agree that they are constructed and that their existence depend directly on our cognitive capacities. Who is right? The thing is: Bunge wants to say that mathematical objects are constructs and, *at the same time*, he wants to claim that they have some sort of autonomy, some sort of independence. For many people, to really exist is to have some sort of independence from other things, at least, from brain processes (or, in a different terminology, from the mind). If mathematical constructs are not brain independent, then they do not really exist. So far, Bunge agrees. But then, they should have properties that come from this dependence. And this is where Bunge hesitates and wants to introduce this pretence. Mathematical objects have properties that clearly go beyond my cognitive capacities and beyond yours, even if you are the best mathematician around these days. Thus, they have some sort of independence after all. Yet, they are dependent too. What are we to make of this?

Second, and this is a related point, there is this nice metaphor: "Ideas are shadows of things". It is another way of talking about ontological dependence. A shadow does not exist by itself. It arises only in very specific circumstances. Do we pretend that a shadow has an independent existence? We don't.[30] But a shadow is connected to a real thing and, thus, we can assert its real existence. Furthermore, its properties follow from this connection. We can determine its length, its surface, its form, etc., by looking at the real object it is connected to: all we need is some knowledge of optics, some knowledge of geometry and we are done. For other properties, we might need some other information about physics and the real thing it is connected to. Couldn't we do the same with constructs? They are thought by persons, real persons. They are therefore connected with real things. Why would they not exist like shadows? Because we do not see them? Because we do not see the connection to the real thing? What if we could? Given the evolution of cognitive neuroscience, we might.[31] Is it because even if we could see the connection to a real thing, we could not, in contrast with a shadow, determine the properties of a concept from the properties of the real thing it is related to? It seems reasonable to infer that even if we could determine that someone is thinking about a number from neurophysiological data, it is unlikely that we could infer *which* number that person is thinking about or, which property of this number she is thinking about.

At this juncture, Bunge turns to a very well-known analogy, which earned him to be classified among the *fictionists*.

> Mathematical objects are then ontologically on a par with artistic and mythololological [sic!] creations: they are all *fictions*. The real number system and the triangle inequality axiom do not exist really any more than Don Quijote or Donald Duck. Nor is the difference epistemological, for some mathematical constructs, just as some artistic and mythological fictions, are idealizations of real things or features of real things. Nor, finally, is mathematics distinguished from art and myth by its certainty,

[30] But of course, there are numerous children's stories in which shadows have an autonomous existence.

[31] Robert Thomas has suggested to me that the inverse metaphor might be more revealing: some real things are shadows of ideas. Written language is certainly one of them and so is, of course, systems of written symbols for mathematical constructs, i.e. notational systems. The danger with the latter metaphor is that it might lure us into Plato's cave…

> even though the yearning for final certainty has often been a powerful motivation for mathematical research, particularly in the foundations of mathematics. (Bunge 1985a, pp. 38–39).

We have to be careful with the terminology. Bunge is borrowing the term from Vaihinger's philosophy of the 'as if'.[32] Recall that the full title of Vaihinger's book is: *The philosophy of 'as if': a system of the theoretical, practical and religious fictions of mankind* (Vaihinger and Ogden 1925). It is important to emphasize immediately that a fiction is not understood here as a term that seems to refer to something but that, in fact, does not. Bunge, as it should patently be clear by now, says explicitly that mathematical terms refer genuinely—in fact, Bunge would say that mathematical terms designate constructs, but in the case of mathematical constructs, designation and reference become identical—, albeit not to real things, but to constructs. In fact, fictions do play an important philosophical role here and not one that we usually attribute to them. Instead of quoting Vaihinger, let me bring in a rather unknown figure, a follower of Vaihinger, quoted by Thomas:

> Whether we speak of real or non-real (in the sense of abstract) Being, we regard both as being entirely independent of whether they are thought by us or not. This, we are bound to do; for otherwise we should be thinking, not of the object of thought, but of ourselves and our thought processes. And this is not the case. … And while in the case of concrete Reality, this is real, in the case of non-Reality or abstract Reality, it is fictitious. (von Freytag-Löringhoff, quoted by Thomas (2000, p. 323)

Notice the properties associated with being fictitious. They are precisely the properties Bunge ascribes to mathematics as it is. This is but one usage of fictions and their properties and probably not the one most people would think about when thinking about fictions. I believe that Robert Thomas is correct about this:

> In his sense of the word, which has little to do with stories, Vaihinger is right to say that mathematics is about fictions; he acknowledges that he is following Bentham, and Bunge acknowledges his debt to Vaihinger. But this sense of fiction is not the one that is widely recognized, even among philosophers. (Thomas 2000, p. 324)

The analogy between mathematical objects and fictions has been used by many philosophers, for many different purposes.[33] In some cases, one wanted to try to show how mathematics could be objective despite the fact that its objects were not real or physical in some sense, others wanted to emphasize certain aspects of fictions, e.g. the fact that they are created and appreciated in a certain way, and transfer these properties to mathematical objects. Examining Bunge's form of fictionism more than thirty years ago, Roberto Torretti claimed that Bunge has his cake and wants to eat it too:

> We shall see that the form of mathematical fictionalism that meets this condition, and which I take to be the one actually held by Bunge, does not differ from, say, Gödel's mathematical Platonism by much more than a change of emphasis. (Torretti 1982, p. 400)

Thus, according to Torretti, Bunge would be a fictionist of a very odd type: it is a form of Platonism but without the latter autonomous world of ideas. Here is Bunge's reply:

> The reader who has come this far may wonder how mathematical fictionism differs from Platonism. The difference is that the Platonic philosophy of mathematics is part and parcel of an objective idealist metaphysics, one that postulates the autonomous existence of ideas and their ontological priority. On the other hand mathematical fictionism is not included in any ontology, because it does not regard mathematical objects as self-existing but as fictions. (Bunge 1997, p. 57)

[32] In fact, when I was a student under his supervision, Bunge recommended that I read the latter, which I did with great interest.

[33] The interested reader should read the exhaustive and insightful papers written by Robert Thomas on the issue. See Thomas (2000).

Is this merely a change of emphasis? Bunge does not think so. Bunge reintroduces specifically at this juncture the ontological dependence of mathematical objects: they are not self-existing, their existence depends upon specific brain activities. At the same time, Bunge does want to leave mathematics as it is—since, according to him, all the reforms proposed by constructivists amounted to important amputations of crucial parts of analysis, e.g. the intermediate value theorem[34]—*and* claim that it is a product of humans. Those who object to this approach usually point out that it follows that mathematics is a pure and free creation of the human brain and, as such, should have properties of human creations, that is, limitations of some sort and a large degree of arbitrariness. Furthermore, the objection continues, it seems impossible to explain how mathematics applies so well to the real world.

As a matter of fact, Bunge does accept the conclusion that mathematics is a pure and free creation of the human brain. To wit:

> When introducing or developing an original mathematical idea, the mathematician creates something that did not exist before. As long as he keeps the idea to himself, it remains locked in his brain—for, as a physiological psychologist would say, the idea is a process occurring in the mathematicians's brain. (Bunge 1997, p. 58)

And again:

> A materialist should not feel uneasy about the thesis that mathematical objects are ideal and therefore timeless, as long as it is accompanied by the thesis that mathematics is a human creation. (Bunge 1997, p. 67)

So, why is it that mathematical ideas, creations of the human brain, do not have properties linked to this creation? Why don't they have historical properties, reflecting some peculiar socio-historical aspects of the society in which they were created? Or properties of the mathematicians, of their personality? Why don't they have neurophysiological properties? One cannot simply say: well, look at mathematical theories, they simply do not have any of these properties. I think the issue is much more subtle than what Bunge suggests here. Perhaps we simply do not see these properties as *our* properties anymore. Perhaps it is possible to explain how mathematics came to have these unique features and no others and it has to do with mathematics itself, properly understood. I will come back to this point in the last section.

Although there are, according to Bunge, similarities between mathematics and fictions in general, there are also important dissimilarities. Let us look first at the similarities. As we have seen, truth in mathematics is a matter of reason, it is an internal affair. Likewise, truth in literary works (and, presumably other similar artworks), is a purely internal affair.

> Moreover occasionally we are justified in talking about artistic truth and falsity, as when we say that Don Quixote[35] was generous and Othello's suspicion false. In order to establish the artistic truth or falsity of an artistic fiction we only resort to the work of art in question. That is, artistic truth, like mathematical truth, is internal and therefore context-dependent, that is, it only holds in some context and it need bear no relation to the external world. (Bunge 1997, p. 53)

Bunge contends that with respect to objects, methods, and relations to the real world, "mathematics is closer to art than to science" (Bunge 1997, p. 54). I, for one, am not sure at all that mathematics is that close to art.[36] There are similarities. There are differences. In

[34] Things are, of course, no longer so clear. Moving to a constructivist framework does not amount to an amputation, but an enrichment of mathematics.

[35] Bunge no longer writes 'Don Quijote' as he did in 1977.

[36] I have to be careful here, since I have tried to show elsewhere that some parts of contemporary mathematics should be treated as conceptual scientific *technologies*. See Marquis (1997, 2006). For a long period of time, mathematics was indeed considered to be an art, a *techne* and I *do* believe that this allows us to

fact, Bunge himself gives an impressive list of the crucial differences between mathematical fictions and all others:

1. Far from being totally free inventions, mathematical objects are constrained by laws (axioms, definitions, theorems); consequently they cannot behave "out of character"—e.g., there can be no such thing as a triangular circle, whereas even mad Don Quixote is occasionally lucid;
2. Mathematical objects exist (ideally) either by postulate or by proof, never by arbitrary fiat;
3. Mathematical objects are either theories or referents of theories, whether in the making or full-fledged, whereas myths, fables, stories, poems, sonatas, paintings, cartoons and films are non-theoretical;
4. Mathematical objects and theories are fully rational, not intuitive, let alone irrational (even though there is such thing as mathematical intuition);
5. Mathematical statements must be justified in a rational manner—either by their fruits or by their premises—not by intuition, revelation, or experience;
6. Far from being dogmas, mathematical theories are based on hypotheses that must be repaired or given up if shown to lead to contradiction, triviality, or redundancy;
7. There are no strays in mathematics: formulas belong to theories, and theories are linked together forming supersystems or by being shown to be alternative models of one and the same abstract theory; thus logic employs algebraic methods, and number theory resorts to analysis; on the other hand artistic or mythological fictions are self-sufficient: they need not belong to any coherent system;
8. Mathematics is neither subjective nor objective: it is ontologically noncommittal; but the process of mathematical invention is subjective, and that of proof (or disproof) intersubjective; what is real (concrete) about mathematics is only living mathematicians and active mathematical communities;
9. Some mathematical objects and theories find application in science, technology, and the humanities;
10. Mathematical objects and theories are socially neutral, whereas myth and art often support or undermine the powers that be; and
11. Because it deals in timeless objects, correct mathematics does not age, even though some of it may go out of fashion (Bunge 1985a, pp. 39–40; Bunge 1997, pp. 63–64).

With such a list, one wonders why Bunge wants to keep the analogy at all! Clearly, Bunge wants to talk about fictions in the way Vaihinger did. I suspect that he came to realize that Vaihinger's usage does not correspond to the common usage. When we have the latter usage in mind, it is easy to add to Bunge's list. To mention but one: as Walton has forcefully argued in Walton (1990), when we entertain certain pieces of fictions like novels, we enter a game of make-believing which, mostly when the artwork is well crafted, spark various (pseudo) emotions, sometimes just as strong as the ones we would feel in a real similar situation. Although mathematicians certainly have emotions of some kind towards certain mathematical results, e.g. surprise, awe, wonder, excitement, even an

Footnote 36 continued
make sense of some aspects of mathematical knowledge, for a large portion of mathematics is indeed a form of conceptual *know how*. Thus, I would be willing to claim that mathematics is close to art in this particular sense. But it is almost always possible to find links between various human activities. The question is whether they are revealing of the fields rather than the activities. Thus, there is a huge literature on the links between art and science, some of which are just fabulous and interesting (and the same holds, of course, for mathematics).

aesthetic appreciation, etc., they are of a radically different kind and on a different spectrum altogether.[37] When I read and try to understand a proof of the fundamental theorem of algebra, I am certainly not moved in the same way that I am when I read Cormac McCarthy's novel *The Road*. I am not emotionally involved in the same way. One of the key ingredients here is that I try to understand a proof, not merely verify that it is correct, although in some cases, one does merely verify that a given proof is correct. I don't usually try to understand a novel, at least not in the same manner. I certainly do not try to make sure that it is correct. Notice also that in the very first item, Bunge underlines the fact that mathematical objects are not totally free inventions. Thus, they might be created, but within very strong constraints. In other words, a mathematician always starts from certain given mathematical facts, ideas or theories and the development of his ideas is always constrained by various imperatives. Another surprising aspect of Bunge's fictionism is that it is neutral with respect to materialism:

> As far as mathematical fictionism is concerned, one may hold the world either to be material, or to be spiritual. (Bunge 1997, p. 67)

In my mind, one should be able to place fictionism in a material world in a consistent manner and it seems to me that it should be rather odd in a spiritualist ontology. I would tend to believe that in a spiritualist ontology, mathematical objects would have a real, independent existence. In other words, fictionism ought to have a privilege status in a materialist framework.

I, for one, believe that the analogy is not worth the money. It is simply too limited and it is a mistake to underline what mathematics seem to have in common with constructs in general. I believe that the analogy is, when everything is said and done, superficial. I am puzzled by the claim that the Klein 4-group is, in some ontological sense, on a par with Mickey Mouse. I seriously doubt that a general theory of fictions would be revealing about mathematical knowledge. I doubt that it would account for mathematics as such. I agree with Robert Thomas's diagnosis:

> … a philosophical discussion of the analogy between mathematics and fiction will probably subsume one under the other or both under some more general notion. The latter, which is what I am trying to do, is preferable to what we shall see happens, which is to subsume the one under the other, taking fiction to be the more general category itself instead of seeking a more general category among texts of which mathematics and fiction can be instances, preserving their distinctiveness from each other. This distinctiveness, …, is important. There are immense logical complications to fiction, largely because of its complex interactions with the everyday world, that no one concerned with mathematics is likely to feel attracted to. (Thomas 2000, p. 321)

After all, what we want to understand is not mathematics as a form of fiction, but mathematics as it evolved and as it is[38]: peculiar, magnificent, powerful, fascinating, beautiful, highly technical, computational,[39] formal and abstract. Bunge has clearly understood that foundational studies, e.g., proof theory, model theory, computability theory—be they done in a set theoretical framework or a categorical framework—have

[37] Although this is highly controversial, it is hard not to foray in the direction explored by the mathematician Ioan James in (2003) who explored the links between various mild forms of Asperger's syndrome and particularly brilliant mathematicians and scientists whose work involved mathematics in a crucial manner. If this is correct, there are some links between brain properties and mathematical ability.

[38] Granted, sometimes we understand by comparing two things, by bringing forward the similarities and the differences, each contributing to our understanding. In this case, the comparison is simply too weak. But we might want to consider mathematics as a sort of artifacts, which also comprises fictions.

[39] As a good mathematician friend of mine once told me: no matter how abstract and conceptual your work is, at some point, if your doing mathematics, you have to calculate something.

provided us with important insights into the nature of mathematics. But I am afraid that he has pushed aside the humane, historical dimension of mathematics too quickly out of the picture. The challenge is to see how the *specific* features of mathematic can be accounted for by this humane, social and historical dimensions. If this is possible, it would constitute at the same time a blow at any form of relativism.

7 Dependent Existence and Autonomy

The basic problem is this: mathematical constructs, like all constructs, are products of brains. But mathematical constructs, unlike other constructs, have peculiar properties that are easily explained if we attribute to them an independent existence. Bunge's answer, as we have just seen, is simply that we pretend that they have this autonomy. We treat them *as if* they were real entities. However, this pretense is certainly not conscious nor can we easily switch to a different position. In the case of artistic fictions, we can remind ourselves that they are creations of talented and hard-working human beings, we can and do sometimes want to know how they were created, when, by whom, etc., and we accept that they are creations of human brains. In the case of mathematics, we tend to resist this conclusion, even when we know the history of various concepts, e.g., the history of complex numbers. Most mathematicians would claim that mathematical objects seem to have an autonomous, independent existence and that mathematical theories are true of these independently existing objects. Numbers do not change, they cannot be in different states. Nor can a specific algebraic equation, say $x^2 + 1 = 0$, be in different states.[40] My knowledge of the natural numbers seems to be basically the same as the knowledge of a professional mathematician, although the latter knows a great deal more about them than I do, as it ought to be expected. We seem to be strongly compelled into this posture by the very nature of mathematical entities.

In the end, I don't think we have to pretend anything about mathematical existence. We do not have to define what it is to exist for a mathematical construct. There is absolutely no mystery there. Mathematicians have a perfectly clear notion of mathematical existence within a theory. But this is not the end of the story. What we have to explain is why we believe that some construct seems to exist in some deeper, more philosophical sense. And the route to follow here, in my opinion, is not to concentrate on a specific kind of existence or an analogy with fictions, but rather on the features exhibited by these concepts that lure us into believing that they ought to exist in some deep philosophical sense.

In an emergent materialist framework, it seems reasonable to conjecture that these properties of mathematics emerge from features of our cognitive make-up and both our evolutionary and social history as cognitive agents, features that are more basic and more difficult to unearth.[41] I believe that this is the fundamental challenge. Needless to say, I cannot provide a complete answer here. I will only sketch what I take to be some of the salient ingredients at the moment.

[40] But of course, as Bunge points out, its meaning can vary from one context to another.

[41] There has to be an analogy with language in general here. Human languages have very specific and unique features which used to be explained also by our supposedly unique faculty of reason, like we would like to account for the unique features of mathematics. But it may very well be that these unique features of language have roots in our cognitive make-up with its specific evolutionary history. See for instance, Hauser et al. (2002).

The ability to approximate the size of collections seems to be fundamentally the same in all cultures, despite various obvious differences in notational systems representing quantities and even in cultures where there is no notational system available for representing quantities (see Lemer et al. 2004 for instance). Indeed, many neuropsychologists now postulate the existence of what is called by Stanislas Dehaene a basic "number sense", a preverbal ability to estimate the size of collections that we would have inherited from our evolutionary history (see Barth et al. 2006; Dehaene 2009, 2011; Dehaene et al. 2004; Dehaene and Brannon 2011; Izard et al. 2009; Jordan and Brannon 2006; Piazza and Dehaene 2004). More importantly, this core system would also play a role in our acquisition and the development of formal arithmetic, which is clearly learned. Thus, this basic neurological system would underlie our concept of (natural) number and it would be a universal feature of human beings, and, in fact, not only of human beings (see Cantlon and Brannon 2006; Cantlon and Brannon 2007; Stocklin et al. 1999). Furthermore, many studies indicate that this core neurological system integrates notational independent numerical representations (see Cantlon et al. 2009; Dehaene 2009).

A similar core system, also independent of language, seems to underly the development of (Euclidean) geometry (see Dehaene et al. 2006; Izard et al. 2011).

Now, if these findings are correct, we can certainly infer that they can account for one important aspect of mathematical knowledge. Indeed, if our numerical and geometrical abilities lie in neurological systems that are independent of language, we can expect that our experience of this knowledge will have a quality that goes beyond language and, in some sense, explicit consciousness. This would account in part for our feeling that mathematics is something that lies beyond and behind our conscious experience, that it is something that we discover. And, indeed, in a very specific sense, we do.

Thus, our cognitive make-up could account for some of the properties of mathematics. But some key properties of mathematical concepts might contribute to these convictions in their own right. It is clear that the ability studied by neuropsychologists are fundamental, but they are far from mathematical theories and mathematical properties of these objects.

I will limit myself to two additional aspects that come into play. The first one is directly related to the construction of a theory of mathematical objects. A mathematical theory establishes properties of a given type of mathematical objects, e.g., of natural numbers. But what are the relevant properties of these objects? How does one determine that it is indeed a relevant property of these objects? Although this is a basic ingredient of the construction of a theory, it is usually implicitly assumed by mathematicians, i.e., they "know" what properties are relevant.

As we all know and as we have emphasized already, the development of symbolic notation and of notational *systems* occupy a crucial place in the history of mathematics and in mathematical knowledge. The systematic nature of these notational system constitute a key component of mathematics.

8 Concluding Remarks

Being in a emergent materialist framework, Bunge asserts that mathematical constructs, concepts, algorithms and theories, are products of brains. As such, they are like any other construct. However, as Bunge is well aware, mathematical constructs seem to possess unique properties.

To conclude, I believe that in a systemic emergent materialist framework, our goal, as philosophers, is to provide a clear conceptual analysis of the nature of mathematical knowledge so that precise factual cognitive socio-neuropsychological theories about it can be constructed and tested. In many ways, Mario Bunge has laid out the foundations for this work to be possible. Philosophical theories cannot only be right, they can also be useful and fruitful.

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References

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. (2006). Non-symbolic arithmetic in adults and young children. *Cognition, 98*(3), 199–222.

Beeson, M. J. (1985). *Foundations of constructive mathematics, volume 6 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in mathematics and related areas (3)]*. Metamathematical studies. Berlin: Springer.

Benacerraf, P. (1973). Mathematical truth. *The Journal of Philosophy, 70*(19), 661–679.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology*. Translated from the French by W. Mays. Dordrecht: D. Reidel.

Bishop, E., & Bridges, D. (1985). *Constructive analysis, volume 279 of Grundlehren der Mathematischen Wissenschaften [Fundamental principles of mathematical sciences]*. Berlin: Springer.

Bunge, M. A. (1961). Analyticity redefined. *Mind, 70*(278), 239–245.

Bunge, M. A. (1974). *Treatise on basic philosophy: Interpretation and truth* (Vol. 2). Dordrecht: D. Reidel.

Bunge, M. A. (1977). *Treatise on basic philosophy: The furniture of the world* (Vol. 3). Dordrecht: D. Reidel.

Bunge, M. A. (1983). *Treatise on basic philosophy: Exploring the world, volume 7, Part I*. Dordrecht: D. Reidel.

Bunge, M. A. (1985a). *Treatise on basic philosophy: Philosophy of science and technology, volume 7, Part I of treatise on basic philosophy*. Dordrecht: D. Reidel.

Bunge, M. A. (1985b). *Treatise on basic philosophy: Philosophy of science and technology, volume 7, Part II of epistemology and methodology*. Dordrecht: D. Reidel.

Bunge, M. A. (1997). Moderate mathematical fictionism. In E. Agazzi, & G. Darvas (Eds.), *Philosophy of mathematics today* (pp. 51–71). Dordrecht: Kluwer.

Bunge, M. A. (2003). *Emergence and convergence: Qualitative novelty and the unity of knowledge*. Toronto: University of Toronto Press.

Bunge, M. A. (2010). *Matter and mind: A philosophical inquiry, volume 287 of Boston studies in the philosophy of science*. Berlin: Springer.

Burgess, J. P., Rosen, G. (1997). *A subject with no object*. Strategies for nominalistic interpretation of mathematics. New York: The Clarendon Press/Oxford University Press.

Cantlon, J. F., & Brannon, E. M. (2006). Shared system for ordering small and large numbers in monkeys and humans. *Psychological Science, 17*(5), 401.

Cantlon, J. F., Brannon, E. M. (2007). Basic math in monkeys and college students. *PLoS Biology, 5*(12), e328.

Cantlon, J. F., Libertus, M. E., Pinel, P., Dehaene, S., Brannon, E. M., & Pelphrey, K. A. (2009). The neural development of an abstract concept of number. *Journal of Cognitive Neuroscience, 21*(11), 2217–2229.

Cellucci, C., & Gillies, D. (Eds.). (2005). *Mathematical reasoning and heuristics*. London: King's College Publications.

Chihara, C. (2007). The Burgess-Rosen critique of nominalistic reconstructions. *Philosophia Mathematica, 15*(1), 54–78.

Chihara, C. (2008). The existence of mathematical objects. In B. Bold, & R. A. Simons (Eds.), *Proof and other dilemmas, MAA spectrum* (pp. 131–156). Washington, DC: Mathematical Association of America.

Chihara, C. (2010). New directions for nominalist philosophers of mathematics. *Synthese, 176*(2), 153–175.

Chihara, C. S. (2004). *A structural account of mathematics*. New York: The Clarendon Press/Oxford University Press.

Corfield, D. (2003). *Towards a philosophy of real mathematics*. Cambridge: Cambridge University Press.

Dehaene, S. (2009). The case for a notation-independent representation of number. *Behavioral and Brain Sciences, 32*(3–4), 333–335.

Dehaene, S. (2011). The number sense: How the mind creates mathematics. New York: Oxford University Press (rev. and updated edition).
Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core knowledge of geometry in an amazonian indigene group. *Science, 311*(5759), 381.
Dehaene, S., Molko, N., Cohen, L., & Wilson, A. J. (2004). Arithmetic and the brain. *Current Opinion in Neurobiology, 14*, 218–224.
Dehaene, S., & Brannon, E. (2011). *Space, time and number in the brain: Searching for the foundations of mathematical thought: An attention and performance series volume* (1st ed.). London: Academic Press.
Detlefsen, M. (2005). Formalism. In S. Shapiro, (Ed.), *The Oxford handbook of philosophy of mathematics and logic, Chap. 8* (pp. 236–317). Oxford: Oxford University Press.
Feferman, S. (2004). Tarski's conception of logic. *Annals of Pure and Applied Logic, 126*(1–3), 5–13.
Feferman, S. (2008). Tarski's conceptual analysis of semantical notions. In *New essays on Tarski and philosophy* (pp. 72–93). Oxford: Oxford University Press.
Feferman, S. (2010). Set-theoretical invariance criteria for logicality. *Notre Dame Journal of Formal Logic, 51*(1), 3–20.
Ferreirós, J. (2009). Hilbert, logicism, and mathematical existence. *Synthese, 170*(1), 33–70.
Ferreirós, J., & Gray, J. J. (Eds). (2006). *The architecture of modern mathematics: Essays in history and philosophy*. Oxford: Oxford University Press.
Hauser, M., Chomsky, N., & W. Tecumseh Fitch. (2002). The faculty of language: What is it, who has it, and how did it evolve? *Science, 298*, 1569–1579.
Izard, V., Pica, E., Spelke, S., & Dehaene, S. (2011). Flexible intuitions of euclidean geometry in an amazonian indigene group. *Proceedings of the National Academy of Sciences of the United States of America, 108*(24), 9782–9787.
Izard, V., Sann, C., Spelke, E. S., & Streri, A. (2009). Newborn infants perceive abstract numbers. *Proceedings of the National Academy of Sciences of the United States of America, 106*(25), 10382.
James, I. (2003). Singular scientists. *Journal of the Royal Society of Medicine, 96*(1), 36.
Jordan, K., & Brannon, E. (2006). The multisensory representation of number in infancy. *Proceedings of the National Academy of Sciences of the United States of America, 103*(9), 3486–3489.
Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an amazonian indigene group. *Science, 306*(5695), 499–503.
Leng, M. (2005). Revolutionary fictionalism: A call to arms. *Philosophia Mathematica, 13*(3), 277–293.
Leng, M. (2007). What's there to know? A fictionalist account of mathematical knowledge. In *Mathematical knowledge* (pp. 84–108). Oxford: Oxford University Press.
Leng, M. (2010). *Mathematics and reality*. Oxford: Oxford University Press.
Maddy, P. (1997). *Naturalism in mathematics*. New York: The Clarendon Press/Oxford University Press.
Mancosu, P. (2008). *The philosophy of mathematical practice*. Oxford: Oxford University Press.
Martin-Löf, Per (1998). An intuitionistic theory of types. In G. Sambin, & J. M. Smith (Eds.), *Twenty-five years of constructive type theory (Venice, 1995), volume 36 of Oxford logic guides* (pp. 127–172). New York: Oxford University Press.
Marquis, J.-P. (1997). Abstract mathematical tools and machines for mathematics. *Philosophia Mathematica, 5*(3), 250–272.
Marquis, J.-P. (2006). A path to the epistemology of mathematics: Homotopy theory. In J. J. Gray, & J. Ferreiros (Eds.), (pp. 239–260). Oxford: Oxford University Press.
Marquis, J.-P. (2009). From a geometrical point of view. New York: Springer.
Mautner, F. I. (1946). An extension of Klein's Erlanger program: Logic as invariant-theory. *American Journal of Mathematics, 68*, 345–384.
McGee, V. (1996). Logical operations. *Journal of Philosophical Logic, 25*(6), 567–580.
Piazza, M., & Dehaene, S. (2004). From number neurons to mental arithmetic: The cognitive neuroscience of number sense. In M. S. Gazzaniga (Eds.), *The cognitive neurosciences* (3rd ed., pp. 865–877).
Quine, W. V. O. (1951). Main trends in philosophy: Two dogmas of empiricism. *The Philosophical Review, 60*(1), 20–43.
Serfati, M. (2005). *La révolution symbolique. Transphilosophiques*. Paris: Éditions PÉTRA.
Sher, G. (1991). *The bounds of logic: A generalized viewpoint. A Bradford book*. Cambridge, MA: MIT Press.
Sher, G. (2003). A characterization of logical constants *is* possible. *Theoria (San Sebastián), 18*(47), 189–198.
Stocklin, P., Brannon, E., & Terrace, H. (1999). Monkey numeration. *Science New Series, 283*(5409), 1851–1852.
Tarski, A. (1986). What are logical notions? *History and Philosophy of Logic, 7*(2), 143–154.

Thomas, R. S. D. (2000). Mathematics and fiction. I: Identification. *Logique et Analyse (N.S.), 43*(171–172), 301–340.
Thomas, R. S. D. (2002). Mathematics and fiction. II: Analogy *Logique et Analyse (N.S.), 45*(177–178), 185–228.
Timmermans, B. (1995). *La résolution des problèmes de Descartes à Kant*. Paris: PUF.
Torretti, R. (1982). Three kinds of mathematical fictionalism. In J. Agassi, R. S. Cohen, & M. A. Bunge (Eds.), *Scientific philosophy today: Essays in honor of Mario Bunge* (Vol. 67, pp. 399–414). Dordrecht, Holland: D. Reidel
Troelstra, A. S., & van Dalen, D. (1988a). *Constructivism in mathematics: An introduction. Vol. I, volume 121 of Studies in logic and the foundations of mathematics*. Amsterdam: North-Holland Publishing Co.
Troelstra, A. S., & van Dalen, D. (1988b). *Constructivism in mathematics: An introduction. Vol. II, volume 123 of Studies in logic and the foundations of mathematics*. Amsterdam: North-Holland Publishing Co.
Vaihinger, H., & Ogden, C. K. (1925). *The philosophy of 'as if': A system of the theoretical, practical and religious fictions of mankind*. New York: Harcourt, Brace and Company.
Walton, K. L. (1990). *Mimesis as make-believe: On the foundations of the representational arts*. Cambridge, MA: Harvard University Press.