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# MARITIME COLLISION AVOIDANCE AS A DIFFERENTIAL GAME

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T. MILOH and S.D. SHARMA

Collision avoidance during a two-ship encounter in the open sea is treated as a problem of optimal control using the theory of differential games. Each ship is assumed to have two controls corresponding to rudder angle and engine setting. Different versions of the game are obtained under assumptions such as (a) one ship evades, while the other stands on, (b) both ships evade in collaboration, and (c) one ship evades while being pursued by the other. Optimal evasive maneuvers and limiting conditions for ship encounters beyond which collision becomes unavoidable are determined analytically. Numerical examples are given for encounters between two ships, one of which has a higher speed and the other a smaller turning radius. Results are presented in a graphical form suitable for display on a radar screen.

## INTRODUCTION

This paper is an offshoot of a broader project seeking to determine how exactly the frequency of collisions, and hence the economy and safety of ship operation, depends on the maneuvering capabilities of ships among other things, see Krappinger (1972). In order to solve this problem it has been found necessary to develop a mathematical model of collision avoidance and occurrence. A natural by-product of this effort was the determination of optimal evasive maneuvers and of limiting conditions for ship encounters beyond which collision cannot be avoided, a concept we adopted from Kenan (1972) and Webster (1974). However, instead of their numerical trial-and-error approach we have used an analytical technique based on the theory of *Differential Games* as formulated by Isaacs (1965). Although this theory was originally developed for solving military problems of guided pursuit, the same techniques can be profitably applied also to problems of collision avoidance, as pointed out explicitly by Isaacs himself in the preface to his book. In fact we have done no more than to apply several modified versions of Isaacs' *Game of Two Cars*. As the participants of a Ship Control Systems Symposium[2] may be expected to be interested in both collision avoidance and differential games, which have much in common with optimal system control, this would seem to be a suitable occasion to present a summary of our work. Those interested in a fuller account are referred to an institute report by Miloh (1974) and to a previous paper addressed to an audience of navigators, cf. Miloh and Sharma (1975). Meanwhile, we have discovered that work along similar lines has been done elsewhere by Vincent et al. (1972) and Merz (1973). However, since Vincent was concerned mainly with aircraft collision avoidance and Merz treated only the special case of two identical ships, our work seems to complement rather than duplicate theirs.

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[2]This paper is a slightly revised version of a paper originally presented at the Fourth Ship Control Systems Symposium, The Hague, Netherlands, 27-31 Oct. 1975.

## COLLISION AVOIDANCE MODEL

### Rules of the Road

In order to be able to formulate a mathematical model of maritime collision avoidance let us consider how an encounter with another ship in the open sea is handled by a ship master. The complex chain of decision making subject to the International Rules of the Nautical Road has been ably summarized in logic flow diagrams by Luse (1972) and Kwik (1973), from whom our Fig. 1 is adapted. Whenever another ship enters the range of observation of own ship we speak of an encounter, which may be said to last until the other ship again passes out of our range of observation. In general, both ships will hold course and speed during the encounter so that the range will steadily decrease upto the closest point of approach (CPA) and then increase. The range at CPA is called the miss distance. If the miss distance predicted from continuous or successive observations of range and relative bearing at the beginning of the encounter is less than a certain amount regarded as safe, the encounter is said to constitute a risk of collision. Depending upon the type of ships involved and upon whether it is a meeting, crossing or overtaking situation, either both ships are burdened, i.e. required to evade, or one is burdened while the other is privileged, i.e. required to hold course and speed (also called *standing on*) upto the so-called *last minute* (also called ships in extremis), when it is also allowed and required to evade. The last minute is presumed to have arrived when it becomes clear that the burdened ship cannot avoid collision by its action alone. Obviously, a collision is always preceded by a last minute situation and, in general, can occur only when both ships have failed to take appropriate action.

### Barriers and Critical Maneuvers

Consider, as depicted in Fig. 2, the initial or steady-state phase of a two-ship encounter during which both ships stand on. This is the phase of situation assessment as expertly described by Luse (1972). Let us draw an imaginary circle of radius $r_m$ about the center of own ship O, such that a miss distance less than $r_m$ would constitute an undesirable event in some suitable sense. A burdened ship in the open sea might choose an $r_m$ of two or three miles to avoid any hazardous approach, whereas a privileged ship in congested waters, not being free to take early action, may have to choose an $r_m$ on the order of a ship length so that a miss distance less than $r_m$ would be tantamount to a physical collision. Since the particular interpretation is of no consequence to the following analysis, which can be carried out for any numerical value of $r_m$, we shall for the sake of brevity call the undesirable event a *collision* and the circle of radius $r_m$ a *terminal* circle.

Consider now the set of all possible encounters with fixed initial velocities but arbitrary relative positions of the two ships. In relative coordinates attached to ship O the ship A will seem to move with relative velocity $V_r$. Let us draw tangents to the terminal circle in a direction parallel but opposite to velocity $V_r$. The inside of the band formed by these two tangents (from B and C) we call the *risk* zone, and the outside we call the *safe* zone.For it is clear from Fig. 2 that a risk of collision in the above sense exists if and only if A is initially observed inside this band. Of course, this risk can, in general, be obviated by a timely evasive maneuver of either ship. It can be shown that there exists a curve BD such that if A is still beyond this curve, a miss distance larger than $r_m$ can be attained by a hard right turn of O, indicated by $O_R$ in Fig. 2. Similarly, there is a limiting curve CE for a hard left turn of O. Hence, if right and left turns were the only permissible maneuvers, we could call the area enclosed by the arcs BC, CF and FB the *collision* zone. If O is responsible for collision avoidance, it must evade before A reaches the boundary BFC. For if O waits until A (initially inside the risk zone) has already penetrated the collision zone, the risk transforms into certainty and collision becomes imminent, even though some further time may elapse before A finally hits the terminal circle irrespective of how O now tries to evade. We may therefore call a boundary such as BFC[4] the *barrier* and the evasive maneuvers that would prevent collision marginally from initial positions on the barrier the *critical* maneuvers.

Evidently, a family of barriers and critical maneuvers determined for systematically varied values of the minimum desired miss distance would constitute a complete set of optimal evasive maneuvers for maximizing miss distance starting from any initial position. It is also clear that the shape and size of the collision and risk zones will depend on the maneuvering capabilities and objectives of the ships involved. Typically, one ship would evade, while the other stands on. However, a best case analysis assuming optimal evasion in collaboration and a worst case analysis assuming optimal evasion in face of optimal pursuit (presumably owing to ignorance or error) might also be meaningful in the context of collision avoidance. We shall see that by formulating the problem as a differential game various combinations can be treated by a single uniform approach.

## DIFFERENTIAL GAME FORMULATION

### Basic Concepts

The essential requisite of a differential game is a dynamic system controlled by two or more players. State variables characterize the current state of the system. Its dynamic behavior is expressed by kinematic equations relating the time rate of change of state variables to the state variables themselves and to control variables, which are at the volition of the players. Quantities which do not change with time during a particular play of the game, but which we might wish to vary from play to play are called parameters of the game. The end of any play is marked by predefined terminal conditions of the system. The game is characterized by a payoff comprising a terminal component as a function of the terminal conditions and/or an integral component as a function of the path along which the terminal conditions are reached. The objective of an individual player is to maximize (or minimize) the payoff. A function relating a control variable to the state variables is called a strategy. All players are assumed to have complete information on the current state of the system and the range of controls available to all other players. If all players play their mutually optimal strategies, there exists for every starting state of the system a conceptually predetermined unique payoff called the value. The theoretical solution of the game comprises the optimal strategies, the optimal paths leading to the terminal conditions and the value as a function of the starting states of the system. A continuous payoff function yields a game of degree with a continuous value function in state space. A payoff function defined to have only discrete values yields a game of kind with the state space subdivided into zones of different value separated by barriers.

### Kinematic Equations

Our dynamic system consists of two ships O and A maneuvering on a sea surface assumed to be homogeneous, isotropic, unbounded and undisturbed. We choose to describe it by the following set of kinematic equations (KE) involving five state variables, four control variables and eight parameters:

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[4] If it is required that O be able to avoid collision while retaining the choice of a right *or* left turn, e.g. in congested or restricted waters, then the curve BEFDC will have to be regarded as the barrier.

$$\mathrm{d}x/\mathrm{d}t = V_a\cos\theta + y\phi_1 V_o/R_o - V_o \quad (1)$$

$$\mathrm{d}y/\mathrm{d}t = V_a\sin\theta - x\phi_1 V_o/R_o \quad (2)$$

$$\mathrm{d}\theta/\mathrm{d}t = \psi_1 V_a/R_a - \phi_1 V_o/R_o \quad (3)$$

$$\mathrm{d}V_o/\mathrm{d}t = \phi_2 T_o/m_o - V_o|V_o|k_o/m_o \quad (4)$$

$$\mathrm{d}V_a/\mathrm{d}t = \psi_2 T_a/m_a - V_a|V_a|k_a/m_a \quad (5)$$

The state variables are the rectangular coordinates $(x, y)$ and the course angle $\theta$ of A relative to O, and the absolute forward speeds $V_o$, $V_a$ of O and A. Whenever convenient we shall also use the polar coordinates (range $r$ and bearing $\alpha$) instead of $(x, y)$. The two are related by

$$x = r\cos\alpha, \qquad y = r\sin\alpha \quad (6)$$

The four fixed parameters for each ship are the minimum turning radius ($R_o$, $R_a$), the maximum propulsive thrust ($T_o$, $T_a$), the effective mass ($m_o$, $m_a$) and the coefficient of resistance ($k_o$, $k_a$). The two controls alloted to each ship ($\phi_1$, $\phi_2$ for O; $\psi_1$, $\psi_2$ for A) can be interpreted as normalized rates of turn and thrusts respectively, being idealizations of the two basic controls (rudder angle and engine setting) available to any ship. The control variables may instantaneously assume any value within the standard interval $[-1, 1]$. Eq. (1-3) are purely kinematic. Eq. (4-5) imply the assumption that resistance to forward motion is proportional to speed squared; all other hydrodynamic and inertial effects are ignored.

Payoff and Termination

We shall consider two alternative but essentially equivalent ways of defining payoff and termination in the game of collision avoidance. (I) We can define the closest approach at first pass, characterized by

$$\mathrm{d}r/\mathrm{d}t = 0, \qquad \mathrm{d}^2r/\mathrm{d}t^2 > 0, \quad (7)$$

as the terminal condition and use the terminal range $r_f$ as payoff yielding a game of degree. (II) We can define collision, characterized by

$$r = r_m, \quad (8)$$

as the terminal condition and use a discrete payoff (say +1 for collision avoidance and -1 for collision occurrence) yielding a game of kind. In the latter case not every point of the terminal surface can be reached from the outside. The useable part is defined by the condition of positive penetration velocity, i.e. negative range rate:

$$\mathrm{d}r/\mathrm{d}t \leqslant 0 \quad (9)$$

The boundary of the useable part (BUP) is delineated by the equality sign in Eq. (9) which again yields the condition of closest approach, Eq. (7). Since in a game of kind only the marginal plays ending at the BUP are of concern, we conclude that in either case the only terminal conditions of interest are those given by Eq. (7). The terminal parameter $r_f$ in our game of degree is identical with the game parameter $r_m$ in our game of kind. Substitution of the KE yields a definite relation between the terminal bearing $\alpha_f$ and the terminal course angle $\theta_f$ on the BUP:

$$\sin\alpha_f = \varepsilon(V_{of} - V_{af}\cos\theta_f)/V_{rf}, \quad \cos\alpha_f = \varepsilon(V_{af}\sin\theta_f)/V_{rf}, \quad \varepsilon = \pm 1 \tag{10}$$

For any given $\theta_f$ and terminal speed ratio $V_{af}/V_{of}$ we obtain two solutions $\alpha_f^u$, $\alpha_f^l$ corresponding to the upper and lower sign in Eq. (10) such that if we go around the terminal circle once in the clockwise sense, the useable part begins at $\alpha_f^l$ (lower BUP) and ends at $\alpha_f^u$ (upper BUP). Note that in the exceptional case $V_{af} = V_{of}$ and $\theta_f = 0$ any value of $\alpha_f$ satisfies Eq. (7) so that the entire terminal circle becomes the useable part!

## Player Objectives

To complete the formulation of the differential game we must now specify the objectives of our two players O and A. We have already seen that in a typical encounter one of the two ships will be obligated to evade and the other to stand on. However, the *best* case of evasion in collaboration and the *worst* case of evasion in face of pursuit are also worth studying. We can cover all possible combinations by considering the following nine versions of the game:

$$O_E\,A_E,\ O_N\,A_E,\ O_P\,A_E;\ O_E\,A_N,\ O_N\,A_N,\ O_P\,A_N;\ O_E\,A_P,\ O_N\,A_P,\ O_P\,A_P.$$

Here E stands for evasion, N for not maneuvering (i.e. standing on) and P for pursuit. For instance, $O_E\,A_E$ implies evasion by O and A in collaboration. More exactly, the subscript E implies that the ship chooses its controls so as to maximize payoff (i.e. range at CPA), N implies keeping controls fixed so as to hold absolute course and speed, and P implies choosing controls so as to minimize payoff (i.e. range at CPA). Note that only the versions $O_P\,A_E$ and $O_E\,A_P$ are genuine games with a conflict of objectives. The others are degenerate or one-sided games which may also be treated as ordinary problems of optimal control. Game $O_N\,A_N$ is trivial and $O_P\,A_P$ is a case of rendezvous rather than collision avoidance.

One might legitimately argue that in normal operation collision avoidance is a side condition rather than the primary objective of a ship so that return to original course rather than CPA should be regarded as the terminal condition and that evasion should aim at minimizing time loss rather than maximizing miss distance. That would indeed generate a useful game for the timely evasive maneuver of a burdened ship. However, this paper is concerned mainly with last minute maneuvers, where collision avoidance is the primary objective and hence miss distance is an appropriate payoff.

# ANALYTICAL SOLUTION

## Main Equation

Following Isaacs (1965, p. 67) optimal play is governed by the equation

$$\mathrm{M}\left\{ W_x \mathrm{d}x/\mathrm{d}t + W_y \mathrm{d}y/\mathrm{d}t + W_\theta \mathrm{d}\theta/\mathrm{d}t + W_{V_o} \mathrm{d}V_o/\mathrm{d}t + W_{V_a} \mathrm{d}V_a/\mathrm{d}t \right\} = 0 \tag{11}$$

where the operator M reflects the players' objectives. Explicitly, M is

$\max\limits_{\phi_1,\phi_2}\ \max\limits_{\psi_1,\psi_2}$ for game $0_E A_E$; $\max\limits_{\phi_1,\phi_2}$ for game $0_E A_N$; $\max\limits_{\phi_1,\phi_2}\ \min\limits_{\psi_1,\psi_2}$ for game $0_E A_P$ etc.

In the game of degree $W$ stands for the unknown value function with the subscripts denoting partial derivatives with respect to the state variables. In the game of kind $W_x$, $W_y$ etc. can be interpreted as the components of the outward normal vector on the barrier surface in the state space. Since $dx/dt$, $dy/dt$ etc. are the components of the vector along which play proceeds in state space, Eq. (11) implies that in a game of terminal payoff optimal paths lie along surfaces of constant value in a game of degree and along barrier surfaces in a game of kind. It is called the Main Equation (ME) of differential games by Isaacs. Its analogue in control theory is called the Bellman Equation, see Bryson and Ho (1969, p. 135). Its solution comprises the solution of the game.

Adjoint Equations

The ME is a first-order partial differential equation of the Hamilton-Jacobi type with {} as the Hamiltonian and $W_x$, $W_y$ etc. as costate variables. It can be solved by integrating its characteristic equations which consist of the KE and the adjoint equations obtained by differentiating the ME with respect to the state variables and rearranging the terms, see Isaacs (1965, p. 80):

$$dW_x/d\tau = -W_y\bar{\phi}_1 V_o/R_o \tag{12}$$

$$dW_y/d\tau = W_x\bar{\phi}_1 V_o/R_o \tag{13}$$

$$dW_\theta/d\tau = (W_y\cos\theta - W_x\sin\theta)\ V_a \tag{14}$$

$$dW_{V_o}/d\tau = (W_x y - W_y x - W_\theta)\ \bar{\phi}_1/R_o - W_x - 2W_{V_o}|V_o|k_o/m_o \tag{15}$$

$$dW_{V_a}/d\tau = W_x\cos\theta + W_y\sin\theta + W_\theta\bar{\psi}_1/R_a - 2W_{V_a}|V_a|k_a/m_a \tag{16}$$

Here the derivatives have been taken with respect to retrograde time $\tau$ to emphasize an idea fundamental to game theory, namely the *retrogression* principle, which requires that the characteristic equations be solved retrogressively, i.e. starting from the terminal conditions and working backward into state space. The bars over $\phi$ and $\psi$ indicate that optimal controls satisfying the ME are to be used.

Optimal Controls

Substitution of the KE (1-5) into the ME (11) immediately yields the following expressions for the optimal controls as functions of the state and costate variables:

$$\bar{\phi}_1 = \gamma\,\mathrm{sgn}\left\{(W_x y - W_y x - W_\theta)\ V_o/R_o\right\} \tag{17}$$

$$\bar{\phi}_2 = \gamma\,\mathrm{sgn}\left\{W_{V_o}T_o/m_o\right\} \tag{18}$$

$$\bar{\psi}_1 = \delta\,\mathrm{sgn}\left\{W_\theta\ V_a/R_a\right\} \tag{19}$$

$$\bar{\psi}_2 = \delta\,\mathrm{sgn}\left\{W_{V_a}T_a/m_a\right\} \tag{20}$$

with

$$\gamma \;=\; +1 \text{ if O evades and } -1 \text{ if O pursues,} \tag{21}$$

$$\delta \;=\; +1 \text{ if A evades and } -1 \text{ if A pursues,} \tag{22}$$

where we have made use of the fact that the control variables were normalized to lie within the standard interval $[-1, 1]$. If a ship stands on, instead of optimal controls we just have fixed controls:

$$\bar{\phi}_1 \;=\; 0 \text{ and } \bar{\phi}_2 \;=\; k_o|V_o|V_o/T_o, \qquad \text{if O stands on,} \tag{23}$$

$$\bar{\psi}_1 \;=\; 0 \text{ and } \bar{\psi}_2 \;=\; k_a|V_a|V_a/T_a, \qquad \text{if A stands on.} \tag{24}$$

Eq. (17-22) imply that the optimal controls are at least piecewise *constant* and, in general, the extreme permissible values, with the possible exception of singular cases where the expressions in braces vanish. This is a direct consequence of our choice of the dynamic model such that the control variables occur linearly in the KE (1-5) and is known as the *bang-bang* principle in control theory. It greatly simplifies the integration of the characteristic equations.

## Optimal Terminal Maneuvers

Since the game has to be solved retrogressively we first seek to evaluate the optimal controls at the terminal conditions. We have a four-parametric terminal manifold[5]:

$$0 \leqslant r_f < \infty, \quad 0 \leqslant \theta_f \leqslant 2\pi, \quad 0 \leqslant V_{of} \leqslant \sqrt{T_o/k_o}, \quad 0 \leqslant V_{af} \leqslant \sqrt{T_a/k_a} \tag{25}$$

as $\alpha_f$ follows from Eq. (10). The terminal values of the state and costate variables have the parametric representations:

$$x \;=\; r_f\cos\alpha_f, \quad y \;=\; r_f\sin\alpha_f, \quad \theta \;=\; \theta_f, \quad V_o \;=\; V_{of}, \quad V_a \;=\; V_{af} \tag{26}$$

$$W_x \;=\; \cos\alpha_f, \quad W_y \;=\; \sin\alpha_f, \quad W_\theta \;=\; 0, \quad W_{V_o} \;=\; 0, \quad W_{V_a} \;=\; 0 \tag{27}$$

Substituting Eq. (26-27) into Eq. (17-20) shows to our disgrace that the arguments of the signum functions are all zero! The situation can be saved, however, by replacing the expressions in braces by their retrograde time derivatives with the final result:

$$\bar{\phi}_1 \;=\; \gamma\,\mathrm{sgn}(-\sin\alpha_f) \;=\; \gamma\varepsilon\,\mathrm{sgn}(\cos\theta_f - V_{of}/V_{af}) \tag{28}$$

$$\bar{\phi}_2 \;=\; \gamma\,\mathrm{sgn}(-\cos\alpha_f) \;=\; -\gamma\varepsilon\,\mathrm{sgn}(\sin\theta_f) \tag{29}$$

$$\bar{\psi}_1 \;=\; \delta\,\mathrm{sgn}[\sin(\alpha_f - \theta_f)] \;=\; \delta\varepsilon\,\mathrm{sgn}(\cos\theta_f - V_{af}/V_{of}) \tag{30}$$

$$\bar{\psi}_2 \;=\; \delta\,\mathrm{sgn}[\cos(\alpha_f - \theta_f)] \;=\; \delta\varepsilon\,\mathrm{sgn}(\sin\theta_f) \tag{31}$$

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[5]For the sake of simplicity we assume positive terminal speeds. A speed reversal during the play would then imply a negative initial speed which can also be interpreted as a positive speed along the opposite course.

Recalling that $(\pi+\alpha-\theta)$ is the relative bearing of O as seen from A, the optimal terminal maneuvers turn out to have a very simple and intuitively appealing interpretation: At CPA the evader is turning and accelerating *away* from his target, while the pursuer is turning and accelerating *toward* his target! In other words, if the target is to port, the evader is turning to starboard and vice versa; if the target is forward of abeam, the evader is applying backward thrust and vice versa. Unfortunately, no such simple rules exist in terms of conditions prevailing *before* the maneuver is executed, and hence the need to solve the game retrogressively. In passing we note that from the above we may expect singular behavior when the target is dead ahead, abeam or abaft at CPA. This is indeed what happens. So-called singular surfaces originate from such points as we shall presently see.

## Optimal Paths

We could now determine optimal paths by integrating the characteristic equations (1-5 and 12-16), working retrogressively from the terminal conditions (26-27). The union of all optimal paths for a given value of terminal parameter $r_f$ would produce a curved surface in state space that can be interpreted as a surface of constant value $r_f$ in our game of degree or as a barrier of miss distance $r_f$ in our game of kind. Different parts of the barrier emanating from the lower and upper BUP (and even from different segments of the same BUP) will in general intersect and enclose a finite collision zone in state space. The intersections are so-called *dispersal* curves separating regions of different optimal strategies (e.g. left turns on one side, right turns on the other). If Eq. (17-20) produce discontinuous controls along some optimal paths, we may have *switching* curves on barriers implying multi-step maneuvers.

For the sake of closed form integration we shall now sacrifice some generality of our kinematic model by assuming *constant* forward speeds, which is probably not too unrealistic in view of the short duration of a collision avoidance maneuver and the rather low accelerations of which ships are capable. (A deceleration is generally considered undesirable in practice due to the attendant loss of rudder effectiveness.) In any case, assuming constant forward speeds $(V_o, V_a)$ and thanks to constant optimal controls $(\bar{\phi}_1, \bar{\psi}_1)$ the characteristic equations can be analytically integrated to yield the following retrograde path equations:

$$\theta = \theta_f + (\bar{\phi}_1 V_o/R_o - \bar{\psi}_1 V_a/R_a)\tau \tag{32}$$

$$x = r_f\cos(\alpha_f+\tau\bar{\phi}_1 V_o/R_o)+R_o\sin(\tau\bar{\phi}_1 V_o/R_o)/\bar{\phi}_1+R_a\left\{\sin\theta-\sin(\theta_f+\tau\bar{\phi}_1 V_o/R_o)\right\}/\bar{\psi}_1 \tag{33}$$

$$y = r_f\sin(\alpha_f+\tau\bar{\phi}_1 V_o/R_o)+R_o\left\{1-\cos(\tau\bar{\phi}_1 V_o/R_o)\right\}/\bar{\phi}_1-R_a\left\{\cos\theta-\cos(\theta_f+\tau\bar{\phi}_1 V_o/R_o)\right\}/\bar{\psi}_1 \tag{34}$$

$$W_x = \cos(\alpha_f+\tau\bar{\phi}_1 V_o/R_o) \tag{35}$$

$$W_y = \sin(\alpha_f+\tau\bar{\phi}_1 V_o/R_o) \tag{36}$$

$$W_\theta = R_a\left\{\cos(\alpha_f-\theta_f)-\cos(\alpha_f-\theta_f+\tau\bar{\psi}_1 V_a/R_a)\right\}/\bar{\psi}_1 \tag{37}$$

Here $\alpha_f$ is to be substituted from Eq. (10) and $\bar{\phi}_1$, $\bar{\psi}_1$ are to be taken for the appropriate game from Eq. (28, 30) in conjunction with Eq. (21, 22); if one of the ships stands on, the limit $\phi_1 \to 0$ or $\psi_1 \to 0$ can be taken analytically. The barrier is now a curved surface in the reduced three-dimensional $(x, y, \theta)$

state space. For its graphical representation on plane paper (or on a radar screen for that matter) it seems convenient to compute cross-sections at constant $\theta$. Moreover, it is numerically expedient to use nondimensional coordinates and retrograde time:

$$\bar{x} = x/R_o, \quad \bar{y} = y/R_o, \quad \bar{\tau} = \tau V_o/R_o \tag{38}$$

as well as nondimensional game and terminal parameters:

$$\zeta = R_a/R_o, \quad \eta = V_a/V_o, \quad \bar{r}_f = r_f/R_o \tag{39}$$

After the necessary substitutions the barrier cross-sections generated by Eq. (32-34) with $\bar{\tau}$ as the parameter are found to be

$$\bar{x} = \varepsilon\bar{r}_f\left\{\eta\sin(\theta+\mu\bar{\psi}_1\bar{\tau})-\sin(\bar{\phi}_1\bar{\tau})\right\}/\omega + \sin(\bar{\phi}_1\bar{\tau})/\bar{\phi}_1 + \zeta\left\{\sin\theta-\sin(\theta+\mu\bar{\psi}_1\bar{\tau})\right\}/\bar{\psi}_1 \tag{40}$$

$$\bar{y} = \varepsilon\bar{r}_f\left\{\cos(\bar{\phi}_1\bar{\tau})-\eta\cos(\theta+\mu\bar{\psi}_1\bar{\tau})\right\}/\omega + \left\{1-\cos(\bar{\phi}_1\bar{\tau})\right\}/\bar{\phi}_1 - \zeta\left\{\cos\theta-\cos(\theta+\mu\bar{\psi}_1\bar{\tau})\right\}/\bar{\psi}_1 \tag{41}$$

where we have used two redundant symbols

$$\mu = \eta/\zeta, \quad \omega = \left\{1+\eta^2-2\eta\cos\left[\theta+(\mu\bar{\psi}_1-\bar{\phi}_1)\bar{\tau}\right]\right\}^{1/2} \tag{42}$$

in order to abbreviate the expressions. Note that valid barrier cross-sections are generated only in the time interval $0 < \bar{\tau} < \bar{\tau}_m$, where $\bar{\tau}_m$ is the smallest value of $\bar{\tau}$ at which the argument of the signum function in Eq. (17) or (19) vanishes after having been nonzero.

<u>Secondary Paths</u>

In general we have exactly two optimal paths leading to any value of $\theta_f$ in the interval $[0, 2\pi]$, i.e. one to lower BUP ($\varepsilon = -1$) and the other to upper BUP ($\varepsilon = +1$). However, there exist singular values of $\theta_f$ from which emanate (retrogressively) whole families of paths which we may call secondary paths for the sake of distinction from the primary paths given above.

We first consider the singular case mentioned following Eq. (10):

$$\eta = 1, \quad \theta_f = 0, \quad 0 \leqslant \alpha_f \leqslant 2\pi \tag{43}$$

Path equations (32-34) still apply, but for obvious reasons barrier cross-sections ($\theta$ = const) should be calculated using $\alpha_f$ rather than $\theta_f$ or $\tau$ as the parameter.

Next we recall having anticipated singular turning controls when the relative bearing of target at CPA becomes zero or $\pi$. Evidently, this can happen only to the slower ship. Accordingly, we distinguish the cases $\eta \gtrless 1$. If $\eta > 1$, Eq. (28) predicts singular behavior of ship O at

$$\sin\alpha_f = 0, \quad \theta_f = \pm\theta_s \quad \text{with} \quad \theta_s = \arccos(1/\eta) \tag{44}$$

Closer scrutiny reveals that there are no optimal paths (but dispersal surfaces) leading to the singular points $\theta_f = \gamma\varepsilon\theta_s$, whereas there are two families of optimal paths (so-called *tributaries*) leading to the singular points $\theta_f = -\gamma\varepsilon\theta_s$ via so-called *universal* surfaces (US) by means of two-step maneuvers ($\bar{\phi}_1 = \pm 1$ along tributaries and zero along US). The barrier cross-sections ($\theta$ = const) generated by $\phi_1$-tributaries are found to be

$$\bar{x} = (\bar{\tau}_u - \gamma\bar{r}_f)\cos(\bar{\phi}_1\bar{\tau}_t) + \sin(\bar{\phi}_1\bar{\tau}_t)/\bar{\phi}_1 + \zeta\left\{\sin\theta - \sin(\bar{\phi}_1\bar{\tau}_t - \gamma\varepsilon\theta_s)\right\}/\bar{\psi}_1 \tag{45}$$

$$\bar{y} = (\bar{\tau}_u - \gamma\bar{r}_f)\sin(\bar{\phi}_1\bar{\tau}_t) + \left\{1 - \cos(\bar{\phi}_1\bar{\tau}_t)\right\}/\bar{\phi}_1 - \zeta\left\{\cos\theta - \cos(\bar{\phi}_1\bar{\tau}_t - \gamma\varepsilon\theta_s)\right\}/\bar{\psi}_1 \tag{46}$$

with the retrograde time $\bar{\tau}$ divided between tributary and US as

$$\bar{\tau}_t = (\theta + \gamma\varepsilon\theta_s + \bar{\psi}_1\bar{\tau}\eta/\zeta)/\bar{\phi}_1, \qquad \bar{\tau}_u = \bar{\tau} - \bar{\tau}_t \tag{47}$$

both of which must be positive. Of course, $\bar{\psi}_1$ depends on the game and follows from Eq. (30).

Similarly if $\eta < 1$, Eq. (30) predicts singular behavior of A at

$$\sin(\alpha_f - \theta_f) = 0, \qquad \theta_f = \pm\theta_s \quad \text{with} \quad \theta_s = \arccos\eta \tag{48}$$

Now there are dispersal surfaces leading to $\theta_f = -\delta\varepsilon\theta_s$ and tributaries via US leading to $\theta_f = \delta\varepsilon\theta_s$. The barrier cross-sections generated by $\psi_1$-tributaries are

$$\bar{x} = (\delta\bar{r}_f - \eta\bar{\tau}_u)\cos(\delta\varepsilon\theta_s + \bar{\phi}_1\bar{\tau}) + \sin(\bar{\phi}_1\bar{\tau})/\bar{\phi}_1 + \zeta\left\{\sin\theta - \sin(\bar{\phi}_1\bar{\tau} + \delta\varepsilon\theta_s)\right\}/\bar{\psi}_1 \tag{49}$$

$$\bar{y} = (\delta\bar{r}_f - \eta\bar{\tau}_u)\sin(\delta\varepsilon\theta_s + \bar{\phi}_1\bar{\tau}) + \left\{1 - \cos(\bar{\phi}_1\bar{\tau})\right\}/\bar{\phi}_1 - \zeta\left\{\cos\theta - \cos(\bar{\phi}_1\bar{\tau} + \delta\varepsilon\theta_s)\right\}/\bar{\psi}_1 \tag{50}$$

$$\text{with} \quad \bar{\tau}_t = \zeta(\bar{\phi}_1\bar{\tau} - \theta + \delta\varepsilon\theta_s)/(\eta\bar{\psi}_1), \qquad \bar{\tau}_u = \bar{\tau} - \bar{\tau}_t \tag{51}$$

Here $\bar{\psi}_1 = \pm 1$ and $\bar{\phi}_1$ depends on the game according to Eq. (28). This essentially completes the analytical solution of our game.

## NUMERICAL EXAMPLES

### Choice of Parameters

Lack of space prohibits the presentation of comprehensive computations with a systematic variation of all relevant parameters. The following numerical examples have been chosen to illustrate just a few salient features of the solution. They all deal with encounters between two ships one of which is twice as fast as the other but also has twice as large a minimum turning radius. The results are presented in a graphical form similar to Fig. 2 showing calculated barrier cross-sections for selected values of relative course angle $\theta$, but with the miss distance $r_f$ varied in steps from 1/3 to twice the minimum turning radius of the faster ship. Nondimensional scales have been used so that the graphs apply to any combination of absolute numbers within the above constraints. For convenience of interpretation the faster ship may be regarded as a container carrier ($L$ = 300 m, $V$ = 30 kn, $R$ = 900 m) and the slower as a tanker ($L$ = 300 m, $V$ = 15 kn, $R$ = 450 m) with the miss distance $r_f$ varied from 300 m to 1800 m, i.e. from one ship length (necessary to avoid physical collision) up to about a mile (considered appropriate for a safe pass). The games treated are $0_E A_N$ (own ship evades, target stands on), $O_E A_E$ (own ship and target evade in collaboration), $O_E A_P$ (own ship evades in face of pursuit by target) and $O_N A_E$ (own ship stands on, target evades) as these are considered typical problems of collision avoidance.

## General Features

Fig. 3 shows barrier cross-sections at the extreme values $\theta = 0^o$ and $180^o$ for the game $O_E A_N$ (which is probably the most common task in collision avoidance) with the faster ship evading. As one might expect intuitively, the collision zone extends forward of abeam, is smallest for overtaking ($\theta = 0^o$) and largest for head-on meeting ($\theta = 180^o$). The barrier cross-sections are nearly semi-circular at large ranges but become elongated at short ranges, i.e. maneuvering capability is of increasing importance for evasion at short ranges. Here the $x$-axis is a dispersal curve separating areas of different optimal strategy. If target A is to port, evader O must turn right and vice versa.

Before passing on to the next figures a few general comments are in order. First, the critical (or optimal, depending on interpretation) maneuvers are marked in the figures by the self-explanatory symbols L, N and R. For instance, the mark $O_R A_N$ denotes that while A stands on, O excutes a hard right turn beginning at the instant A hits the barrier and terminating at CPA[6]. The marks LN and RN denote two-step maneuvers, i.e. the ship so marked executes a hard left or right turn respectively beginning at the barrier and terminating at the US, at which time it switches to straight course and holds it at least upto CPA. Second, note that although the barrier consists entirely of optimal paths, its cross-sections with $\theta$ = const are, in general, not optimal paths but only starting points of optimal paths which leave the plane $\theta$ = const on their way to a point $\theta_f$ on the BUP. Moreover, barriers are not static, but change with time when either ship maneuvers during an encounter. Finally, note that barrier cross-sections need be calculated only for $0 \leqslant \theta \leqslant \pi$. By virtue of symmetry, cross-sections for negative $\theta$ are obtained by taking mirror images about the $x$-axis and interchanging L and R.

## Advantage of Speed

Fig. 4 has been chosen mainly to illustrate the complicated strategies required of the slower ship (whether evading or pursuing) as compared to the relatively simple strategies of the faster ship in a two-ship encounter. Consider first Fig. 4 (top) showing barrier cross-sections at $\theta = 90^o$ with the faster ship evading and the slower standing on. In conformity with intuition we still have only two areas of different strategies, although the collision zone is now tilted and the dispersal curve (dashed line D) is neither a straight line nor pointing ahead. Now look at Fig. 4 (bottom) showing essentially the same encounter but now with the slower ship evading and the faster standing on. The scales have been so chosen that the two figures are directly comparable. Not only is the collision zone larger when the slower ship is burdened (despite its smaller turning radius), but also it must discriminate five different regions (including two requiring two-step maneuvers RN) separated by dashed lines $D_1$-$D_4$.

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[6]If from CPA both ships hold their current courses, the range cannot decrease again. In fact it will always increase except in the special case $V_a = V_o$, $\theta_f = 0$. Original courses may be resumed after awaiting a safe separation. The required amount of turn (i.e. change of course angle upto CPA or US) is also indicated in degrees for each ship. Lines of equal turn (short dashed lines in the figures) happen to be straight lines orthogonal to the barrier cross-sections (solid lines).

Here $D_2$ and $D_4$ are true dispersal curves formed by intersecting optimal paths. $D_1$, $D_3$ only mark the transition between barrier segments formed by primary and secondary paths. $D_2$ separates regions of similar strategy leading to upper and lower BUP. Of special interest are also singular points $E_1$-$E_3$ with three different but equivalent optimal strategies. The inner barriers have fewer different segments and can be obtained by a local orthogonal shift of the outer barriers by virtue of Eq. (35-36).

## Collaboration versus Conflict

Fig. 5 is designed to display the difference between collaborative and conflicting maneuvers. Barrier cross-sections at $\theta = 90^\circ$ are shown with the faster ship O always evading, and the slower ship A either evading (top) or pursuing (bottom). As expected, compared to the case of A standing on (Fig. 4 top) the collision zone for collaboration is smaller, while that for pursuit is larger. That the differences are small means that the slower ship is rather ineffective while the faster ship commands the game. Note also that there are two genuine dispersal curves $D_1$, $D_2$ in the game $O_E A_E$. But in game $O_E A_P$ only $D_3$ is a true dispersal curve, $D_1$ marks the transition from primary paths to secondary paths and $D_2$ from left-turn tributaries to right-turn tributaries. Hence $D_2$ is the intersection of a US with the plane $\theta$ = const and represents points at which the optimal strategy of A is simply to stand on up to CPA. Finally, we note that in games of pursuit the collision zone is not necessarily closed unless the evader is faster, the exact criterion for closure being known only for the special case $r_f = 0$, see Cockayne (1967).

## Slower Ship's Dilemma

The previous figures all show the collison zone from the evader's point of view. However, for an objective determination of the "last minute" according to the Rules of the Road the stand-on vessel must consult the barriers for the game $O_N A_E$. This is shown in Fig. 6 with O as the faster ship. Comparison with Fig. 4 (top) shows that the faster stand-on ship can afford to let the slower burdened ship approach upto the "last minute" and still avoid collision by own action alone. If we reverse the situation we see that the slower stand-on ship cannot afford the same since the collision zone of game $O_E A_N$ is then larger than for $O_N A_E$. Hence if the faster burdened ship does not do her the favor of collaborating, the slower stand-on ship may be forced either to break the "last minute" rule or to risk collision. This might be called the slower ship's dilemma. The Rules of the Road were apparently designed for similar ships for which the problem does not arise.

## CONCLUSIONS

It has been shown how the powerful analytical theory of differential games can be applied to determine mathematically optimal evasive maneuvers for two-ship encounters in the open sea, using a simple but realistic kinematic model of maneuvering. Various generalizations of this model and of side conditions (e.g. more than two ships, restricted waters etc.) are conceivable and much work remains to be done before the problem of maritime collision avoidance can be considered completely solved.

Perhaps the most obvious criticism of our model is that it ignores two typical dynamical effects of a real ship maneuver: the time lag $\Delta t$ from a rudder command upto the onset of an actual turn, and the loss of speed $\Delta V$ due to increased resistance in a turn. Pending further calculations using a more elaborate dynamic model we recommend the following approximate way of accounting for these effects: Just pretend that the target on the radar screen is at a distance $V\Delta t$ in advance of its true position and use barriers based on a reduced speed $(V-\Delta V)$ instead of the approach speed $V$.

Comparing our results with the traditional work in the field of anti-collision maneuvers such as that of Calvert (1960) and Jones (1971) we find that the problem is much deeper than previously thought. Optimal evasive maneuvers depend not only on relative bearing of target, but also on its range, course angle, speed ratio, maneuvering capability and objectives. This naturally raises the question of how all this information is to be acquired in practice onboard. Lacking mutual communication, even the accurate determination of the other ship's speed and course by radar is a formidable problem in face of noise. Fortunately, this problem is being tackled by various people, see e.g. Strobel and Richter (1974/75).

The theory of differential games is full of surprises. We close with one final oberservation in apparent contradiction with intuition. It is generally believed that the so-called perfect collision course (absolutely steady initial bearing) is the most crucial condition for collision avoidance, cf. Kenan (1972). Simple inspection of calculated barriers shows that this is not necessarily so, specially for the slower ship, see Fig. 4-6. For almost any $\theta$ (expect zero and $\pi$) the longest critical range occurs along a bearing other than that corresponding to a perfect collision.

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## LIST OF SYMBOLS

### Abbreviations

| | |
|---|---|
| A | Another ship |
| BUP | Boundary of useable part |
| CPA | Closespt point of approach |
| E | Evasion (used as subscript to identify game) |
| KE | Kinematic equations |
| L | Hard left turn upto CPA (used as subscript to identify maneuver) |
| LN | Two-step maneuver: first L upto US, then N upto CPA ( " ) |
| ME | Main Equation |
| N | No maneuver, i.e. straight course at constant speed |
| O | Own ship (also origin of relative coordinates) |
| P | Pursuit (used as subscript to identigy game) |
| R | Hard right turn upto CPA (used as subscript to identify maneuver) |
| RN | Two-step maneuver: first R upto US, then N upto CPA ( " ) |
| UP | Useable part of terminal surface |
| US | Universal surface |

### Variables

| | |
|---|---|
| $k$ | Coefficient of ship resistance |
| $L$ | Length of ship |
| $m$ | Mass of ship (including hydrodynamic mass) |
| $R$ | Minimum turning radius of ship |
| $r$ | Range between A and O; polar coordinate of A |
| $\bar{r}$ | Nondimensional range, see Eq. (39) |
| $r_m$ | Minimum required range |
| $T$ | Maximum thrust of ship propeller |
| $t$ | Physical time |
| $x$ | Distance of A from O measured along velocity of O, see Fig. 1 |
| $\bar{x}$ | Nondimensional coordinate, see Eq. (38) |
| $y$ | Distance of A from O normal to velocity of O, see Fig. 1 |
| $\bar{y}$ | Nondimensional coordinate, see Eq. (38) |
| $V$ | Absolute speed of ship |
| $V_r$ | Speed of A relative to O |
| $W$ | Value function |
| $\alpha$ | Relative bearing of A taken clockwise from velocity of O |
| $\gamma$ | Game parameter (+1 if O evades and -1 if O pursues) |
| $\delta$ | Game parameter (+1 if A evades and -1 if A pursues) |
| $\varepsilon$ | Path parameter (+1 if leading to upper BUP and -1 to lower BUP) |
| $\zeta$ | Ratio of minimum turning radii, see Eq. (39) |
| $\eta$ | Ratio of ship speeds, see Eq. (39) |
| $\theta$ | Relative course angle of A taken clockwise from velocity of O |
| $\theta_s$ | Singular value of terminal course angle, see Eq. (44, 48) |
| $\tau$ | Retrograde time, i.e. time to reach CPA |
| $\bar{\tau}$ | Nondimensional retrograde time, see Eq.(38) |
| $\tau_t$ | Time travelled along tributary to reach US |
| $\tau_u$ | Time travelled along US to reach CPA |
| $\phi$, $\psi$ | Control variable of ship O, A |
| $\phi_1$, $\psi_1$ | Normalized rate of turn of O, A (positive to right) |
| $\phi_2$, $\psi_2$ | Normalized thrust of O, A (positive for acceleration) |
| $\bar{\phi}$, $\bar{\psi}$ | Optimal value of control $\phi$, $\psi$ |

### Subscripts

| | | Apply to |
|---|---|---|
| a | Of ship A | $k,L,m,R,T,V$ |
| f | Final (or terminal) value, i.e. value at CPA | $r,\alpha,\theta,V$ |
| o | Of ship O | $k,L,m,R,T,V$ |

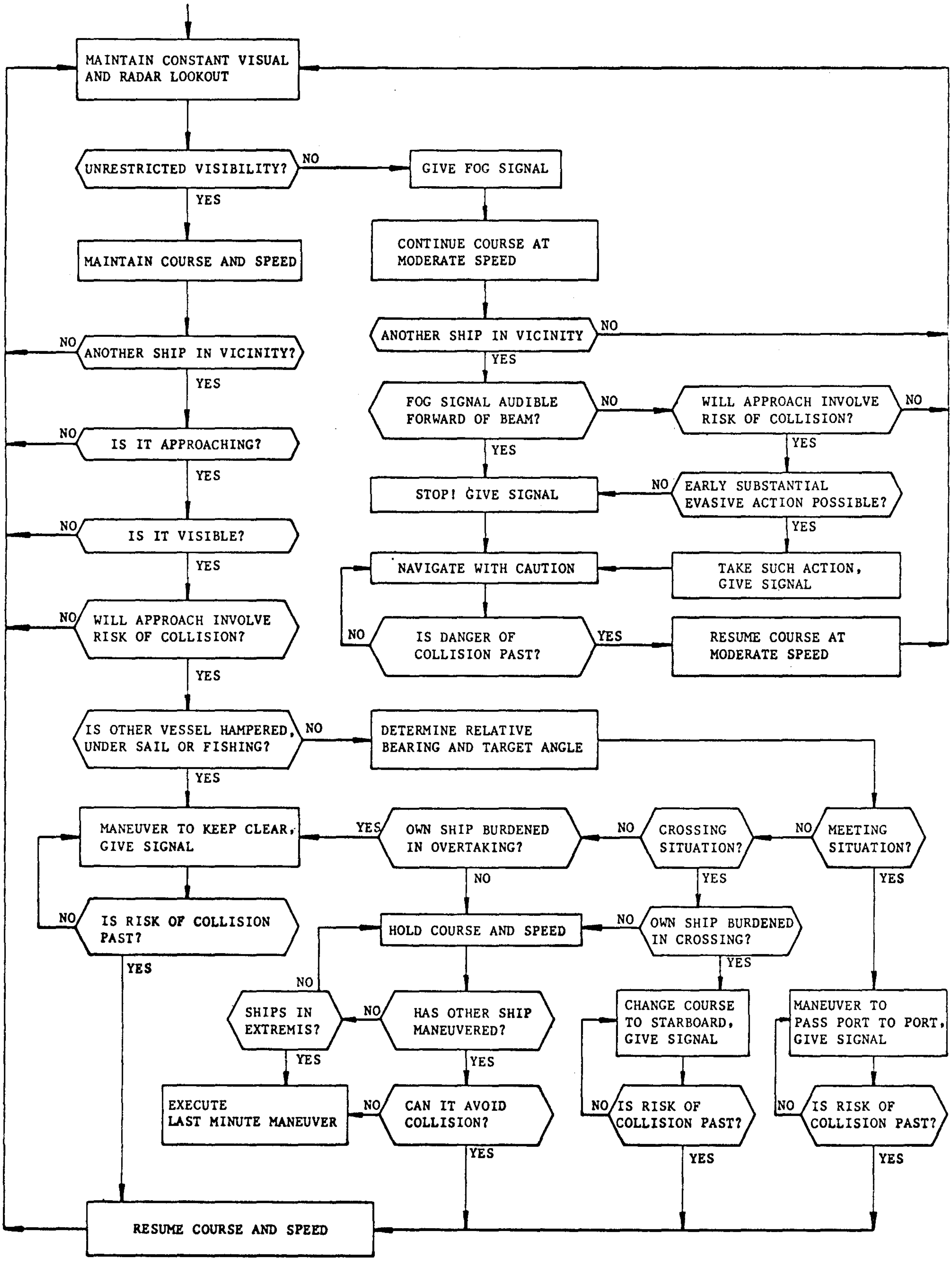


Fig. 1 - Logic Flow Diagram for Two-Ship Encounter in Open Sea following International Rules of the Nautical Road

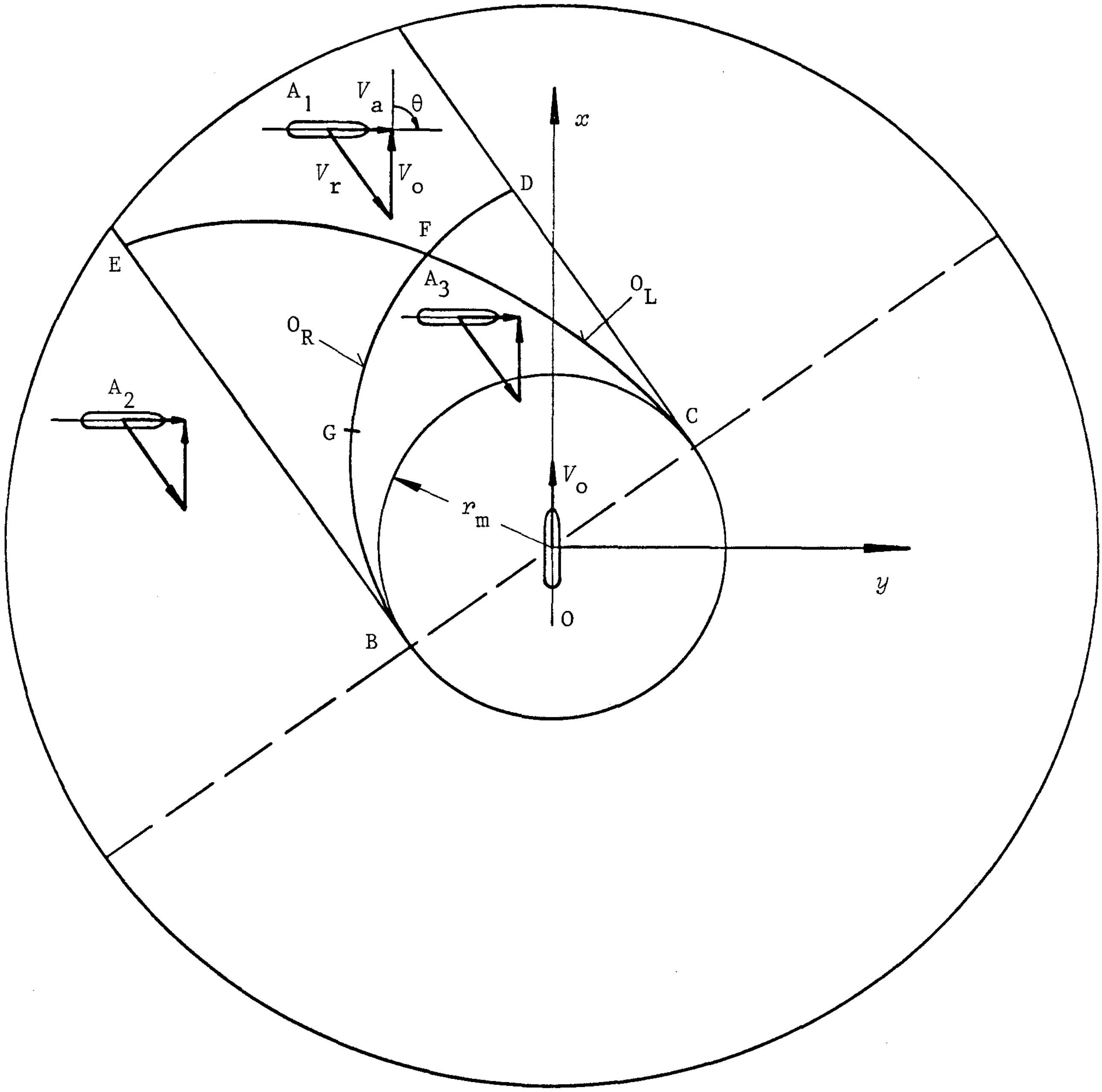


Fig. 2 - Sample Illustration of Barriers and Critical Maneuvers for Collision Avoidance in a Two-Ship Encounter

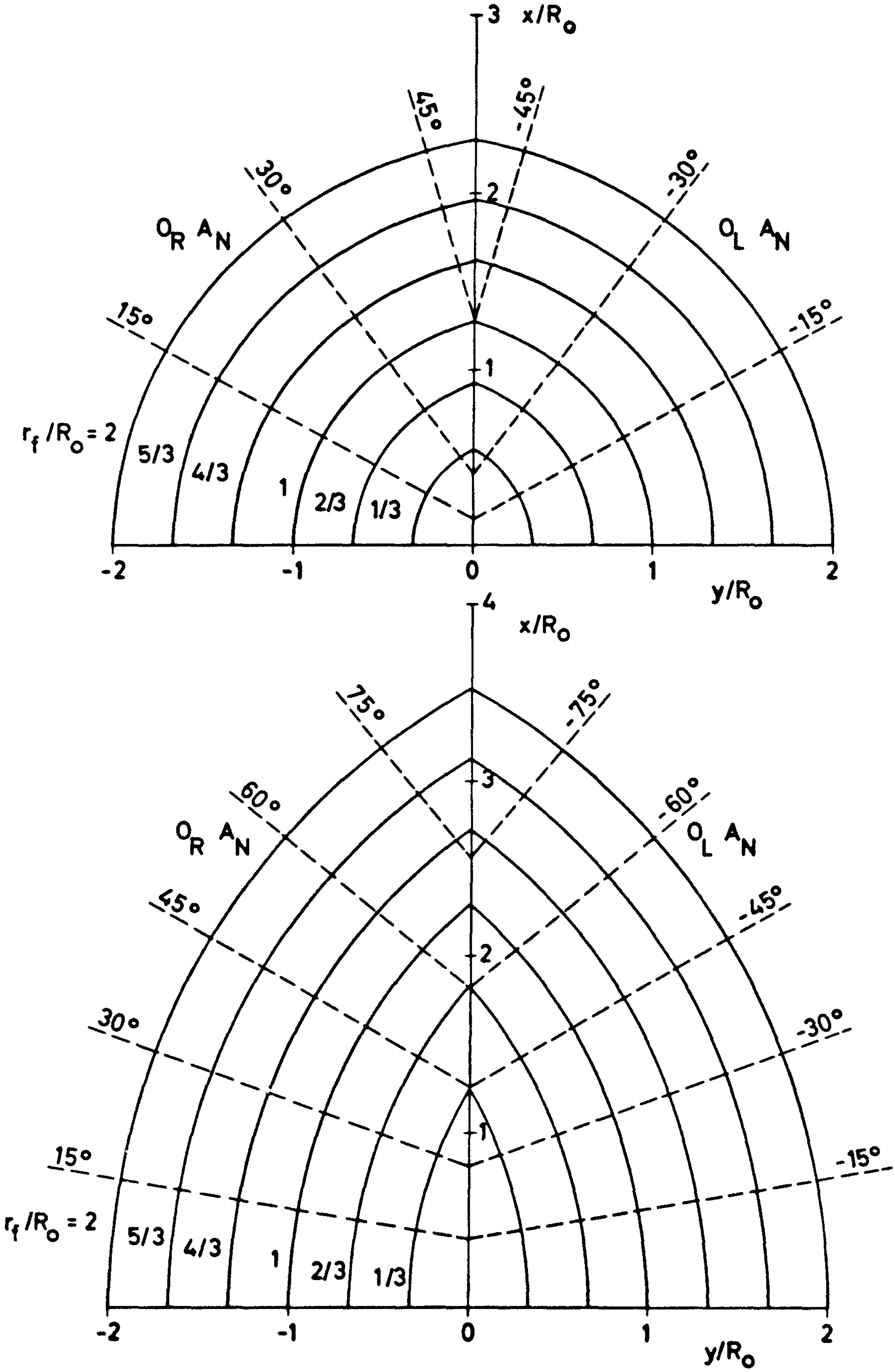


Fig. 3 - Barrier Cross-Sections at $\theta = 0^{o}$ (top) and $\theta = 180^{o}$ (bottom) for Game $O_E\,A_N$ with $V_a = V_o/2$ and $R_a = R_o/2$

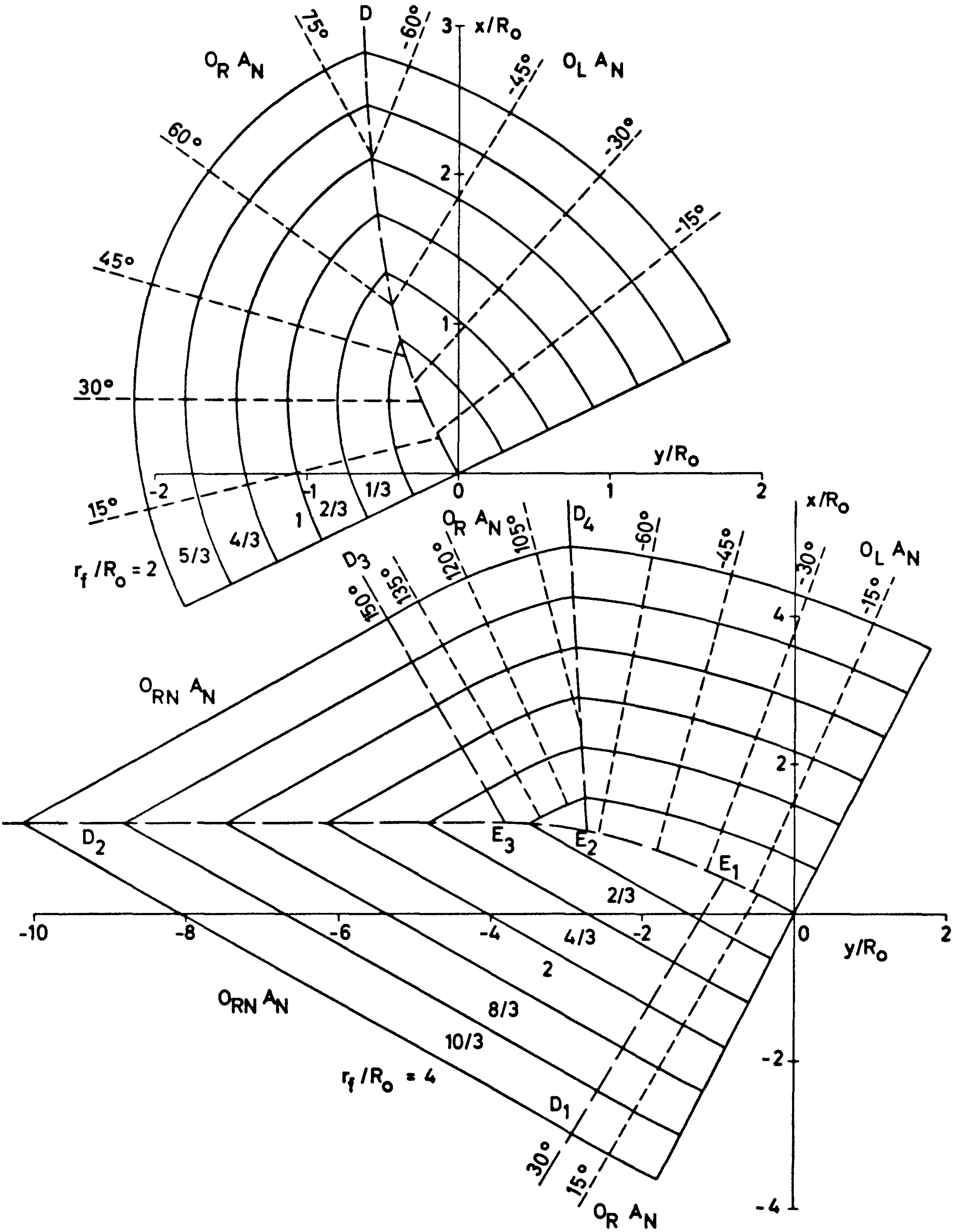


Fig. 4 - Barrier Cross-Sections at $\theta = 90^{\circ}$ for Game $O_E A_N$ with $V_a = V_o/2$, $R_a = R_o/2$ (top) and $V_a = 2V_o$, $R_a = 2R_o$ (bottom)

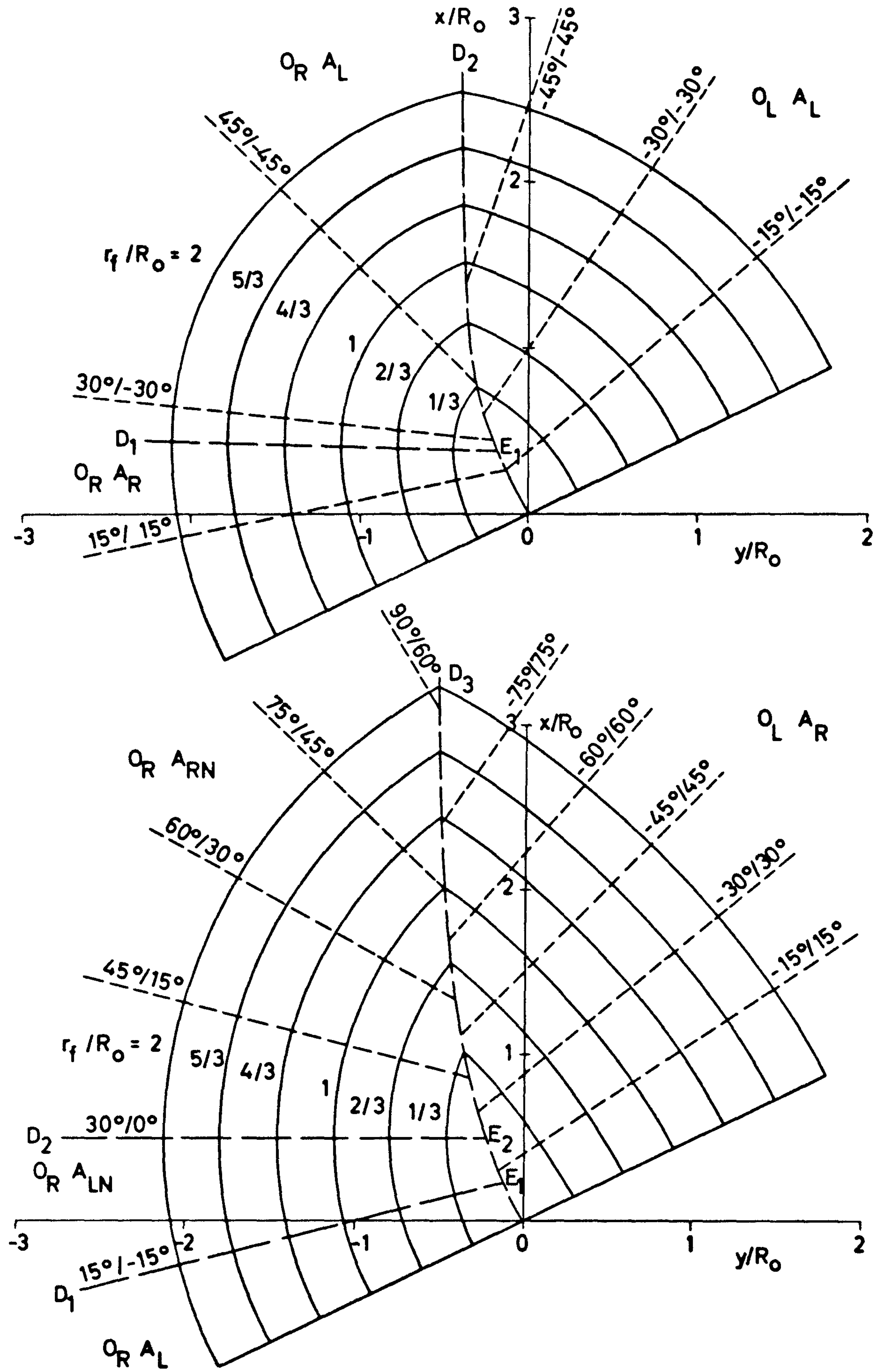


Fig. 5 - Barrier Cross-Sections at $\theta = 90^{\circ}$ for Game $O_E A_E$ (top) and $O_E A_P$ (bottom), each time with $V_a = V_o/2$, $R_a = R_o/2$

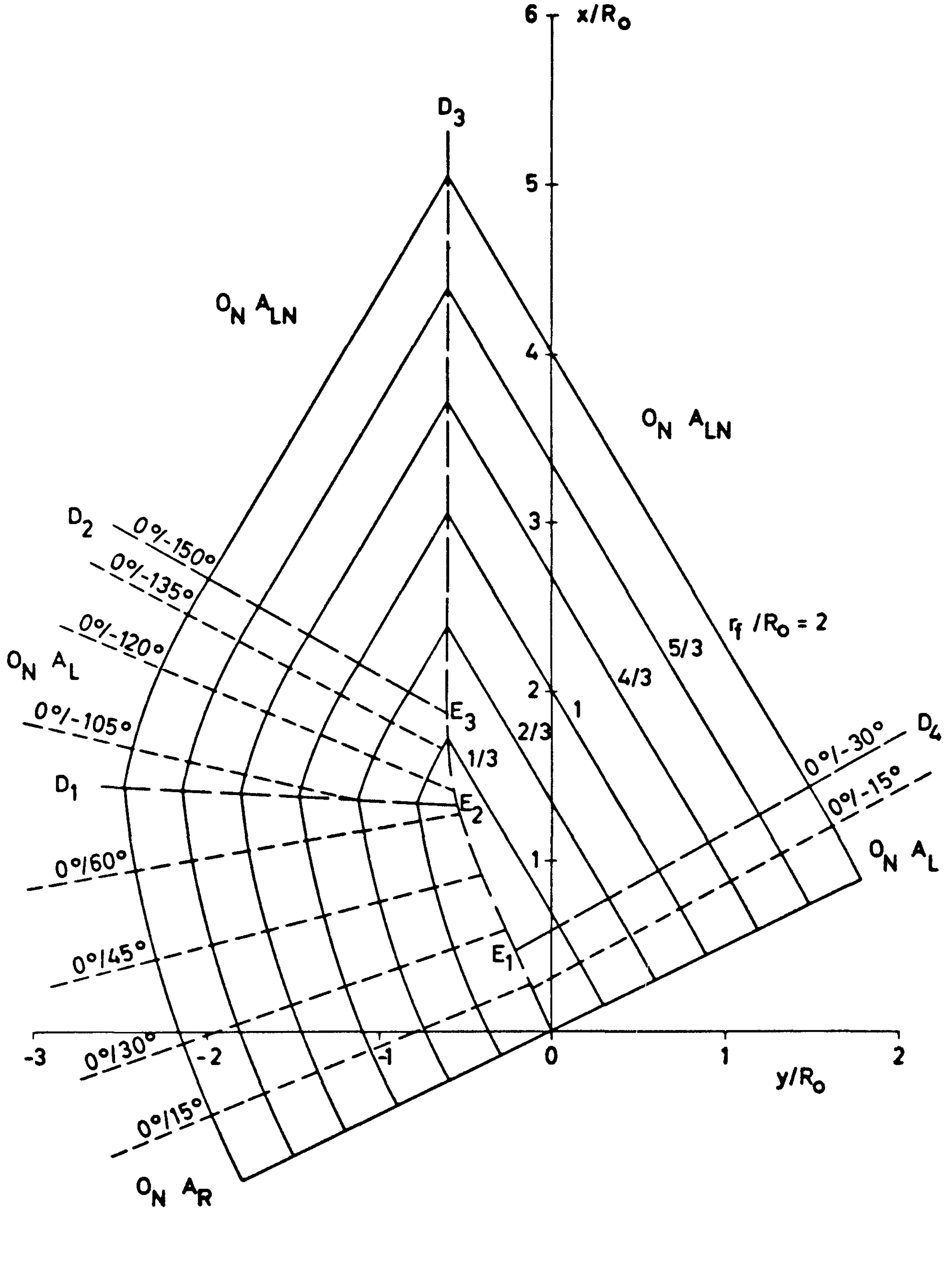


Fig. 6 - Barrier Cross-Sections at $\theta = 90^{o}$ for Game $O_N A_E$ with $V_a = V_o/2$ and $R_a = R_o/2$