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# Market Segmentation and Product Technology Selection for Remanufacturable Products

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# Market Segmentation and Product Technology Selection for Remanufacturable Products

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**Abstract**

Remanufacturing is a production strategy whose goal is to recover the residual value of used products. Used products can be remanufactured at a lower cost than the initial production cost, but remanufactured products are valued less than new products by consumers. The choice of production technology influences the value that can be recovered from a used product. In this paper, we solve the joint pricing and production technology selection problem faced by a manufacturer who considers introducing a remanufacturable product in a market that consists of heterogeneous consumers. Our analysis discusses the market and technology drivers of product remanufacturability and identifies some phenomena of managerial importance that are typical of a remanufacturing environment.

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# 1 Introduction and Literature Review

Remanufacturing is a production strategy whose goal is to recover the residual value of used products by reusing components that are still functioning well. Remanufactured products are obtained by collecting used products and replacing worn-out components by new ones (Thierry et al. 1995). The remanufacturing literature focuses mainly on logistics, production planning and inventory control (Fleischmann et al. 1997), but these considerations constitute only one facet of the managerial issues surrounding remanufacturing. Consider the tire manufacturing and retreading industry, for example. The casing (the inner structure of the tire) may be reusable even after the tread (the outer layer) wears out. The remanufacturing activity consists of "retreading," a process that replaces the worn tread by a new one. By law, retreaded tires have to be marked on the side-wall (Commission of The European Communities 2000), which allows consumers to distinguish between new and retreaded tires. Typically, retreads are perceived to have lower quality than new tires (Préjean 1989). The retreadability of tires can be influenced by the manufacturer (BIPAVER 1998, Bozarth 2000a), via the choice of material and production technology, but increased retreadability requires a higher production cost. Tire manufacturers face these issues in making production technology and product pricing decisions.

Similar considerations are relevant in a variety of industries. Klausner et al. (1998) describe the remanufacturing of electrical motors. Most electrical motors last longer than the product that they power. Products containing remanufactured electrical motors can be sold to low-end consumers at a discounted price. Whether a used motor can be remanufactured depends on the usage pattern, but this is unobservable by the manufacturer. Integrating an Electronic Data Log (EDL) into the motor at additional cost makes it easier to assess whether the motor is remanufacturable or not, and may also facilitate the remanufacturing operation. The question is whether it is worth incurring the extra cost of installing an EDL on new motors.

Xerox has invested in the remanufacturability of its copiers (Vietor 1993) and has been successful in marketing its remanufactured product line. With the digitalization of the copier, Xerox faces a new challenge: The cost of software upgrades required in remanufacturing the used copier may be too high to be recovered given the low willingness-to-pay of consumers for the remanufactured copier.

In this paper, we address the key managerial issues faced by a manufacturer who considers producing a remanufacturable product, where consumers are heterogeneous in their willingness to pay and where they value remanufactured products less than new products. The most fundamental question is whether producing a remanufacturable product is profitable. A remanufacturable product is typically more costly to produce than a single-use product. The revenue potential of the remanufactured product is questionable when it is valued less than the new product by consumers. On the other hand, the remanufactured product is cheaper to manufacture and creates the opportunity to sell to the low-end consumers. In this context, the key questions facing the manufacturer are the following: Does the opportunity to reach low-end consumers outweigh the high cost of producing a remanufacturable product? What are the key drivers determining the profitability of offering a product portfolio consisting of a new and a remanufactured product? How do the costs and consumer perceptions impact the value proposition of remanufacturability?

The relative size of the low-end and high-end consumer populations differs across markets. An important issue in this context is understanding the impact of the characteristics of the target market: Does the decision to remanufacture depend on the consumer profiles? What pricing strategy and production technology choice best fit the target market?

The combination of new and remanufactured products creates a unique product portfolio in the sense that the remanufactured product exists only due to previous sales of the new product. Thus, a decrease in demand for new products results in a decrease in the availability of remanufactured products. It is useful to understand the implications of this dependence for planning and marketing purposes. For example, how does the remanufacturing cost impact the desired production volume and mix? From a marketing perspective, an important managerial question is how to position the new product: Is it valued for the immediate margin it creates or for the future value stream that it has the potential to generate? What should be the pricing policy that reflects the role of the new product?

Producing a remanufacturable product may invite independent remanufacturers to compete with the manufacturer on the lower end of the market. Since the manufacturer can typically control the remanufacturability level and therefore control the supply of remanufacturable products to independent remanufacturers, an important question that arises in this situation is how competition with independent remanufacturers should be taken into account when determining the remanufacturability level?

Many products that are considered for remanufacturing (e.g. consumer durables, household appliances) exhibit a well-pronounced product life cycle, i.e. they gradually diffuse through the market. With a product remanufacturing option, the diffusion of the whole product portfolio with both new and remanufactured products needs to be considered. We address the following questions: How does product diffusion impact the value and level of remanufacturability? How is new and remanufactured product capacity management impacted by diffusion?

The literature on remanufacturing has focused primarily on operational issues that arise in inventory management and production control as a result of the return flows of used products. These issues include disassembly (Guide and Srivastava 1998), MRP for product recovery (Inderfurth 1998), scheduling and shop floor control (Guide et al. 1997) and inventory management (van der Laan et al. 1999, Toktay et al. 2000, Inderfurth 2002). Fleischmann (2000) considers reverse logistics network design. In these papers, price, demand rate, and remanufacturability level are assumed to be exogenous, and consumers do not differentiate between new and remanufactured products. The focus is on determining the cost-minimizing operating policy or system design for a given remanufacturability level and price. Our paper complements this literature by determining the remanufacturability level and the optimal prices using a market model that reflects how remanufactured products are perceived by consumers. Other researchers modelling market-related issues in remanufacturing are Savaşkan et al. (2004) who determine the optimal collection channel configuration of a monopolist manufacturer, and Groenevelt and Majumder (2001a,b) who investigate the impact of competition between a manufacturer who also performs remanufacturing activities, and a local remanufacturer.

The literature on market segmentation (Mussa and Rosen 1978, Moorthy 1984) studies the optimal pricing of independent products that are differentiated by quality in a market of heterogeneous consumers whose valuations of quality vary. In a remanufacturing setting, there is a dependence between the two products: The supply of used products that can be remanufactured depends on past sales volumes of new products and the level of remanufacturability. Ferrer (2000) solves the market segmentation problem for a fixed remanufacturability level. He finds that remanufacturing is not viable if the resulting cost savings are not high enough to price the remanufactured product above its marginal cost. We consider the simultaneous determination of product prices and the production technology for a general consumer profile. One of our results complements Ferrer's findings by determining under which circumstances his pure cost savings analysis is sufficient to determine the viability of remanufacturing.

The literature on competition in remanufacturing is quite recent. Goenevelt and Majumder (2001a, b) study price competition between an OEM and a local remanufacturer, taking the return fraction of products as exogenously determined. We introduce the level of remanufacturability as a key variable that can be determined by the OEM. In addition, we allow the price of remanufacturable products to be endogenously determined via a market-clearing mechanism.

The literature on product diffusion does not incorporate remanufacturing considerations, in particular, the joint diffusion of new and remanufactured products, even though many products that are considered for remanufacturing exhibit diffusion characteristics. In order to study this, we consider a product with a finite life duration, which triggers a remanufacturing opportunity and possible repeat purchases, in addition to modelling the consumer choice between new and remanufactured products and the supply constraint. In Bass (1969), each customer purchases a product with infinite life duration exactly once. Mesak and Berg (1995) and Kamakura and

Balasubramanian (1987) analyze the diffusion of a single product with a finite life duration and consequent repeat sales. Ho et al. (2002) analyze a model where the diffusion of a single product with infinite life duration is supply-constrained. Kouvelis and Mukhopadhyay (1999) study, in a competitive setting, the diffusion of a product with infinite life duration that is controlled by price and quality. In our setting, the sales of remanufactured products are constrained by the supply of used products, an endogenous constraint on the diffusion of remanufactured products. We develop a monopoly model that simultaneously incorporates repeat purchases, supply-constrained diffusion and substitution. Diffusion is controlled by the prices of new and remanufactured products and the level of remanufacturability.

The remainder of this paper is structured as follows: In §2, we introduce the basic model in which the monopolist determines the remanufacturability level of the new product and segments the market between new and remanufactured products. In §3, we solve and interpret the optimal solution to the manufacturer's problem to answer the questions raised in the introduction. §4 extends our monopoly model to the case where remanufacturers compete on the remanufactured product market. In §5, we study product life cycle issues that arise when new and remanufactured products diffuse gradually through the market. In §6, we discuss the implications of our results for the integrated management of product lines with new and remanufactured products. We conclude with directions for future research.

# 2 The Model

We introduce our assumptions concerning the production technology, the cost structure, the consumer preferences, the industry structure, and the decision-making framework in §2.1 and formulate the manufacturer's optimization problem in §2.2.

## 2.1 Assumptions

**Production Technology Choice.** Motivated by the examples of §1, we assume that the manufacturer controls the level of remanufacturability through the choice of production technology. We model the remanufacturability level, denoted by $q$, as the fraction of products that can be remanufactured after one period of use. The manufacturer can choose any remanufacturability level $q \in [0, 1]$. If the remanufacturability level is set to zero, this is a "single-use" product and cannot be remanufactured. Used remanufacturable products require a remanufacturing operation before being sold as remanufactured products. We assume that a remanufacturable product can be remanufactured at most once.

**Cost Structure.** The technology choice impacts both up-front costs independent of subsequent production volume such as R&D expenditure, as well as costs that are a function of the production volume. We model the impact of technology choice on the former costs by means of a fixed cost $k(q)$ incurred before production starts. We assume that $k(q)$ is a convex increasing function of $q$ with $k(0) = 0$. We model the dependence of the new-product unit manufacturing cost and used-product unit remanufacturing cost on the technology choice by $c_n(q)$ and $c_r(q)$, respectively. We assume that $c_n(q)$ is a convex increasing function of $q$ and that $c_r(q)$ is a non-increasing function of $q$: A higher level of remanufacturability requires a higher new product unit manufacturing cost (due to the use of better materials, more precise production processes, addition of a data logger, etc.), and at the same time, can result in a lower unit remanufacturing cost (due to easier disassembly, less testing etc.).

**Consumer preferences.** Consumers typically differ in their willingness-to-pay. For this reason, we associate with each consumer its willingness-to-pay for a new product, $\theta$, also called its 'type'. We refer to consumers with a low (high) willingness-to-pay for new products as 'low-end' ('high-end') consumers. We assume that $\theta$ is distributed on $[0, 1]$ according to a function $F$, where $F(\theta)$ denotes the volume of consumers with willingness-to-pay in $[0, \theta]$ and is a strictly increasing and continuous function with $F(0) = 0$ and $F(1) = 1$. Markets differ in the relative concentration of consumers with different levels of willingness-to-pay. Introducing a general structure for $F$ allows us to capture this variety. In particular, we consider a class $\mathcal{F}^{\kappa}$ of distributions of the form $F(\theta) = 1 - (1 - \theta)^{\kappa}$, where $\kappa \in (0, \infty)$. The uniform distribution frequently used in the market segmentation literature is a special case of this distribution obtained by setting $\kappa = 1$.

Typically, remanufactured products are valued less than new products by consumers. For example, Xerox studies showed that the presence of used components in a remanufactured product decreased the consumer's willingness-to-pay for this product (Vietor). Furthermore, retreaded tires are typically bought by budget-conscious consumers (Alford 2001). The retread industry has also been plagued with image problems (Préjean). To model this, we assume that the willingness-to-pay of consumer type $\theta$ for a remanufactured product is $\eta(\theta)$, which is a non-negative monotonically increasing function, not greater than $\theta$, over $[0, 1]$. We refer to $(F, \eta)$ as the 'consumer profile.'

Let $p_N$ and $p_R$ denote the prices of new and remanufactured products, respectively. We model the net utility that a consumer of type $\theta$ derives from buying a new product, a remanufactured product, and no product, by $\theta - p_N$, $\eta(\theta) - p_R$, and 0, respectively. In a given period, consumers choose which product to buy based on the utility that they derive in that period from this purchase.

**Industry Structure.** Our main analysis and discussion (§3) is for an industry in which the manufacturer holds a monopoly in the markets for new *and* remanufactured products. This assumption is reasonable if the manufacturer has a proprietary remanufacturing technology (e.g. MRT retread technology developed by Michelin) that would limit the formation of a market for used remanufacturable products, and the supply of used but remanufacturable products is controlled by the manufacturer (e.g., Michelin's retread company, Pneu Laurent, operates a fleet of over two hundred vehicles collecting used Michelin tires from dealers). Nevertheless, independent competing remanufacturers abound in this industry, as in other industries (Groenevelt and Majumder 2001a,b). To capture the impact of competition in the remanufactured product market on the remanufacturability level chosen by the manufacturer, we consider an industry in which the manufacturer holds a monopoly in the market for new products, and independent remanufacturers compete on the remanufactured product market. This variant is analyzed in §4.

**The Decision-Making Framework.** The manufacturer's goal is to maximize the net present value of introducing a remanufacturable product, calculated over the life-cycle of this product, by determining the level of remanufacturability and a sequence of prices for the new and remanufactured products.

To model this, we develop a discrete-time, infinite-horizon, discounted profit optimization problem. Each period corresponds to a period of use of the product by a consumer, after which the product needs to be remanufactured for further use. This period may range from several weeks (e.g. single-use cameras) to several months (e.g. tires). The product life duration is thus exactly one period. Let $\beta$ denote the discount factor over this time period. Thus, the longer the time on the market, the lower the discount factor should be. In §5, we introduce a more general product life duration to study the diffusion of new and remanufactured products.

We assume that the level of remanufacturability is determined at time 0 since it is the initial technology choice that determines this value for all subsequent periods. Product prices are allowed to be time-dependent. Recall that the supply of used products that can be remanufactured in each period is constrained by the historical sales of new products and the level of remanufacturability. Starting without a supply of used products induces a transient period during which this supply is built up. We allow the manufacturer to carry inventory of used remanufacturable products. In order to keep the focus on the technology selection and market segmentation issues, we do not consider associated holding costs.

The infinite-horizon assumption is particularly appropriate when the period of use of a product is short relative to the total life-cycle of the product on the market, as is the case with tires and electrical motors, for example. Moreover, the infinite horizon analysis provides some insight into problems with a finite, but sufficiently long, horizon. Finally, the infinite-horizon framework lends itself to an approximate analysis of the optimal technology choice based on the stationary solution and allows us to derive a number of comparative statics results.

## 2.2 Formulation of the Monopolist's Optimization Problem

**The Single-Period Profit.**

Recall that $p_N$ and $p_R$ denote the prices of new and remanufactured products, respectively, and define $p \doteq (p_N, p_R)$, where $p \in \mathcal{S} \doteq \{(p_N, p_R) \in \mathbb{R}_+^2 : 0 \leq p_N \leq 1, 0 \leq p_R \leq \eta(p_N)\}$. Then $\Omega_N(p) \doteq \{\theta \in [0,1] : \theta - p_N \geq \eta(\theta) - p_R\}$ is the set of consumer types who purchase a new product. $\Omega_R(p)$ is defined analogously as the set of consumer types who purchase a remanufactured product.

Let $n$ and $r$ denote the volume of consumers who purchase new and remanufactured products, respectively, and define $\nu \doteq (n, r)$. Then $n = \int_{\Omega_N(p)} dF(\theta)$ and $r = \int_{\Omega_R(p)} dF(\theta)$. By construction, $\nu \in \mathcal{D} \doteq \left\{(n,r) \in \mathbb{R}_+^2 : n + r \leq 1\right\}$. Since $F$ is strictly increasing, the mapping $p \to \nu(p)$ is one-to-one. Therefore, the inverse mapping $\nu \in \mathcal{D} \to p(\nu) \in \mathcal{S}$ is well defined. We can now define $R(\nu) \doteq np_N(\nu) + rp_R(\nu)$, the revenue of the monopolist who prices so as to create demand $\nu$. Some properties of the revenue function are developed in §1 in the Appendix under mild restrictions on $\eta(\theta)$. Finally, let $\pi(\nu, q) \doteq R(\nu) - c_n(q)\, n - c_r(q)\, r$; this is the profit obtained in a generic period under the decision $(\nu, q)$.

**The Infinite-Horizon Optimization Problem.**

Let $s_t \doteq (s_{N,t}, s_{R,t})$ be the sales of new and remanufactured products in period $t$. Let $I_t$ denote the supply of used products that can be remanufactured that remain in stock at the beginning of period $t$ from returns in previous periods. Then, $I_0 = 0$ and $s_{R,0} = 0$ since no used products exist initially and $I_t = \sum_{k=1}^{t} qs_{N,t-k} - s_{R,t-k}$. If the price in period $t$, $p_t$, is chosen such that the resulting demand for remanufactured products is greater than the available inventory ($r_t > I_t$), the manufacturer can only sell $s_{R,t} = \min(r_t, I_t) = I_t$. In this case, the manufacturer can increase both $p_{R,t}$ and $p_{N,t}$ in such a way that the demand for remanufactured products decreases to $I_t$ and the demand for new products remains the same. Since the sales volumes remain identical while both prices increase, a higher profit can be realized in this manner. Therefore, it will never be optimal to price such that $r_t > I_t$; at optimality, $r_t \leq I_t$ and $s_{R,t} = r_t$. In addition, as a result of our assumptions on production capacity, production lead times and raw material supply, any volume of new products can be satisfied, that is, $s_{N,t} = n_t$. Thus, we can formulate our problem in terms of demand volumes and define the feasible region such that the demand for remanufacturable products in each period is less than or equal to the available supply of used remanufacturable products. We define an *implementable* path $\mathcal{P}$ starting with initial remanufacturable product inventory $I$ (denoted by $\mathcal{P} \in \mathcal{I}(I)$) as $\mathcal{P} \doteq \{\nu_t, t \geq 0 | \nu_t \in \mathcal{D},\ I_0 = I,\ I_t = I_{t-1} + qn_{t-1} - r_{t-1}\ \forall t \geq 1, \text{ and } r_t \leq I_t\ \forall t \geq 0\}$. The analysis in the remainder of this paper will use $n_t$ and $r_t$ as decision variables.

Let $V_\beta(I;q)$ denote the optimal $\beta$-discounted infinite-horizon profit of the manufacturer for a given remanufacturability level $q$ under the initial condition $I_0 = I$, i.e., $V_\beta(I;q) \doteq \max\limits_{\mathcal{P} \in \mathcal{I}(I)} \sum_{t=0}^{\infty} \beta^t \pi(\nu_t, q)$.

In this paper we analyze this problem for $I_0 = 0$. We define $V_\beta(q) \doteq V_\beta(0; q)$, that is,

$$V_\beta(q) \doteq \max_{\mathcal{P} \in \mathcal{I}(0)} \sum_{t=0}^{\infty} \beta^t \pi(\nu_t, q). \tag{1}$$

The optimal solution to this problem is the path of new and remanufactured demands $\nu_t^*$, to which corresponds a unique optimal price path $p_t^*$.

The technology selection problem of a manufacturer with a monopoly position in both markets for new and remanufactured products is then

$$\max_{q \in [0,1]} V_\beta(q) - k(q). \tag{2}$$

# 3 Analysis

We characterize the optimal solution of the monopolist's optimization problem in §3.1. In §3.2, we derive a sufficient condition under which it is optimal for the manufacturer to invest in the remanufacturability of its product. §3.3 focuses on the stationary solution to explore the characteristics of the optimal portfolio. In particular, we investigate the dependence of the optimal remanufacturability level on the consumer profile (§3.3.1), we characterize new and remanufactured product margins (§3.3.2) and we establish properties of the demand mix as a function of the remanufacturing cost (§3.3.3).

## 3.1 A Characterization of the Optimal Solution

Let $c(q) \doteq c_n(q) + \beta q c_r(q)$, $\overline{v}(q) \doteq \frac{\partial R(0,0)}{\partial n} + q\beta \frac{\partial R(0,0)}{\partial r}$, $\tilde{\nu} \doteq (\tilde{n}, \tilde{r}) \doteq \arg\max_{\nu \in \mathcal{D}} \pi(\nu, q)$ and $n_{su} \doteq \arg\max_{0 \le n \le 1} \pi(n, 0, 0)$. Here, $c(q)$ and $\overline{v}(q)$ are the marginal present cost incurred and revenue realized, respectively, when manufacturing and selling the new product now, and remanufacturing the resulting used but remanufacturable products and selling them one period later; $\tilde{\nu}$ is the optimal single-period demand (sales) volumes unconstrained by availability of used remanufacturable products, and $n_{su}$ is the optimal demand (sales) volume of single-use products (products which have remanufacturability level $q = 0$). If $\tilde{\nu} \in int(\mathcal{D})$, then $\frac{\partial R(\tilde{\nu})}{\partial n} = c_n(q)$ and $\frac{\partial R(\tilde{\nu})}{\partial r} = c_r(q)$. If $0 < n_{su} < 1$, then $\frac{\partial R(n_{su},0)}{\partial n} = c_n(0)$.

Some assumptions that are used in the following analysis, but were not discussed in §2 are listed in §2 of the Appendix. We do not repeat any assumptions in the statement of each result; it is implicit that they hold throughout the analysis.

**Lemma 1** *Let $q \in [0,1]$ and $I_0 = I$. Then (i) $V_\beta(I; q)$ is concave nondecreasing in $I$; (ii) There exists a unique optimal path $\{\nu_t^*, t \ge 0\}$; call it $\mathcal{P}_q^*(I_0)$.*

Let $n^*(I)$ and $r^*(I)$ denote the unique maximizers of the right-hand side in the Bellman Equation

$$v(I;q) = \max_{\nu \in \mathcal{D}, r \leq I} \pi(\nu, q) + \beta v(I + qn - r; q). \tag{3}$$

Define the policy function $g$ such that $g(I) \doteq I + qn^*(I) - r^*(I)$. The optimal path starting with initial condition $I_0 = 0$, denoted by $\mathcal{P}_q^*$, is found by applying $g$ recursively to $I_t$ starting with $I_0 = 0$. Then $n_t^* = n^*(I_t)$, $r_t^* = r^*(I_t)$ and $I_{t+1} = g(I_t)$ $\forall t$. We characterize properties of the optimal path and/or of $V_\beta(q)$ as follows: Lemma 2 identifies a necessary and sufficient condition for $V_\beta(q) > 0$. Subject to this condition, Lemma 3 characterizes the optimal path and $V_\beta'(q)$ when $q\tilde{n} \geq \tilde{r}$ and Proposition 1 builds on Lemmas 6 and 7 (in §3 of the Appendix) to characterize the optimal path and to derive $V_\beta'(q)$ when $q\tilde{n} < \tilde{r}$. In particular, Lemma 6 derives the shape of the policy function $g$ and Lemma 7 shows that $I_t \to I_\infty$ and $\nu_t^* \to \nu_\infty$, the stationary solution.

**Lemma 2** *$V_\beta(q) > 0$ if and only if $c(q) < \overline{v}(q)$.*

**Lemma 3** *Let $q > 0$. If $c(q) < \overline{v}(q)$ and $q\tilde{n} \geq \tilde{r}$, then $\mathcal{P}_q^* = \{(n_s(q), 0), (\tilde{n}, \tilde{r}), (\tilde{n}, \tilde{r}), (\tilde{n}, \tilde{r}) ...\}$, where $n_s \doteq \arg\max_{0 \leq n \leq 1} \pi(n, 0, q)$. In addition, $V_\beta'(q) < 0$.*

Lemmas 2 and 3 imply that $q^* \in Q \doteq \{q \in [0,1] | c(q) < \overline{v}(q) \text{ and } q\tilde{n} < \tilde{r}\}$. For the remainder of the paper, we work with $q \in Q$.

**Proposition 1** *Let $\mathcal{P}_q^* = \{\nu_t^*, t \geq 0\} \in \mathcal{I}(0)$ be the optimal path found when solving (1) for a fixed $q$. Then, it satisfies*

$$\frac{\partial R(\nu_t)}{\partial n} + \beta q \frac{\partial R(\nu_{t+1})}{\partial r} = c(q) \ \ \forall t. \tag{4}$$

*In addition,*

$$V_\beta'(q) = \sum_{t=0}^{\infty} \beta^t \left( \beta \left( \frac{\partial R(\nu_{t+1}^*)}{\partial r} - c_r(q) \right) n_t^* - c_n'(q) n_t^* - c_r'(q) r_t^* \right). \tag{5}$$

Equations (4) and (5) have the following economic interpretation: Increasing the volume of new products during a single period results in an immediate increase in revenues from new products, $\frac{\partial R(\nu_t^*)}{\partial n}$, and an increase in revenues from remanufactured products in the next period on a fraction $q$ of new products $\frac{\partial R(\nu_{t+1}^*)}{\partial r}$. Equation (4) equates the marginal increase in revenues over two periods with the marginal increase in cost over two periods $(c(q))$. Equation (5) calculates the effect of increasing $q$ by $dq$. This change has an effect over all periods: An increase in unit manufacturing and remanufacturing costs in period $t$ by $(c_n'(q) n_t^* + c_r'(q) r_t^*) dq$, but also an increase in revenues in the next period, due to an increase $dr_{t+1} = n_t^* dq$ in available remanufacturable units which generates an additional profit $\left( \frac{\partial R(\nu_{t+1}^*)}{\partial r} - c_r(q) \right) n_t^* dq$.

We build on Equations (4) and (5) to derive the results in the next subsections.

### 3.2 Whether to Produce a Remanufacturable Product

In this subsection, we discuss the conditions under which the solution $q^*$ to (2) is strictly positive, that is, the manufacturer invests in the remanufacturability of his product. Let us define $\Delta \doteq V_\beta'(0) - k'(0)$. We call $\Delta$ the 'remanufacturing potential': If $\Delta$ is positive, then, it is profitable to produce a remanufacturable product.

**Proposition 2** *It is optimal to produce a remanufacturable product (i.e. $q^* > 0$) if the following condition is satisfied:*

$$\Delta = \frac{1}{1-\beta}\left(\beta\left\{\frac{\partial R\left(n_{su},0\right)}{\partial r} - c_r\left(0\right)\right\} - c_n'\left(0\right)\right) n_{su} - k'\left(0\right) > 0. \tag{6}$$

*If the consumer profile is linear, i.e. $\eta(\theta) = (1-\delta)\theta$, (6) reduces to:*

$$\Delta = \frac{1}{1-\beta}\left(\beta\left\{(1-\delta)c_n(0) - c_r\left(0\right)\right\} - c_n'\left(0\right)\right) n_{su} - k'\left(0\right) > 0. \tag{7}$$

From now on, we focus on a linear consumer profile represented by $(F, \delta)$. We refer to $\delta$ as the "perceived depreciation" of the remanufactured product. We will now discuss the impact of all technology and market related parameters on the remanufacturing potential.

#### 3.2.1 Factors Directly Influencing the Remanufacturing Potential

From (6), we observe that the remanufacturing potential $\Delta$ increases as the product becomes cheaper to remanufacture ($c_r(0)$ decreases), or the marginal increase in the unit cost $c_n'(0)$ decreases, or the marginal increase in the fixed cost $k'(0)$ decreases. These factors all relate to incremental costs associated with moving from a single-use product to a remanufacturable product. Note also that $\Delta$ increases in the discount factor, $\beta$, which is influenced by the length of time the new product stays on the market before it returns to the manufacturer (the length of one period in our model). Furthermore, from (7), we observe that $\Delta$ increases as $\delta$, the perceived depreciation factor, decreases. These parameters are direct drivers of the remanufacturing potential.

#### 3.2.2 Factors Indirectly Influencing the Remanufacturing Potential

The manufacturing cost $c_n(0)$ has a direct positive impact on $\Delta$. In addition, both $c_n(0)$ and the consumer profile $(F, \delta)$ play a role in determining $\Delta$ via the term $n_{su}$. In order to gain insight into the role of the consumer profile, we focus on the class $\mathcal{F}^\kappa$ of distributions of the form $F(\theta) = 1-(1-\theta)^\kappa$, where $\kappa \in (0, \infty)$. Figure 1 plots the density $f(\theta)$ for four different values of $\kappa$. Observe that as $\kappa$ increases, the mass of consumers shifts from high-valuation consumers to low-valuation consumers.

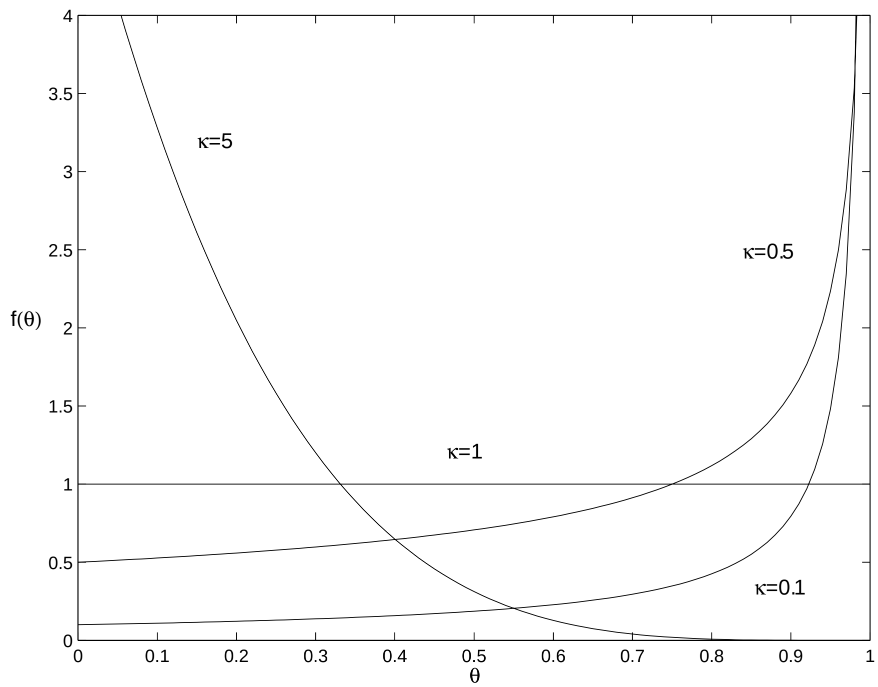


Figure 1: $f(\theta) = \kappa (1-\theta)^{\kappa-1}$ for $\kappa = 0.1, 0.5, 1$ and 5

**Proposition 3** *Let $F \in \mathcal{F}^{\kappa}$. If $\beta \{(1-\delta) c_n(0) - c_r(0)\} > c_n'(0)$, then $\frac{d\Delta}{d\kappa} < 0$ and $\frac{d\Delta}{dc_n(0)} \lesseqgtr 0$.*

As expected, when the mass of consumers shifts towards the lower end of the spectrum ($\kappa$ increases), the optimal sales volume of single use products, $n_{su}$, decreases. Therefore, the remanufacturing potential decreases. Note that the distribution of consumer types $F(\theta)$ impacts the sign of the remanufacturing potential only when $k'(0) > 0$, and not when $k'(0) = 0$.

It is interesting to note that increasing the cost of single-use products impacts the remanufacturing potential in two opposing ways. On one hand, through the term $(1-\delta)c_n(0)$, the remanufacturing potential increases as the unit production cost increases. The intuition is the following: When $q = 0$, the optimal path is $n_t^* = n_{su}$ $\forall t$. It follows that $\frac{\partial R(\nu_t^*)}{\partial r} = \frac{\partial R(n_{su},0)}{\partial r}$ in (5) for $q = 0$. For the consumer profile $(F, \delta)$, it can be shown that $\frac{\partial R(n,0)}{\partial r} = (1-\delta)\frac{\partial R(n,0)}{\partial n}$ for any $n \in [0,1]$. By the definition of $n_{su}$, $\frac{\partial R(n_{su},0)}{\partial n} = c_n(0)$, and we obtain the term $\frac{\partial R(n_{su},0)}{\partial r} = (1-\delta) c_n(0)$ in $\Delta$. It may seem counter-intuitive that the marginal *cost* $c_n(0)$ contributes to the marginal *profit* ($\Delta$). However, this makes sense in a remanufacturing context since remanufacturing is a strategy that exploits the reduction in production cost. Therefore, all else being equal, a higher production cost to start with (without remanufacturing), makes remanufacturing more attractive.

On the other hand, the sales of single-use products, $n_{su}$, decrease as $c_n(0)$ increases. Therefore, in the presence of a fixed cost, it may be that expensive single-use products are not profitable to remanufacture, because the sales of single-use products is too limited to generate a profitable

| Parameter, increasing | Interpretation | Impact on potential |
|---|---|---|
| $c_r(0)$ | remanufacturing cost | negative |
| $k'(0)$ | increase in fixed cost | negative |
| $\beta$ | discount factor/sojourn time on market | positive |
| $\delta$ | perceived depreciation | negative |
| $c_n'(0)$ | increase in unit new product costs | negative |
| $c_n(0)$ | single use production cost | pos./neg. |
| $\kappa$ | consumer profile | negative |

Table 1: Determinants of profitability of remanufacturing

market for remanufactured products. On the other hand, when $k'(0) = 0$, sufficiently expensive single-use products will have a positive remanufacturing potential.

### 3.2.3 Summary of Factors Influencing the Remanufacturing Potential.

We summarize the direct and indirect drivers of profitability of remanufacturing in Table 1. A necessary, but not sufficient, condition for a positive remanufacturing potential can be found between $\{\}$ in (7): If $(1-\delta)\,c_n(0) - c_r(0) < 0$, then the lowest consumer type to whom a new product can be sold without a loss ($\theta = c_n(0)$) has a willingness-to-pay $(1-\delta)\,c_n(0)$ for the remanufactured product that is lower than the production cost of the remanufactured product ($c_r(0)$) and remanufacturing is not viable. Therefore, the condition $(1-\delta)\,c_n(0) - c_r(0) > 0$ is necessary (but not sufficient) to assure a positive remanufacturing potential. In the emerging operations literature on remanufacturing, pure production cost savings $c_n(0) - c_r(0)$ are often assumed to drive remanufacturing activities (Klausner et al., Savaşkan et al., Ferrer). Condition (7) generalizes this construct significantly to take into account the characteristics of the consumer profile, discounting, and the incremental cost of providing remanufacturability.

## 3.3 Characteristics of the Optimal Product Portfolio

In this section, we answer the questions raised in the introduction about the integrated management of a product line consisting of new and remanufactured products.

### 3.3.1 Fitting the Level of Remanufacturability to the Consumer Profile

In §3.2, we showed how the remanufacturing potential (which determines whether to invest in remanufacturability) depends on the production costs and consumer profile $F \in \mathcal{F}^\kappa$. In this subsection, we wish explore how the optimal *level* of remanufacturability changes as a function of the consumer profile, that is, we wish to characterize $q^*$ as a function of $\kappa$. Let $q^*_\kappa$ be the solution of (2) parametrized by $\kappa$. An asymptotic analysis as $\beta \to 1^-$ yields a characterization of $q^*_\kappa$ for $\beta \approx 1$.

**Proposition 4** *If $F \in \mathcal{F}^\kappa$ and $\beta \approx 1$, then for a consumer profile with a higher concentration on the lower end of the market ($\kappa_a > \kappa_b$), the optimal level of remanufacturability is higher: $q^*_{\kappa_a} > q^*_{\kappa_b}$.*

When there is a larger mass of low-end consumers ($\kappa$ large), there is a higher volume of potential buyers to be captured by offering a remanufactured product. A bigger supply of remanufactured products can be generated by building a higher level of remanufacturability into the new product. This would of course increase the fixed cost $k(q)$, but when the discount factor is very high, the initial fixed cost is inconsequential. Therefore, $q^*_\kappa$ increases monotonically in $\kappa$.

At lower discount factors, $k(q)$ becomes consequential. To explore the impact of $k(q)$, we approximate the solution to (1) by linearizing (4) around $\nu_\infty$ and find a closed-form expression approximating $V'_\beta(q)$. Let $\widetilde{q}_\kappa$ denote the approximately optimal remanufacturability level obtained using this expression. Figure 2 plots $\widetilde{q}_\kappa$ for $\beta = 0.7$ and for different levels of fixed costs of the form $k(q) = kq$. Recall from Proposition 2 that the remanufacturing potential decreases in $\kappa$ and $k'(0)$, becoming negative after a threshold value of either factor. This is observed in Figure 2: For each $\kappa$, there is a threshold value of $k$ beyond which $q^* = 0$; and for each fixed cost level, there is a threshold value of $\kappa$ beyond which $q^* = 0$.

Without a fixed cost ($k = 0$), we observe that $\widetilde{q}_\kappa$ monotonically increases as a function of $\kappa$ as Proposition 4 leads us to expect. For $k > 0$, $\widetilde{q}_\kappa$ first increases and then decreases in $\kappa$. This can be explained as follows: As $\kappa$ increases, reaching the low-end consumers by building a higher level of remanufacturability becomes more attractive, so the optimal remanufacturability level increases. However, since the fixed cost also increases in the remanufacturability level, there is a point beyond which the fixed cost dominates and the optimal remanufacturability level starts to decrease in $\kappa$. We can thus conclude that the optimal remanufacturability level is the highest for *medium levels of market heterogeneity*. For markets with high concentrations of customers either on the high end, or, on the low end, the optimal remanufacturability level is low. This result highlights that building a high remanufacturability product is particularly suitable when catering to a diverse market.

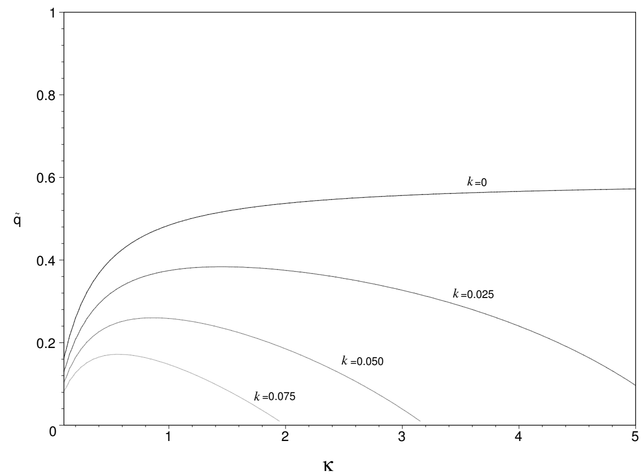


Figure 2: $\widetilde{q}_{\kappa}$ for $k = 0$, 0.025, 0.050 and 0.075 where $k(q) = kq$, for $\beta = 0.7$, $c_n(q) = 0.25 - 0.05\ln(1-q)$, $c_r = 0$, and $\delta = 0.2$.

### 3.3.2 The Role of the New Product

Due to the interdependence of new and remanufactured products, a decrease in demand for new products results in a decrease in the availability of remanufactured products. We investigate the implications of this dependence on pricing strategy. Let $p^*_\infty$ be the price vector that corresponds to the optimal stationary demand volumes.

**Proposition 5** *Let $F \in \mathcal{F}^\kappa$ and $\beta \approx 1$. There exists a consumer profile for which $p^*_{N,\infty} < c_n(q^*)$ Remanufactured product margins are always non-negative.*

Proposition 5 reveals the dual role of new products: They have the potential of generating profits in their own right. At the same time, they generate a volume of used products that are sold at a profit after being remanufactured. In fact, the manufacturer may choose to produce some new products only for the future value that they generate through their sale as remanufactured products, although he sells them at a loss.

In numerical experiments, we observe that as $\kappa$ increases, the optimal prices of both new and remanufactured products decrease at an increasing rate. Combined with the fact that $c_n(q)$ increases in $q$, we see that the optimal new product margin erodes rapidly in $\kappa$.

Figure 3 illustrates this phenomenon. We plot the optimal price of new and remanufactured products against the optimal level of remanufacturability for different values of $\kappa$ (ranging from 0.1 to 25), with $c_n(q) = 0.2 - 0.05\ln(1-q)$ (on the right) and $c_n(q) = 0.35 - 0.05\ln(1-q)$ (on the left). Note that the curves are downward sloping: when the market shifts towards the lower end, the optimal prices of both new and remanufactured products decrease, while the optimal level of remanufacturability increases. The rate of increase of $q^*$ decreases and the rates of decrease of $p_N^*$ and $p_R^*$ increase as $\kappa$ increases. In particular, for high values of $\kappa$, $q^*$ increases only marginally, but optimal prices vary significantly. The optimal price of a new product drops below the unit production cost for some values of $\kappa$. This occurs in markets characterized by a high level of $\kappa$, corresponding to a large enough concentration of low-valuation consumers: To tap into the low-valuation market, the manufacturer needs to generate a high volume of remanufactured products, sometimes sacrificing margins on the new product to do so. This effect is enhanced when the cost of manufacturing is high (compare the plot on the left with the plot on the right of Figure 3) because pricing above cost significantly limits the demand for new products in this case. We observe that the range of $\kappa$ values for which there is a negative margin on new products is larger in this case. The magnitude of the loss increases in $\kappa$. We observed this in other numerical examples: The negative margin occurs for new products with a high unit production cost and markets with high concentrations of low-end consumers.

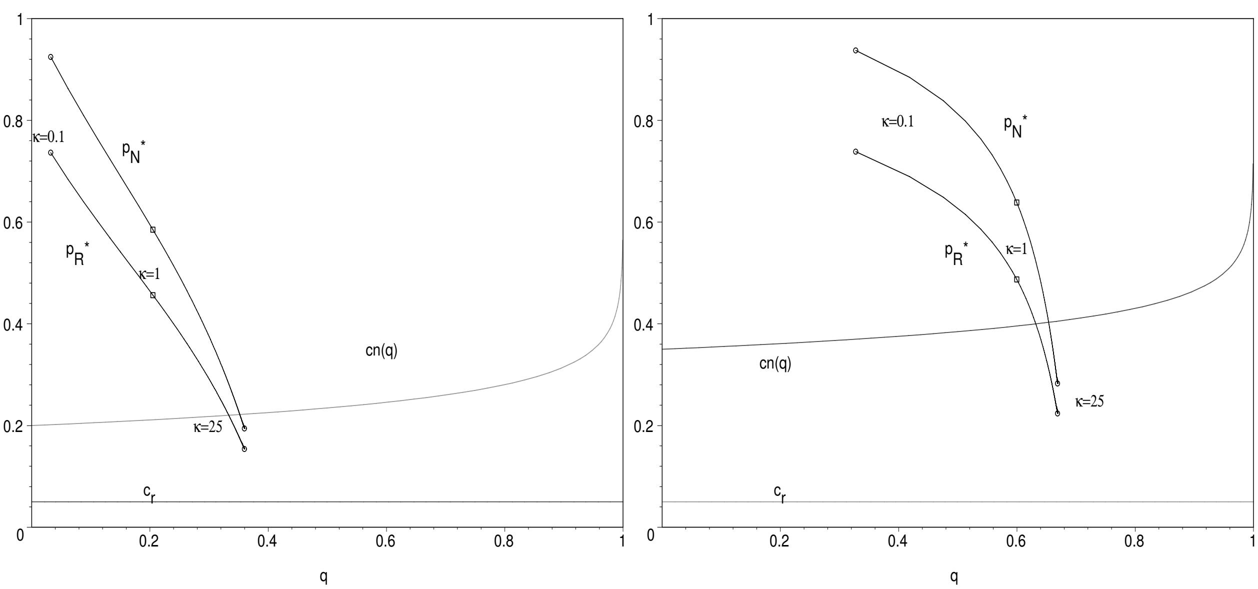


Figure 3: Graphs of $(q^*(\kappa), p_N^*(\kappa))$, $(q^*(\kappa), p_R^*(\kappa))$, $c_n(q)$ and $c_r$ with $\delta = 0.2$, $c_r = 0.05$ and $c_n(q) = 0.2 - 0.05ln(1-q)$ (left) and $c_n(q) = 0.35 - 0.05ln(1-q)$ (right) for values of $\kappa$ ranging from 0.1 to 25.

For a product whose value increases and/or whose cost decreases over time, Dhebar and Oren (1985), Padmanabhan and Bass (1993), and Whang (1995) investigate the evolution of profit margins in a variety of contexts. They show that setting thin or negative margins initially may be optimal; the low-value product is the "loss leader." In contrast, in a remanufacturing context, the loss leader is the high value product.

It is interesting to note that most manufacturer/remanufacturers manage the manufacturing and remanufacturing operations separately. In particular, these operations are typically part of separate profit centers. Our analysis reveals that this practice of focusing on the profits obtained from new and remanufactured products separately can be counterproductive, and that considering the total product line as part of the same profit center may lead to higher profitability for the firm.

#### 3.3.3 The Impact of the Remanufacturing Cost

We explore the sensitivity of the optimal remanufacturability level and sales volumes to the unit remanufacturing cost. Let $c_r(q) = c_{r0} + c_{r1}(q)$.

**Proposition 6** *Let $F \in \mathcal{F}^\kappa$ and $\beta \approx 1$. If $q^* > 0$, then $\frac{dq^*}{dc_{r0}} < 0$, $\frac{dr^*_\infty}{dc_{r0}} < 0$ and $\frac{dn^*_\infty}{dc_{r0}} \gtrless 0$.*

This result states that if the remanufacturing cost is lower, a higher remanufacturability level will be provided and a higher volume of remanufacturable products will be sold at optimality, which is intuitive. In addition, a lower remanufacturing cost may result in either a decrease or an increase in new product sales, which is less intuitive. Since new and remanufactured products are substitutes, we would have expected that higher demand for remanufactured products would go hand in hand with lower demand for new products. However, Proposition 6 shows that a decrease in the remanufacturing cost may lead to an *increase* in the optimal sales volume of the new product. In other words, the two products may exhibit the characteristics of complementary products, although they are substitutes.

In Figure 4, on the left side, we plot optimal prices and production volumes for $c_r$ varying from 0 to 0.08. This is an illustration of the phenomenon of Proposition 6: As $c_r$ increases, $r^*$ *and* $n^*$ both initially decrease, but for higher values of $c_r$, we see that $n^*$ increases and $r^*$ continues to decrease as $c_r$ increases. In Figure 4, on the right side, we plot the prices for new and remanufactured products for the same example. The price of new products decreases as $c_r$ decreases. Keeping all else equal, this would result in less demand for remanufactured products since some people who previously bought remanufactured would now buy new. However, this is not the case: The price of remanufactured products *also decreases*, in such a way that the net effect is a parallel *increase* in demand for remanufactured products.

Why does this counterintuitive complementarity phenomenon occur for low values of $c_r$? A decrease of $c_r$ makes a remanufactured product more attractive with respect to a new product.

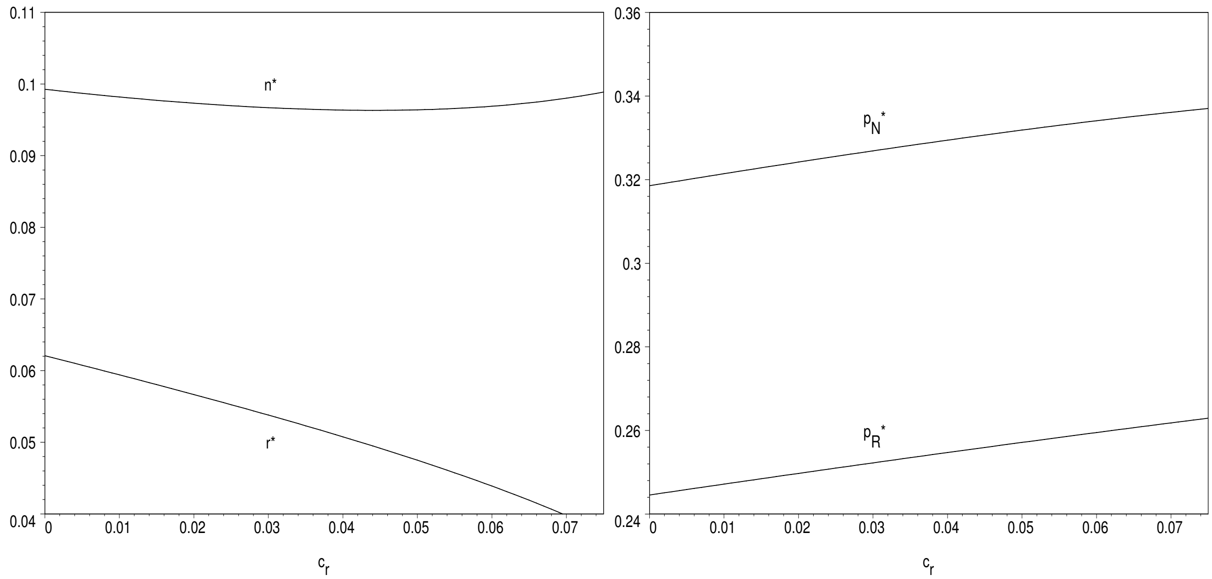


Figure 4: $n^*$ and $r^*$ as a function of $c_r$, with $\kappa = 5$, $\delta = 0.2$ and $c_n(q) = 0.25 - 0.05ln(1-q)$ (left) and $p_N^*$ and $p_R^*$ as a function of $c_r$, with $\kappa = 5$, $\delta = 0.2$ and $c_n(q) = 0.25 - 0.05ln(1-q)$ (right).

Indeed, the optimal (stationary) pricing will be such that the demand for remanufactured products will increase. However, this requires a larger volume of used, remanufacturable products to be available due to the supply constraint. There are two levers that the manufacturer can use to ensure this: The first is to increase the level of remanufacturability. The second is to decrease the price of both new and remanufactured products in such a way that demand for *both* new and remanufactured products increases. If the increase in manufacturing cost due to an increase in the remanufacturability level is relatively low, the manufacturer will chose to increase the supply of used, remanufacturable products by increasing the level of remanufacturability (and $n^*$ decreases). Otherwise, the supply of used remanufacturable products is generated by pricing such that $n^*$ increases. Since $c_n(q)$ is convex, the former effect is seen at low levels of remanufacturability, and the latter, at high levels. Since $q^*$ increases as the remanufacturing cost decreases, the former effect is seen at high remanufacturing costs, and the latter, at low remanufacturing costs.

Remanufacturing is often touted as a strategy that has positive environmental consequences (Thierry et al.). According to this logic, improvements in technologies enabling remanufacturing would be desirable. Consider a situation where the product constitutes an environmental hazard. Proposition 6 states that there are situations in which improving the efficiency of the remanufacturing technology (decreasing the remanufacturing cost) has a perverse environmental impact: The manufacturer now sells a larger volume of new products than before.

# 4 Extension: Competition in the Remanufactured Product Market

In our analysis, we assumed so far that the manufacturer is a monopolist in both the market for new and for remanufactured products. As discussed earlier, it is not uncommon for several independent firms to remanufacture another manufacturer's product. In order to investigate whether and at what level remanufacturability is provided by the manufacturer in such an environment, we develop a model where the manufacturer produces only the new product (and has a monopoly position in that market), and used products are remanufactured by $N$ independent competing remanufacturers. These remanufacturers buy used remanufacturable products from consumers on a perfectly competitive market in the sense that in every period, the price of the used remanufacturable product is such that the market clears. We therefore need to analyze an infinite-horizon, discounted-profit $N+1$ player game. From the Folk Theorem (Fudenberg and Tirole 1991), we know that there may be multiple equilibria for such games. For a clear comparison with the full monopoly case, we select equilibria in which each player's action in each period depends only on the supply of used remanufacturable products available at the beginning of that period, $I$, and not on the history of the game. This is similar to a Markov Perfect equilibrium refinement in stochastic games.

Consumers take the residual value of the new product into account when purchasing it. In order to keep the game tractable, we assume that the consumers sell their used remanufacturable product at the end of its useful life at the prevailing market price, $p_U$. In other words, they do not strategically keep their product in stock in order to sell it at a higher price in a future period. We assume that the price of used remanufacturable products depends only on the supply of such products, $I$, denoted by $p_U(I)$. Since a fraction $q$ of all used products will be remanufacturable in the next period, the discounted residual value to the customer of a new product purchased in the current period is $\beta q p_U(I')$, where $I'$ is the supply of used remanufacturable products that will become available in the next period. If the quoted price for a new product is $\tilde{p}_N$, then, the net discounted acquisition cost of a new product is $\tilde{p}_N - \beta q p_U(I')$ for a consumer. Thus, given prices $(\tilde{p}_N, \tilde{p}_R)$ in the current period and $p_U$ in the next period, the market demand $\nu$ for the current period solves $p(\nu) = (\tilde{p}_N - \beta q p_U, \tilde{p}_R)$, where $p \doteq (p_N(\nu), p_R(\nu))$ is as defined in §2.2.

The manufacturer determines the technology, $q$, and a sales policy $n(I)$, which depends only on the supply of remanufacturable products in the beginning of the period. The remanufacturers, $i \in \mathcal{N} \doteq \{1, ..., N\}$, compete with each other in quantities: For a given $\mathbf{r} = (r_i)_{i \in \mathcal{N}}$ and a fixed $n$, the market price of remanufactured products is $p_R\left(n, \sum_{i \in \mathcal{N}} r_i\right)$. In our Markov perfect equilibrium, each manufacturer determines a policy $r_i(I)$.

Let $\nu(I) \doteq \left(n(I), \sum_{i\in\mathcal{N}} r_i(I)\right)$. Then, remanufacturer $i$'s single-period profit is $\pi_{R,i}(\nu(I), r_i(I), I, q) \doteq (p_R(\nu(I)) - c_r(q) - p_U(I))r_i(I)$, which includes the cost $p_U(I)$ of purchasing used remanufacturable products on the market. The manufacturer's profit in that period is $\pi_M(\nu(I), I, q) \doteq (p_N(\nu(I)) + \beta q p_U(I') - c_n(q))n(I)$, with $I' = I + qn(I) - \sum_{i\in\mathcal{N}} r_i(I)$.

For a given $p_U(.)$, each player chooses in equilibrium a policy that maximizes his discounted profits over the infinite horizon, given the other players' policies. We denote this set of policies by $n^e_{p_U(.)}(I)$ and $\left(r^e_{p_U(.),i}(I)\right)_{i\in\mathcal{N}}$. The market-clearing price of used remanufacturable products, $p^e_U(I)$, is such that either $\sum_{i\in\mathcal{N}} r^e_{p^e_U(.),i}(I) < I$ and $p^e_U(I) = 0$, or, $\sum_{i\in\mathcal{N}} r^e_{p^e_U(.),i}(I) = I$ and $p^e_U(I) > 0$. Let $n^e(I)$ denote $n^e_{p^e_U(.)}(I)$, $r^e_i(I)$ denote $r^e_{p^e_U(.),i}(I)$, and $\nu^e(I)$ denote $\left(n^e(I), \sum_{i\in\mathcal{N}} r^e_i(I)\right)$. The discounted profit for the manufacturer and remanufacturers, for a given level of technology, $q$, are $V^{c,M}_\beta(q) \doteq \sum_{t=0}^\infty \beta^t \pi_M(\nu^e(I_t), I_t, q)$ and $V^{c,R,i}_\beta(q) \doteq \sum_{t=0}^\infty \beta^t \pi_{R,i}(\nu^e(I_t), r^e_i(I_t), I_t, q)$, respectively, with $I_0 = 0$ The market clearing price, $p^e_U(I)$, assures that the resulting path, $\mathcal{P}^c \doteq \{\nu^e(I_0), \nu^e(I_1), ...\}$, is implementable; $\mathcal{P}^c \in \mathcal{I}(0)$.

Proposition 7 develops a sufficient condition under which the manufacturer prefers to build a remanufacturable product.

**Proposition 7** *In a market with independent competing remanufacturers, it is optimal for the manufacturer to produce a remanufacturable product (i.e. $q^* > 0$) if $\Delta > 0$.*

We show here that $\Delta > 0$, which is a sufficient condition in the total monopoly scenario, is also sufficient in this scenario. The economic intuition behind this result is that a market for remanufactured products increases the residual value to the buyers of new products. This allows the manufacturer to charge a higher price for new products and make a higher profit than with a single-use product.

In order to determine the optimal remanufacturability level in presence of competition with independent remanufacturers, we have to characterize the equilibrium. This task is non-trivial for a general consumer profile $(F, \delta)$, but we are able to analyze the special case of a uniform distribution of consumer types. In Figure 5, we display the discounted profits for the manufacturer as a function of the level of remanufacturability, for different levels of competition on the market for remanufactured products.

Keeping all else equal, a manufacturer is better off without competition on the market for remanufactured products. The economic intuition is the following: As the manufacturer does not make any profit on remanufacturing its products, his incentive to produce a remanufacturable product is driven by the residual value of a used remanufacturable product. With increased competition on the market for remanufactured products, the prices of both remanufactured products and used

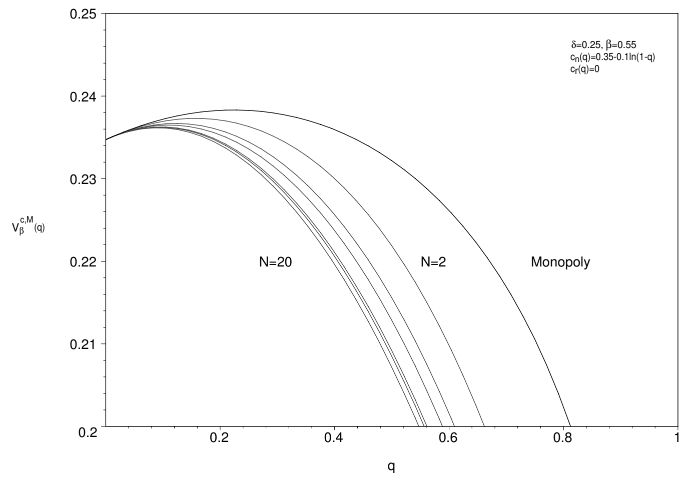


Figure 5: Equilibrium discounted profits for the manufacturer in a market $N = 2, ..., 20$ remanufacturers and optimal discounted profits of a monopolist manufacturer-remanufacturer.

remanufacturable products decrease. This limits the price the manufacturer can charge for the new product, and his profit net of fixed cost from investing in remanufacturability decreases. As a result, the remanufacturability level that is optimal for the manufacturer decreases. Therefore, any legislator encouraging competition for remanufactured products should take into account that the level of remanufacturability of the new product will decrease with competition. These findings also help us understand why some manufacturers of printer cartridges have developed a 'killer chip' that is inserted in the cartridge and that records the amount of remaining ink. The chip cannot be reset when it is re-filled with ink, unless a secret code is known. This makes virtually all refilling by independent remanufacturers impossible. According to environmental activists, '*bad news for the environment as re-use is far better than the landfilling or recycling of empty cartridges. It will also be bad news for consumers because re-filled cartridges are much cheaper than buying new ones*' (Friends of the Earth, 2004).

# 5 Extension: Progressive Market Penetration of New and Remanufactured Products

In the previous sections, we assumed that the total market size is constant over time. In this section, we focus on the progressive penetration of a potential market over time and study the integrated dynamic management of a portfolio of new and remanufactured products over the product life cycle. We capture the progressive market penetration of new and remanufactured products using a variant of the Bass diffusion model, while maintaining the two essential features of our basic model: (a) substitution between new and remanufactured products and (b) a constraint on the diffusion of remanufactured products due to the limited supply of used, remanufacturable products. In addition, we incorporate repeat purchases, which are due to the finite life duration of the product. As outlined in the literature review, the diffusion literature has dealt with these aspects separately.

**Market penetration of new and remanufactured products.** We normalize the total potential market size to 1. We denote the *penetrated market size* in period $t$ by $M_t \in [0,1]$. Following the Bass diffusion model, the potential market size is determined follows:

$$m_t \doteq M_t - M_{t-1} = (a + b i_{n,t-1}) (1 - M_{t-1}), \text{ for } t \geq 1 \tag{8}$$

with $M_0 = 0$. In each period, the fraction of new potential consumers that is added from the untapped market, $\frac{m_t}{1-M_{t-1}}$, is a constant, $a > 0$, the 'innovation' coefficient, plus a term proportional to the installed base of new products, $i_{n,t}$, with $b > 0$, the 'imitation' coefficient. The word-of-mouth propagation is driven in this diffusion process by the installed base of new products, which is the number of consumers that actually own a product in period $t$. The installed base is smaller than the cumulative volume of new product sales as some consumers may have repeatedly purchased a new product. This modification to the Bass diffusion models allows us to capture the impact of repeat purchases in our analysis.

**Product life duration.** We assume that each product (new as well as remanufactured) has a life duration distribution characterized by $\mathbf{h} = (h_1, h_2, ..., h_L)$ with $\sum_{\tau=1}^{L} h_\tau = 1$: After $\tau$ periods of use, a fraction $h_\tau$ of new products require a remanufacturing operation for further use, where $\tau = 1, 2, \ldots, L$. Thus, the average life duration of a product is $\Lambda = \sum_{\tau=1}^{L} \tau h_\tau$. Similarly, after $\tau$ periods of use, a fraction $h_\tau$ of remanufactured products reach the end of their useful life. Due to the finite life duration of the product, each consumer may make repeat purchases during the life cycle of the product.

**Customer purchasing behavior.** In period $t$, $m_t$ potential consumers enter the market. These consumers have types that are distributed over $[0,1]$ according to $F(\theta)$, with $\int_0^1 dF(\theta) = 1$. Depending on their type, the newly entered consumers either ($i$) decide to buy a new product, ($ii$) decide to buy a remanufactured product or ($iii$) decide not to buy a product. For tractability, we assume that a fraction $h_\tau$ of consumers who do not buy any product reappear on the market $\tau$ periods later. In this case, we can imagine that the customer buys an 'outside' product with a life duration of $\mathbf{h}$. With this assumption, the contribution of period $t$'s new potential consumers to period $t + \tau$'s 'repeat' consumers is $h_\tau m_t$ *and* they are distributed according to $F(\theta)$.

**Sales volumes.** Letting $R_t$ be total volume of repeat purchasers in period $t$, we can write the sales of new and remanufactured product as follows:

$$n_t = (R_t + m_t) \int_{\Omega_N(p_t)} dF(\theta) \text{ and } r_t = (R_t + m_t) \int_{\Omega_R(p_t)} dF(\theta) \text{ with } R_t = \sum_{\tau=1}^{\min(L,t)} h_\tau (R_{t-\tau} + m_{t-\tau}), \tag{9}$$

with $R_0 = m_0 = 0$. To explain this equation, in period $t$, there are $m_t$ first-time and $R_t$ repeat potential consumers for a new or remanufactured product. They are distributed according to $F(\theta)$. The prices $p_t$ determine which fraction of the total volume $R_t + m_t$ of customers purchases a new or remanufactured product. The volume of repeat consumers is determined by the fraction $h_\tau$ of potential and repeat consumers in period $t - \tau$ whose new, remanufactured or outside product reaches the end of its useful life at the beginning of period $t$.

**Installed consumer base.** In each period, the installed base of new products is increased by new product sales of the current period and decreased by the products sold in previous periods that reach their end of life:

$$i_{n,t+1} = i_{n,t} + n_t - \sum_{\tau=1}^{\min(L,t)} h_\tau n_{t-\tau}. \tag{10}$$

**Remanufacturing constraint.** In each period, remanufactured product sales are constrained by the availability of remanufacturable products. The supply of remanufacturable products available depends on any possible leftover supply from the previous period and the volume of used products that return from the market and are remanufacturable. Let $I_t$ be the volume of remanufacturable products that are available at the end of period $t$. Then

$$I_t = I_{t-1} + q \sum_{\tau=1}^{\min(L,t-1)} h_\tau n_{t-\tau} - r_{t-1} \text{ with } 0 \leq r_t \leq I_t. \tag{11}$$

Similarly as for the base model, we define an *implementable diffusion* path $\mathcal{P}_d$ starting with initial remanufacturable product inventory $I$ (denoted by $\mathcal{P}_d \in \mathcal{I}(I)$) as a path that is generated by means of an underlying diffusion process: $\mathcal{P}_d \doteq \{(\nu_t, p_t, m_t, R_t, i_{n,t}), t \geq 0 | \nu_t \in \mathcal{D},\ I_0 = I,$ (8), (9), (10), (11) $\forall t \geq 1$, and $r_t \leq I_t\ \forall t \geq 0\}$.

**Capacity adjustment costs.** We assume that the cost of expanding/contracting manufacturing or remanufacturing capacity is proportional to the square of the capacity change: $K(\nu_t, \nu_{t-1}) = K_n (n_t - n_{t-1})^2 + K_r (r_t - r_{t-1})^2$. Such a quadratic cost structure is a well-established way of modelling capacity considerations (e.g. Holt et al. 1955) and allows us to keep the model tractable.

**Time varying perceived depreciation.** We allow for the possibility of a perceptional change concerning remanufactured products over time, as the product is sold and used on the market. We model this by letting the perceived depreciation decrease (relative willingness-to-pay for the remanufactured product increase) as a function of the cumulative sales of remanufactured products, denoted by $S_{r,t}$, where $S_{r,t} = S_{r,t-1} + r_t$. The initial value of the perceived depreciation is denoted by $\delta_0$. We use the following relationship between perceived depreciation in period $t$ and the cumulative sales of remanufactured products, $S_{r,t}$: $\delta_t = \delta_0 - (1 - \exp(-\lambda S_{r,t}))(\delta_0 - \delta)$. Note that perceived depreciation converges to $\delta$. We assume $\delta > 0$. This reflects that there may be inherent concerns about quality that cannot be fully overcome by customer use, or a perception by customers that the fair price of a remanufactured product is below that of a new product as it contains used parts. Also note that this formulation allows the remanufacturability level to influence the convergence rate: A higher level of remanufacturability allows the manufacturer to sell more remanufactured products so that $S_{r,t}$ increases faster, resulting in a faster convergence of perceived depreciation to $\delta$.

**The model.** In summary, for a given level of remanufacturability, we find the price path that maximizes the following objective function:

$$V_\beta(q) \doteq \max_{\mathcal{P}_d \in \mathcal{I}(0)} \sum_{t=0}^{\infty} \beta^t \left(\pi(\nu_t, q) - K(\nu_1, \nu_{t-1})\right),$$

with $n_{-1} = r_{-1} = 0$. Then we optimize with respect to the remanufacturability level $q \in [0, 1]$, as in (2). Note that if $M_0 = 0$, $a = 1$ and $b = 0$, then $M_t = 1$ for $t \geq 1$, and we have instantaneous market penetration, which together with $L = h_1 = 1$ and $K(\nu_t, \nu_{t-1}) = 0$, was the subject of previous sections.

## 5.1 Insights from Numerical Experiments

Recall that our model incorporates repeat sales, a product remanufacturing option and capacity adjustment costs. In this subsection, we first study diffusion dynamics of new and remanufactured products (§5.1.1). Sections 5.1.2 to 5.1.4 then investigate the impact of product diffusion, capacity considerations, and time-varying perceived depreciation on the value and extent of remanufacturability.

Figures are included to facilitate the understanding of the phenomena described, using the following parameters: $\delta = 0.2$, $\beta = 0.97$, $c_n(q) = 0.5 - 0.05 \ln(1 - q)$, $c_r = 0$, $k(q) = 0$, $a = 0.01$ and $h_\tau$ following a symmetric beta distribution over $[1, 20]$ with parameter 2.5. We report results from numerical experiments with a time horizon of 200 periods, but show only the first 180 periods, cutting off the end-of-horizon effect. All figures show the diffusion curves for the optimal price path and the optimal level of remanufacturing.

### 5.1.1 Diffusion Dynamics

*The Impact of Repeat Sales.* To understand the impact of repeat sales, we first describe diffusion dynamics with no remanufacturing ($q = 0$) and a long life duration ($\Lambda \approx \infty$), which precludes repeat purchases (Bass 1969). The optimal sales pattern of the product is determined by innovators' and imitators' behavior: In the early phase of the product life cycle, the sales are driven by innovators who purchase the product for the first time. When the imitation effect is sufficiently high, innovators are joined by imitators early in the product life cycle. This leads to an increase in sales volumes over the early product life cycle. However, as the product diffuses through the market, the remaining number of potential purchasers decreases. Therefore, the volume of purchases decreases and drops to zero when the product is fully diffused through the market. Thus, there exists a period during which sales peak. This is the familiar diffusion curve obtained in Bass (1969).

If the product life duration is finite, first-time purchases will be followed by repeat purchases, with $\Lambda$ periods on average between repeat purchases. Depending on the timing of the first-time sales peak and the duration of the product life, we observe two situations (see Figure 6): ($i$) when the first-time sales peak occurs after the average time to the first repeat purchase, $\Lambda$, and ($ii$) when the first-time sales peak occurs before the average time to the first repeat purchase, $\Lambda$. In Figure 6(i), observe that the peak occurs after period $\Lambda$. Thus, by the first-time sales peak, the replacement market has already started. When first-time sales drop, repeat sales constitute the bulk of the sales. Total sales (i.e. first time sales and repeat sales) steadily grow until a steady regime is reached. In Figure 6(ii), the first-time sales peak occurs before period $\Lambda$. This implies that repeat sales will mostly occur after the first peak, and the first-time sales peak will coincide with the total sales peak, as observed in this figure. Since there is some spread on the product life duration (reflected in $h_\tau$), the pattern of repeat sales will have a smoothed peak. In sum, the sales pattern will oscillate, with an amplitude that dampens until a steady regime is reached.

*Joint Diffusion of New and Remanufactured Products.* Let us first assume that capacity adjustment costs are 0. With remanufacturing as an option ($q \geq 0$) (see top row in Figure 7), we observe that as soon as remanufacturable products are available, the remanufactured product diffuses jointly through the market with the new product. In the left panel ($b = 0.2$), sales of new and remanufactured products increase steadily, until a steady regime is reached. Note that at all times, new and remanufactured products are complementary products, i.e. sales of new and remanufactured products increase simultaneously. In the right panel ($b = 2$), sales of remanufactured products, shifted by $\Lambda$, also oscillate with an amplitude that dampens until a steady regime is reached. Note that new and remanufactured products may behave as complements (between periods 10 and 20) as well as substitutes (around period 20).

Now let us incorporate capacity adjustment costs. The bottom row of Figure 7 shows the

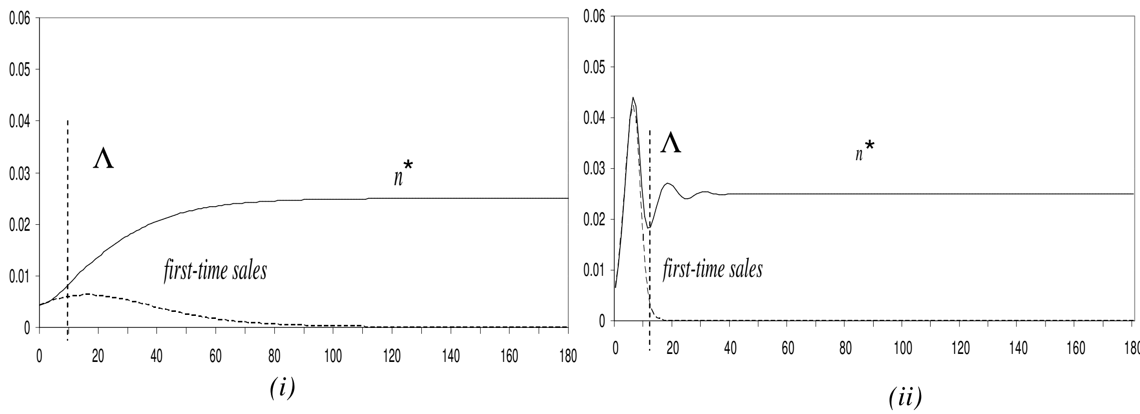


Figure 6: *Optimal sales path of non-remanufacturable new products ($q = 0$) under diffusion. The dashed line represents first-time sales and the solid line represents total sales. In (i), $b = 0.2$ (slow diffusion); in (ii), $b = 2$ (fast diffusion).*

optimal sales paths of new and remanufactured products in case of expensive capacity adjustment costs ($K_n = K_r = 500$). Note that the effect for slowly diffusing products (left panel) is minimal, as capacity is monotonically built up for both new and remanufactured products even in the absence of capacity adjustment costs. For fast diffusing products (right panel), the capacity adjustment costs have a significant impact. Two effects are traded off in determining the optimal capacity investment path: capacity investment cost and discounted revenues. Following the optimal uncapacitated sales path would take into account the fast diffusion of the product and maximize sales revenues, but would entail high capacity adjustment costs, due to the peaks. On the other hand, building capacity very gradually would forego revenues from innovators and imitators in the early periods, although it would limit capacity expansion costs. As a result, the optimal rate of capacity expansion may be faster than in the slow diffusion case, but smoother than the low-cost case.

#### 5.1.2 The Impact of Product Diffusion on Extent and Value of Remanufacturing

Given these sales patterns, we now discuss the impact of product diffusion on the optimal level of remanufacturability and on discounted profits.

**Observation 1** *Faster diffusing products lead to a higher optimal remanufacturability level and*

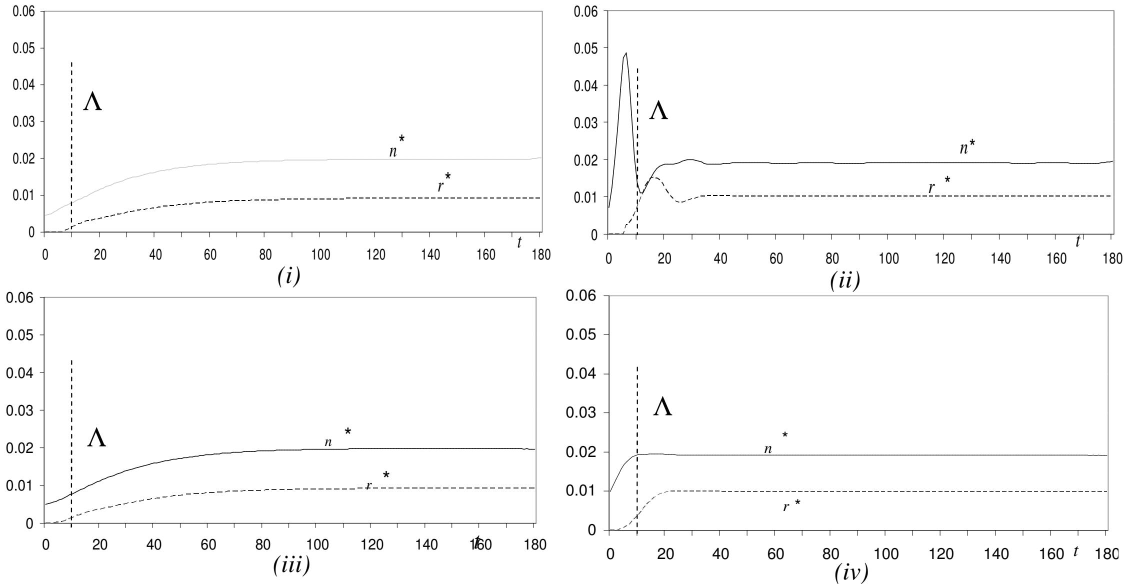


Figure 7: *Optimal sales path of a portfolio of new and remanufactured products with* $q = q^* > 0$ *without capacity adjustment costs (top row) and with capacity adjustment costs (bottom row) for imitation coefficient* $b = 0.2$ *(left column) and* $b = 2$ *(right column).*

*are more profitable.*

For an example, see the curves in Figure 8. When products diffuse faster through the market ($b$ higher), the revenues of new and remanufactured products become available earlier. Under a discounting scheme, this effect will create incentives to make new products more remanufacturable. Obviously, more (discounted) profits will be made. Note, however, that faster diffusion creates more oscillations. In the following subsection, we discuss the impact of these oscillations on the extent and value of remanufacturing when capacity adjustment is expensive.

### 5.1.3 The Effect of Capacity Adjustment Costs on the Extent and Value of Remanufacturing

**Observation 2** *Capacity adjustment costs may decrease the remanufacturability of fast-diffusing products and increase that of slow-diffusing products. Profitability decreases in capacity adjustment costs.*

For an illustration, see the curves in Figure 8. When products diffuse through the markets fast, the resulting sales oscillations make capacity management expensive, which offsets the positive

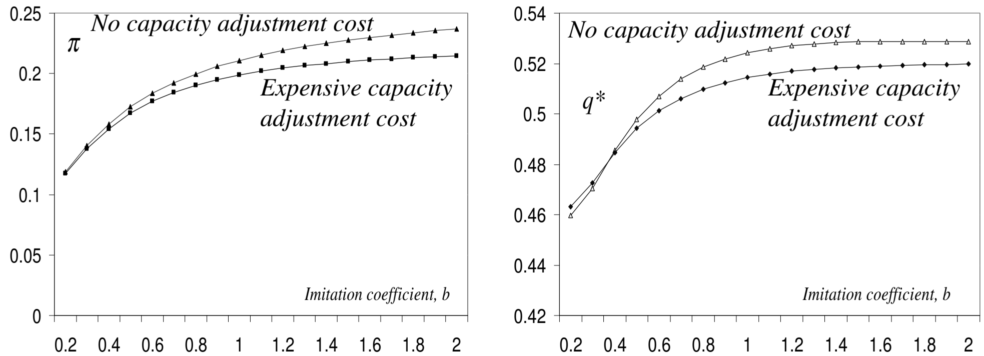


Figure 8: *Value (left) and extent (right) of remanufacturability as a function of the imitation coefficient.*

effect of obtaining revenues from new and remanufactured product markets early. The optimal capacity expansion plan limits the sales volumes to less than what they would have been with low capacity investment costs. Thus, there is no incentive to invest in remanufacturability because this would only increase the sales potential of remanufactured products, which the manufacturer cannot capitalize on. In contrast, when products diffuse slowly, it may be the case that companies invest in remanufacturability more as capacity costs increase. This can be seen in Figure 8 for $b = 0.2$ and 0.3. This increase in remanufacturability may seem counter-intuitive, but can be understood as follows: As new product sales are higher than remanufactured product sales, the total investment into new product capacity is higher. This makes selling new products relatively more expensive. In this case, a strategy to increase profits is to increase the remanufacturability level and generate more revenues from the market for remanufactured products.

#### 5.1.4 The Impact of Time-Varying Depreciation on the Extent and Value of Remanufacturing

In this subsection, we study the impact of time-varying perceived depreciation by letting $\delta_0 = 0.6$ and $\delta = 0.2$. In order to separate different effects, we set the capacity costs at $K_n = K_r = 0$.

**Observation 3** *An increasing speed of convergence results in a higher optimal remanufacturability level.*

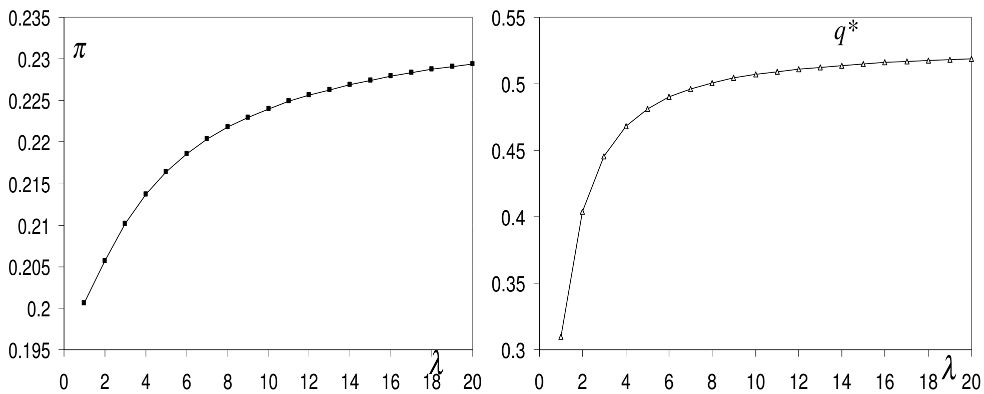


Figure 9: *Value (left) and extent (right) of remanufacturability as a function of the increase rate in perceived depreciation.*

For an illustration, see Figure 9. This can be understood as follows: A higher speed of convergence $\lambda$ allows capturing earlier profits from remanufactured products. This increases the incentive to invest in remanufacturability. In turn, a higher level of remanufacturability results in a faster increase in cumulative sales of remanufactured products $S_{r,t}$, which accelerates the convergence of the perceived depreciation. These two effects, acting in the same direction, incentivize the manufacturer to choose a higher remanufacturability level.

# 6 Discussion and Conclusion

In this paper, we develop insights for managers who consider producing a remanufacturable product. Our model captures some of the key elements driving the decision to introduce a remanufacturable product, and the subsequent management of the total product line; in particular, we focus on the market drivers and technology enablers. Motivated by examples from industry, we consider a market where a remanufactured product is valued less than a new product and is targeted to the lower end of the market. The proportion of used products that can be remanufactured can be increased by applying a more expensive production technology.

To our knowledge, this is the first paper to address the integrated market segmentation and production technology choice problem in a remanufacturing setting. We investigate how these choices are driven by the characteristics of the market and the cost structure. The existing literature focuses mainly on operational issues or deals with technology selection (Klausner et al.) and market segmentation (Ferrer) separately. These two dimensions are strongly coupled via the dependence of the remanufactured product supply on previous new product sales. Our analysis reveals the implications of this dependence. Our key results are summarized below.

We study which characteristics of the consumer profile and the production technology make remanufacturing a profitable strategy. We find that high production costs of the single-use product, low remanufacturing costs and low incremental costs to make a single-use product remanufacturable are the key technology drivers. The consumer profile plays a role in the determination of the profitability of remanufacturing: The more consumers are concentrated on the lower end of the market, the lower the remanufacturing potential. In addition, the consumer profile and the fixed cost jointly interact to determine the optimal remanufacturability level: If the fixed cost is higher, the optimal remanufacturability level is lower, and the market at which this level is attained has more high-valuation consumers. These results highlight that the consumer profile is a crucial element in determining the potential for remanufacturing and the optimal remanufacturability level. Therefore, it would be very useful in practice to invest in understanding the market well before launching a remanufacturable product.

We characterize a specific role of the new product in the portfolio of new and remanufactured products: New products may be sold in order to generate a supply of remanufactured products, on which the profit is made. This role is illustrated by the finding that it may be optimal to sell new products below unit cost. This suggests that manufacturers who also have remanufacturing operations may benefit from managing both new and remanufactured product lines as part of the same profit center. Furthermore, a decrease in the unit remanufacturing cost may lead to an increase in the new product sales volume, in order to supply remanufactured products in response to increased demand for them. This phenomenon has implications for legislation that provides subsidies for remanufacturing in order to reduce the total disposal volume.

We analyze two extensions to the basic model. In the first extension, we investigate whether the manufacturer would produce a remanufacturable product, and if so, what remanufacturability level he would choose if used products were remanufactured and sold by independent competing remanufacturers. We find that the same condition as in the monopoly case is sufficient for the introduction of a remanufacturable product to be profitable, but the optimal level of remanufacturability offered by the manufacturer is lower than in the monopoly model and decreases as the number of competing remanufacturers increases.

In the second extension, we study the value and the extent of remanufacturability when products diffuse gradually through the market. We find interesting product life cycle considerations for portfolios with both new and remanufactured products: The value and extent of product remanufacturing increases as products penetrate the market faster. Furthermore, due to product life cycle dynamics, capacity management decisions become critical and interact with technology selection decisions. We find that the impact of higher capacity adjustment costs on the incentives to invest in remanufacturability depend on the rate of diffusion. With fast diffusion, higher capacity adjustment costs reduce the level of remanufacturability. The opposite may occur with slow diffusing products. Finally, we consider the situation where relative willingness-to-pay for the remanufactured product increases over the product life cycle to its highest possible level as a function of the cumulative sales of remanufactured products. We find that the incentive to invest in remanufacturability increases with the speed of convergence.

We conclude with a discussion of the generality and applicability of our results. We assumed that there is no cost to dispose of the used products. A unit disposal cost $d$ can be easily accommodated in our model. If the manufacturer is responsible for disposal, as the European WEEE directive (2003) on producer responsibility stipulates for example, since all products will eventually be disposed of at the manufacturer's expense, the problem can be reformulated with a modified production cost that includes the disposal cost. If the manufacturer disposes only of returned unusable products, and the consumer disposes of used remanufactured products, a similar reformulation is obtained where both the manufacturing cost and the remanufacturing cost are modified. This is because the maximum price the manufacturer can charge for the remanufactured product to consumer type $\theta$ is then $(1-\delta)\theta - \beta d$; in other words, the disposal cost is implicitly borne by the manufacturer. The results we obtained are valid if the modified costs satisfy the cost-related assumptions made in the analysis.

We assumed that the costs of manufacturing and remanufacturing are constant over the life cycle of the product, but in practice, a new technology may become available for either operation. The sensitivity analysis of §3.3.3 gives us some insight concerning how the optimal product portfolio would be impacted by a potential decrease in the remanufacturing cost. In particular, our results indicate that the manufacturer would choose to build a higher remanufacturability level into the product if he anticipates that the remanufacturing cost will go down in the future. A complete model that captures beliefs about how costs may evolve over time could potentially be developed to rigorously model this phenomenon.

We would like to end with the caveat that although the framework that we developed captures several important elements that factor into the determination of a remanufacturing strategy, in practice, the market decisions and technology choices are much more complex. Comprehensive decision support tools to help managers evaluate various options would therefore be very useful. We hope that our research stimulates such research and development.

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# Market Segmentation and Product Technology Selection for Remanufacturable Products: Appendix


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## 1 Some properties of the revenue function

**Lemma 4** *(i) The Hessian $H$ of the revenue function $R(\cdot)$ is of the form $H = \begin{bmatrix} a+b & a \\ a & a \end{bmatrix}$. (ii) If $F \in \mathcal{F}^k$ and $n > 0$, then $\eta'(\theta)(1+\kappa) - \eta''(\theta) > 0$ and $(1-\eta'(\theta))(1+\kappa) - \eta''(\theta) > 0 \ \forall \ \theta \in [0,1)$ imply $a < 0$ and $b < 0$.*

**Proof**. (i) Recall that $p_N$ and $p_R$ denote the prices of new and remanufactured products, respectively, and that the net utility that a consumer of type $\theta$ derives from buying a new product, a remanufactured product, and no product, is $\theta - p_N$, $\eta(\theta) - p_R$, and 0, respectively. In a given period, consumers choose which product to buy based on the utility that they derive in that period from this purchase. Without loss of generality, we only consider cases where $p_R \leq \eta(p_N)$; if $p_R$ were larger than $\eta(p_N)$, no remanufactured products would be sold, and the price $p_R$ could be reduced to the level $\eta(p_N)$ without affecting the demand for either product. Let $p$ denote the vector $(p_N, p_R)$. Then $\Omega_N(p) \doteq \{\theta \in [0,1] : \theta - p_N \geq (1-\delta)\,\theta - p_R\}$ is the set of consumer types who purchase a new product. $\Omega_R(p)$ is defined analogously as the set of consumer types who purchase a remanufactured product. Define the marginal consumers $\theta_l(p)$ and $\theta_h(p)$ such that $\theta_l$ is indifferent between buying no product and buying a remanufactured product, and $\theta_h$ is indifferent between buying a remanufactured product and a new product. Since $\eta(\theta)$ is a strictly increasing function, $\Omega_R(p) = [\theta_l(p), \theta_h(p))$ and $\Omega_N(p) = [\theta_h(p), 1]$, where $\theta_l(p)$ and $\theta_h(p)$ satisfy

$$\eta(\theta_l) = p_R \text{ and } \theta_h - \eta(\theta_h) = p_N - p_R. \tag{A-1}$$

Taking the derivative of these two equalities with respect to $p_N$ and $p_R$ gives $\frac{\partial \theta_l}{\partial p_R} = \frac{1}{\eta'(\theta_l)}$, $\frac{\partial \theta_l}{\partial p_N} = 0$, $\frac{\partial \theta_h}{\partial p_R} = \frac{-1}{1-\eta'(\theta_h)}$ and $\frac{\partial \theta_h}{\partial p_N} = \frac{1}{1-\eta'(\theta_h)}$.

Let $n$ and $r$ denote the volume of consumers who purchase new and remanufactured products, respectively, and define $\nu \doteq (n, r)$. Then $n = \int_{\Omega_N(p)} dF(\theta) = \int_{\theta_h}^{1} f(\theta)\, d\theta = 1 - F(\theta_h)$ and $r = \int_{\Omega_R(p)} dF(\theta) = \int_{\theta_l}^{\theta_h} f(\theta)\, d\theta = F(\theta_h) - F(\theta_l)$.

Taking the derivative of these two equalities with respect to $n$ and $r$ gives

$$\begin{cases} 1 = -\frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_N}{\partial n} + \frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_R}{\partial n} \\ 0 = -\frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_N}{\partial r} + \frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_R}{\partial r} \\ 0 = \frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_N}{\partial n} - \left( \frac{1}{1-\eta'(\theta_h)} f(\theta_h) + \frac{1}{\eta'(\theta_l)} f(\theta_l) \right) \frac{\partial p_R}{\partial n} \\ 1 = \frac{1}{1-\eta'(\theta_h)} f(\theta_h) \frac{\partial p_N}{\partial r} - \left( \frac{1}{1-\eta'(\theta_h)} f(\theta_h) + \frac{1}{\eta'(\theta_l)} f(\theta_l) \right) \frac{\partial p_R}{\partial r} \end{cases}$$

The simultaneous solution of these four equations yields $\frac{\partial p_N}{\partial n} = -\frac{\eta'(\theta_l)}{f(\theta_l)} - \frac{1-\eta'(\theta_h)}{f(\theta_h)}$ and $\frac{\partial p_N}{\partial r} = \frac{\partial p_R}{\partial r} = \frac{\partial p_R}{\partial n} = -\frac{\eta'(\theta_l)}{f(\theta_l)}$.

Since $R(\nu) = np_N(\nu) + rp_R(\nu)$, we obtain, using the chain rule, that

$$\begin{aligned} &\left[ \begin{array}{cc} \frac{\partial R}{\partial n}, & \frac{\partial R}{\partial r} \end{array} \right] = \\ &\left[ \begin{array}{cc} p_N + n\frac{\partial p_N}{\partial n} + r\frac{\partial p_R}{\partial n} & , p_R + n\frac{\partial p_N}{\partial r} + r\frac{\partial p_R}{\partial r} \end{array} \right] = \\ &\left[ \begin{array}{cc} p_N - \left( \frac{1-\eta'(\theta_h)}{f(\theta_h)} + \frac{\eta'(\theta_l)}{f(\theta_l)} \right)(1 - F(\theta_h)) - \frac{\eta'(\theta_l)}{f(\theta_l)}(F(\theta_h) - F(\theta_l)), & p_R - \frac{\eta'(\theta_l)}{f(\theta_l)}(1 - F(\theta_l)) \end{array} \right]. \end{aligned}$$

Define $G_N(\theta) \doteq \theta - \frac{1-F(\theta)}{f(\theta)}$ and $G_R(\theta) \doteq \eta(\theta) - \eta'(\theta)\frac{1-F(\theta)}{f(\theta)}$. Using A-1, we find

$$\left[ \begin{array}{cc} \frac{\partial R}{\partial n}, & \frac{\partial R}{\partial r} \end{array} \right] = \left[ \begin{array}{cc} G_N(\theta_h) - G_R(\theta_h) + G_R(\theta_l), & G_R(\theta_l) \end{array} \right]. \tag{A-2}$$

Taking the derivative of $\frac{\partial R(\nu)}{\partial n}$ and $\frac{\partial R(\nu)}{\partial r}$ with respect to $r$ and $n$, and following similar steps, we obtain the elements of the Hessian $H$:

$$\frac{\partial^2 R}{\partial n^2} = \left(G'_N(\theta_h) - G'_R(\theta_h)\right)\frac{\partial \theta_h}{\partial n} + G'_R(\theta_l)\frac{\partial \theta_l}{\partial n} \text{ and } \frac{\partial^2 R}{\partial r^2} = \frac{\partial^2 R}{\partial r \partial n} = G'_R(\theta_l)\frac{\partial \theta_l}{\partial r}. \tag{A-3}$$

Since $\frac{\partial \theta_l}{\partial n} = \frac{\partial \theta_l}{\partial r} = -\frac{1}{f(\theta_l)}$, we see that $H = \begin{bmatrix} a+b & a \\ a & a \end{bmatrix}$ with $a = -G'_R(\theta_l)\frac{1}{f(\theta_l)}$ and $b = (G'_N(\theta_h) - G'_R(\theta_h))\frac{\partial \theta_h}{\partial n}$.

(ii) We now specialize this matrix to the case $F(\theta) = 1 - (1-\theta)^{\kappa}$, or $f(\theta) = \kappa(1-\theta)^{\kappa-1}$. For this distribution,

$$G_N(\theta) = \theta - \frac{1-\theta}{\kappa}, \quad G_R(\theta) = \eta(\theta) - \eta'(\theta)\frac{1-\theta}{\kappa} \tag{A-4}$$

so $G'_N(\theta) = \frac{1}{\kappa}+1$ and $G'_R(\theta) = \eta'(\theta)\left(\frac{1}{\kappa}+1\right) - \eta''(\theta)\frac{1-\theta}{\kappa}$. Since $n = 1 - F(\theta_h)$, we have $\theta_h = 1 - n^{\frac{1}{\kappa}}$. Therefore, $\frac{\partial \theta_h}{\partial n} = -\frac{1}{\kappa}(1-\theta_h)^{1-\kappa}$. Substituting, we find

$$\begin{aligned} a &= -\left(\eta'(\theta_l)\left(\frac{1}{\kappa}+1\right) - \eta''(\theta_l)\frac{1-\theta_l}{\kappa}\right)\frac{1}{\kappa}(1-\theta_l)^{1-\kappa} \\ b &= -\left(\left(\frac{1}{\kappa}+1\right)(1-\eta'(\theta_h)) + \eta''(\theta_h)\frac{1-\theta_h}{\kappa}\right)\frac{1}{\kappa}(1-\theta_h)^{1-\kappa}. \end{aligned}$$

$n > 0 \Leftrightarrow \theta_h < 1$. In this case, the last term in both expressions is positive since $\theta_l \leq \theta_h$. When the conditions $\eta'(\theta)(1+\kappa) - \eta''(\theta) > 0$ and $(1-\eta'(\theta))(1+\kappa) - \eta''(\theta) > 0$ hold $\forall\, \theta \in [0,1)$, this is sufficient for the first term in both expressions to be positive. We conclude that $a < 0$ and $b < 0$. In particular, if $\eta(\theta) = (1-\delta)\theta$ with $\delta \in (0,1)$, then $\eta'(\theta) = 1-\delta$ and $\eta''(\theta) = 0$, so these conditions are satisfied. ■

In the remainder of the paper, we will assume $\eta(\theta)$ is such that $a < 0$ and $b < 0$ for the family of distributions $F \in \mathcal{F}^k$ when $n > 0$. In this case, $H$ has the following properties:

$$|H| > 0, \ \frac{\partial^2 R}{\partial n^2} < 0 \ (R(\cdot) \text{ is strictly concave.}) \tag{Property 1}$$

$$\frac{\partial^2 R}{\partial n \partial r} < 0 \ (\text{New and remanufactured products are imperfect substitutes.}) \tag{Property 2}$$

$$\frac{\partial^2 R}{\partial r^2} < 0 \tag{Property 3}$$

$$\left|\frac{\partial^2 R}{\partial n \partial r}\right| < \left|\frac{\partial^2 R}{\partial n^2}\right| \tag{Property 4}$$

$$\frac{1}{2}\frac{\partial^2 R}{\partial r^2} > \frac{\partial^2 R}{\partial n \partial r} \tag{Property 5}$$

## 2 Assumptions

A useful condition for characterizing the optimal path is that the solution to (3) is interior to the feasible region. In addition, to characterize $q^*$ using first order conditions, we need $q^* < 1$. To this end, we introduce Assumption 1 and Assumption 2, and we provide conditions on model parameters that assure that Assumption 1 holds if $F \in \mathcal{F}^{\kappa}$ (Lemma 5). We will assume throughout the derivations that in addition to those assumptions already introduced in the text, the following assumptions hold:

$$\text{The maximizer } (n^*, r^*) \text{ of (3) satisfies } n^* + r^* < 1. \tag{Assumption 1}$$

$$c_n(0) < 1,\ c_r(q) < 1\ \forall\ q \text{ and } \exists\ \overline{q} < 1 : c(q) < \overline{v}(q) \text{ for } q < \overline{q}. \tag{Assumption 2}$$

$$c''(q) > 0 \text{ and } c_n'(q) + \beta q c_r'(q) > 0. \tag{Assumption 3}$$

$$c_n'(q) \text{ and } c_r'(q) \text{ are finite for all } q \in [0,1]. \tag{Assumption 4}$$

$$F \in \mathcal{F}^\kappa. \tag{Assumption 5}$$

$$\eta'(\theta)(1+\kappa) - \eta''(\theta) > 0,\quad (1-\eta'(\theta))(1+\kappa) - \eta''(\theta) > 0,\quad \eta'(\theta) < 1\ \ \forall\ \theta \in [0,1). \tag{Assumption 6}$$

**Lemma 5** *For a consumer profile with $F \in \mathcal{F}^\kappa$, there exists $\kappa_0 > 0$ such that Assumption 1 is satisfied for all $\kappa \in (0, \kappa_0)$.*

**Proof**. The Lagrangian has the form

$$L = \pi(n,r) + \beta v(I + qn - r) + \mu_n n + \mu_r r + \lambda(1-n-r) + \tau(I-r).$$

$(r^*, n^*, \lambda^*, \mu_n^*, \mu_r^*, \tau^*)$ satisfy the K-T conditions:

$$\begin{cases} \frac{\partial \pi}{\partial n} + q\beta v'(I + qn - r) + \mu_n - \lambda = 0 \\ \frac{\partial \pi}{\partial r} - \beta v'(I + qn - r) + \mu_r - \lambda - \tau = 0 \\ \lambda(1-n-r) = 0, \mu_n n = 0, \mu_r r = 0, \tau(I-r) = 0 \\ \lambda \geq 0, \mu_n \geq 0, \mu_r \geq 0, \nu \geq 0 \end{cases}$$

Assume that $r^* + n^* = 1$ and $\mu_n^* = \mu_r^* = 0$Then $\frac{\partial \pi}{\partial n}(n^*, 1-n^*) + q\beta v'(I + qn^* - r^*) - \lambda = 0$, or $\lambda = G_N(\theta_h) - G_R(\theta_h) + G_R(0) - c_n(q) + q\beta v'(I + qn^* - r^*)$. Define $y(\theta_h) \doteq G_N(\theta_h) - G_R(\theta_h) + G_R(0) = \theta_h - \frac{1-\theta_h}{\kappa} - \eta(\theta_h) + \eta'(\theta_h)\frac{1-\theta_h}{\kappa} + \eta(0) - \frac{\eta'(0)}{\kappa}$. Let $\bar{\theta}$ be the maximizer of $y$. If $y(\bar{\theta}) - c_n(0) + v'(0) < 0$, then $\lambda < 0$ for all $\theta_h \in [0,1]$, then $\lambda < 0$, which is inconsistent with the K-T conditions. This condition can be rewritten as $\kappa < a \doteq \frac{(1-\bar{\theta})(1-\eta'(\bar{\theta})) + \eta'(\bar{\theta})}{[v'(0) + \bar{\theta} + \eta(0) - \eta(\bar{\theta}) - c_n(0)]^+}$. Since Assumption 2 holds and $\eta'(\bar{\theta}) < 1$ with Assumption 6, $a$ exists.

Assume that $r^* + n^* = 1$ and $\mu_n^* = 0$, but $\mu_r^* > 0$. Then $n^* = 1$, $r^* = 0$, and $\tau^* = 0$, and $\frac{\partial \pi}{\partial n}(1,0) + q\beta v'(I+q) - \lambda = 0$, or $\lambda = -\frac{1}{\kappa} - c_n(q) + q\beta v'(I+q)$. If $\kappa < \hat{a} \doteq \frac{1}{[v'(0) - c_n(0)]^+}$, then $\lambda < 0$, which is inconsistent with the K-T conditions. With Assumption 2, $\hat{a}$ exists.

Assume that $r^* + n^* = 1$ and $\mu_r^* = 0$, but $\mu_n^* > 0$. (i) Assume $I \geq 1$. Then $n^* = 0$, $r^* = 1$, and $\tau^* = 0$. In this case, $\frac{\partial \pi}{\partial r}(0,1) - \beta v'(I-1) - \lambda = 0$, or $\lambda = \eta(0) - \frac{\eta'(0)}{\kappa} - c_r(q) - \beta v'(I-1)$. If $\kappa < \tilde{a} \doteq \frac{\eta'(0)}{\eta(0)}$, then $\lambda < 0$, which is inconsistent with the K-T conditions. (ii) Assume $I < 1$. Then $r^* = 1$ contradicts the K-T condition $r^* \leq I$.

Assume that $r^* + n^* = 1$ and $\mu_r^* > 0$, $\mu_n^* > 0$. Then $n^* = r^* = 0$, which contradicts the assumption $r^* + n^* = 1$.

We conclude that if $\kappa < \kappa_0 \doteq min(a, \hat{a}, \tilde{a})$, then $r^* + n^* < 1$. ■

# 3 Proofs

**Proof of Lemma 1.** Smith and McCardle (2002) consider a Markov decision process of the form $v_k^*(x_k) = sup_{a_k \in A_k}\{r_k(a_k, x_k) + \delta_k E[v_{k-1}^*(\tilde{x}_{k-1}(a_k, x_k))]\}$ for $k > 0$, $v_0^*(x_0) = 0$. Here, $a_k$ and $x_k \in \Theta$ are the decision variable and the state variable, respectively, in period $k$. The authors establish conditions under which the properties of the reward functions $r_k$ are inherited by the value function $v_k^*$. In particular, they prove the following:

*Proposition 5* (Smith and McCardle 2002). Let $U$ be the set of functions on $X$ satisfying a $C3$ (closed convex cone) property $P$ and let $P^*$ be a joint extension of $P$ on $A$x$\Theta$. If, for all $k$, a) the reward functions $r_k(a_k, x_k)$ satisfy $P^*$ and b) the transitions $\tilde{x}_{k-1}(a_k, x_k)$ satisfy $P^*(\succsim_U)$, then each $v_k^*$ satisfies $P$ and $\lim_{k\to\infty} v_k^*$, if it exists, also satisfies $P$.

The corresponding variables in our problem are the following: $a = \nu$, $A = \mathcal{D} \cap \{r \leq I\}$, $x = I$, $\Theta = [0, \infty)$, $\delta = \beta$, and $\tilde{x} = I + qn - r$. We would like to show that the value function is concave. Concavity is a $C3$ property. The authors show that for convex action and state spaces $A$ and $\Theta$, joint concavity on $A$x$\Theta$ is a joint extension of concavity on $\Theta$. In our problem, the reward function $r_k(\nu_k, I_k) = \pi(\nu_k)$ and is independent of $I_k$. Since $\pi$ is concave, condition (a) is satisfied. Since our recursion is deterministic and the transition function is linear, it trivially satisfies condition (b). In addition, since we have a discounted-cost formulation with a bounded reward function, $V_\beta = \lim_{k\to\infty} v_k$ exists. We conclude that $V_\beta(I; q)$ is a concave function of $I$. Note that if the path $\{(n_t, r_t), t \geq 0\}$ is feasible for $I_0 = I$, then it is feasible for any $I_0 > I$. Thus, $V_\beta(I; q)$ is non-decreasing in $I$. This concludes the proof of part (i). The Hessian $\tilde{H}$ of $f(\nu, I)$ with respect to $\nu$ satisfies $\tilde{H}_{11} < 0$ and $|\tilde{H}| > 0$ so $f$ is strictly concave in $\nu$. A unique maximizer of $f$ on $A$ therefore exists for all $t$ and the optimal path $\{\nu_t^* = (n_t^*, r_t^*), t \geq 0\}$ is unique. ■

**Proof of Lemma 2. Part (i):** We want to show that if $c(q) < \overline{v}(q)$, then there exists a feasible path with a positive discounted profit. We proceed by constructing such a path. Pick $\varepsilon > 0$ and consider the path $\mathcal{P}_\varepsilon = \{(\varepsilon, 0), (\varepsilon, q\varepsilon), (\varepsilon, q\varepsilon), ...\} \in \mathcal{I}(0)$. Using the Taylor series expansion of $R(\nu)$ around the point (0,0), we can write $R(\varepsilon, 0) = R(0,0) + \varepsilon\frac{\partial R(0,0)}{\partial n} + o(\varepsilon) = \varepsilon\frac{\partial R(0,0)}{\partial n} + o(\varepsilon)$ and $R(\varepsilon, q\varepsilon) = R(0,0) + \varepsilon\frac{\partial R(0,0)}{\partial n} + q\varepsilon\frac{\partial R(0,0)}{\partial r} + o(\varepsilon) = \varepsilon\left(\frac{\partial R(0,0)}{\partial n} + q\frac{\partial R(0,0)}{\partial r}\right) + o(\varepsilon)$, where we used the fact that $R(0,0)=0$. Define $\tilde{V}_\beta(\varepsilon;q) \doteq \sum_{\{\nu_t\in\mathcal{P}_\varepsilon,t\geq 0\}}\beta^t\pi(\nu_t,q)$. Then,

$$\begin{aligned}
\tilde{V}_\beta(\varepsilon;q) &= R(\varepsilon,0)-c_n(q)\varepsilon+\sum_{t=1}^{\infty}\beta^t\left(R(\varepsilon,q\varepsilon)-c_n(q)\varepsilon-qc_r(q)\varepsilon\right)\\
&= \varepsilon\frac{\partial R(0,0)}{\partial n}+o(\varepsilon)-c_n(q)\varepsilon+\sum_{t=1}^{\infty}\beta^t\left(\varepsilon\left(\frac{\partial R(0,0)}{\partial n}+q\frac{\partial R(0,0)}{\partial r}\right)+o(\varepsilon)-c_n(q)\varepsilon-qc_r(q)\varepsilon\right)\\
&= \varepsilon\left(\frac{\partial R(0,0)}{\partial n}+\beta q\frac{\partial R(0,0)}{\partial r}\right)+\varepsilon\sum_{t=1}^{\infty}\beta^t\left(\frac{\partial R(0,0)}{\partial n}+\beta q\frac{\partial R(0,0)}{\partial r}\right)\\
&\quad -\varepsilon\left(c_n(q)+\beta qc_r(q)+\sum_{t=1}^{\infty}\beta^t\left(c_n(q)+\beta qc_r(q)\right)\right)+o(\varepsilon)\\
&= \varepsilon\sum_{t=0}^{\infty}\beta^t\left(\frac{\partial R(0,0)}{\partial n}+\beta q\frac{\partial R(0,0)}{\partial r}\right)-\sum_{t=0}^{\infty}\beta^t\left(c_n(q)+\beta qc_r(q)\right)+o(\varepsilon)\\
&= \frac{\varepsilon}{1-\beta}\left(\overline{v}(q)-c(q)\right)+o(\varepsilon)
\end{aligned}$$

To show that $\tilde{V}_\beta(\varepsilon;q)>0$ for some $\varepsilon>0$, consider $\frac{\partial\tilde{V}_\beta}{\partial\varepsilon}=\lim_{\varepsilon\to 0+}\frac{\tilde{V}_\beta(\varepsilon;q)}{\varepsilon}=\frac{1}{1-\beta}\left(\overline{v}(q)-c(q)\right)+\lim_{\varepsilon\to 0+}\frac{o(\varepsilon)}{\varepsilon}=\frac{1}{1-\beta}\left(\overline{v}(q)-c(q)\right)$. Since the latter expression is strictly positive if $c(q)<\overline{v}(q)$ we conclude that there exists an $\varepsilon>0$ such that $\tilde{V}_\beta(\varepsilon;q)>0$ if $c(q)<\overline{v}(q)$**.**

**Part (ii):** We now show that if $c(q)\geq\overline{v}(q)$, then $V_\beta(q)=0$ (and it is optimal not to sell anything). To do this, we first show that there exists a feasible path with zero discounted profit. Next, we find a non-positive upper bound on the discounted profit on any path under the condition $c(q)\geq\overline{v}(q)$.

The discounted profit on the path $\{(0,0),(0,0),(0,0),...\}\in\mathcal{I}(0)$ is 0. Therefore $V_\beta(q)\geq 0$. Take any $\mathcal{P}\in\mathcal{I}(0)$. By the definition of $\mathcal{I}(0)$, $\sum_{t=0}^{T}(qn_t-r_{t+1})\geq 0$ for any $T$. We will now show by induction that $\sum_{t=0}^{T}\beta^t(qn_t-r_{t+1})\geq 0$.

We will first establish that $\sum_{t=0}^{T}\beta^t(qn_t-r_{t+1})\geq\beta^T\sum_{t=0}^{T}(qn_t-r_{t+1})$ by induction on $T$. For $T=0$, this condition holds with equality. For any $T\geq 1$, assume that $\sum_{t=0}^{T-1}\beta^t(qn_t-r_{t+1})\geq\beta^{T-1}\sum_{t=0}^{T-1}(qn_t-r_{t+1})$ (induction step). Then

$$\begin{aligned}
\sum_{t=0}^{T}\beta^t(qn_t-r_{t+1}) &= \sum_{t=0}^{T-1}\beta^t(qn_t-r_{t+1})+\beta^T(qn_t-r_{t+1})\\
&\geq \beta^{T-1}\sum_{t=0}^{T-1}(qn_t-r_{t+1})+\beta^T(qn_t-r_{t+1})\\
&= \beta^T\sum_{t=0}^{T-1}(qn_t-r_{t+1})+\beta^T(qn_t-r_{t+1})\\
&= \beta^T\sum_{t=0}^{T}(qn_t-r_{t+1}),
\end{aligned}$$

where the inequality follows from the induction hypothesis and the next step from the fact that $\beta < 1$ multiplies a positive term. This proves the induction hypothesis. Since $\sum_{t=0}^{T}(qn_t - r_{t+1}) \geq 0$, we conclude that $\sum_{t=0}^{T}\beta^t(qn_t - r_{t+1}) \geq 0$, or $q\sum_{t=0}^{T}\beta^t n_t \geq \sum_{t=1}^{T}\beta^{t-1}r_t$. Multiplying the inequality by $\beta$ and taking the limit for $T \to \infty$, we obtain

$$\sum_{t=1}^{\infty}\beta^t r_t \leq q\beta\sum_{t=0}^{\infty}\beta^t n_t. \tag{A-6}$$

We can use this property to derive an upper bound on $V_{\beta,\mathcal{P}}$, the discounted profits for path $\mathcal{P}$. The profits are:

$$\begin{aligned}
V_{\beta,\mathcal{P}} &= R(\nu_0) - c_n(q)n_0 + \sum_{t=1}^{\infty}\beta^t\left(R(\nu_t) - c_n(q)n_t - c_r(q)r_t\right) \\
&\leq \frac{\partial R(0,0)}{\partial n}n_0 - c_n(q)n_0 \\
&\quad + \sum_{t=1}^{\infty}\beta^t\left(\frac{\partial R(0,0)}{\partial n}n_t + \frac{\partial R(0,0)}{\partial r}r_t - c_n(q)n_t - c_r(q)r_t\right) \\
&= \left(\frac{\partial R(0,0)}{\partial n} - c_n(q)\right)n_0 \\
&\quad + \sum_{t=1}^{\infty}\beta^t\left(\left(\frac{\partial R(0,0)}{\partial n} - c_n(q)\right)n_t + \left(\frac{\partial R(0,0)}{\partial r} - c_r(q)\right)r_t\right) \\
&\leq \left(\frac{\partial R(0,0)}{\partial n} - c_n(q)\right)\sum_{t=0}^{\infty}\beta^t n_t + \left(\frac{\partial R(0,0)}{\partial r} - c_r(q)\right)q\beta\sum_{t=0}^{\infty}\beta^t n_t \\
&= \left(\frac{\partial R(0,0)}{\partial n} - c_n(q) + q\beta\left(\frac{\partial R(0,0)}{\partial r} - c_r(q)\right)\right)\sum_{t=0}^{\infty}\beta^t n_t \\
&= \left(\overline{v}(q) - c(q)\right)\sum_{t=0}^{\infty}\beta^t n_t.
\end{aligned}$$

where the first inequality holds because of the concavity of $R(\nu)$ and the second inequality holds by (A-6). We conclude that $c(q) \geq \overline{v}(q)$ implies $V_\beta(q) = 0$, which is achieved on the path $\{(0,0),(0,0),(0,0),...\}$. ∎

**Proof of Lemma 3.** Consider the path $\mathcal{P}_q = \{(n_s,0),(\tilde{n},\tilde{r}),(\tilde{n},\tilde{r}),(\tilde{n},\tilde{r})...\} \in \mathcal{I}(0)$ and let $V_{\beta,\mathcal{P}}(q)$ denote corresponding discounted profit on this path. We have that $\frac{\partial R(\tilde{n},\tilde{r})}{\partial n} = c_n(q)$ and $\frac{\partial R(n_s,0)}{\partial n} = c_n(q)$. Hence, $\frac{\partial R(n_s,0)}{\partial n} = \frac{\partial R(\tilde{n},\tilde{r})}{\partial n}$. By Property 1 and Property 2, $\frac{\partial R(n,r)}{\partial n}$ is a strictly decreasing function of $r$ and $n$. Therefore, either $\tilde{n} = n_s$ and $\tilde{r} = 0$ or $\tilde{n} < n_s$ and $\tilde{r} > 0$. Then, $I_1 = qn_s \geq q\tilde{n} \geq \tilde{r}$. As $q\tilde{n} \geq \tilde{r}$, we have that $I_{t+1} = I_t + q\tilde{n} - \tilde{r} \geq I_t$ $\forall t$. Since $I_1 \geq \tilde{r}$, it follows that $I_t \geq \tilde{r}$ $\forall t$ and $\mathcal{P} \in \mathcal{I}(0)$. By the definition of $n_s$ and $(\tilde{n},\tilde{r})$, the elements of policy $\mathcal{P}$ achieve the optimal profit over the feasible region in each period starting with the initial condition $I_0 = 0$. Therefore, $\mathcal{P}$ achieves the optimal profit: $V_{\beta,\mathcal{P}}(q) = V_\beta(q)$.

We will show that the derivative of $V_{\beta,\mathcal{P}}(q)$ with respect to $q$ is negative. $V'_{\beta,\mathcal{P}}(q) = \sum_{t=0}^{\infty} \beta^t \frac{\partial \pi(\nu_t,q)}{\partial q}$. $\frac{\partial \pi(n_s,0,q)}{\partial q} = \frac{\partial \pi}{\partial n}\frac{\partial n_s}{\partial q} + \frac{\partial \pi}{\partial q} = \frac{\partial \pi}{\partial q} = -c'_n(q)n_s$, where we used $\frac{\partial \pi(n_s,0)}{n} = 0$. $\frac{\partial \pi(\tilde{\nu}_t,q)}{\partial q} = \frac{\partial \pi}{\partial n}\frac{\partial \tilde{n}}{\partial q} + \frac{\partial \pi}{\partial r}\frac{\partial \tilde{r}}{\partial q} + \frac{\partial \pi}{\partial q} = \frac{\partial \pi}{\partial q} = -c'_n(q)\tilde{n} - c'_r(q)\tilde{r}\ \forall t \geq 1$ where we used $\left(\frac{\partial \pi}{\partial n}, \frac{\partial \pi}{\partial r}\right) = (0,0)$ by the definition of $(\tilde{n}, \tilde{r})$. Summing, we find that $V'_{\beta,\mathcal{P}}(q) = -c'_n(q)\left(n_s + \tilde{n}\frac{\beta}{1-\beta}\right) - c'_r(q)\,\tilde{r}\frac{\beta}{1-\beta}$; Assumption 4 assures that the sum converges. Since $n_s \geq \tilde{n}$, $V'_{\mathcal{P},\beta}(q) \leq -c'_n(q)\,\tilde{n}\frac{1}{1-\beta} - c'_r(q)\,\tilde{r}\frac{\beta}{1-\beta}$. By assumption, $c'_n(q) > 0$. If, in addition, $c'_r(q) = 0$, then $V'_{\mathcal{P},\beta}(q) < 0$. If $c'_r(q) < 0$, then $V'_{\mathcal{P},\beta}(q) \leq (-c'_n(q) - \beta q c'_r(q))\,\tilde{n}\frac{1}{1-\beta}$ since $\tilde{r} \leq q\tilde{n}$. Invoking Assumption 3, we again obtain $V'_{\mathcal{P},\beta}(q) < 0$. ■

**Lemma 6** *Assume $c(q) < \overline{v}(q)$. Recall that the policy function $g(I) = I + qn^*(I) - r^*(I)$ and consider the interval $X = [0, g(0)]$. Then, $g(I) > 0$ for $I \in X$ and either*
*(i) $g'(I) < 0$, $|g'(I)| < 1$ and $r^*(I) = I$ for $I \in X$, or*
*(ii) there exists $\underline{I} \in X$ such that $g'(I) < 0$, $|g'(I)| < 1$ and $r^*(I) = I$ for $I \in [0, \underline{I}]$, and $g'(I) \geq 0$, $r^*(I) < I$ for $I \in (\underline{I}, g(0)]$. In addition, if $q\tilde{n} < \tilde{r}$, $g(I) < I$ on $(\underline{I}, g(0)]$.*

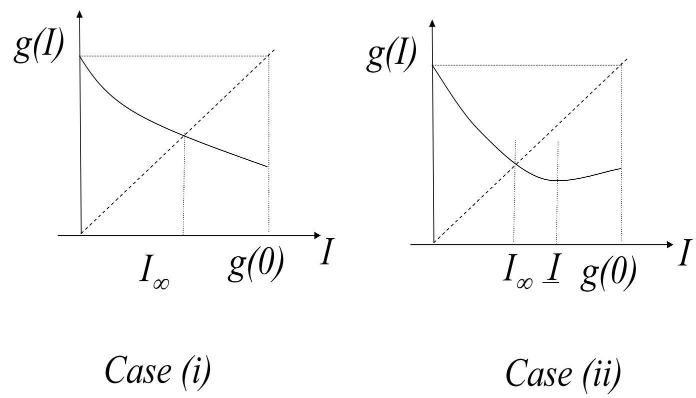


Figure 10: Policy function for Lemma 6

**Proof** In this lemma, we prove that the policy function $g$ has one of the two forms shown in Figure 6 (decreasing or U-shaped on $[0, g(0)]$, which is the relevant interval, as we shall show.

For notational simplicity, we suppress the dependence of $\pi(\nu, q)$ and $V_\beta(I, q)$ on $q$ in this proof.

We make the following conjecture concerning $(n^*(I), r^*(I))$: There exists an interval $[0, \tilde{I}]$ on which $n^*(I) > 0$ and $r^*(I) = I$, i.e. there exists an interval over which the constraint $r \leq I$ in

(3) is binding. Assuming that the conjecture is correct, we characterize $n^*(I)$ and $g(I)$ in Step 1a. We validate the conjecture in Step 1b, and determine whether it holds for the entire interval $[0, g(0)]$. If so, then part (i) is proven with $\tilde{I} = g(0)$; otherwise, we determine the exact interval $[0, \underline{I}]$ for which the conjecture holds. We make a second conjecture concerning $(n^*(I), r^*(I))$: On $(\underline{I}, g(0)]$, $n^*(I) > 0$ and $0 < r^*(I) < I$ . Steps 2a and 2b characterize $g'$ under these assumptions and validate this conjecture, respectively, proving part (ii).

**Step 1a: Characterization of** $n^*(I)$ and $g(I)$ **under the conjecture** $r^*(I) = I$ **and** $n^*(I) > 0$ on $[0, \tilde{I}]$.

Consider the maximization problem in (3). By Lemma Assumption 1, $r^*(I) + n^*(I) < 1$. In addition, we conjectured that $n^*(I) > 0$, so $n^*(I)$ is an interior solution. Finally, we conjectured a boundary solution $r^*(I) = I$. Therefore, $n^*(I)$ and $r^*(I)$ jointly satisfy

$$\frac{\partial \pi(n^*(I), I)}{\partial n} + \beta q \frac{\partial V_\beta(qn^*(I))}{\partial I} = 0. \tag{A-7}$$

$$\frac{\partial \pi(n^*(I), I)}{\partial r} - \beta \frac{\partial V_\beta(qn^*(I))}{\partial I} > 0. \tag{A-8}$$

We now characterize $g(I)$ using A-7: Taking the derivative of (A-7) with respect to $I$ and using the chain rule, we can calculate $n' \doteq \frac{dn^*(I)}{dI}$:

$$\frac{\partial^2 \pi(n^*(I), I)}{\partial n^2} n' + \frac{\partial^2 \pi(n^*(I), I)}{\partial n \partial r} + \beta q^2 V_\beta''(qn^*(I)) n' = 0.$$

We find $n' = -\frac{\frac{\partial^2 \pi(n^*(I),I)}{\partial n \partial r}}{\frac{\partial^2 \pi(n^*(I),I)}{\partial n^2} + \beta q^2 V_\beta''(qn^*(I))} < 0$ where the inequality follows by Property 1, Property 2 and the concavity of $V_\beta$. Since $g(I) = qn^*(I)$ under the conjecture, $g'(I) = qn' < 0$ and $|n'| = \frac{-\frac{\partial^2 \pi(n,I)}{\partial n \partial r}}{-\frac{\partial^2 \pi(n,I)}{\partial n^2} - \beta q^2 V_\beta''(qn,I)} \leq \frac{-\frac{\partial^2 \pi(n,I)}{\partial n \partial r}}{-\frac{\partial^2 \pi(n,I)}{\partial n^2}} < 1$ (the former step because $V_\beta''(qn) \leq 0$ and the latter step by Property 4). Thus $|g'(I)| < 1$. To summarize, we have shown that $g'(I) < 0$, $|g'(I)| < 1$ and $n^*(I)$ strictly decreases if we assume that $r^*(I) = I$ and $n^*(I) > 0$.

**Step 1b: Validation of the conjecture**

We now need to show that there exists a range $[0, \tilde{I}]$ on which $n^*(I) > 0$ and A-8 is satisfied. Let us take the derivative of the two terms in A-8 with respect to $I$. We find $\frac{\partial}{\partial I}\left(\frac{\partial \pi(n^*(I),I)}{\partial r}\right) = \frac{\partial^2 \pi}{\partial n \partial r} n' + \frac{\partial^2 \pi}{\partial^2 r} = \frac{|H| + \beta q \frac{\partial^2 \pi}{\partial^2 r} V_\beta''(qn^*)}{\frac{\partial^2 \pi}{\partial n^2} + \beta q V_\beta''(qn^*)} < 0$ and $\frac{\partial}{\partial I}\left(\beta V_\beta'(qn^*(I))\right) = \beta q V_\beta''(qn^*(I)) n' > 0$. A-8 holds at $I = 0$. As the first term in (A-8) strictly decreases in $I$ and the second term strictly increases in $I$, one of the following two cases is true on $[0, g(0)]$: either (i) A-8 holds $\forall I \in [0, g(0)]$ or (ii) there exists $\underline{I} \in (0, g(0)]$ for which A-8 is satisfied at equality. Expressing this more precisely, we have one of the following two cases:

(i) $\frac{\partial \pi(n^*(g(0)), g(0))}{\partial r} - \beta V_\beta'(qn^*(g(0))) > 0$. Then $g'(I) < 0$, $|g'(I)| < 1$ and $n^*(I)$ strictly decreases

on $X = [0, g(0)]$. Since $|g'(I)| < 1$, we have $g(g(0)) > 0$, that is, $n^*(g(0)) > 0$ (since $g = qn^*$ on this range) which, together with the fact that $n^*(I)$ strictly decreases on this range, validates the conjecture that $n^*(I) > 0$ over $[0, g(0)]$.
(ii) There exists $\underline{I} \doteq \sup\left(I : \frac{\partial \pi(n^*(I),I)}{\partial r} - \beta V'_\beta(qn^*(I)) > 0\right) < g(0)$ The conjecture that $r^*(I) = I$ on $[0, \tilde{I}]$ is validated with $\tilde{I} = \underline{I}$ since A-8 holds on $[0, \underline{I})$, with equality holding only at $I = \underline{I}$. Then $g'(I) < 0$, $|g'(I)| < 1$ and $n^*(I)$ strictly decreases on $[0, \underline{I}]$.

**Step 2a: Characterization of $g'(I)$ under the conjecture $0 < r^*(I) < I$ and $n^*(I) > 0$ on $(\underline{I}, g(0)]$**

We now complete case (ii) by characterizing $(n^*(I), r^*(I))$ over $(\underline{I}, g(0)]$ We conjecture that $0 < r^*(I) < I$ and $n^*(I) > 0$ for $I \in (\underline{I}, g(0)]$. In this case, $n^*(I)$ and $r^*(I)$ satisfy the first order conditions of the right hand side of (3):

$$\frac{\partial \pi(n^*(I), r^*(I))}{\partial n} + \beta q \frac{\partial V_\beta(g(I))}{\partial I} = 0. \tag{A-9}$$

$$\frac{\partial \pi(n^*(I), r^*(I))}{\partial r} - \beta \frac{\partial V_\beta(g(I))}{\partial I} = 0. \tag{A-10}$$

We also know that $V_\beta, n^*, r^*$ jointly satisfy $V_\beta(I) = \pi(n^*(I), r^*(I)) + \beta V_\beta(g(I))$ with $g(I) = I + qn^*(I) - r^*(I)$. Taking the derivative of this expression with respect to $I$ gives $V'_\beta(I) = \frac{\partial \pi}{\partial n} n' + \frac{\partial \pi}{\partial r} r' + \beta V'_\beta(g(I)) g'(I)$ (with $r' \doteq \frac{dr^*(I)}{dI}$). Taking the derivative of $g(I)$ and evaluating it at $(n^*(I), r^*(I))$ gives $g'(I) = 1 + qn' - r'$. Substituting $g'(I)$ in the previous expression, collecting terms, and simplifying using A-9 and A-10, we find

$$V'_\beta(I) = \beta V'_\beta(g(I)). \tag{A-11}$$

$$V''_\beta(I) = \beta V''_\beta(g(I))\, g'(I). \tag{A-12}$$

By Lemma 1, $V'_\beta(I) \geq 0$. First consider the case $V'_\beta(I) = 0$. By (A-11), if $V'_\beta(I) = 0$, we have $V'_\beta(g(I)) = 0$. Then, by A-9 and A-10 and the definition of $\tilde{\nu}$, $(n^*(I), r^*(I)) = (\widetilde{n}, \widetilde{r})$. Thus, $g(I) = I + q\widetilde{n} - \widetilde{r}$, and $g'(I) = 1 > 0$. Note that if $q\widetilde{n} < \widetilde{r}$, then $g(I) < I$.

Next consider the case $V'_\beta(I) > 0$. By (A-11), if $V'_\beta(I) > 0$, then, as $\beta < 1$, we must have that $V'_\beta(g(I)) > V'_\beta(I)$, and by the concavity of $V_\beta$, $g(I) < I$. We conclude that if $q\widetilde{n} < \widetilde{r}$, then $g(I) < I$ on $(\underline{I}, g(0)]$.

To characterize $g'(I)$ when $V'_\beta(I) > 0$, consider the following three subcases:

(a) $V''_\beta(g(I)) < 0$ and $V''_\beta(I) < 0$. Since $0 < \beta < 1$, and A-12 must hold, $g'(I) > 0$.

(b) $V''_\beta(g(I)) < 0$ and $V''_\beta(I) = 0$. Since $\beta > 0$ and A-12 must hold, $g'(I) = 0$.

(c) $V''_\beta(g(I)) = 0$. Taking the derivative of A-9 and A-10 with respect to $I$ and using $V''_\beta(g(I)) = 0$ gives a system of two equations in the two unknowns $n'(I)$ and $r'(I)$ whose only solution is $n'(I) = 0$ and $r'(I) = 0$. In this case, $g'(I) = 1 > 0$.

Thus, we conclude that $g'(I) \geq 0$ on $(\underline{I}, g(0)]$.

**Step 2b: Validation of the conjecture**

We will first determine the sign of $n'$ and $r'$. Taking the derivative of (A-9) and (A-10) with respect to $I$, we obtain

$$\begin{array}{c}\left(\frac{\partial^2\pi}{\partial n^2} + \beta q^2 \frac{\partial^2 V_\beta}{\partial I^2}\right) n' + \left(\frac{\partial^2\pi}{\partial n \partial r} - \beta q \frac{\partial^2 V_\beta}{\partial I^2}\right) r' = -q\beta \frac{\partial^2 V_\beta}{\partial I^2} \\ \left(\frac{\partial^2\pi}{\partial r \partial n} - \beta q \frac{\partial^2 V_\beta}{\partial I^2}\right) n' + \left(\frac{\partial^2\pi}{\partial r^2} + \beta \frac{\partial^2 V_\beta}{\partial I^2}\right) r' = \beta \frac{\partial^2 V_\beta}{\partial I^2}\end{array} \tag{A-13}$$

from which we solve for $(n', r')$:

$$\left[\begin{array}{c} n' \\ r' \end{array}\right] = \frac{\beta \frac{\partial^2 V_\beta}{\partial I^2}}{|H| + \beta\left(\frac{\partial^2\pi}{\partial n^2} + q^2 \frac{\partial^2\pi}{\partial r^2} + 2q \frac{\partial^2\pi}{\partial n \partial r}\right) \frac{\partial^2 V_\beta}{\partial I^2}} \left[\begin{array}{c} -q \frac{\partial^2\pi}{\partial r^2} - \frac{\partial^2\pi}{\partial n \partial r} \\ q \frac{\partial^2\pi}{\partial n \partial r} + \frac{\partial^2\pi}{\partial n^2} \end{array}\right].$$

The signs of each term can easily be determined: $\frac{\partial^2 V_\beta}{\partial I^2} \leq 0$, $|H| + \beta\left(\frac{\partial^2\pi}{\partial n^2} + q^2 \frac{\partial^2\pi}{\partial r^2} + 2 \frac{\partial^2\pi}{\partial n \partial r} q\right) \frac{\partial^2 V_\beta}{\partial I^2} \geq 0$, $-q \frac{\partial^2\pi}{\partial r^2} - \frac{\partial^2\pi}{\partial n \partial r} > 0$ and $q \frac{\partial^2\pi}{\partial n \partial r} + \frac{\partial^2\pi}{\partial n^2} < 0$ by Lemma 1, Property 2, Property 3 and Property 4. Therefore, $n' \leq 0$, $r' \geq 0$ and $|n'| = -n'$. Showing that $|n'| < 1$ is equivalent to showing that

$$\beta\left(q(1-q) \frac{\partial^2\pi}{\partial r^2} + (1-2q) \frac{\partial^2\pi}{\partial n \partial r} - \frac{\partial^2\pi}{\partial n^2}\right) \frac{\partial^2 V_\beta}{\partial I^2} < |H|. \tag{A-14}$$

With Property 5 we obtain

$$q(1-q) \frac{\partial^2\pi}{\partial r^2} + (1-2q) \frac{\partial^2\pi}{\partial n \partial r} > \frac{\partial^2\pi}{\partial n \partial r},$$

and with Property 4, we obtain further that

$$q(1-q) \frac{\partial^2\pi}{\partial r^2} + (1-2q) \frac{\partial^2\pi}{\partial n \partial r} > \frac{\partial^2\pi}{\partial n^2}.$$

Together with $\frac{\partial^2 V_\beta}{\partial I^2} < 0$ and $|H| > 0$, we have that (A-14) is satisfied. We conclude that $|n'| < 1$.

Since $r^*(\underline{I}) = \underline{I} > 0$ and we have proven that $r' \geq 0$ on $(\underline{I}, g(0)]$, we conclude that $r^*(I) > 0$ on $(\underline{I}, g(0)]$. However, since we have proven that $n' < 0$, it is not as immediate that $n^*(I) > 0 \;\; \forall I \in (\underline{I}, g(0)]$. On the other hand, as $|n'| < 1$ both on $(\underline{I}, g(0)]$ and on $[0, \underline{I}]$ (by part i), we have that $n^*(g(0)) > 0$. Because $n' < 0$ on [0,g(0)], we conclude that $n^*(I) > 0$ on $(\underline{I}, g(0)]$ is also verified. ■

**Lemma 7** *Define* $\nu_\infty \doteq (n_\infty, r_\infty) \in \mathcal{D}$ *simultaneously satisfying* $\frac{\partial R(\nu_\infty)}{\partial n} + q\beta \frac{\partial R(\nu_\infty)}{\partial r} = c(q)$ *and* $qn_\infty = r_\infty$. *If* $c(q) < \overline{v}(q)$ *and* $q\tilde{n} < \tilde{r}$, *then,*
*(i) There exists a unique* $I_\infty \in [0, g(0)]$ *such that* $I_\infty$ *solves* $g(I) = I$. *In addition,* $g'(I_\infty) < 0$ *and* $|g'(I_\infty)| < 1$. *Moreover,* $r_\infty = I_\infty$.
*(ii) The region* $[g(\underline{I}), \underline{I}]$ *is a capture region, that is,* $I_t \in [g(\underline{I}), \underline{I}]$ *implies that* $I_{t+1} = g(I_t) \in [g(\underline{I}), \underline{I}]$.
*(iii) Starting with* $I_0 = 0$, *there exists a* $T_q \geq 0$ *such that* $r_t^* = I_t \; \forall t \geq T_q$. *Moreover,* $lim_{t\to\infty} I_t = I_\infty$.

**Proof** Part (i): We start by conjecturing that there exists $I_\infty$ such that $g(I_\infty) = I_\infty$, $g'(I_\infty) < 0$ and $|g'(I_\infty)| < 1$. Then, $I_\infty = g(I_\infty) = I_\infty + qn^*(I_\infty) - r^*(I_\infty)$, implying that

$$qn^*(I_\infty) = r^*(I_\infty). \tag{A-15}$$

By Lemma 6, we know that $g'(I_\infty) \leq 0$ and $|g'(I_\infty)| < 1$ if and only if $r^*(I_\infty) = I_\infty$, so under the above conjecture, $I_\infty$ satisfies A-7:

$$\frac{\partial \pi\left(n^*(I_\infty), I_\infty\right)}{\partial n} + \beta q \frac{\partial V_\beta\left(qn^*(I_\infty)\right)}{\partial I} = 0. \tag{A-16}$$

Also recall from Lemma 6 that $V_\beta(I) = \pi\left(n^*(I), I\right) + \beta V_\beta\left(qn^*(I)\right))$ for any $I$ such that $r^*(I) = I$. Using the chain rule, $V'_\beta(I) = \frac{\partial \pi}{\partial n}\frac{\partial n^*}{\partial I} + \frac{\partial \pi}{\partial r} + \beta q V'_\beta(qn^*(I))\frac{\partial n^*}{\partial I} = \frac{\partial n^*}{\partial I}(\frac{\partial \pi}{\partial n} + \beta q V'_\beta(qn^*(I))) + \frac{\partial \pi}{\partial r} = \frac{\partial \pi}{\partial r}$ where the last equality follows by substituting A-7. In particular, under the above conjecture, this equality holds for $I = I_\infty$, yielding

$$V'_\beta(I_\infty) = \frac{\partial \pi(n^*(I_\infty), I_\infty)}{\partial r} \tag{A-17}$$

In addition, since $I_\infty = r^*(I_\infty) = qn^*(I_\infty)$,

$$V'_\beta(qn^*(I_\infty)) = \frac{\partial \pi(n^*(I_\infty), qn^*(I_\infty))}{\partial r}. \tag{A-18}$$

Substituting this equation in A-16, we find:

$$\frac{\partial \pi\left(n^*(I_\infty), qn^*(I_\infty)\right)}{\partial n} + q\beta \frac{\partial \pi\left(n^*(I_\infty), qn^*(I_\infty)\right)}{\partial r} = 0. \tag{A-19}$$

Consider $N\left(n\right) \doteq \frac{\partial \pi(n,qn)}{\partial n} + q\beta\frac{\partial \pi(n,qn)}{\partial r}$. As $N'\left(n\right) = \frac{\partial^2 \pi(n,qn)}{\partial n^2} + q\frac{\partial^2 \pi(n,qn)}{\partial n \partial r} + q\beta\left(\frac{\partial^2 \pi(n,qn)}{\partial n \partial r} + q\frac{\partial^2 \pi(n,qn)}{\partial r^2}\right) < 0$ since all terms are negative, the solution to (A-19), if it exists, is unique. Now, we show that such a solution exists. $N\left(0\right) = \overline{v}\left(q\right) - c(q)$ by definition. Since $\overline{v}\left(q\right) > c(q)$ we have $N(0) > 0$. Note that $n \in [0, \frac{1}{1+q}]$, as $n + qn \leq 1$. Thus, if we show that $N(\frac{1}{1+q}) < 0$, then, we will have proven that $N\left(n\right) = 0$ has a unique solution $\bar{n}$ such that $\bar{n} > 0$ and $\bar{n} + q\bar{n} < 1$. The proof of Lemma 5 develops conditions on $\kappa$ that satisfy $\frac{\partial \pi(n,1-n)}{\partial n} + q\beta\frac{\partial \pi(n,1-n)}{\partial r} < 0$. Noting that at $n = \frac{1}{1+q}$, $r = 1 - n$, we conclude that $\frac{\partial \pi(\frac{1}{1+q},\frac{q}{1+q})}{\partial n} + q\beta\frac{\partial \pi(\frac{1}{1+q},\frac{q}{1+q})}{\partial r} < 0$, which can be rewritten as $N(\frac{1}{1+q}) < 0$. We further conclude that $n^*(I_\infty)$ solving A-19 is unique and that $(n^*(I_\infty), qn^*(I_\infty)) \in int(\mathcal{D})$. In addition, since $\frac{dn^*(I)}{dI} < 0$ in this range (as shown in the proof of part (i) in Lemma 6), $I_\infty$ is unique.

Recall that $\nu_\infty \doteq (n_\infty, r_\infty)$ satisfying $\frac{\partial \pi(\nu_\infty)}{\partial n} + q\beta\frac{\partial \pi(\nu_\infty)}{\partial r} = 0$, $qn_\infty = r_\infty$ and $\nu_\infty \in \mathcal{D}$ simultaneously. Comparing A-15 and A-19 with the definition of $\nu_\infty$, and noting that $(n^*(I_\infty), qn^*(I_\infty)) \in int(\mathcal{D})$, we conclude that $n_\infty = n^*(I_\infty)$ and $r_\infty = r^*(I_\infty)$ under the conjecture. In addition, since $I_\infty = r^*(I_\infty)$ under the conjecture, we have that $r_\infty = I_\infty$.

We now need to show that the conjecture is true. By Lemma 6, $I_\infty$ satisfies the conjecture if $\frac{\partial \pi(n^*(I_\infty),I_\infty)}{\partial r} - \beta V'_\beta\left(qn^*\left(I_\infty\right)\right) > 0$, or, using A-15 and substituting A-18, if $\left(1-\beta\right)\frac{\partial \pi(n^*(I_\infty),qn^*(I_\infty))}{\partial r} >$

0, or, since $\beta > 0$,

$$\frac{\partial\pi\left(n^*\left(I_\infty\right), qn^*\left(I_\infty\right)\right)}{\partial r} > 0. \tag{A-20}$$

Let the functions $\underline{r}(n)$ and $\overline{r}(n)$ be defined by $\frac{\partial\pi(n,\underline{r}(n))}{\partial n} = 0$ and $\frac{\partial\pi(n,\overline{r}(n))}{\partial r} = 0$. Consider Figure 11. We will show that if $q\tilde{n} < \tilde{r}$, then the solution to (A-19) lies on the segment determined by $r = qn$, $r < \overline{r}(n)$, and $r > \underline{r}(n)$ (marked in bold in Figure 11), and that on this segment, $\frac{\partial\pi(n,qn)}{\partial r} > 0$. In this case, $I_\infty$ satisfies the conjecture.

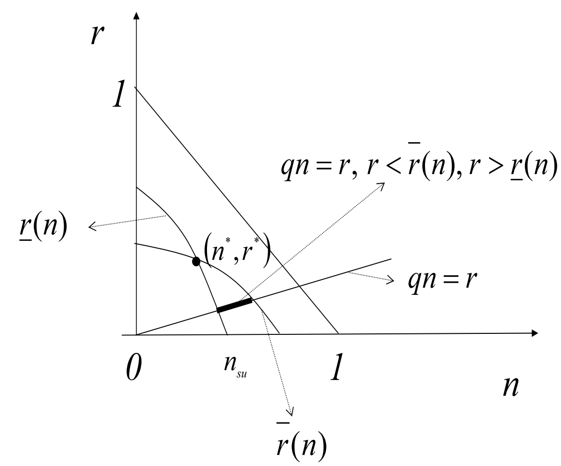


Figure 11: $r < \overline{r}(n)$, and $r > \underline{r}(n)$ for Lemma 7

Since $R(\nu)$ is concave, $(\tilde{n}, \tilde{r})$ is the unique intersection point of $\underline{r}(n)$ and $\overline{r}(n)$. Note that by differentiating $\frac{\partial\pi(n,\underline{r}(n))}{\partial n} = 0$ and $\frac{\partial\pi(n,\overline{r}(n))}{\partial r} = 0$, we obtain $\frac{\partial^2\pi(n,\underline{r}(n))}{\partial n^2} + \frac{\partial^2\pi(n,\underline{r}(n))}{\partial n\partial r}\frac{\partial\underline{r}(n)}{\partial n} = 0$ and $\frac{\partial^2\pi(n,\overline{r}(n))}{\partial n\partial r} + \frac{\partial^2\pi(n,\overline{r}(n))}{\partial r^2}\frac{\partial\overline{r}(n)}{\partial n} = 0$. Evaluating these expressions at $n = \tilde{n}$ and using the fact that $\underline{r}(\tilde{n}) = \overline{r}(\tilde{n}) = \tilde{r}$, we find $\frac{\partial\overline{r}(\tilde{n})}{\partial n} = -\frac{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n\partial r}}{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial r^2}}$ and $\frac{\partial\underline{r}(\tilde{n})}{\partial n} = -\frac{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n^2}}{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n\partial r}}$. It follows that

$$\left|\overline{r}'(\tilde{n})\right| < \left|\underline{r}'(\tilde{n})\right| \Leftrightarrow \frac{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n\partial r}}{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial r^2}} < \frac{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n^2}}{\frac{\partial^2\pi(\tilde{n},\tilde{r})}{\partial n\partial r}} \Leftrightarrow 0 < \frac{\partial^2\pi^*}{\partial n^2}\frac{\partial^2\pi^*}{\partial r^2} - \frac{\partial^2\pi^*}{\partial n\partial r}\frac{\partial^2\pi^*}{\partial n\partial r} = |H|$$

which is true by Property 1. Therefore, $\underline{r}(n)$ crosses $\overline{r}(n)$ from above, validating Figure 11. Thus, for $n > \tilde{n}$, we have $\underline{r}(n) < \overline{r}(n)$. As $\tilde{r} > q\tilde{n}$, it follows that the line $r = qn$ lies below the point $(\tilde{n}, \tilde{r})$, further validating Figure 11. Furthermore, as $\frac{\partial^2\pi}{\partial n\partial r} < 0$ we have that $\frac{\partial\pi(n,r)}{\partial n} < 0$ for $r > \underline{r}(n)$ and $\frac{\partial\pi(n,r)}{\partial r} > 0$ for $r < \overline{r}(n)$. Therefore, on the segment determined by $r = qn$, $r < \overline{r}(n)$, and $r > \underline{r}(n)$, marked in bold in Figure 11, we have that $\frac{\partial\pi(n,r)}{\partial n} < 0$ and $\frac{\partial\pi(n,r)}{\partial r} > 0$. A-19 and A-20 being simultaneously satisfied means that the first term of (A-19) must be negative and the second term must be positive. Therefore, the solution $(n_\infty, qn_\infty)$ to (A-19) lies on the segment determined by $r = qn$, $r < \overline{r}(n)$, and $r > \underline{r}(n)$. Thus, from (A-20), it follows that $I_\infty$ satisfies the conjecture.

We have shown that there exists a unique solution to $g(I) = I$ on $[0, \underline{I}]$. To complete the proof, we need to show that $g(I) = I$ has a unique solution on $[0, g(0)]$. When $q\tilde{n} < \tilde{r}$, Lemma 6 shows that $g(I) < I$ for $I \in (\underline{I}, g(0)]$. In addition, $g'(I) < 0$ for $I \in [I_\infty, \underline{I}] \subset [0, \underline{I}]$. So $g(I) < I$ for $I \in (I_\infty, g(0)]$ under the condition $q\tilde{n} < \tilde{r}$ and the equality $g(I) = I$ admits only one solution on $[0, g(0)]$.

Part (ii): Let $\{I_t\}$ be the sequence obtained by starting with $I_0 = 0$ and applying $g$ successively, i.e., $I_t = g(I_{t-1})$. Define $L \doteq [g(\underline{I}), \underline{I}] \subset [0, g(0)]$. $L = [g(\underline{I}), I_\infty) \cup \{I_\infty\} \cup (I_\infty, \underline{I}]$. Pick $I_t \in L$. If $g(\underline{I}) \leq I_t < I_\infty$, $I_{t+1} = g(I_t) \leq g(g(\underline{I})) < g(\underline{I}) + (\underline{I} - g(\underline{I})) = \underline{I}$ where the first inequality follows because $g$ is strictly decreasing in this region, and the second inequality follows because, in addition, $|g'| < 1$ in this region. If $I_t = I_\infty$, then $I_{t+1} = I_\infty$ by the definition of $I_\infty$. If $I_\infty < I_t \leq \underline{I}$, then $g(\underline{I}) \leq I_{t+1} = g(I_t) < I_t$ where the first inequality follows because $g(\underline{I})$ is the minimum value that the function $g$ attains on $[0, g(0)]$, and the second inequality follows since $g(I) < I$ in this region. Putting it all together, we conclude that $g(\underline{I}) \leq I_{t+1} < \underline{I}$. In other words, $I_t \in [g(\underline{I}), \underline{I}]$ implies that $I_{t+1} \in [g(\underline{I}), \underline{I})$. Therefore, if there exists a finite time $T_q$ such that $I_{T_q} \in L$, then $I_t \in L \ \forall t \geq T_q$.

Part (iii): We will prove this result separately for cases (i) and (ii) in Lemma 6. For case (i), consider $g(g(0))$. Because $|g'| < 1$, $g(g(0)) < g(0)$. If $0 \leq I_t \leq g(0)$, then $0 < g(g(0)) \leq I_{t+1} = g(I_t) \leq g(0)$, where the first inequality follows because $g > 0$ and the last two inequalities follow because $g$ is strictly decreasing. The interval $[0, g(0)]$ is therefore a capture region and $I_t \in [0, g(0)] \ \ \forall t$ starting with $I_0 = 0$. By the properties of $r(I)$ in case (i), $r_t^* = I_t \ \ \forall t \geq 0$ and $T_q = 0$. In case (ii), $g(I_0) = g(0) > \underline{I}$. We need to prove that there exists a $T_q$ such that $I_t > \underline{I}$ if $1 \leq t \leq T_q - 1$, and $I_{T_q} \in [g(\underline{I}), \underline{I}]$. To prove this, let us start by supposing that no such $T_q$ exists. Then $\underline{I} < I_t \leq g(0) \ \ \forall t \geq 1$. In this region, $I_{t+1} = g(I_t) < I_t$, so $\{I_t\}$ is a strictly decreasing sequence. Because this sequence is in the bounded interval $\underline{I} < I_t \leq g(0)$, it must converge to $\underline{I}$, that is, $lim_{t\to\infty} I_t = \underline{I}$. However, $\underline{I}$ cannot be a limit point of this sequence since $g(\underline{I}) < \underline{I}$. By contradiction, it cannot be true that $\underline{I} < I_t \leq g(0) \ \ \forall t \geq 1$, and there exists a finite $T_q$ such that $I_{T_q} \leq \underline{I}$. In addition, since $g$ increasing on $(\underline{I}, g(0)]$, $\ g(I) \geq g(\underline{I}) \ \ \forall I \in (\underline{I}, g(0)]$, so $I_{T_q} = g(I_{T_q - 1}) \geq g(\underline{I})$. We therefore conclude that there exists a finite $T_q$ such that $I_{T_q} \in [g(\underline{I}), \underline{I}]$. By (ii), $I_t \in [g(\underline{I}), \underline{I}] \ \forall t \geq T_q$ and $r_t^* = I_t$ by the properties of $r^*(I)$ on $[0, \underline{I}]$ in case (ii) of Lemma 6. Finally, since $I_\infty \leq min(g(0), \underline{I})$ is the unique point such that $g(I) = I$, $lim_{t\to\infty} I_t = I_\infty$. ∎

**Proof of Proposition 1:** The proof of this proposition draws on Lemma 6 which characterizes the policy function $g$ when $c(q) < \overline{v}(q)$ and on Lemma 7 that characterizes the optimal path when $c(q) < \overline{v}(q)$ and $q\tilde{n} < \tilde{r}$.

**Derivation of (4).** From Lemmas 2 and 3, it follows that we can restrict our attention to cases where $q \in Q$ and from Lemma 7, it follows that we can split problem (1) in two parts when $q\tilde{n} < \tilde{r}$. In particular, for $t \geq T_q$, we can focus on solutions of the form $r_t = I_t$ and $I_{t+1} = g(I_t) = I_t + qn_t - r_t = qn_t$. For simplicity, the feasible region is suppressed in the maximization problems below.

$$V_\beta(q) \doteq \max \sum_{t=0}^{T_q-1} \beta^t \pi(n_t, r_t, q) + \beta^{T_q} V_\beta\left(I_{T_q}, q\right), \tag{A-21}$$

where

$$\begin{aligned} V_\beta\left(I_{T_q}, q\right) &\doteq \max \sum_{\tau=0}^{\infty} \beta^\tau \pi\left(n_{T_q+\tau}, I_{T_q+\tau}, q\right) \\ &= \max\left(\pi\left(n_{T_q}, I_{T_q}, q\right) + \sum_{\tau=0}^{\infty} \beta^{\tau+1} \pi\left(n_{T_q+\tau+1}, qn_{T_q+\tau}, q\right)\right) \end{aligned} \tag{A-22}$$

and

$$I_{T_q} = qn_0 + \sum_{t=1}^{T_q-1} (qn_t - r_t) \tag{A-23}$$

We now establish (4) in two steps, first for the case where $t \geq T_q$ and then for the case $0 \leq t < T_q$. In the case that $T_q = 0$, the part $0 \leq t < T_q$ can be omitted.

Taking the derivative of $V_\beta(I, q)$ with respect to $I$ and evaluating it at $I = I_{T_q}$, we obtain:

$$\frac{\partial}{\partial I} V_\beta\left(I_{T_q}, q\right) = \frac{\partial \pi\left(n^*_{T_q}, I_{T_q}, q\right)}{\partial r} \tag{A-24}$$

Taking the derivative of the sum in the right-hand side of (A-22) with respect to $n_{T_q+\tau}$ and using the chain rule for $\tau \geq 0$, we obtain first order conditions that are satisfied by the optimal sequence $\nu^*_t, t \geq T_q$:

$$\beta^\tau \frac{\partial \pi\left(n^*_{T_q+\tau}, r^*_{T_q+\tau}\right)}{\partial n} + \beta^{\tau+1} q \frac{\partial \pi\left(n^*_{T_q+\tau+1}, r^*_{T_q+\tau+1}\right)}{\partial r} = 0, \text{ for } \tau = 0, 1, ... \tag{A-25}$$

or, equivalently,

$$\frac{\partial R\left(n^*_{T+\tau}, r^*_{T+\tau}\right)}{\partial n} + \beta q \frac{\partial R\left(n^*_{T+\tau+1}, r^*_{T+\tau+1}\right)}{\partial r} = c(q), \text{ for } \tau = 0, 1, ... \tag{A-26}$$

We have thus established (4) for $t \geq T_q$. Let us now turn to $t < T_q$. Taking the derivative of the sum in A-21 with respect to $n_t,\ t = 0, 1, ..T_q - 1$, we obtain $\beta^t \frac{\partial \pi(n_t, r_t)}{\partial n} + \beta^{T_q} \frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial I} \frac{\partial I_{T_q}}{\partial n_t} = \beta^t \frac{\partial \pi(n_t, r_t)}{\partial n} + q\beta^{T_q} \frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial I},\ t = 0, 1, ..T_{q-1}$, where the equality follows because $\frac{\partial I_{T_q}}{\partial n_t} = q$ by the definition of $I_{T_q}$ in A-23.

Similarly, taking the derivative of the sum in A-21 with respect to $r_t,\ t = 1, 2, ..T_{q-1}$, we obtain $\beta^t \frac{\partial \pi(n_t, r_t)}{\partial r} + \beta^{T_q} \frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial I} \frac{\partial I_{T_q}}{\partial r_t} = \beta^t \frac{\partial \pi(n_t, r_t)}{\partial r} - \beta^{T_q} \frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial I},\ t = 1, 2, ..T_{q-1}$.

The optimal sequence $\{\nu_t^*, t = 0, 1, \ldots, T_q - 1\}$ satisfies the first order conditions obtained by equating these expressions to 0. Using $r_0^* = 0$, A-24 and dividing through by $\beta^t$, we can write:

$$\begin{cases} \frac{\partial \pi(n_t^*, r_t^*, q)}{\partial n} + q\beta^{T_q - t}\frac{\partial \pi\left(n_{T_q}^*, r_{T_q}^*, q\right)}{\partial r} = 0, & t = 0, 1, ..., T_q - 1 \\ \frac{\partial \pi(n_t^*, r_t^*, q)}{\partial r} - \beta^{T_q - t}\frac{\partial \pi\left(n_{T_q}^*, r_{T_q}^*, q\right)}{\partial r} = 0, & t = 1, 2, ..., T_q - 1 \end{cases}$$

Substituting the second set of equalities into the first set, we find

$$\frac{\partial \pi\left(n_t^*, r_t^*, q\right)}{\partial n} + q\frac{\partial \pi\left(n_t^*, r_t^*, q\right)}{\partial r}, t = 0, 1, 2, \ldots, T_q - 1. \tag{A-27}$$

From the second set of equalities, we find that

$$\beta^{T_q - t}\frac{\partial \pi\left(n_{T_q}^*, I_{T_q}, q\right)}{\partial r} = \frac{\partial \pi\left(n_t^*, r_t^*, q\right)}{\partial r} = \beta\frac{\partial \pi\left(n_{t+1}^*, r_{t+1}^*, q\right)}{\partial r}, t = 1, ..., T_q - 1 \tag{A-28}$$

Substituting in (A-27), we obtain

$$\frac{\partial \pi\left(n_t^*, r_t^*, q\right)}{\partial n} + q\beta\frac{\partial \pi\left(n_{t+1}^*, r_{t+1}^*, q\right)}{\partial r}, t = 0, 1, 2, \ldots, T_q - 1$$

Thus, we have

$$\frac{\partial R\left(n_t^*, r_t^*\right)}{\partial n} + \beta q\frac{\partial R\left(n_{t+1}^*, r_{t+1}^*\right)}{\partial r} = c\left(q\right) \text{ for } t = 0, 1, ..., T_q - 1, \tag{A-29}$$

Together with (A-26), we obtain (4).

**Derivation of (5).** We again proceed in two parts. For $t \geq T_q$, we focus on solutions of the form $r_t = I_t$ and $I_{t+1} = g(I_t) = I_t + qn_t - r_t = qn_t$. It is convenient to define the following recursive relationship for $t \geq T_q$: $V_\beta\left(I_t, q\right) = \max\limits_{0 \leq n_t + I_t \leq 1} \left[\pi\left(n_t, I_t, q\right) + \beta V_\beta\left(qn_t, q\right)\right]$. Let $n^*(I_t) = \underset{0 \leq n_t + I_t \leq 1}{argmax} \left[\pi\left(n_t, I_t, q\right) + \beta V_\beta\left(qn_t, q\right)\right]$. By Assumption 1, we are assured that $n^*$ is an interior solution and satisfies the first-order condition $\frac{\partial\left[\pi(n_t, I_t, q) + \beta V_\beta(qn_t, q)\right]}{\partial n} = 0$. We can further write $V_\beta\left(I_t, q\right) = \pi\left(n_t^*, I_t, q\right) + \beta V_\beta\left(qn_t^*, q\right) \;\; \forall t \geq T_q$.

Taking the derivative of this expression with respect to $q$ for each $t \geq T_q$, and simplifying using the first order conditions satisfied by $n_t^*$, we obtain

$$\beta^\tau\frac{\partial V_\beta\left(I_{T_q+\tau}, q\right)}{\partial q} - \beta^{\tau+1}\frac{\partial V_\beta\left(I_{T_q+\tau+1}, q\right)}{\partial q} = \beta^\tau\frac{\partial \pi\left(n_{T_q+\tau}^*, I_{T_q+\tau}, q\right)}{\partial q} + \beta^{\tau+1}\frac{\partial V_\beta\left(I_{T_q+\tau+1}, q\right)}{\partial I}n_{T_q+\tau}^*$$

for $\tau \geq 0$. Adding these equalities over $\tau \geq 0$ and taking the limit as $\tau \to \infty$, we obtain

$$\frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial q} = \sum_{\tau=0}^{\infty}\beta^\tau\left(\frac{\partial \pi\left(n_{T_q+\tau}^*, I_{T_q+\tau}, q\right)}{\partial q} + \beta\frac{\partial V_\beta\left(I_{T_q+\tau+1}, q\right)}{\partial I}n_{T_q+\tau}^*\right).$$

By definition, $\pi(n^*_{T_q+\tau}, I_{T_q+\tau}, q) = R(n^*_{T_q+\tau}, I_{T_q+\tau}) - c_n(q)n^*_{t+\tau} - c_r(q)r^*_{t+\tau}$, so $\frac{\partial\pi(n^*_{T+\tau}, I_{T+\tau}, q)}{\partial q} = -c'_n(q)n^*_{t+\tau} - c'_r(q)r^*_{t+\tau}$. In addition, $\frac{\partial\pi(n^*_{T+\tau}, I_{T+\tau}, q)}{\partial r} = \frac{\partial R(n^*_{T+\tau}, I_{T+\tau})}{\partial r} - c_r(q)$. Using (A-24),

$$\frac{\partial}{\partial I} V_\beta\left(I_{T_q+\tau+1}, q\right) = \frac{\partial\pi\left(n^*_{T_q+\tau+1}, I_{T_q+\tau+1}, q\right)}{\partial r}. \tag{A-30}$$

Putting these two together, we find

$$\frac{\partial V_\beta\left(I_{T_q}, q\right)}{\partial q} = \sum_{\tau=0}^{\infty} \beta^\tau \left(\left(\beta \frac{\partial R\left(n^*_{T_q+\tau+1}, r^*_{T_q+\tau+1}, q\right)}{\partial r} - \beta c_r(q) - c'_n(q)\right) n^*_{T_q+\tau} - c'_r(q)\, r^*_{T_q+\tau}\right) \tag{A-31}$$

We now turn to $t < T_q$. Let us first rewrite A-21: $V_\beta(q) = \sum_{t=0}^{T_q-1} \beta^t \pi(n^*_t, r^*_t, q) + \beta^{T_q} V_\beta\left(I_{T_q}, q\right) = \sum_{t=0}^{T_q-1} \beta^t \pi(n^*_t, r^*_t, q) + \beta^{T_q} V_\beta\left(qn^*_0 + \sum_{t=1}^{T_q-1}(qn^*_t - r^*_t), q\right)$. We find

$$\frac{\partial V_\beta(q)}{\partial q} = \sum_{t=0}^{T_q-1} \beta^t \frac{\partial\pi(n^*_t, r^*_t, q)}{\partial q} + \beta^{T_q} \frac{\partial V_\beta(I_{T_q}, q)}{\partial I} \sum_{t=0}^{T_q-1} n^*_t + \beta^{T_q} \frac{\partial V_\beta(I_{T_q}, q)}{\partial q}$$

where we used $\frac{d\left(I_0 + \sum_{t=0}^{T_q-1}(qn^*_t - r^*_t), q\right)}{dq} = \sum_{t=0}^{T_q-1} n^*_t$. Collecting terms, and using A-24 and A-31, we obtain

$$\begin{aligned}\frac{\partial V_\beta(q)}{\partial q} &= \sum_{t=0}^{T_q-1} \beta^t \left(\left(\beta^{T_q-t} \frac{\partial\pi\left(n^*_{T_q}, I_{T_q}, q\right)}{\partial r} - c'_n(q)\right) n^*_t - c'_r(q)\, r^*_t\right) \\ &\quad + \beta^{T_q} \sum_{\tau=0}^{\infty} \beta^\tau \left(\left(\beta \frac{\partial\pi\left(n^*_{T_q+\tau+1}, I_{T_q+\tau+1}, q\right)}{\partial r} - c'_n(q)\right) n^*_{T_q+\tau} - c'_r(q)\, r^*_{T_q+\tau}\right)\end{aligned}$$

or with (A-28) we obtain:

$$\begin{aligned}\frac{\partial V_\beta(q)}{\partial q} &= \sum_{t=0}^{T_q-1} \beta^t \left(\left(\beta \frac{\partial\pi(n^*_{t+1}, r^*_{t+1}, q)}{\partial r} - c'_n(q)\right) n^*_t - c'_r(q)\, r^*_t\right) \\ &\quad + \beta^{T_q} \sum_{\tau=0}^{\infty} \beta^\tau \left(\left(\beta \frac{\partial\pi\left(n^*_{T_q+\tau+1}, I_{T_q+\tau+1}, q\right)}{\partial r} - c'_n(q)\right) n^*_{T_q+\tau} - c'_r(q)\, r^*_{T_q+\tau}\right)\end{aligned}$$

Finally, we can rewrite the previous expression using the definition $\pi(n, r, q) = R(n, r) - c_n(q)\, n - c_r(q)\, r$:

$$\begin{aligned}\frac{\partial V_\beta(q)}{\partial q} &= \sum_{t=0}^{T_q-1} \beta^t \left(\left(\beta \frac{\partial R(n^*_{t+1}, r^*_{t+1})}{\partial r} - \beta c_r(q) - c'_n(q)\right) n^*_t - c'_r(q)\, r^*_t\right) \\ &\quad + \beta^{T_q} \sum_{\tau=0}^{\infty} \beta^\tau \left(\left(\beta \frac{\partial R\left(n^*_{T_q+\tau+1}, I_{T_q+\tau+1}\right)}{\partial r} - \beta c_r(q) - c'_n(q)\right) n^*_{T_q+\tau} - c'_r(q)\, r^*_{T_q+\tau}\right).\end{aligned}$$

Note that the terms in the latter expression have the same structure for $0 \le t < T_q$ as well as for $T_q \le t$. Thus, we have obtained (5):

$$\frac{\partial V_\beta(q)}{\partial q} = \sum_{t=0}^{\infty} \beta^t \left( \left( \beta \frac{\partial R\left(n^*_{t+1}, r^*_{t+1}\right)}{\partial r} - \beta c_r(q) - c'_n(q) \right) n^*_t - c'_r(q)\, r^*_t \right).$$

■

**Proof of Proposition 2**:

The function $V_\beta(q)$ was defined exclusive of the initial fixed investment cost $k(q)$; the discounted profit at time 0 equals $V_\beta(q) - k(q)$. A sufficient condition for the existence of a $q^* > 0$ is therefore $V'_\beta(0) - k'(0) > 0$.

Let us evaluate $V'_\beta(0)$. As $q = 0$, and $I_0 = 0$, we have that $r^*_t = 0\ \forall t \ge 0$. Evaluating (4) at $q = 0$ gives $\frac{\partial R(\nu^*_t)}{\partial n} = c(0)\ \forall t \ge 0$. By the definition of $n_{su}$, $\nu^*_t = (n_{su}, 0)\ \forall t \ge 0$. Evaluating (5) at $q = 0$ and $\nu^*_t = (n_{su}, 0)\ \forall t \ge 0$, we find

$$V'_\beta(0) = \frac{1}{1-\beta} \left( \beta \left( \frac{\partial R(n_{su}, 0)}{\partial r} - c_r(0) \right) - c'_n(0) \right) n_{su}.$$

For a linear consumer profile with $\eta(\theta) = (1-\delta)\,\theta$, it can easily be shown that $\frac{\partial R(n,0)}{\partial r} = (1-\delta)\frac{\partial R(n,0)}{\partial n}$ (see (A-2) and (A-4)). As by definition $\frac{\partial R(n_{su},0)}{\partial n} = c_n(0)$, we obtain (7). ■

**Proof of Proposition 3.** We first derive $\frac{dn_{su}}{d\kappa}$. If $\beta\{(1-\delta)\,c_n(0) - c_r(0)\} > c'_n(0)$, the sign of $\frac{d\Delta}{d\kappa}$ is identical to the sign of $\frac{dn_{su}}{d\kappa}$. By definition, $n_{su}$ is determined by $\frac{\partial R(n_{su},0)}{\partial n} = c_n(0)$. For $F(\theta) = 1 - (1-\theta)^\kappa \in \mathcal{F}^\kappa$, the marginal consumer, $\theta_{su}$, is defined by $n_{su} = (1-\theta_{su})^\kappa$. From A-2, we know that $\frac{\partial R(n_{su},0)}{\partial n} = G_N(\theta_{su})$. Using A-4, we obtain $\theta_{su} - \frac{1-\theta_{su}}{\kappa} = c_n(0)$ or $\theta_{su} = \frac{1+c_n(0)\kappa}{1+\kappa}$. Plugging $\theta_{su}$ into $n_{su} = (1-\theta_{su})^\kappa$, we obtain $n_{su} = \left(\frac{(1-c_n(0))\kappa}{1+\kappa}\right)^\kappa$. Finally, $\frac{dn_{su}}{d\kappa} = \left(\frac{(1-c_n(0))\kappa}{1+\kappa}\right)^\kappa \left(\ln\left(\frac{(1-c_n(0))\kappa}{1+\kappa}\right) + \frac{1}{1+\kappa}\right)$. Since the first term is non-negative, we obtain that $\frac{dn_{su}}{d\kappa} < 0$ if and only if $c_n(0) > 1 - \frac{1+\kappa}{\kappa}e^{-1/(1+\kappa)}$. As $0 > 1 - \frac{1+\kappa}{\kappa}e^{-1/(1+\kappa)}$ and $c_n(0) \ge 0$, the latter condition is always satisfied. This completes our proof that $\frac{dn_{su}}{d\kappa} < 0$, or, that $\frac{d\Delta}{d\kappa} < 0$ if $\beta\{(1-\delta)\,c_n(0) - c_r(0)\} > c'_n(0)$. We now consider $\frac{d\Delta}{dc_n(0)}$. $\frac{d\Delta}{dc_n(0)} = \beta(1-\delta)n_{su}(c_n(0)) + (\beta\{(1-\delta)\,c_n(0) - c_r(0)\} - c'_n(0))\frac{dn_{su}}{dc_n(0)}$. The first term is positive. $\frac{dn_{su}}{dc_n(0)} = -\kappa(\frac{(1-c_n(0))\kappa}{1+\kappa})^{\kappa-1}\frac{\kappa}{1+\kappa} < 0$. We conclude that depending on the magnitude of $\frac{dn_{su}}{dc_n(0)}$, $\frac{d\Delta}{dc_n(0)}$ could be either positive or negative. ■

**Proof of Proposition 4.** This proof is done in two steps. First, we characterize an approximation of $q^*$ for $\beta \approx 1$. Second, we show how this approximation depends on $\kappa$. In previous results developed for a given remanufacturability level $q$, $\nu^*_t$ was determined for $q$, but this dependence was suppressed in the notation. From now on, we work with $q^*$, so $\nu^*_t$ is determined for $q^*$. We again suppress this dependence in the notation.

**Step 1a. Approximate characterization of $q^*$:**

Since $r_0^* = 0$, we can rewrite (5), evaluated at $q^*$, as

$$\frac{\partial V_\beta(q^*)}{\partial q} = (1-\beta)\sum_{t=0}^{\infty}\beta^t\left(\left(\beta\frac{\partial R\left(n_{t+1}^*, r_{t+1}^*\right)}{\partial r} - \beta c_r(q^*) - c_n'(q^*)\right)n_t^* - \beta c_r'(q^*)\, r_{t+1}^*\right).$$

Multiplying this equation by $1-\beta$, and separating it into two parts, we obtain

$$\begin{aligned}(1-\beta)\frac{\partial V_\beta(q^*)}{\partial q} &= (1-\beta)\sum_{t=0}^{T_{q-1}}\beta^t\left(\left(\beta\frac{\partial R\left(n_{t+1}^*, r_{t+1}^*\right)}{\partial r} - \beta c_r(q^*) - c_n'(q^*)\right)n_t^* - \beta c_r'(q^*)\, r_{t+1}^*\right)\\ &\quad + (1-\beta)\sum_{t=T_q}^{\infty}\beta^t\left(\left(\beta\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r} - \beta c_r(q^*) - c_n'(q^*)\right)n_t^* - \beta q^* c_r'(q^*)\, n_t^*\right).\end{aligned}$$

We find that $lim_{\beta\to 1^-}(1-\beta)\frac{\partial V_\beta(q^*)}{\partial q}$

$$= lim_{\beta\to 1^-}(1-\beta)\sum_{t=T_q}^{\infty}\beta^t\left(\left(\beta\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r} - \beta c_r(q^*) - c_n'(q^*)\right)n_t^* - \beta q^* c_r'(q^*)\, n_t^*\right)$$

since the first term above is finite. $q^*$ satisfies the first-order condition $\frac{\partial V_\beta(q)}{\partial q} = k'(q)$, or, $(1-\beta)\frac{\partial V_\beta(q^*)}{\partial q} = (1-\beta)k'(q^*)$. Then $lim_{\beta\to 1^-}(1-\beta)\frac{\partial V_\beta(q^*)}{\partial q} = 0$. From the previous expression, we have

$$lim_{\beta\to 1^-}(1-\beta)\sum_{t=T_q}^{\infty}\beta^t\left(\left(\beta\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r} - \beta c_r(q^*) - c_n'(q^*)\right)n_t^* - \beta q^* c_r'(q^*)\, n_t^*\right) = 0, \text{ or,}$$

$$lim_{\beta\to 1^-}(1-\beta)\sum_{t=T_q}^{\infty}\beta^t\left(\beta\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r} - c'(q^*)\right)n_t^* = 0. \quad \text{(A-32)}$$

We approximate $\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r}$ around $\nu_\infty^* = (n_\infty^*, q^* n_\infty^*) \doteq \nu^*(I_\infty(q^*))$ using Taylor series expansion:

$$\frac{\partial R\left(n_{t+1}^*, q^* n_t^*\right)}{\partial r} = \frac{\partial R_\infty}{\partial r} + \frac{\partial^2 R_\infty}{\partial r\partial n}\left(n_{t+1}^* - n_\infty\right) + q^*\frac{\partial^2 R_\infty}{\partial r^2}\left(n_t^* - n_\infty^*\right) + o(||\nu_{t+1}^* - \nu_\infty^*||), \quad \text{(A-33)}$$

where $R_\infty \doteq R(\nu_\infty^*)$. Substituting into A-32 and collecting terms, we find

$$\begin{aligned}0 &= lim_{\beta\to 1^-}(1-\beta)\left(\beta\frac{\partial R_\infty}{\partial r} - c'(q^*)\right)\sum_{t=T_q}^{\infty}\beta^t n_t^*\\ &\quad + lim_{\beta\to 1^-}(1-\beta)\sum_{t=T_q}^{\infty}\beta^{t+1}\left(\frac{\partial^2 R_\infty}{\partial r\partial n}\left(n_{t+1}^* - n_\infty^*\right) + q^*\frac{\partial^2 R_\infty}{\partial r^2}\left(n_t^* - n_\infty^*\right) + o(||\nu_{t+1}^* - \nu_\infty^*||)\right)n_t^*.\end{aligned}$$

From Lemma 6, we have that at $\underline{I}$, (A-7) is satisfied and (A-8) is satisfied with equality, therefore $\frac{\partial\pi(n^*(\underline{I}),\underline{I})}{\partial n} + q\frac{\partial\pi(n^*(\underline{I}),\underline{I})}{\partial r} = 0$. Remember from Lemma 7 that $\frac{\partial\pi(n^*(I_\infty),I_\infty)}{\partial n} + q\beta\frac{\partial\pi(n^*(I_\infty),I_\infty)}{\partial r} = 0$ with

$I_\infty = qn^*(I_\infty)$ (A-19). Therefore, as $\beta \to 1-$, $\underline{I} \to I_\infty$. Thus for $t \geq T_q$, $\left(n^*_{t+1}, q^* n^*_t\right) \to (n^*_\infty, q^* n^*_\infty)$ as $\beta \to 1^-$ and the second term in parentheses converges to 0 in the last equality. Since $n^*_t > 0\ \forall t$, expressing the dependence of $q^*$ on the parameter $\beta$ with the expression $q^*_\beta$, we conclude that the following must hold for the first term to also be equal to 0:

$$\lim_{\beta \to 1^-} \beta \frac{\partial R_\infty\left(n^*_\infty, q^*_\beta n^*_\infty\right)}{\partial r} - c'\left(q^*_\beta\right) = 0.$$

Therefore, for $\beta \approx 1$, $\beta \frac{\partial R_\infty\left(n^*_\infty, q^*_\beta n^*_\infty\right)}{\partial r} \approx c'(q^*_\beta)$. Let $\tilde{q}^*$ be such that

$$\beta \frac{\partial R_\infty\left(n^*_\infty, \tilde{q}^* n^*_\infty\right)}{\partial r} = c'(\tilde{q}^*). \tag{A-34}$$

$\tilde{q}^*$ approximates the optimal remanufacturability level $q^*$. With (A-19) evaluated at $\tilde{q}^*$, we obtain

$$\frac{\partial R_\infty\left(n^*_\infty, \tilde{q}^* n^*_\infty\right)}{\partial n} + \tilde{q}^* \beta \frac{\partial R_\infty\left(n^*_\infty, \tilde{q}^* n^*_\infty\right)}{\partial r} = c\left(\tilde{q}^*\right),$$

which, with (A-34) gives

$$\frac{\partial R_\infty\left(n^*_\infty, \tilde{q}^* n^*_\infty\right)}{\partial n} = c\left(\tilde{q}^*\right) - \tilde{q}^* c'\left(\tilde{q}^*\right). \tag{A-35}$$

In conclusion, (A-34) and (A-35) approximately determine $(q^*, n^*_\infty)$. This completes our approximate characterization of $q^*$.

We will now specialize this characterization to $F \in F^\kappa$. The marginal consumers $\left(\theta_l^{\infty,*}, \theta_h^{\infty,*}\right)$ are defined by $n^*_\infty = \left(1 - \theta_h^{\infty,*}\right)^\kappa$ and $r^*_\infty = \left(1 - \theta_l^{\infty,*}\right)^\kappa - \left(1 - \theta_h^{\infty,*}\right)^\kappa$ for $F \in \mathcal{F}^\kappa$. We can rewrite (A-34) and (A-35) as a function of $\left(\theta_l^{\infty,*}, \theta_h^{\infty,*}, q^*\right)$:

$$\beta G_R\left(\theta_l^{\infty,*}\right) = c'\left(q^*\right) \text{ and } \delta G_N\left(\theta_h^{\infty,*}\right) + G_R\left(\theta_l^{\infty,*}\right) = c\left(q^*\right) - q^* c'\left(q^*\right).$$

Furthermore, we can write A-15 as a function of $\left(\theta_l^{\infty,*}, \theta_h^{\infty,*}, q^*\right)$: $\frac{(1-\theta_l^{\infty,*})^\kappa}{(1-\theta_h^{\infty,*})^\kappa} = 1 + q^*$. Defining $c_l\left(q\right) \doteq \frac{c'(q)}{\beta}$ and $c_h\left(q\right) \doteq c\left(q\right) - \left(1 + \beta q\right) \frac{c'(q)}{\beta}$, we obtain that the triple $\left(\theta_l^{\infty,*}, \theta_h^{\infty,*}, q^*\right)$ satisfies the following conditions:

$$(1-\delta) G_N\left(\theta_l\right) = c_l\left(q\right) \text{ and } \delta G_N\left(\theta_h\right) = c_h\left(q\right) \tag{A-36}$$

$$\frac{1-\theta_l}{1-\theta_h} = (1+q)^{\frac{1}{\kappa}}. \tag{A-37}$$

With Lemma 4, we have that $G_N\left(\theta\right) = \theta - \frac{1-\theta}{\kappa}$. Substituting in (A-36), solving for $(\theta_l, \theta_h)$ and substituting these expressions into A-37, we find that the following condition that must be satisfied by $q^*$:

$$\frac{1 - \frac{c_l(q)}{1-\delta}}{1 - \frac{c_h(q)}{\delta}} = (1+q)^{\frac{1}{\kappa}}. \tag{A-38}$$

This completes our approximate characterization of $q^*$ for $F \in \mathcal{F}^\kappa$ and $\beta \approx 1$.

**Step 2. Characterizing $q^*$ as a function of $\kappa$.**

If $\Delta > 0$, then $q^* \in (0, \bar{q})$. It can easily established that $c_l(q)$ strictly increases in $q$. This follows from $c_l'(q) = \frac{c''(q)}{\beta}$ and from $c''(q) > 0$ (Assumption 3). Similarly, it can easily be established that $c_h(q)$ strictly decreases in $q$. This follows from $c_h'(q) = -(1+\beta q)\frac{c''(q)}{\beta}$ and from $c''(q) > 0$. As $c_l(q)$ strictly increases and $c_h(q)$ strictly decreases in $q$, the left hand side of A-38 strictly decreases in $q$. The right hand side strictly increases with $q$. Therefore, $q^*$ solving A-38 is unique.

Let us denote the dependence of $q^*$ on $\kappa$ explicitly with the notation $q^*_\kappa$. Note that increasing $\kappa$ decreases the right-hand side of A-38 but does not affect its left-hand side. Therefore, $q^*_\kappa$ increases in $\kappa$. ■

**Proof of Proposition 5.** In this proof, we use the approximate characterization of $q^*$ developed in the proof of Proposition 4 for $\beta \approx 1$. In particular, we use that $q^*$ solves (A-38). Remember from (A-1) that we can write $p_N$ and $p_R$ as a function of $\theta_l$ and $\theta_h$: $p_R = (1-\delta)\theta_l$ and $p_N = (1-\delta)\theta_l + \delta\theta_h$. The profit margin on the remanufactured product at the solution $(n^*_\infty, q^* n^*_\infty)$ is thus $M^*_r \doteq (1-\delta)\theta_l^{\infty,*} - c_r(q^*)$. By A-36, $\theta_l^{\infty,*} = G_N^{-1}\left(\frac{c'(q^*)}{(1-\delta)\beta}\right)$. For $F \in \mathcal{F}^\kappa$, we have $G_N^{-1}(c) = \frac{\frac{1}{\kappa}+c}{\frac{1}{\kappa}+1} > c$ for $c < 1$. As from A-38, it follows that $\frac{c'(q^*)}{(1-\delta)\beta} < 1$, we obtain that $G_N^{-1}\left(\frac{c'(q^*)}{(1-\delta)\beta}\right) > \frac{c'(q^*)}{(1-\delta)\beta}$, which yields $(1-\delta)\theta_l^* - c_r(q^*) > \frac{c_n'(q^*)}{\beta} + q^* c_r'(q^*) > 0$, where the last inequality follows from Assumption 3.

The profit margin on the new product at the solution $(n^*_\infty, q^* n^*_\infty)$ is $M^*_n \doteq (1-\delta)\theta_l^{\infty,*} + \delta\theta_h^{\infty,*} - c_n(q^*)$. With (A-36) and for $F \in \mathcal{F}^\kappa$, we obtain $\theta_l^{\infty,*} = G_N^{-1}\left(\frac{c'(q^*)}{(1-\delta)\beta}\right) = \frac{\frac{c'(q^*)}{(1-\delta)\beta}+\frac{1}{\kappa}}{1+\frac{1}{\kappa}}$ and $\theta_h^{\infty,*} = G_N^{-1}\left(\frac{c(q^*)-\left(\frac{1}{\beta}+q^*\right)c'(q^*)}{\delta}\right) = \frac{\frac{c(q^*)-\left(\frac{1}{\beta}+q^*\right)c'(q^*)}{\delta}+\frac{1}{\kappa}}{1+\frac{1}{\kappa}}$. We can rewrite $M^*_n$ as

$$\begin{aligned} M^*_n &= \frac{\frac{c'(q^*_\kappa)}{\beta} + c(q^*_\kappa) - \left(\frac{1}{\beta}+q^*_\kappa\right)c'(q^*_\kappa) + \frac{1}{\kappa}}{1+\frac{1}{\kappa}} - c_n(q^*_\kappa) \\ &= \frac{1 - \kappa q^*_\kappa\left(c_n'(q^*_\kappa) + q^*_\kappa \beta c_r'(q^*_\kappa)\right) - c_n(q^*_\kappa)}{1+\kappa} \end{aligned}$$

From this equation, we see that the margin on the new product becomes negative for large enough values of $\kappa$, and that an increase in the cost of the new product works in the same direction as an increase in $\kappa$. ■

**Proof of Proposition 6.** In this proof, we use the approximate characterization of $q^*$ developed in the proof of Proposition 4 for $\beta \approx 1$. Let $c_r(q) = c_{r0} + c_{r1}(q)$, with $c_{r1}(0) = 0$. We calculate $\frac{dq^*}{dc_{r0}}$ (step 1), $\frac{dn^*_\infty}{dc_{r0}}$ and $\frac{dr^*_\infty}{dc_{r0}}$ (step 2).

**Step 1: Calculation of $\frac{dq^*}{dc_{r0}}$**

Let us first introduce some notation: $c_1(q) \doteq c_n(q) + \beta q c_{r1}(q)$, $c_{l1}(q) \doteq \frac{c_1'(q)}{\beta}$ and $c_{h1}(q) \doteq c_1(q) - (1+\beta q)\frac{c_1'(q)}{\beta}$. Recall $c(q) \doteq c_n(q) + \beta q c_r(q)$, $c_l(q) \doteq \frac{c'(q)}{\beta}$ and $c_h(q) \doteq c(q) - (1+\beta q)\frac{c'(q)}{\beta}$. Some algebraic manipulation yields

$$c(q) = c_1(q) + \beta q c_{r0},\ c_l(q) = c_{r0} + c_{l1}(q) \text{ and } c_h(q) = -c_{r0} + c_{h1}(q).$$

Plugging in these expressions for $c_l(q)$ and $c_h(q)$ in condition (A-38), we obtain that $q^*$ satisfies

$$\frac{1 - \frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}} = (1+q^*)^{\frac{1}{\kappa}}. \tag{A-39}$$

Differentiation of both sides of A-39 with respect to $c_{r0}$ gives:

$$\frac{-\frac{1+c_{l1}'(q^*)\frac{dq^*}{dc_{r0}}}{1-\delta}\left(1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}\right) + \frac{-1+c_{h1}'(q^*)\frac{dq^*}{dc_{r0}}}{\delta}\left(1 - \frac{c_{r0}+c_{l1}(q^*)}{1-\delta}\right)}{\left(1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}\right)^2} = \frac{1}{\kappa}(1+q^*)^{\frac{1}{\kappa}-1}\frac{dq^*}{dc_{r0}}$$

From the latter equation, we can solve for $\frac{dq^*}{dc_{r0}}$:

$$\frac{dq^*}{dc_{r0}} = -\frac{\frac{1}{1-\delta}\left(1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}\right) + \frac{1}{\delta}\left(1 - \frac{c_{r0}+c_{l1}(q^*)}{1-\delta}\right)}{\frac{1}{\kappa}(1+q^*)^{\frac{1}{\kappa}-1}\left(1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}\right)^2 + \frac{c_{l1}'(q^*)}{1-\delta}\left(1 - \frac{-c_{r0}+c_{h1}(q^*)}{\delta}\right) - \frac{c_{h1}'(q^*)}{\delta}\left(1 - \frac{c_{r0}+c_{l1}(q^*)}{1-\delta}\right)}$$

or, making use of (A-39):

$$\frac{dq^*}{dc_{r0}} = -\frac{\frac{1}{1-\delta} + \frac{1}{\delta}(1+q^*)^{\frac{1}{\kappa}}}{\frac{1}{\kappa}\frac{1-\frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1+q} + \frac{c_{l1}'(q^*)}{1-\delta} - \frac{c_{h1}'(q^*)}{\delta}(1+q^*)^{\frac{1}{\kappa}}}. \tag{A-40}$$

Since $c(q)$ is convex by Assumption 3, $c_1(q)$ is convex. Taking the derivative of $c_{l1}(q)$ and $c_h(q)$ with respect to $q$ we obtain

$$c_{l1}'(q) = \frac{c_1''(q)}{\beta} > 0 \text{ and } c_{h1}'(q) = -(1+\beta q)\frac{c_1''(q)}{\beta} < 0. \tag{A-41}$$

With A-39 and A-41, we observe that the sign of each of the terms in the previous expression is positive. We conclude that $\frac{dq^*}{dc_{r0}} < 0$.

**Step 2: Calculation of $\frac{dn_\infty^*}{dc_{r0}}$ and $\frac{dr_\infty^*}{dc_{r0}}$.**

Define the marginal consumers $(\theta_l^*, \theta_h^*)$ such that $n_\infty^* = (1-\theta_h^*)^\kappa$, $r_\infty^* = (1-\theta_l^*)^\kappa - (1-\theta_h^*)^\kappa$ and $r_\infty^* = q^* n_\infty^*$. Then, we need to calculate $\frac{dn_\infty^*}{dc_{r0}} = \kappa(1-\theta_h^*)^{\kappa-1}\frac{d\theta_h^*}{dc_{r0}}$ and $\frac{dr_\infty^*}{dc_{r0}} = \kappa(1-\theta_h^*)^{\kappa-1}\frac{d\theta_h^*}{dc_{r0}} - \kappa(1-\theta_l^*)^{\kappa-1}\frac{d\theta_l^*}{dc_{r0}} = \kappa(1-\theta_h^*)^{\kappa-1}\left(\frac{d\theta_h^*}{dc_{r0}} - (1+q)^{\frac{\kappa-1}{\kappa}}\frac{d\theta_l^*}{dc_{r0}}\right)$ where the last equality follows from A-38.

Using A-36, we can solve for $\theta_l^*, \theta_h^*$:

$$\theta_h^* = \frac{\frac{-c_{r0}}{\delta} + \frac{c_{h1}(q^*)}{\delta} + \frac{1}{\kappa}}{1+\frac{1}{\kappa}} \Rightarrow \frac{d\theta_h^*}{dc_{r0}} = \frac{\frac{-1}{\delta} + \frac{c_{h1}'(q^*)}{\delta}\frac{dq^*}{dc_{r0}}}{1+\frac{1}{\kappa}} \tag{A-42}$$

$$\text{and } \theta_l^* = \frac{\frac{c_{r0}}{1-\delta} + \frac{c_{l1}(q^*)}{(1-\delta)} + \frac{1}{\kappa}}{1+\frac{1}{\kappa}} \Rightarrow \frac{d\theta_l^*}{dc_{r0}} = \frac{\frac{1}{1-\delta} + \frac{c_{l1}'(q)}{1-\delta}\frac{dq^*}{dc_{r0}}}{1+\frac{1}{\kappa}} \tag{A-43}$$

Note that $\frac{-1}{\delta} < 0$ and $\frac{c'_{h1}(q)}{\delta}\frac{dq^*}{dc_{r0}} > 0$, therefore, the sign of $\frac{d\theta_h^*}{dc_{r0}}$ is indeterminate. As the sign of $\frac{dn_\infty^*}{dc_{r0}}$ is the same as the sign of $\frac{d\theta_h^*}{dc_{r0}}$, we find that $\frac{dn_\infty^*}{dc_{r0}} \lessgtr 0$. One can easily find examples of both cases.

Using (A-42) and (A-43) we find

$$\frac{d\theta_h^*}{dc_{r0}} - (1+q)^{\frac{\kappa-1}{\kappa}}\frac{d\theta_l^*}{dc_{r0}} = \frac{\frac{-1}{\delta} + \frac{c'_{h1}(q^*)}{\delta}\frac{dq^*}{dc_{r0}} - (1+q^*)^{\frac{\kappa-1}{\kappa}}\frac{1}{1-\delta} - (1+q^*)^{\frac{\kappa-1}{\kappa}}\frac{c'_{l1}(q^*)}{1-\delta}\frac{dq^*}{dc_{r0}}}{1+\frac{1}{\kappa}}.$$

The sign of this expression determines the sign of $\frac{dr_\infty^*}{dc_{r0}}$. We can substitute (A-40) in the previous expression and obtain

$$\frac{d\theta_h^*}{dc_{r0}} - (1+q)^{\frac{\kappa-1}{\kappa}}\frac{d\theta_l^*}{dc_{r0}} = \frac{-\left(1-\delta+\delta(1+q^*)^{\frac{\kappa-1}{\kappa}}\right)\frac{1-\frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1+q^*} + q^*\kappa\left(c'_{h1}(q^*)+c'_{l1}(q^*)\right)}{\left(1+\frac{1}{\kappa}\right)\delta(1-\delta)\left(\frac{1-\frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1+q^*} - \frac{c'_{h1}(q^*)}{\delta}(1+q^*)^{\frac{1}{\kappa}}\kappa + \frac{c'_{l1}(q^*)}{1-\delta}\kappa\right)}.$$

Recall that $c_{l1}(q) \doteq \frac{c'_1(q)}{\beta}$ and $c_{h1}(q) \doteq c_1(q) - (1+\beta q)\frac{c'_1(q)}{\beta}$. Substituting the following equality $c'_{h1}(q^*) + c'_{l1}(q^*) = c'_1(q^*) - \beta\frac{c'_1(q^*)}{\beta} - (1+\beta q^*)\frac{c''_1(q^*)}{\beta} + \frac{c''_1(q^*)}{\beta} = -q^*c''_1(q^*)$ in the latter expression, we obtain

$$\frac{d\theta_h^*}{dc_{r0}} - (1+q)^{\frac{\kappa-1}{\kappa}}\frac{d\theta_l^*}{dc_{r0}} = -\frac{\left(1-\delta+\delta(1+q^*)^{\frac{\kappa-1}{\kappa}}\right)\frac{1-\frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1+q^*} + q^{*2}\kappa c''_1(q^*)}{\left(1+\frac{1}{\kappa}\right)\delta(1-\delta)\left(\frac{1-\frac{c_{r0}+c_{l1}(q^*)}{1-\delta}}{1+q} - \frac{c'_{h1}(q^*)}{\delta}(1+q^*)^{\frac{1}{\kappa}}\kappa + \frac{c'_{l1}(q^*)}{1-\delta}\kappa\right)} < 0.$$

Noting that $c'_{h1}(q^*) < 0$, $c'_{l1}(q^*) > 0$ (by A-41), we conclude by inspection of all terms that $\frac{dr_\infty^*}{dc_{r0}} < 0$.
■

**Proof of Proposition 7.**

This proof is structured as follows: First, we fix $p_U(I)$ and calculate the dynamic Cournot competition between the manufacturer and remanufacturers by determining $\left(r^e_{p_U(.),i}(I)\right)_{i\in\mathcal{N}}$ and $n^e_{p_U(.)}(I)$. As the remanufacturers are symmetric, we have that $r^e_{p_U(.),i}(I) = r^e_{p_U(.)}(I)\ \forall i \in \mathcal{N}$ (step 1). Second, we determine the 'market clearing' price, $p^e_U(I)$, such that the obtained Cournot equilibrium satisfies $Nr^e_{p^e_U(.)}(I) = I$ for $p^e_U(.) > 0$, or $Nr^e_{p^e_U(.)}(I) < I$ for $p^e_U(.) = 0$ (step 2). In step 3, we take the derivative of the equilibrium value function of the manufacturer with respect to $q$, at $q = 0$. In this way, we obtain $\Delta^e$ and note that it is exactly the same as $\Delta$.

**Step 1: Cournot competition between the manufacturer and $N$ remanufacturers.**

Fixing $p_U(I)$ and $(r_i(I))_{i\in\mathcal{N}}$, the manufacturer's problem can be written as the following DP:

$$\begin{aligned} V_{p_U(.),N}(I) &= \max_{0\le n\le 1} n\left(p_N\left(n,\sum_{i=1}^N r_i(I)\right) + \beta q p_U\left(I+qn-\sum_{i=1}^N r_i(I)\right) - c_n(q)\right) \\ &\quad +\beta V_{p_U(.),N}\left(I+qn-\sum_{i=1}^N r_i(I)\right) \end{aligned} \tag{A-44}$$

i.e. the manufacturer chooses for every $I$ a new product quantity of $n$. Fixing $p_U(I)$, $n(I)$ and $(r_j(I))_{j\in\mathcal{N},j\neq i}$, the remanufacturer's problem can be written as the following DP:

$$\begin{aligned} V_{p_U(.),R,i}(I) &= \max_{r_i} r_i \left( p_R \left( n(I), r_i + \sum_{j=1,j\neq i}^{N} r_j(I) \right) - p_U(I) - c_r(q) \right) \\ &\quad + \beta V_{p_U(.),R,i} \left( I + qn(I) - r_i - \sum_{j=1,j\neq i}^{N} r_j(I) \right) \end{aligned} \quad \text{(A-45)}$$

i.e. each remanufacturer chooses for every $I$ a remanufacturing quantity $r_i$. For a fixed $p_U(I)$, let $n^e_{p_U(.)}(I)$ be the maximizer of (A-44), with $(r_i(I))_{i\in\mathcal{N}} = \left(r^e_{p_U(.),i}(I)\right)_{i\in\mathcal{N}}$ and let be $r^e_{p_U(.),i}(I)$ the maximizer of (A-45), for $n(I) = n^e_{p_U(.)}(I)$ and $(r_j(I))_{j\in\mathcal{N},j\neq i} = \left(r^e_{p_U(.),j}(I)\right)_{j\in\mathcal{N},j\neq i}$. Then, $\left(n^e_{p_U(.)}(I), \left(r^e_{p_U(.),i}(I)\right)_{i\in\mathcal{N}}\right)$ determine the dynamic Cournot equilibrium for a fixed $p_U(I)$. $n^e_{p_U(.)}(I)$ satisfies the FOC of (A-44) with respect to $n$:

$$\begin{aligned} 0 &= p_N \left( n, \sum_{i=1}^{N} r^e_{p_U(.),i}(I) \right) - c_n(q) + \beta q p_U \left( I + qn - \sum_{i=1}^{N} r^e_{p_U(.),i}(I) \right) \\ &\quad + n \left( \frac{\partial p_N \left( n, \sum_{i=1}^{N} r^e_{p_U(.),i}(I) \right)}{\partial n} + \beta q^2 p'_U \left( I + qn - \sum_{i=1}^{N} r^e_{p_U(.),i}(I) \right) \right) \\ &\quad + \beta q \frac{d}{dI} V_{p_U(.),N} \left( I + qn - \sum_{i=1}^{N} r^e_{p_U(.),i}(I) \right) \end{aligned}$$

and $r^e_{p_U(.),i}(I)$ satisfies the FOC of (A-45) with respect to $r$:

$$\begin{aligned} 0 &= p_R \left( n^e_{p_U(.)}(I), r + \sum_{j=1,j\neq i}^{N} r^e_{p_U(.),j}(I) \right) - c_r(q) - p_U(I) \\ &\quad + r \frac{\partial p_R \left( n^e_{p_U(.)}(I), r + \sum_{j=1,j\neq i}^{N} r^e_{p_U(.),j}(I) \right)}{\partial r} \\ &\quad - \beta \frac{d}{dI} V_{p_U(.),R} \left( I + q n^e_{p_U(.)}(I) - r - \sum_{j=1,j\neq i}^{N} r^e_{p_U(.),j}(I) \right) \end{aligned}$$

Using the symmetry of (A-45), let us suppress the index $i$ and use $r^e_{p_U(.)}(I)$ instead. The previous equations reduce then to:

$$\begin{aligned} 0 &= p_N \left( n^e_{p_U(.)}(I), N r^e_{p_U(.)}(I) \right) - c_n(q) + \beta q p_U \left( I + q n^e_{p_U(.)}(I) - N r^e_{p_U(.)}(I) \right) \\ &\quad + n \left( \frac{\partial p_N \left( n^e_{p_U(.)}(I), N r^e_{p_U(.)}(I) \right)}{\partial n} + \beta q^2 p'_U \left( I + q n^e_{p_U(.)}(I) - N r^e_{p_U(.)}(I) \right) \right) \\ &\quad + \beta q \frac{d}{dI} V_{p_U(.),N} \left( I + q n^e_{p_U(.)}(I) - N r^e_{p_U(.)}(I) \right) \end{aligned} \quad \text{(A-46)}$$

and

$$\begin{aligned} 0 &= p_R\left(n^e_{p_U(.)}(I), Nr^e_{p_U(.),j}(I)\right) - c_r(q) - p_U(I) \\ &+r\frac{\partial p_R\left(n^e_{p_U(.)}(I), Nr^e_{p_U(.),j}(I)\right)}{\partial r} - \beta\frac{d}{dI}V_{p_U(.),R}\left(I + qn^e_{p_U(.)}(I) - Nr^e_{p_U(.),j}(I)\right) \qquad \text{(A-47)} \end{aligned}$$

Let $V^e_{p_U(.),N}(I)$ and $V^e_{p_U(.),R,i}(I)$ denote the value functions in A-44 and A-45 evaluated at $\left(n^e_{p_U(.)}(I), \left(r^e_{p_U(.),i}(I)\right)_{i\in\mathcal{N}}\right)$. Taking the derivative of these functions with respect to $I$, we find

$$\begin{aligned} \frac{d}{dI}V^e_{p_U(.),N}(I) &= n^e_{p_U(.)}(I)\frac{\partial p_N\left(n^e_{p_U(.)}(I), \sum_{i=1}^N r^e_{p_U(.),i}(I)\right)}{\partial r}\sum_{i=1}^N\frac{d}{dI}r^e_{p_U(.),i}(I) \\ &+\beta qn^e_{p_U(.)}(I)\,p_U'\left(I + qn^e_{p_U(.)}(I) - \sum_{i=1}^N r^e_{p_U(.),i}(I)\right)\left(1 - \sum_{i=1}^N\frac{d}{dI}r^e_{p_U(.),i}(I)\right) \\ &+\beta\frac{d}{dI}V_{p_U(.),N}\left(I + qn^e_{p_U(.),i}(I) - \sum_{i=1}^N r^e_{p_U(.),i}(I)\right)\left(1 - \sum_{i=1}^N\frac{d}{dI}r^e_{p_U(.),i}(I)\right); \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dI}V^e_{p_U(.),R,i}(I) &= r^e_{p_U(.),i}(I)\{\frac{\partial p_R\left(n^e_{p_U(.)}(I), r^e_{p_U(.),i}(I) + \sum_{j\neq i} r^e_{p_U(.),j}(I)\right)}{\partial n}\frac{d}{dI}n^e_{p_U(.)}(I) \\ &+\frac{\partial p_R\left(n^e_{p_U(.)}(I), r^e_{p_U(.),i}(I) + \sum_{j\neq i} r^e_{p_U(.),j}(I)\right)}{\partial r}\sum_{j\neq i}\frac{d}{dI}r^e_{p_U(.),j}(I) - p_U'(I)\} \\ &+\beta\frac{d}{dI}V_{p_U(.),R,i}\left(I + qn^e_{p_U(.)}(I) - r^e_{p_U(.),i}(I) - \sum_{j\neq i} r^e_{p_U(.),j}(I)\right) * \\ &\left(1 + q\frac{d}{dI}n^e_{p_U(.)}(I) - \sum_{j\neq i}\frac{d}{dI}r^e_{p_U(.),j}(I)\right) \end{aligned}$$

Again, with symmetry with respect to $i \in \mathcal{N}$, we obtain:

$$\begin{aligned} \frac{d}{dI}V^e_{p_U(.),N}(I) &= n^e_{p_U(.)}(I)\frac{\partial p_N\left(n^e_{p_U(.)}(I), Nr^e_{p_U(.)}(I)\right)}{\partial r}Nr^{e\prime}_p(I) \\ &+\beta\{qn^e_{p_U(.)}(I)\,p_U'\left(I + qn^e_{p_U(.)}(I) - Nr^e_{p_U(.)}(I)\right) \\ &+\frac{d}{dI}V_{p_U(.),N}\left(I + qn^e_{p_U(.)}(I) - Nr^e_{p_U(.)}(I)\right)\left(1 - N\frac{d}{dI}r^e_{p_U(.)}(I)\right)\} \qquad \text{(A-48)} \end{aligned}$$

$$\begin{aligned} \frac{d}{dI}V^e_{p_U(.),R}(I) &= r^e_{p_U(.)}(I)\{\frac{\partial p_R\left(n^e_{p_U(.)}(I), Nr^e_{p_U(.)}(I)\right)}{\partial n}\frac{d}{dI}n^e_{p_U(.)}(I) \\ &+\frac{\partial p_R\left(n^e_{p_U(.)}(I), Nr^e_{p_U(.)}(I)\right)}{\partial r}(N-1)\frac{d}{dI}r^e_{p_U(.)}(I) - p_U'(I)\} \\ &+\beta\frac{d}{dI}V_{p_U(.),R}\left(I + qn^e_{p_U(.)}(I) - Nr^e_{p_U(.)}(I)\right)\left(1 + q\frac{d}{dI}n^e_{p_U(.)}(I) - (N-1)\frac{d}{dI}r^e_{p_U(.)}(I)\right) \qquad \text{(A-49)} \end{aligned}$$

We thus have four equations (A-44), (A-45), (A-48) and (A-49) determining the four unknowns $n^e_{p_U(.)}(I)$, $r^e_{p_U(.)}(I)$, $\frac{d}{dI}V^e_N(I)$ and $\frac{d}{dI}V^e_R(I)$, for a given $p_U(I)$.

**Step 2: Equilibrium $p^e_U(I)$.**

As mentioned above, the 'market clearing' price, $p^e_U(I)$ is such that the obtained Cournot equilibrium satisfies $Nr^e_{p^e_U(.)}(I) = I$ for $p^e_U(.) > 0$, or $Nr^e_{p^e_U(.)}(I) < I$ for $p^e_U(.) = 0$. Let $(n^e(I), r^e(I))$ denote $(n^e_{p_U(.)}(I), r^e_{p_U(.)}(I))$ and $(V^{e\prime}_R(I), V^{e\prime}_N(I))$ denote $\left(\frac{d}{dI}V^e_{p^e_U(.),R}(I), \frac{d}{dI}V^e_{p^e_U(.),N}(I)\right)$.

Consider $p^e_U(I) > 0$ and $Nr^e(I) = I$. Substituting these in (A-44), (A-45), (A-48) and (A-49), we obtain:

$$\begin{cases} p_N(n^e(I), I) - c_n(q) + \beta q p^e_U(qn^e(I)) + n^e(I)\left(\frac{\partial p_N(n^e(I),I)}{\partial n} + \beta q^2 p^{e\prime}_U(qn^e(I))\right) + \beta q V^{e\prime}_N(qn^e(I)) = 0 \\ p_R(n^e(I), I) - c_r(q) - p^e_U(I) + \frac{I}{N}\frac{\partial p_R(n^e(I),I)}{\partial r} - \beta V^{e\prime}_R(qn^e(I)) = 0 \\ V^{e\prime}_N(I) = n^e(I)\frac{\partial p_N(n^e(I),I)}{\partial r} \\ V^{e\prime}_R(I) = \frac{I}{N}\left(\frac{\partial p_R(n^e(I),I)}{\partial n}n^{e\prime}(I) + \frac{\partial p_R(n^e(I),I)}{\partial r}\frac{N-1}{N} - p^{e\prime}_U(I)\right) + \beta V^{e\prime}_R(qn^e(I))\left(\frac{1}{N} + qn^{e\prime}(I)\right) \end{cases} \tag{A-50}$$

The solution to this set of four equations determines the four unknowns $n^e(I)$, $p^e_U(I)$, $V^{e\prime}_N(I)$ and $V^{e\prime}_R(I)$. In the next step, we study the case of $I = 0$ and $q = 0$. Under these conditions, we have that $\sum_{i=1}^N r^e_i(0) = 0$. We will use (A-50), and validate that $p^e_U(0) > 0$.

**Step 3: Derivative of $V^e_N(I, q)$ with respect to $q$ evaluated at $q = 0$.**

In this step, we reintroduce the dependence of all previous expressions with respect to $q$. Taking the partial derivative of $V^e_N(I, q)$ with respect to $q$ yields

$$\begin{aligned} \frac{\partial V^e_N(I,q)}{\partial q} &= n^e(I)\left(-c'_n(q) + \beta p^e_U(qn^e(I)) + \beta q n^e(I) p^{e\prime}_U(qn^e(I))\right) \\ &\quad + \beta n^e(I)\frac{\partial V^e_N(I + qn^e(I) - Nr^e(I), q)}{\partial I} + \beta\frac{\partial V^e_N(I + qn^e(I) - Nr^e(I), q)}{\partial q}. \end{aligned}$$

Evaluated for $I = 0$ and $q = 0$, can write the previous expression as

$$(1-\beta)\frac{\partial V^e_N(0,0)}{\partial q} = n^e(0)\left(\beta p^e_U(0) - c'_n(q)\right) + \beta n^e(0)\frac{\partial V^e_N(0,0)}{\partial I}. \tag{A-51}$$

Substituting $I = 0$ and $q = 0$ in the first equation of (A-50), we obtain $p_N(n^e(0), 0) - c_n(0) + n^e(0)\frac{\partial p_N(n^e(0),0)}{\partial n} = 0$, which can be rewritten as $\frac{\partial R(n^e(0),0)}{\partial n} = c_n(0)$, and is solved by $n_{su}$ (see definition of $n_{su}$). Substituting $I = 0$ and $q = 0$ in the fourth equation of (A-50), we obtain $V^{e\prime}_R(0) = \beta V^{e\prime}_R(0)\frac{1}{N}$, from which it follows that $V^{e\prime}_R(0) = 0$. Plugging the latter result in the second equation of (A-50), we obtain $p_R(n_{su}, 0) - c_r(0) = p^e_U(0)$.

Using the proof of Lemma 4, we see that $p_R(n_{su}, 0) = (1-\delta)p_N(n_{su}, 0)$ and that $p_N(n_{su}, 0) > \frac{\partial R(n_{su},0)}{\partial n}$. Therefore, $p^e_U(0) = (1-\delta)p_N(n_{su}, 0) - c_r(0) > (1-\delta)\frac{\partial R(n_{su},0)}{\partial n} - c_r(0) = (1-\delta)c_n(0) - c_r(0)$. Thus, if $(1-\delta)c_n(0) > c_r(0)$, then $p^e_U(0) > 0$, which validates our assumption in step 2.

Finally, from the third equation of (A-50), we obtain $V_N^{e\prime}(0) = n_{su}\frac{\partial p_N(n_{su},0)}{\partial r}$. These relationships can be substituted back in (A-51):

$$(1-\beta)\frac{\partial V_N^e(0,0)}{\partial q} = n_{su}\left(\beta\left(p_R(n_{su},0) - c_r(0) + n_{su}\frac{\partial p_N(n_{su},0)}{\partial r}\right) - c_n'(0)\right)$$

or

$$\frac{\partial V_N^e(0,0)}{\partial q} = \frac{1}{1-\beta}n_{su}\left(\beta\left(\frac{\partial R(n_{su},0)}{\partial r} - c_r(0)\right) - c_n'(0)\right)$$

Thus, taking the fixed investment costs into account, we obtain

$$\Delta^e \doteq \frac{\partial V_N^e(0,0)}{\partial q} - k'(0) = \frac{1}{1-\beta}n_{su}\left(\beta\left(\frac{\partial R(n_{su},0)}{\partial r} - c_r(0)\right) - c_n'(0)\right) - k'(0)$$

and we observe that $\Delta^e = \Delta$. ■