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MARKET EXCESS DEMAND FUNCTIONS

BY HUGO SONNENSCHEIN[1]

The purpose of this paper is to investigate the structure of the class of market excess demand functions which can be generated by aggregating individual utility maximizing behavior. Among the results are: (i) in a region of the relative price domain an arbitrary polynomial function can be generated as an excess demand function for a particular commodity, and (ii) for any p in the relative price domain, a given configuration of excess demands and rates of change in excess demand can be generated at p if and only if it is consistent with Walras' Law.

1. INTRODUCTION

THE CONCEPT of a market excess demand function occupies a central role in the explanation of value furnished by all models of the competitive mechanism. It is elementary that these functions must be homogeneous of degree zero in all prices and satisfy Walras' Law. Under a standard set of assumptions they will be continuous (see, for example, [**2**]). Walras' Law together with continuity guarantee that excess demand functions must have a zero, or in more familiar terminology, that equilibrium prices must exist (see, for example, [**2**]). Clearly this zero may be unique (consider the case of a single individual with smooth indifference surfaces). Beyond these facts, and a small number of results from the "stability" literature (see [**4**] and [**9**]), very little is known about the structure of excess demand functions. A sampling of the results presented here and their relationship to the existing literature follows. We assume in this discussion that there are n commodities and that the price of the nth commodity is fixed at unity. Price vectors are points in the positive orthant of an $n - 1$ dimensional Euclidean space.

(1) Can an arbitrary continuous function, defined on a compact subset C of the interior of a positive orthant, be an excess demand function for some commodity in a general equilibrium economy? To my knowledge this question has received little attention, yet specific functional forms for aggregate demand relationships are the starting point for a large body of empirical work. Despite the generally accepted importance of basing statistical studies on consistent theoretical grounds, it is interesting to note that there is no literature exploring whether, for example, a linear aggregate excess demand function is theoretically possible.

[1] The research on which this paper is based was done at the Pennsylvania State University. I appreciate the comments of the people who discussed the results with me at a large number of seminars. However, I am especially indebted to two colleagues: C. Moler, University of Michigan, who got me started on the proof of the lemma in the Appendix; and an anonymous referee, who in addition to suggesting the title, streamlined two theorems and convinced me to dismiss a sideshow.

In Theorem 2 we prove that every polynomial on C is an excess demand function for a specified commodity in some n commodity economy. As a consequence of a classic mathematical result on approximation, this theorem has as a corollary the fact that every continuous real-valued function is approximately an excess demand function.

(2) Conditions are known under which competitive equilibrium is unique (see, for example, [**2**]); however, there are virtually no other results which yield information on what the sets of equilibrium prices can look like.[2] In Theorem 4 we establish that given any compact subset C of the interior of a positive orthant, and almost any finite set P of prices in C, there exists an economy whose set of competitive equilibrium prices in C is P. To the extent that an arbitrary bounded set may be approximated by a finite set of points, this may be stated by saying that every bounded set is approximately a set of equilibrium prices for some economy.

(3) Samuelson in his *Foundations* [**6**] characterized the comparative statics properties of the class of demand functions which arise from individual utility maximizing behavior.[3] If a demand function is derived from utility maximizing behavior, then its matrix of substitution terms must be symmetric and negative definite at each point in the domain of the demand function. Furthermore, if a function of prices and income yields a substitution matrix which is symmetric and negative definite, then it is possible to derive that function as a demand function; that is, the function can be generated from an individual's utility maximizing behavior. To interpret this theorem in a slightly different way: the only relationship among prices, quantities demanded, and rates of change in quantities demanded, that is implied by the utility hypothesis, is that the substitution matrix be symmetric and negative definite when evaluated at each point in the domain of the demand function.

In Theorem 5 we prove a local analogue of the above characterization for the case of aggregate excess demand functions. It states that Walras' Law characterizes the comparative statics properties of the class of excess demand functions which arise from aggregating individual utility maximizing behavior. Informally, we summarize this proposition by saying that Walras' Law is the only local comparative statics theorem of competitive equilibrium analysis. The proof is established by showing that given any price vector p, any $n - 1$ numbers $N(1), N(2), \ldots, N(n-1)$, and any $(n-1)^2$ numbers $a(i,j)$, there exists a collection of consumers (each consumer is a utility function and an initial holding) such that:

(4a) the excess demand for the ith commodity at prices p is $N(i)$, $i = 1, 2, \ldots, n-1$ (the excess demand for the nth commodity is determined by Walras' Law), and

(4b) the partial derivative of the ith excess demand function with respect to the jth price, evaluated at p, is $a(i,j)$, $i, j = 1, 2, \ldots, n-1$ (the partial derivatives of the nth excess demand function are determined by Walras' Law).

[2] A paper by G. Debreu [**1**] is an exception.

[3] See also [**7**]. This result has been generalized by Hurwicz and Uzawa [**3**], and alternative characterizations are established in the revealed preference literature.

In order to illustrate the power of this result, we apply it to demonstrate the existence of a competitive equilibrium which is Hicksian stable but not dynamically stable, and this solves a problem which has been outstanding for over twenty-five years (see [**5** and **8**]).

2. PRELIMINARY NOTIONS

Let $\varDelta$ denote the unit $(n-1)$ simplex; i.e., $\varDelta = \{(x_1, x_2, \ldots, x_n) : x_i \geqslant 0$ for all i and $\Sigma x_i = 1$, where the sum is taken over all i, $1 \leqslant i \leqslant n\}$. $\hat{\varDelta}$ denotes the interior of $\varDelta$, and, for $0 < \delta < 1$, $\varDelta(\delta) = \{(x_1, x_2, \ldots, x_n) \in \varDelta : x_i \geqslant \delta,\ i = 1, 2, \ldots, n\}$. The set of $(n-1)$-tuples of positive real numbers is denoted by Π, and $\Pi(\delta)$, $0 < \delta < 1$, is that subset of Π whose points have all of their co-ordinates between δ and $1/\delta$. The set of vectors in R^n (Euclidean n-dimensional space) whose co-ordinates are all non-negative is denoted by Ω and is called the commodity space. A point in the commodity space is called a bundle. A utility function U is a continuous real-valued function defined on the commodity space satisfying these conditions:

(5) for all bundles x and y, if each co-ordinate of x is larger than the corresponding co-ordinate of y, then $U(x) > U(y)$, and

(6) for all bundles x, $\{x' : U(x') \geqslant U(x)\}$ is convex.

The ith consumer is represented as a utility function U^i and a bundle $\omega^i = (\omega^i_1, \omega^i_2, \ldots, \omega^i_n)$. The bundle ω^i is called the bundle of initial holdings of individual i. Demand functions are defined in the usual way on both $\hat{\varDelta}$ and Π, and the demand function of the ith individual is denoted by h^i. His excess demand function E^i is defined by $E^i = h^i - \omega^i$.

We will consider the case where $E^i(p)$ is a singleton for each $p \in \varDelta$ (respectively $p \in \Pi$). It is easily established that $p \cdot E^i(p) = 0$ (respectively $(p, 1) \cdot E^i(p) = 0$) for all $p \in \hat{\varDelta}$ (respectively $p \in \Pi$). If (U^i, ω^i) is a consumer for each $i \in J$, let $E^J = \Sigma E^i$, where the sum is taken over $i \in J$. $(E^J_1, E^J_2, \ldots, E^J_n) = E^J$ is called the aggregate excess demand function. It maps $\hat{\varDelta}$ (respectively Π) into R^n. One can prove that E^J satisfies Walras' Law; i.e., for each $p \in \hat{\varDelta}$ (respectively $p \in \Pi$) we have $p \cdot E^J(p) = 0$ (respectively $(p, 1) \cdot E^J(p) = 0$).

3. A READER'S GUIDE

The analysis proceeds as follows. Let $0 < \delta < 1$. In Theorem 1 we prove that any real-valued function of one variable, which has a continuous derivative for prices between δ and $1/\delta$, is an excess demand function on $[\delta, 1/\delta]$ for the first commodity in some two person two commodity economy (the excess demand for the second commodity is determined by Walras' Law). By defining an appropriate composite good, this fact is used to establish Lemma 1: Any function of $n-1$ prices, which can be represented as a continuously differentiable function of a linear function of the $n-1$ prices on $\Pi(\delta)$, is the excess demand function on $\Pi(\delta)$ for the ith commodity in some two person economy. In the Appendix we establish a representation of polynomials in several variables as the sum of powers of polynomials that are linear in the variables. This combines with Lemma 1 to yield

Theorem 2, which is the representation of an arbitrary polynomial on $\Pi(\delta)$ as an excess demand function on $\Pi(\delta)$. In Theorem 3 we analyze the relationship between the excess demand functions for the ith and jth commodities implied by the techniques of construction employed through Theorem 2. It is shown that the method is capable of generating precisely those polynomial excess demand functions which satisfy the symmetry condition that the change in excess demand for the ith commodity $(1 \leqslant i \leqslant n-1)$ with respect to a change in the jth price $(1 \leqslant j \leqslant n-1)$ is equal to the change in excess demand for the jth commodity with respect to a change in the ith price (the changes in excess demand for the nth commodity are determined by Walras' Law). This result enables us to prove Theorem 4: Let $p^1, p^2, \ldots, p^m$ be a finite set of points in $\Pi(\delta)$, $\delta > 0$, that have no co-ordinates in common. There exists an economy which has an equilibrium at each p^j, $j = 1, 2, \ldots, m$, and none elsewhere in $\Pi(\delta)$. Finally, by alternating our choice of numeraire, and appealing after each alternation to a very weak form of Theorem 3, we obtain Theorem 5: Walras' Law is the only restriction on the relationship among a given price vector $\bar{p}$, excess demand at $\bar{p}$, and the rates of change in excess demand with respect to price changes evaluated at $\bar{p}$.

4. THE TWO COMMODITY CASE

THEOREM 1: *If f has a continuous derivative on $[\delta, 1/\delta]$ for some δ, $0 < \delta < 1$, then there exist two consumers (U^1, ω^1) and (U^2, ω^2) such that $f = E_1^J$ on $[\delta, 1/\delta]$, $J = \{1, 2\}$.*

PROOF: Write $f = \phi + \psi$ where $\phi(p_1) = ap_1 - b$, $a > 0$, $b > 0$, $2a/b < \delta$, and ψ is positive, strictly decreasing, and has a continuous derivative on $[\delta, 1/\delta]$. We now will construct two individuals whose excess demand functions on $[\delta, 1/\delta]$ are ψ and ϕ respectively. This is all that is necessary, since the excess demand function generated by the union of two individuals is the sum of their excess demand functions.

We turn our attention first to ψ. Let ω_1^1 be any positive number, and $\omega_2^1 - p_1\psi(p_1) > 1$ for all $p_1 \in [\delta, 1/\delta]$. This is done to guarantee that the demand for the second commodity by the first person is positive. Using the fact that ψ satisfies a uniform Lipschitz condition on $[\delta, 1/\delta]$, one can prove that there exists an $\varepsilon > 0$ and an $\eta > 0$ defined on $[\delta, 1/\delta]$ such that no point of the form $(\omega_1^1 + \psi(p_1), \omega_2^1 - p_1\psi(p_1)) \equiv (h_1(p_1), h_2(p_1))$ lies in $S(\bar{p}_1) = \{(x_1, x_2): (\bar{p}_1 - \eta)x_1 + x_2 \leqslant (\bar{p}_1 - \eta)h_1(\bar{p}_1) + h_2(\bar{p}_1)\}$ whenever $0 < (p_1 - \bar{p}_1) < \varepsilon$. Now define $\mu = \min\{\delta, \eta\}/2$. The utility function U^1 is obtained in the following manner:

Given a point $(h_1(\bar{p}_1), h_2(\bar{p}_1))$ on $\sigma = \{(h_1(p_1), h_2(p_1)): p_1 \in [\delta, 1/\delta]\}$, let the points defined in (7a), (7b), and (7c) be assigned the value $h_1(\bar{p}_1)$ under U^1:

(7a) $$A(\bar{p}_1) = \{(h_1(\bar{p}_1), h_2(\bar{p}_1))\},$$

(7b) $$B(\bar{p}_1) = \{(x_1, x_2): x_1(\bar{p}_1 - \mu) + x_2 = h_1(\bar{p}_1)(\bar{p}_1 - \mu) + h_2(\bar{p}_1),\ 0 \leqslant x_2 < h_2(\bar{p}_1)\},$$

(7c) $$C(\bar{p}_1) = \{(x_1, x_2): x_1 = h_1(\bar{p}_1) \text{ and } x_2 > h_2(\bar{p}_1)\}.$$

(The need for $\mu > 0$ in (7b) is to guarantee that the portion of the indifference curve below $h_2(\bar{p}_1)$ can be chosen so that $(h_1(\bar{p}_1), h_2(\bar{p}_1))$ is the unique choice at prices $\bar{p}_1$.)

It must be shown that U^1 is well defined where it has been defined. This will be established if the segments of the iso-U^1 contours below the path σ do not cross, and this will be the case if $\sigma \cap B(\bar{p}_1)$ is empty for each $\bar{p}_1 \in [\delta, 1/\delta]$. The latter fact follows from the mean value theorem and our choice of μ.

U^1 is continuous, satisfies (5) and (6), and is defined on a subset of the commodity space which is bounded above by the iso-U^1 contour corresponding to $U^1 = h_1(\delta)$ and below by the iso-U^1 contour corresponding to $U^1 = h_1(1/\delta)$. It can readily be extended continuously to the entire northeast quadrant in such a way so as to preserve the satisfaction of (5) and (6).

As a consequence of this construction, for all $\bar{p}_1 \in [\delta, 1/\delta]$, $(h_1(\bar{p}_1), h_2(\bar{p}_1))$ is the unique U^1 maximizing bundle corresponding to $\bar{p}_1$. We have thus found a consumer (U^1, ω^1) such that $\psi = E_1^1$ on $[\delta, 1/\delta]$. Next we must construct a consumer (U^2, ω^2) such that $\phi = E_1^2$ on $[\delta, 1/\delta]$. Let $\omega_1^2 > \sup\{\phi(x) : \delta \leqslant x \leqslant 1/\delta\}$, $\omega_2^2 = 1$, and note that $g(p_1) = -p_1\phi(p_1)$ is an increasing function of p_1 on $[\delta, 1/\delta]$. Let $p_2 = 1/p_1$. Observe g is a positive decreasing function of p_2 and has a continuous derivative. By the method employed to construct the first individual, there exists a second individual such that $g(p_2) = E_2^2(p_2)$ on $[\delta, 1/\delta]$. Thus $(-1/p_1)\phi(1/p_1) = g(1/p_1) = g(p_2) = E_2^2(p_2) = E_2^2(1/p_1) = (-1/p_1)E_1^2(1/p_1)$, and so $\phi(p_1) = E_1^2(p_1)$ on $[\delta, 1/\delta]$. Since $f = \psi + \phi$, this completes the proof of the theorem.

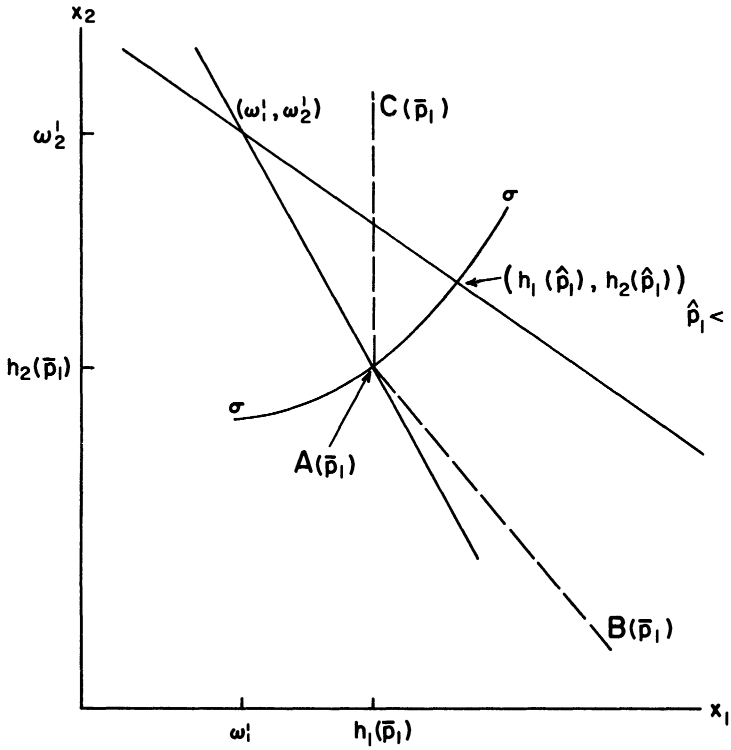

FIGURE 1.

The following two corollaries characterize, for the case of two commodities, the structure of excess demand functions on compact subsets of price space. The first corollary is exact, and a proof is obtained by using the fact that the excess demand functions generated in Theorem 1 satisfy Walras' Law. The second corollary states that an arbitrary continuous function and its companion, which is determined by Walras' Law, are approximately excess demand functions for the first and second commodities in a two commodity economy. This result is an immediate consequence of Theorem 1 and the Weierstrass Approximation Theorem.

COROLLARY 1: *If E_1 has a continuous derivative on $[\delta, 1/\delta]$ for some $0 < \delta < 1$, $(E_1, E_2) = E:[\delta, 1/\delta] \to R^2$, and $(p_1, 1)\cdot E(p_1) = 0$ for all $p_1 \in [\delta, 1/\delta]$, then there exist two consumers (U^1, ω^1) and (U^2, ω^2) such that $E = E^J$ on $[\delta, 1/\delta]$, $J = \{1, 2\}$.*

COROLLARY 2: *If $(E_1, E_2) = E:[\delta, 1/\delta] \to R^2$ is continuous for some $0 < \delta < 1$ and $(p_1, 1)\cdot E(p_1) = 0$ for all $p_1 \in [\delta, 1/\delta]$, then there exists two consumers (U^1, ω^1) and (U^2, ω^2) such that $|E_i(p_1) - E_i^J(p_1)| < \delta$ for all $p_1 \in [\delta, 1/\delta]$ and $i \in \{1, 2\} = J$.*

5. EXCESS DEMAND FOR ONE COMMODITY AS A FUNCTION OF MORE THAN ONE RELATIVE PRICE

Lemma 1 sets the stage for the main result. It proves that any function g of $n-1$ prices, which can be represented as a continuously differentiable function f of a linear function L of the $n-1$ prices on $\Pi(\delta)$, is the excess demand function on $\Pi(\delta)$ for the jth commodity in some two person economy. The technique of proof is to define a composite commodity, whose unit cost at $p \in \Pi(\delta)$ is $L(p)$, and to apply Theorem 1 to construct a two person economy whose demand for the composite commodity is $f(L(p))$.

LEMMA 1: *Let g be defined on $\Pi(\delta)$ and $0 < \delta < 1$. If there exist $n-2$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}$ between zero and one, and a continuously differentiable function f such that $g(p_1, p_2, \ldots, p_{n-1}) = f(\alpha_1 p_1 + \ldots + \alpha_{j-1}p_{j-1} + p_j + \alpha_{j+1}p_{j+1} + \ldots + \alpha_{n-1}p_{n-1})$ for all $(p_1 p_2, \ldots, p_{n-1}) \in \Pi(\delta)$, then there exist two individuals (U^1, ω^1) and (U^2, ω^2) such that $E_j^J(p_1, p_2, \ldots, p_{n-1}) = g(p_1, p_2, \ldots, p_{n-1})$ for all $(p_1, p_2, \ldots, p_{n-1}) \in \Pi(\delta)$, where $J = \{1, 2\}$.*

PROOF: Define a composite commodity x_c to be one unit of commodity j and α_k units of commodity k, $k = 1, 2, \ldots, j-1, j+1, \ldots, n-1$. Let $p_c = \alpha_1 p_1 + \ldots + \alpha_{j-1}p_{j-1} + p_j + \alpha_{j+1}p_{j+1} + \ldots + \alpha_{n-1}p_{n-1}$; p_c is the price of one unit of commodity x_c. By Theorem 1 there exist two consumers $(U^{1'}, \omega^{1'})$ and $(U^{2'}, \omega^{2'})$ such that $E_c^{J'}(p_c) = f(p_c)$ for all $p_c \in [\delta/(n-1), (n-1)/\delta] = [\delta', 1/\delta']$, where $J' = \{1', 2'\}$. $E_c^{J'}$ may be viewed as taking values in $S' = S \cap \Omega$, where S is the subspace of R^n which is spanned by $(\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, 1, \alpha_{j+1}, \ldots, \alpha_{n-1}, 0)$ and $(0, 0, \ldots, 0, 1)$. The utility functions $U_c^{1'}$ and $U_c^{2'}$ are defined only on S'. In order to establish the lemma they must be extended to functions U^1 and U^2 defined on Ω in such a manner so that $E_j^J(p_1, p_2, \ldots, p_{n-1}) = g(p_1, p_2, \ldots, p_{n-1})$ for all $(p_1, p_2, \ldots,$

$p_{n-1}) \in \Pi(\delta)$, where $J = \{1, 2\}$. This is achieved in the following manner. For any point $x = (x_1, x_2, \ldots, x_n) \in \Omega$, let $m = \min\{x_1/\alpha_1, x_2/\alpha_2, \ldots, x_j, \ldots, x_{n-1}/\alpha_{n-1}\}$, and let the U^i utility of x be the $U_c^{i'}$ utility of $\phi(x) = (\alpha_1 m, \alpha_2 m, \ldots, \alpha_{j-1} m, m, \alpha_{j+1} m, \ldots, \alpha_{n-1} m, x_n)$, which is defined since $\phi(x)$ is a point in S'. Finally, let $\omega^i = (\alpha_1 \omega_c^{i'}, \alpha_2 \omega_c^{i'}, \ldots, \alpha_{j-1}\omega_c^{i'}, \omega_c^{i'}, \alpha_{j+1}\omega_c^{i'}, \ldots, \alpha_{n-1}\omega_c^{i'}, \omega_n^{i'})$. Consider now $(p_1, p_2, \ldots, p_{n-1}) \in \Pi(\delta)$, and note that $(p_1, p_2, \ldots, p_{n-1}) \in \Pi(\delta)$ implies $(\alpha_1 p_1 + \ldots + \alpha_{j-1} p_{j-1} + p_{j-1} + \alpha_{j+1} p_{j+1} + \ldots + \alpha_{n-1} p_{n-1}) \in [\delta', 1/\delta']$. Individual i will maximize utility over the set $\{(x_1, x_2, \ldots, x_n) : p_1 x_1 + p_2 x_2 + \ldots + p_{n-1} x_{n-1} + x_n \leqslant p_1 \omega_1^i + p_2 \omega_2^i + \ldots + p_{n-1}\omega_{n-1}^i + \omega_n^i\}$, $i = 1, 2$. But as a consequence of our construction this is equivalent to maximizing $U_c^{i'}$ on $\{(x_c, x_n) : p_c x_c + x_n \leqslant p_c \omega_c^i + \omega_n^i\}$, $i = 1, 2$. Thus, since each unit of the composite commodity c contains one unit of commodity j,

$$\begin{aligned} E_j^J(p_1, p_2, \ldots, p_{n-1}) &= E_c^J(p_1, p_2, \ldots, p_{n-1}) \\ &= E_c^{J'}(\alpha_1 p_1 + \ldots + \alpha_{j-1} p_{j-1} + p_j \\ &\quad + \alpha_{j+1} p_{j+1} + \ldots + \alpha_{n-1} p_{n-1}) \\ &= f(p_c) \\ &= g(p_1, p_2, \ldots, p_{n-1}) \end{aligned}$$

for all $(p_1, p_2, \ldots, p_{n-1}) \in \Pi(\delta)$.

The next result is fundamental for our analysis. Given an arbitrary polynomial g defined on $\Pi(\delta)$, $0 < \delta < 1$, and a number j, $1 \leqslant j \leqslant n - 1$, we will show that there exists an economy such that g is the excess demand for the jth commodity on $\Pi(\delta)$.

THEOREM 2: *If g is a polynomial defined on $\Pi(\delta)$, and $0 < \delta < 1$, then there exists a collection of individuals $\{(U^i, \omega^i)\}$, $i \in J$, such that $E_j^J = g$ on $\Pi(\delta)$.*

PROOF: Assume that g is of degree $q \geqslant 0$, and let $0 < \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_q < 1$. By the theorem proved in the Appendix, g may be written as the sum of $\Sigma\,(i + 1)^{n-2}$ terms of the form

$$c(d_1 p_1 + d_2 p_2 + \ldots + d_{j-1} p_{j-1} + p_j + \ldots + d_{n-1} p_{n-1})^i, \tag{8}$$

where the above sum is taken over $0 \leqslant i \leqslant q$, and the d_k's are chosen from $\{\alpha_0, \alpha_1, \ldots, \alpha_q\}$. By Lemma 1 each term of this form can be generated as the excess demand function for the jth commodity on $\Pi(\delta)$ by two individuals. Since the excess demand function generated by the union of economies is the sum of their excess demand functions, there exists an economy with $2\,\Sigma\,(i + 1)^{n-2}$ individuals which has g as excess demand function for the jth commodity on $\Pi(\delta)$. (In forming the union of economies, individuals with identical tastes and initial holdings are treated as distinct points.)

By applying a theorem on simultaneous interpolation and approximation [**11**], one obtains the following corollary:

COROLLARY 3: *If g is an arbitrary continuous function defined on Π, $0 < \delta < 1$, and $p(1), p(2), \ldots, p(m)$ are m points in $\Pi(\delta)$, then there exists a collection of individuals $\{(U^i, \omega^i), i \in J$, such that $g(p(k)) = E_j^J(p(k)), k = 1, 2, \ldots, m$, and $|g(p) - E_j^J(p)| < \delta$ for all $p \in \Pi(\delta)$.*

6. EXCESS DEMAND FOR SEVERAL COMMODITIES

So far we have been concerned with the following problem: given an arbitrary function of $n - 1$ variables, is it an excess demand function? We now ask a more difficult question: given $n - 1$ functions, each of $n - 1$ variables, are they excess demand functions for the first $n - 1$ commodities in an n commodity economy (the excess demand for the nth commodity is of course determined by Walras' Law)? In Theorem 3 we provide an affirmative answer for a rather restricted class of functions.

THEOREM 3: *Let $0 < \delta < 1$, and $E_1, E_2, \ldots, E_{n-1}$ be $n - 1$ polynomials defined on $\Pi(\delta)$ satisfying $\partial E_s/\partial p_t = \partial E_t/\partial p_s$ $(s, t = 1, 2, \ldots, n - 1)$. Then there exists an n commodity economy $\{(U^i, \omega^i)\}, i \in J$, such that $(E_1^J, E_2^J, \ldots, E_{n-1}^J) = (E_1, E_2, \ldots, E_{n-1})$ on $\Pi(\delta)$.*

PROOF: In order to simplify the exposition we will prove this theorem for $n = 5$; the argument is easily generalized. We also note that it is sufficient to prove this theorem for the case of the given polynomials homogeneous of degree m. Thus we write

$$E_s(p_1, p_2, p_3, p_4) = \sum a_{ijk}^s p_1^{m-i-j-k} p_2^i p_3^j p_4^k, \quad i + j + k \leqslant m, \qquad i, j, k \geqslant 0, \quad s = 1, 2, 3, 4,$$

and observe by the symmetry condition that

$$w_{ijk}^1 = \frac{a_{ijk}^1}{\binom{m}{m-i-j-k, i, j, k}} = \frac{a_{(i-1)jk}^2}{\binom{m}{m-(i-1)-j-k, i-1, j, k}} = w_{(i-1)jk}^2, \tag{9.1}$$

$$w_{ijk}^1 = \frac{a_{ijk}^1}{\binom{m}{m-i-j-k, i, j, k}} = \frac{a_{i(j-1)k}^3}{\binom{m}{m-i-(j-1)-k, i, j-1, k}} = w_{i(j-1)k}^3, \tag{9.2}$$

$$w_{ijk}^1 = \frac{a_{ijk}^1}{\binom{m}{m-i-j-k, i, j, k}} = \frac{a_{ij(k-1)}^4}{\binom{m}{m-i-j-(k-1), i, j, k-1}} = w_{ij(k-1)}^4, \tag{9.3}$$

(9.4) $$w^2_{(i-1)jk} = w^3_{i(j-1)k},$$

(9.5) $$w^2_{(i-1)jk} = w^4_{ij(k-1)}, \quad \text{and}$$

(9.6) $$w^3_{i(j-1)k} = w^4_{ij(k-1)},$$

where the symbols w^s_{ijk} are defined in (9.1)–(9.3).[4]

Let $\alpha_0, \alpha_1, \ldots, \alpha_{m+1}$ be $m+2$ distinct numbers all of which are greater than zero and less than one. Consider the system

(10.1) $$\sum_{\substack{i+j+k\leqslant m\\ i,j,k\geqslant 0}} a^1_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m,$$

(10.2) $$\sum_{\substack{i+j+k\leqslant m\\ i,j,k\geqslant 0}} a^2_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} \alpha_u c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m,$$

(10.3) $$\sum_{\substack{i+j+k\leqslant m\\ i,j,k\geqslant 0}} a^3_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} \alpha_q c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m,$$

(10.4) $$\sum_{\substack{i+j+k\leqslant m\\ i,j,k\geqslant 0}} a^4_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} \alpha_r c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m.$$

Expanding the above expressions by the multinomial formula yields the fact that (10) will be satisfied if there exists $\{c_{uqr}\}$, $0 \leqslant u, q, r \leqslant m+1$, such that

(11.1) $$w^1_{ijk} = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} c_{uqr}\alpha_u^i \alpha_q^j \alpha_r^k = \phi^1_{ijk},$$

(11.2) $$w^2_{ijk} = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} c_{uqr}\alpha_u^{i+1} \alpha_q^j \alpha_r^k = \phi^2_{ijk},$$

(11.3) $$w^3_{ijk} = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} c_{uqr}\alpha_u^i \alpha_q^{j+1} \alpha_r^k = \phi^3_{ijk},$$

(11.4) $$w^4_{ijk} = \sum_{u=0}^{m+1}\sum_{q=0}^{m+1}\sum_{r=0}^{m+1} c_{uqr}\alpha_u^i \alpha_q^j \alpha_r^{k+1} = \phi^4_{ijk}, \qquad i,j,k \geqslant 0,\ i+j+k \leqslant m+1.$$

[4] Assume that (9.1)–(9.3) are satisfied and $i+j+k = m+1$. The argument $w^2_{(i-1)jk} = w^1_{ijk} = w^3_{i(j-1)k}$ is invalid since w^1_{ijk} is not defined. Thus (9.4) is not implied by (9.1)–(9.3). Likewise (9.5) and (9.6) are not implied in such a manner.

Let $\{\phi^s_{ijk}\}$ $(s = 1, 2, 3, 4)$ be defined by the right hand equalities and observe that

$$\phi^1_{ijk} = \phi^2_{(i-1)jk} = \phi^3_{i(j-1)k} = \phi^4_{ij(k-1)}, \quad \text{for all } i, j, k. \tag{12}$$

Define w^1_{ijk} for $i + j + k = m + 1, i, j, k \geqslant 0$, by

$$w^1_{ijk} = w^2_{(i-1)jk}, \tag{13}$$

and recall from (9.4)–(9.6) that $w^2_{(i-1)jk} = w^3_{i(-1)jk} = w^4_{ij(k-1)}$. It follows from (9) and (12) that a solution to (11.1) for all $i, j, k \geqslant 0$ satisfying $i + j + k \leqslant m + 1$ is also a solution to (11.2), (11.3), and (11.4). For if (11.1) is satisfied, then

$$\begin{aligned}
& \qquad\quad \alpha \qquad\qquad\quad \beta \qquad\qquad\quad \gamma \\
w^2_{ijk} &= w^1_{(i+1)jk} = \phi^1_{(i+1)jk} = \phi^2_{ijk}, \\
w^3_{ijk} &= w^1_{i(j+1)k} = \phi^1_{i(j+1)k} = \phi^3_{ijk}, \\
w^4_{ijk} &= w^1_{ij(k+1)} = \phi^1_{ij(k+1)} = \phi^4_{ijk},
\end{aligned} \tag{14}$$

for all $i, j, k \geqslant 0$, $i + j + k \leqslant m$. The equalities below α follow from (9) and (13), the equalities below β follow from (11), and the equalities below γ follow from (12).

However, (11.1) has a solution (see Lemma, Appendix), and thus there exist $(m + 2)^3$ real numbers $\{c_{uqr}\}$ which make all of the equations in (11) hold.

Consider now the $(m + 1)^3$ terms (each of the form $c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m$) in the sum which forms the right hand side of (10.1). For each of those terms there exists, by Lemma 1, two consumers who generate that term as excess demand for the first commodity on $\Pi(\delta)$, $n = 5$. An investigation of the proof of Lemma 1 reveals that these consumers generate

$$\alpha_u c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m, \quad \alpha_q c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m,$$

and

$$a_r c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m,$$

as excess demand for the second, third, and fourth commodities respectively.[5] Thus, the economy $\{(U^i, \omega^i)\}$, $i \in J$, composed of two individuals of the type constructed in Lemma 1 for each of the $(m + 2)^3$ terms in the sum in (10.1), has the functions on the right hand side of (10) as excess demand functions for commodities one, two, three, and four. But we have shown that (11) is satisfied, which implies that the equations in (10) are satisfied, and yields

$$\begin{aligned}
E^J_1(p_1, p_2, p_3, p_4) &= \sum_u \sum_q \sum_r c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m \\
&= \sum a^1_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = E_1(p_1, p_2, p_3, p_4), \\
E^J_2(p_1, p_2, p_3, p_4) &= \sum_u \sum_q \sum_r \alpha_u c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m \\
&= \sum a^2_{ijk} p_1^{m-i-j-k} p_2^i p_3^j p_4^k = E_2(p_1, p_2, p_3, p_4),
\end{aligned}$$

[5] For each unit of commodity one, they demand α_u, α_q, and α_r units of commodities two, three, and four.

$$E_3^J(p_1, p_2, p_3, p_4) = \sum_u \sum_q \sum_r \alpha_q c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m$$
$$= \sum a_{ijk}^3 p_1^{m-i-j-k} p_2^i p_3^j p_4^k = E_3(p_1, p_2, p_3, p_4),$$
$$E_4^J(p_1, p_2, p_3, p_4) = \sum_u \sum_q \sum_r \alpha_r c_{uqr}(p_1 + \alpha_u p_2 + \alpha_q p_3 + \alpha_r p_4)^m$$
$$= \sum a_{ijk}^4 p_1^{m-i-j-k} p_2^i p_3^j p_4^k = E_4(p_1, p_2, p_3, p_4)$$

on $\Pi(\delta)$ $(n = 5)$. This completes the proof of Theorem 3.

It is a well known fact that not all excess demand functions can be linear in prices. An immediate consequence of the preceding analysis is that $n - 1$ of them can in fact be linear and non-constant in the $n - 1$ prices.[6]

This is apparent from the following argument: Consider the quadratic form

$$F(p_1, p_2, \ldots, p_{n-1}) = \sum_{i=1}^{n-1} \sum_{i=1}^{n-1} a_{ij} p_i p_j$$

and the functions $\partial F/\partial p_1, \partial F/\partial p_2, \ldots, \partial F/\partial p_{n-1}$. These $n - 1$ functions are linear and satisfy the conditions of Theorem 3 (since $\partial^2 F/\partial p_i \partial p_j = \partial^2 F/\partial p_j \partial p_i$). Thus there exists an economy which generates them as excess demand functions for the first $n - 1$ commodities.

Unfortunately our methods will not work to construct an economy which obtains arbitrary functions of the form

$$E_1(p_1, p_2) = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3,$$
$$E_2(p_1, p_2) = \beta_1 p_1 + \beta_2 p_2 + \beta_3,$$

as excess demand functions for the first two commodities in a three commodity economy. The fact that it is unknown whether or not there exists an economy which yields arbitrary excess demand functions of this form provides a most powerful illustration of our lack of knowledge concerning the structure of excess demand. We now turn to an investigation of what "equilibrium price sets" can look like.

7. EQUILIBRIUM PRICE SETS

For a given economy $\{(U^i, \omega^i)\}$, $i \in J$, value is determined by those price vectors in Δ (or Π) which are mapped into the zero vector by the function E^J. It is well known that in general more than one price vector will be mapped into the zero vector; i.e., value will in general not be uniquely determined by a static specification of an economy. Theorem 4 determines a class of subsets of $\Pi(\delta)$ which are possible equilibrium price sets for an economy.

[6] The question of whether or not this is possible was brought to my attention several years ago by James Quirk and Rubin Saposnik.

THEOREM 4: *Let $p^1, p^2, \ldots, p^M$ be M arbitrary points in $\Pi(\delta)$, $\delta > 0$, which have no co-ordinates in common. There exists an economy which has an equilibrium at each $p^j, j = 1, 2, \ldots, M$, and none elsewhere in $\Pi(\delta)$.*

PROOF: Consider the polynomial

$$F_1(p) = \prod_{j=1}^{j=M} \sum_{i=1}^{i=n-1} (p_i - p_i^j)^2.$$

This function is non-negative and is zero if and only if $p \in \{p^j\}$. By Theorem 2 there exists an economy $\{(U^j, \omega^j)\}$, $j \in J(1)$, which has F_1 as excess demand for the first commodity on $\Pi(\delta)$.

Consider the M numbers $\{-E_2^{J(1)}(p^j)\}$, $1 \leqslant j \leqslant M$, and let F_2 be a polynomial in p_2 only which takes on the value $-E_2^{J(1)}(p^j)$ at p^j, $1 \leqslant j \leqslant M$. This is possible since the p^j have been chosen so that $p_2^k \neq p_2^j, j \neq k$. By Theorem 3 there exists an economy $\{(U^j, \omega^j)\}$, $j \in J(2)$ such that $E_s^{J(2)} = 0$ on $\Pi(\delta)$ if $s \neq 2, n$, and $E_2^{J(2)} = F_2$ on $\Pi(\delta)$. Let F_3 be a polynomial in p_3 only which takes on the value $-E_3^{J(1)}(p^j)$ at p^j, $1 \leqslant j \leqslant M$. This is possible since the p^j have been chosen so that $p_3^k \neq p_3^j, j \neq k$. By Theorem 3 there exists an economy $\{(U^j, \omega^j)\}$, $j \in J(3)$ such that $E_s^{J(3)} = 0$ on $\Pi(\delta)$ if $s \neq 3, n$, and $E_3^{J(3)} = F_3$ on $\Pi(\delta)$.

Continue this process. The economy $\{(U^j, \omega^j)\}$, $j \in \cup J(i)$ where the union is taken over $0 \leqslant i \leqslant n-1$, has an equilibrium at each point p^j, $1 \leqslant j \leqslant M$; it can have no other equilibria in $\Pi(\delta)$, since the excess demand for the first commodity is not zero anywhere else in $\Pi(\delta)$. This completes the proof. The reader will note that an analog of this result obtains if $\Pi(\delta)$ is replaced by $\Delta(\delta)$.

8. A LOCAL RESULT WITH AN APPLICATION TO STABILITY ANALYSIS

The next theorem establishes that at any given price vector, the first $n-1$ (of n) excess demand functions and their partial derivatives may take on any arbitrary set of values. This is informally summarized by saying that Walras' Law is the only local comparative statics theorem of competitive equilibrium analysis.

THEOREM 5: *Assume $p = (p_1, p_2, \ldots, p_{n-1})$ belongs to Π, $(a(j,k))$ is an arbitrary $(n-1) \times (n-1)$ matrix, and $N(1), N(2), \ldots, N(n-1)$ are arbitrary numbers, then there exists an economy $\{(U^i, \omega^i)\}$, $i \in J$, such that $E_j^J(p) = N(j)$ and $\partial E_j^J(p)/\partial p_k = a(j,k)$, $1 \leqslant j, k \leqslant n-1$.*[7]

PROOF: The proof is given only for $n = 3$ since this illustrates the general method. If we consider commodity two to be the numeraire, i.e., if we fix its price at one, then by Theorem 3, there exists an economy (composed of individuals indexed by $J(2)$) which generates c_1, $-c_1p_1 - c_3p_3$, c_3 as excess demand for

[7] It was demonstrated to me by Professor Paul Samuelson, that for a single individual, the matrix $(\partial E_j/\partial p_k - \partial E_k/\partial p_j)$ is the sum of two matrices of rank one, and hence $(\partial E_j/\partial p_k)$ is not arbitrary. This provides the basis for a rigorous distinction between the single consumer and many consumer cases.

commodities one, two, and three respectively. If p_2 is entered as a variable, then the excess demand function generated by $\{(U^i, \omega^i)\}$, $i \in J(2)$ must be homogeneous of degree zero in p_1, p_2, and p_3, and agree with the non-homogeneous form when $p_2 = 1$. It is thus $c_1, (-c_1p_1 - c_3p_3)/p_2, c_3$. Likewise, there exists an economy (composed of individuals indexed by $J(3)$) which generates excess demand $(-d_2p_2 - d_3p_3)/p_1, d_2, d_3$, and an economy (composed of individuals indexed by $J(1)$) which generates excess demand $e_1, e_2, (-e_1p_1 - e_2p_2)/p_3$. The excess demand for the first two commodities generated by $\{(U^i, \omega^i)\}$, $i \in \cup J(k)$ is $e_1 + c_1 - (d_2p_2 + d_3p_3)/p_1, e_2 + d_2 - (c_1p_1 + c_3p_3)/p_2$. Setting $p_3 = 1$ yields

$$e_1 + c_1 - (d_2p_2 + d_3)/p_1, \quad \text{and} \tag{15}$$

$$e_2 + d_2 - (c_1p_1 + c_3)/p_2, \tag{16}$$

as excess demand for commodities one and two at prices p_1 and p_2. The result follows from first differentiating the above expressions partially with respect to p_1 and p_2, and then noting that by appropriately choosing the c's, d's, and e's, both (15) and (16), and the four expressions derived by differentiation, may be assigned any desired value at any point in Π, $n = 3$.

One use of Theorem 5 is to help answer questions concerning the relationship between various definitions of local stability. For example, although it is known that there are matrices that are Hicksian stable but not local dynamically stable, it has not been shown that there are economies which yield these arrays as matrices of $\partial E_i/\partial p_j$'s at an equilibrium [**5**, pp. 188–9, and **8**]. This problem is resolved in Corollary 4.

COROLLARY 4: *There exists an equilibrium that is Hicksian stable, but not dynamically stable.*

PROOF: The matrix

$$\begin{bmatrix} -1 & +1 & +14 \\ -1 & -1 & +1 \\ 0 & -1 & -1 \end{bmatrix}$$

is Hicksian stable, but not dynamically stable [**5**, p. 173]. By the previous theorem there exists a four commodity economy which has an equilibrium at prices (1, 2, 3). and which generates excess demand functions whose jth co-ordinate function differentiated with respect to p_k, and evaluated at (1, 2, 3), is precisely the j, kth entry of the above matrix, $1 \leqslant j, k \leqslant 3$. Such an economy has an equilibrium which is Hicksian stable but not dynamically stable at prices (1, 2, 3).

9. UNSOLVED PROBLEMS

All of the preceding theorems would be implied by an affirmative answer to *Question A*: Can any $n - 1$ functions in $n - 1$ prices, which have continuous derivatives on $\Pi(\delta)$, be realized as excess demand functions for the first $n-1$ commodities in an n commodity economy?[8]

Also interesting is *Question B*: If Question A is answered in the affirmative, what additional restrictions must be imposed on the given functions so that when $\Pi(\delta)$ is replaced by Π (or $\hat{\Delta}$) in A, an affirmative answer still obtains?

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APPENDIX

A Representation for Polynomials of Several Variables

The purpose of this appendix is to prove the following:

Theorem: *Any polynomial in several variables can be written as a sum of powers of linear polynomials. More explicitly, let integers $m \geqslant 1$, $n \geqslant 0$, and distinct numbers $\alpha(0), \alpha(1), \dots, \alpha(n)$ be given. Consider the $K \equiv (n+1)^m$ linear polynomials $Z_1, Z_2, \dots, Z_K$ of the form $x_0 + \alpha(s)x_1 + \alpha(t)x_2 + \dots + \alpha(u)x_n$, $0 \leqslant s, t, \dots, u \leqslant n$. Then the collection $\{(Z_j)^n\colon j = 1, 2, \dots, K\}$ spans the vector space of homogeneous polynomials of degree n in the $(m+1)$-variables $x_0, x_1, \dots, x_m$.*

The proof of the Theorem is based on the following:

Lemma: *If $\alpha(0), \alpha(1), \dots, \alpha(n)$ are $n+1$ distinct real numbers, and $\{w(i, j, \dots, k)\colon 0 \leqslant i, j, \dots, k \leqslant n\}$, are real numbers, then there exist real numbers $\{c(s, t, \dots, u)\colon 0 \leqslant s, t, \dots, u \leqslant n\}$ such that $w(i, j, \dots, k) = \Sigma\, c(s, t, \dots, u)\alpha(s)^i\alpha(t)^j \dots \alpha(u)^k$ for all $0 \leqslant i, j, \dots, k \leqslant n$, where the above sum is taken over $\{s, t, \dots, u\colon 0 \leqslant s, t, \dots, u \leqslant n\}$.*

Proof: Let m denote the number of arguments of w. For $m = 1$ the result follows immediately from the non-singularity of Vandermonde matrices. We will prove the lemma only for $m = 2$, since the procedure employed is illustrative of the proof of the necessary induction step.

If $\alpha(0), \alpha(1), \dots, \alpha(n)$ are distinct real numbers, and $\{w(i, j)\colon 0 \leqslant i, j \leqslant n\}$ are real numbers, then we must find real numbers $\{c(s, t)\colon 0 \leqslant s, t \leqslant n\}$ such that

$$w(i,j) = \sum_{s=0}^{n}\sum_{t=0}^{n} c(s,t)\alpha(s)^i\alpha(t)^j = \sum_{s=0}^{n}\left[\sum_{t=0}^{n} c(s,t)\alpha(t)^j\right]\alpha(s)^i$$

for all $0 \leqslant i, j \leqslant n$.

Since the lemma holds for $m = 1$, for each j, $0 \leqslant j \leqslant n$, there exist $n + 1$ real numbers $d(j, k)$, $0 \leqslant k \leqslant n$, such that $w(i,j) = \Sigma\, d(s,j)\alpha(s)^i$ for all $0 \leqslant i \leqslant n$, where the sum is taken over $0 \leqslant s \leqslant n$. The proof (for $m = 2$) is completed by applying the lemma (for $m = 1$) $n + 1$ times to obtain real numbers $c(s, t)$ which satisfy $d(s,j) = \Sigma\, c(s, t)\alpha(t)^j$ for all $0 \leqslant s, j \leqslant n$, where the sum is taken over $0 \leqslant t \leqslant n$.

Proof of the theorem: We will prove that the collection $\{(Z_j)^n : j = 1, 2, \dots, K\}$ spans the vector space of homogeneous polynomials of degree n in three variables x, y, and z, since this illustrates the general case of $m + 1$ variables. If P is homogeneous of degree n in x, y, and z, then

$$P(x, y, z) = \sum a(i,j)x^{n-i-j}y^iz^j, \tag{A1}$$

where the sum is taken over $S = \{(i,j): 0 \leqslant i,j \leqslant n, i + j \leqslant n\}$. For $0 \leqslant i,j \leqslant n$, let

$$w(i, j) = \begin{cases} \dfrac{a(i,j)}{\dbinom{n}{n-i-j, i, j}} & \text{if } i + j \leqslant n, \\ 0 & \text{if } i + j > n. \end{cases}$$

[8] Added in proof: The author obtained an affirmative answer to *Question A* for the case of polynomial excess demand in January, 1972.

We must show that

$$\text{(A2)}\qquad \begin{aligned} P(x, y, z) &= \sum c(s, t)(x + \alpha(s)y + \alpha(t)z)^n \\ &= \sum c(s, t)\left(\sum \binom{n}{n - i - j, i, j} \alpha(s)^i \alpha(t)^j x^{n-i-j} y^i z^j\right) \end{aligned}$$

for some collection of real numbers $\{c(s, t): 0 \leqslant s, t \leqslant n\}$, where the sum inside the brackets is taken over S, and the sum outside is taken over $T = \{(s, t): 0 \leqslant s, t \leqslant n\}$. Matching coefficients in (A1) and (A2) we conclude that the Theorem will be established if we can find $\{c(s, t): 0 \leqslant s, t \leqslant n\}$ such that $w(i, j) = \Sigma\, c(s, t)\alpha(s)^i\alpha(t)^j$, for all $0 \leqslant i, j \leqslant n$, where the sum is taken over T. But this follows from the lemma with $m = 2$.

An alternative proof can be found in [**10**].

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