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Markov approach to fatigue crack growth under stochastic load

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ABSTRACT

Under variable amplitude loading the fatigue crack growth rate to the load cycles following an overload is reduced. Under stochastic loading the retardation phases occur randomly with random intensity and duration depending on the last random overload. In the paper the load process is assumed to be stationary and its extremes are modelled as a Markov sequence. The fatigue crack growth appears to form a random sequence of retardation and post- retardation phases. The drift and diffusion parameters of a Markov approximation for the whole crack growth process are evaluated from a numerical simulation within retardation and post-retardation blocks. This mixed, numerical and analytical, approach allows us to efficiently investigate the effect of different load and retardation model assumptions on fatigue structural lifetime.

INTRODUCTION

Load sequence effects are observed in fatigue experiments under variable amplitude loading where the time to reach a given critical length by a macro crack, called the lifetime, strongly depends on the arrangement of sequence of load maxima, e.g. Schijve [1]. The usual change observed in the Mode I of fatigue crack growth is a transient diminution of the crack propagation rate after an overload. The duration of this phenomenon, called retardation of the crack growth, and the magnitude of the retardation effect depend on many factor including specimen geometry, enviromental effects, material properties, the magnitude of the overload and of subsequent extremes. The physical nature of this phenomenon has not been completely explained, yet. Most of the models that are proposed in the literature to predict the fatigue crack growth with regard to the load sequence effects refer to the overload-inducted plastic zone and a diminution of the effective stress intensity factor range after an overload, see e.g. Wanhill & Schijve

[2]. They also assume the plasticity-inducted fatigue crack closure as a dominant cause of fatigue crack growth retardation in Mode I, e.g Shin & Fleck [3]. Such an approach originated also some previous attempts, see e.g Dolinski [4], Winterstein & Veers [5], to model stochastic fatigue crack growth under stochastic loading and to assess some probabilistic characteristics of the structural lifetime when the critical length of a fatigue macro crack determines the structural failure.

FATIGUE CRACK GROWTH LAW

Very wide class of fatigue crack growth laws can be written in a general form

$$\frac{\Delta a}{\Delta n} = F[a; S^+(n), S^-(n)] \tag{1}$$

where a denotes the crack length and n is the number of load cycles representing the time in cycle units. The function $S^+(n)$ and $S^-(n)$ denote, respectively, the stress maximum and minimum in the n-th cycle of the far-field stress applied to a cracked element. Since Elber [6] noticed the crack growth closure phenomenon and pointed out its significance for fatigue crack growth, the effective stress cycle amplitude, $\Delta S_{eff} = S^+ - S_{op}$, is usually considered in fatigue crack equations instead of $\Delta S = S^+ - S^-$ if $S_{op} > S^-$. In the literature there is no universal formula relating S_{op} to the other load cycle parameters. Most of the proposal are based on experimental data, see e.g. Bulloch [7]. The bilinear form, $q(R) = S_{op}/S^+ = \min\{q_0 \cdot (1 + R/|R_0|), R\}$, proposed by Veers [8], is used in the paper with q_0 and R_0 as material parameters. The ratio, $R = S^-/S^+$, defines the stress cycle asymmetry coefficient.

It is observed in fatigue experiments under constatnt amplitude loading with a single overload that the crack opening stress increase transitorily after the overload application and then return to its pre-overload value, e.g Reynolds [9]. It lessens the effective stress amplitude, ΔS_{eff}, and the fatigue crack growth, eventually. In some retardation models, Wheeler [10], Willenborg et al. [11], the retarded growth of fatigue crack after an overload is assumed to continue as long as the plastic zones, $r_Y(a, S^+)$, due to the current maxima following the overload are contained in the plastic zone, $r_Y(a_{ol}, S_{ol})$, produced by the overload. In order to specify the retardation intensity Veers [8] introduced the so-called reset stress, S_r. It determines the stress levels that is necessary to reset the maximum extend of the overload plastic zone, $r_Y(a_{ol}, S_{ol})$. Applying the equality, $a_{ol}+r_Y(a_{ol}, S_{ol}) = a+r_Y(a, S_r)$, where a is the current crack length, and assuming Irwin's [12] estimate of the plastic zone range the reset stress can be calculated as follows

$$S_r = S_r(a, a_{ol}, S_{ol}) = \frac{\sigma_Y}{Y(a)} \cdot \sqrt{\frac{a_{ol} + r_y(a_{ol}, S_{ol}) - a}{a}} \tag{2}$$

where σ_Y is the yield stress of material and $Y(a)$ denotes a dimensionless function depending on the crack and specimen geometry.

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Generalising the concept of the crack opening stress, $S_{op} = q \cdot S^+$, at constant amplitude loading for variable amplitude loading where $S_{op} = q \cdot S_r$ the effective minimum of a cycle, $S_{eff}{}^-(n)$, can be explicitly written as

$$S_{eff}{}^-(n) = S_{eff}{}^-(a, a_{ol}, S_{ol}; S^-(n), S^+(n)) = \tag{3}$$
$$= \begin{cases} q \cdot S^+(n) \text{ if } S^+ > S_r \text{ and } S^- < q \cdot S^+ \\ q \cdot S_r(a, a_{ol}, S_{ol}) \text{ if } S_r > S^+ > q \cdot S_r \text{ and } S^- < q \cdot S_r \\ S^-(n) \text{ if } S_r > S^+ > q \cdot S_r \text{ and } S^- \geq q \cdot S_r \\ S^+(n) \text{ if } S^+ \leq q \cdot S_r \end{cases}$$

Substituting the real cycle minimum, $S^-(n)$, with the effective minimum, Eq. (3), in Eq. (1) we obtain the fatigue crack growth equation

$$\frac{\Delta a}{\Delta n} = F[a; S^+(n), S^-_{eff}(a, a_{ol}, S_{ol}; S^-(n), S^+(n))] \tag{4}$$

which involves some load sequence effect due to the presence of parameters, a_{ol} and S_{ol}, associated with the last overload. Eq. (4) requires a cycle-by-cycle summation in calculation of the fatigue crack length for deterministic variable amplitude loading.

STOCHASTIC LOADING AND RETARDATION

For stochastic loading every maximum can likely be an averload. For the retardation model described in the previous section the necessary and sufficient condition for a maximum, $S_k{}^+$, to be an overload, S_{ol}, is the inequality

$$S_k{}^+ = S_{ol} \text{ iff } S_{k-1}{}^+ \leq S_k{}^+ \text{ and } S_{k+1}{}^+ \leq S_r(a_k = a_{ol}, a_{k+1}; S_k{}^+) \tag{5}$$

with a_i correspond whit a maximum S_i and so on. The maximum S_{ol} starts a retardation phase which continues as long as

$$S_i{}^+ \leq S_r(a_{ol}, a_i; S_{ol}) \text{ for } i > k \tag{6}$$

If the condition given in Eq. (6) for continuation of the retardation phase is not satisfied for a maximum, $S_j{}^+$ say, this maximum can be the next overload or it can start a post-retardation phase which will continue as long as

$$S_{j-1}{}^+ \leq S_j{}^+ \leq S_{j+1}{}^+ \tag{7}$$

If the contition of continuation of the post-retardation phase is not satisfied for a maximum, $S_l{}^+$ say, this maximum is assumed to start the next retardation phase. This scheme can be extended on the whole fatigue crack propagation process which appear to alternately consist of retardation and post retardation phases. Every both successive phases, retardation and post-retardation ones, will be consider as block starting and terminating with overloads.

The stress extremes are the only load parameters involved in fatigue crack growth equations. Therefore, just their probabilistic characteristics are desired to predict the structural lifetime due to the macro crack propagation. Unfortunately, a full stochastic description of sequence of extremes can be obtained in very specific cases of stochastic process only. Recently, Frendahl & Rychlik [13] have shown on a very wide numerical simulation basis that a homogeneous Markov chain is a good approximation of the random sequence of extremes of stationary Gaussian and some non-Gaussian processes with various spectral characteristics. In practical application the length of correlation, n_{corr} = a dozen of cycles or so, of the sequence of extremes appears much shorter than the length of the blocks, N_B = several dozen of cycles, consisting of retardation and post-retardation phases. Then the number of cycles to failure, N_F = several thousands cycles or more, is much longer than N_B's. This property, $n_{corr} \ll N_B \ll N_F$, suggest to apply the Markov approximation for the fatigue crack growth equation, Eq. 4, see e.g. Tichonov & Mironov [14]. A very great lifetime cycle number makes justifiable a continuous approximation to find the statistical moments of N_F from the sequence of recurrent differential equations

$$\eta(a_0) \cdot \bar{N}_n'(a_0) + \frac{1}{2} \cdot \sigma^2(a_0) \cdot \bar{N}_n''(a_0) = -n \cdot \bar{N}_{n-1}(a_0) \quad \text{with} \quad \bar{N}_0(a_0) = 1 \qquad (8)$$

where $\bar{N}_n(a_0) = \bar{N}_n(0, a_0, a_F) = E[{N_F}^n]$ denotes the n-th statistical moment of the lifetime for a crack starting at $a = a_0$ and terminating at $a = a_F$. The absorbing boundary condition are assumed as $\bar{N}_n(0,0,a_F) = \bar{N}_n(0, a_F, a_F) = 0$. For continuous Markov process the parameters $\eta(a)$ and $\sigma^2(a)$, denote the drift and diffusion coefficients. For Markov sequence they are considered as the the mean and the second moments of the crack length increment in a cycle: $\eta(a) = E[A(i+1) - A(i) | A(i) = a]$ and $\sigma^2(a) = E[(A(i+1) - A(i))^2 | A(i) = a]$. The following section deals with an approach to determine these parameters.

SIMULATION PROCEDURE

Following the procedure proposed in Frendahl & Rychlik [13] we discretize the extremes into a finite number of levels, $u_1, u_2 < ... < u_n$. Samples of stochastic process obtained from numerical simulation or given as records from observations are statistically analysed to estimate the transition probability matrices from a maximum to a subsequent minimum, $\mathbf{P}$, and from a minimum to a subsequent maximum, $\hat{\mathbf{P}}$. The assumption of Markovian character and the transition probability matrices define completly the random sequence of extremes. They suffice to propose a simple numerical procedure that allows us to follow all probabilistic characteristics associated with fatigue crack propagation after a given overload, $S_{ol} = u_k$, applied to the specimen with a crack of length $a = a_{ol}$. Simulation begins with assumption of a crack length, a_{ol}, and an overload, $S_{ol} = u_k$. We than discretize the crack length, $a_i = a_{ol} + i \cdot \Delta a$, with an appropriate discretization parameter Δa. Successive calculation (cycle after cycle) of the transition probabilities from the initial state, $a = a_{ol}$, to any state, $a = a_i$, involves an appropriate multiplication of the transition probability matrices, $\mathbf{P}$ and $\hat{\mathbf{P}}$, with accounting for the inequality condition given by Eqs. (5), (6) and (7) and using Eq. (4) for the crack length increment due to a current stress cycle. The calculation directly provides the joint probability distribution, $P[A_k(a_{ol}) = a_i \cap I_k(a_{ol}) = i | S_{ol} = u_k]$, of the crack length, $A_k(a_{ol})$, at the end of the retardation and post-retardation block, which has started at $a = a_{ol}$, and the number of cycles, $I_k(a_{ol})$, within the block given the overload, $S_{ol} = u_k$, applied to the structural element with a crack of length, $a = a_{ol}$. Repeating the calculation for all value of maxima as overloads we release the overload condition as follow

$$P[A_k(a_{ol}) = a_i \cap I_k(a_{ol}) = i] = \sum_{k=1}^{n} P[A_k(a_{ol}) = a_i \cap I_k(a_{ol}) = i | S_{ol} = u_k] \cdot P[S_{ol} = u_k] \tag{9}$$

The calculation shows that the length of a block $\Delta A_k(a_{ol}) = A_k(a_{ol}) - a_{ol} \ll a_{ol}$ for most cases of practical engineering interest. It justifies the following averaging procedure

$$\widehat{\Delta A}^m(a_{ol}) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{(a_i - a_{ol})^m}{n} \cdot P[A_k(a_{ol}) = a_i \cap I_k(a_{ol}) = n] \quad \text{for} \quad m = 1, 2 \tag{10}$$

which leads to the mean (m=1) and the second (m=2) moment of the crack length increment per cycle averaged over the retardation and post-retardation block starting at $a = a_{ol}$. Repeating the computation for several a_{ol}'s we release the start condition due to the summation

$$\mu(a; m) = \sum_{a_{ol} < a} \widehat{\Delta A}^m(a_{ol}) \cdot \frac{P[A_k(a_{ol}) \geq a]}{\sum_{a_{ol} < a} P[A_k(a_{ol}) \geq a]} \quad \text{with} \quad m = 1, 2 \tag{11}$$

Now, $\eta(a) = \mu(a; 1)$ and $\sigma^2(a) = \mu(a; 2)$. They are considered, respectively, as empirical drift and diffusion parameters in Eq. (8) enabling the calculation of lifetime moments.

NUMERICAL EXAMPLE

This numerical example deals with an infinite metal sheet where a central part-through crack of initial size, a_0=0.127 mm is present. The crack propagates up to a final length, a_f=3.810 mm, where failure is assumed to take place. A Paris-Erdogan law, a special case of Eq. (1), is adopted to describe the fatigue crack growth, $da/dn = C \cdot [(S^+ - S_{eff}^-) \cdot Y(a) \cdot \sqrt{\pi a}]^m$. For the simple structural situation under investigation we have $Y(a) = 1$ while $C = 4.13 \cdot 10^{-13}$ MPa units and m=3.5. The parameters involved in the definition of the reset stress (Eq. (2)) and the effective minimum (Eq. (3)) are: σ_Y=137.6 MPa; R_0=-5 and q_0=.496. Zero mean stationary Gaussian load $S(t)$ are selected to show the applicability of the proposed approach. Each loads has a rectangular one-side power spectra density function in the frequency interval $[\alpha, \beta]$ and constant standard deviation equal to 30.76 MPa. The lower limit α of the power spectrum density function is keep costant to 10 Hz. For $\beta - \infty$, the bandwidth parameter (m_i are the spectral moments) $\epsilon = \sqrt{1 - m_2/(m_0 \cdot m_4)}$ tends to 0.667.
Realizations of the load process were performed for the following set of the upper

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limit β: 11, 13, 20, 40, 50, 100 (Hz.) by using the sequential simulation method due to Veers [8], which simulates the sequential random variable which make up successive peaks and valley. It means that only the significant peaks and ranges are generated.

Given a realization of the stress process the extremes are discretized into 51 equally spaced levels and the transition matrices $\mathbf{P}$ and $\hat{\mathbf{P}}$ are generated by statistical analysis. The simulation procedure outlined in the previous section produces then the joint probability distribution function of the crack length after the end of the retardation and post-retardation phase and the number of cycles to the next overloading. This result is shown in Fig. (1) for β=40 Hz. The drift and diffusion coefficients are then numerically evaluated and finally the first and second moment of the mean number of the duty cycles to failure is computed by solving the recurrent differential equation given in Eq. (8) applying the appropriate boundary conditions. The results for the mean value of the number of cycles to failure is given in Fig. (2) as a function of ϵ in order to shown the effect of the bandwidth parameter on the structural lifetime. In the same picture it is also reported the structural lifetime without sequence effects to illustrate the importance of retardation phenomenon on fatigue crack growth.

CONCLUSIONS

This paper deals with the problem of fatigue crack growth retardation due to load sequence effect.

An original simulation procedure is introduced to evaluate the drift and diffusion coefficient of the Markov approximation of the fatigue crack growth.

It avoids the solution of a complicated system of stochastic differential equations for the probabilistic description of the structural lifetime.

The proposed method is very flexible since it allows us to consider a very wide class of Gaussian and non-Gaussian processes with different spectral characteristics.

Further research is in progress to insert the effect of randomness of material and geometrial parameters in the proposed simulation scheme.

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REFERENCES

1. Schijve, J., Effect of load sequence on crack propagation under random and program loading, *Engng. Fract. Mech.*, vol. 5, pp. 269-280, 1973.
2. Wanhill, R.J.H., Schijve, J., in *Fatigue Crack Growth Under Variable Amplitude Loading*, ed. Petit, J. et al., Elsevier, London, p. 326, 1988.
3. Shin, C.S., Fleck, N.A., Overload retardation in structural steel, *Fatigue Fract. Engng. Mater. Struct.*, vol. 9, No. 5, pp. 379-393, 1987.

Transactions on Engineering Sciences vol 6, © 1994 WIT Press, www.witpress.com, ISSN 1743-3533

4. Dolinski, K., Fatigue crack growth with retardation under stationary stochastic loading, *Engng. Fract. Mech.*, vol. 27, No. 3, pp. 279-290, 1987.
5. Winterstein, S.R., Veers, P.S., Diffusion models of fatigue crack growth with sequence effects due to stationary random loading, *Proc. of the 5-th Int. Conf. on Struct. Safety and Reliability, ICOSSAR '89*, pp. 1523-1530, 1989.
6. Elber, W., The significance of fatigue crack closure, in: *Damage Tolerance in Aircraft Structures*, ASTM STP 486, pp. 230-242, 1971.
7. Bulloch, J.H., The influence of mean stress or R-ratio on the fatigue crack threshold characteristics of steels - A review, *Int. J. Pres. Ves. & Piping*, vol. 47, No. 3, pp 263-292.
8. Veers, P.J., Fatigue crack growth due to random loading, *SAND87-2037*, Sandia National Laboratories, 1987.
9. Reynolds, A.P., Constant amplitude and post-overload fatigue crack growth behaviour in PM aluminium alloy AA 8009, *Fatigue Fract. Engng. Mater. Struct.*, vol. 15, No. 6, pp. 551-562, 1992.
10. Wheeler, O.E., Spectrum loading and crack growth, *J. Basic Engng.*, vol. 94, pp. 181-186, 1972.
11. Willenborg, J., Engle, R.M., Wood, H.A., A crack growth retardation model using an effective stress concept, *Report TM-71-1-FBR*, Wrigth-Patterson Air Force Basis, Ohio, 1971.
12. Irwin, G.R., Fracture mode transition for a crack traversing a plate, *J. Basic Engng.*, Trans. ASME, vol. 82, 1960.
13. Frendahl, M., Rychlik, I., Rainflow analysis - Markov method, Univ. of Lund, Dept. of Mathematical Statistics, *Res. Rep.*, 3, 1992.
14. Tichonov, V.I., Mironov, M.A., *Markov Process*, Sov. Radio, 1977 (in Russian).
15. Casciati, F., Fravelli, L., *Fragility analysis of complex structural systems*, Research Studies Press, Taunton, U.K., 1991.

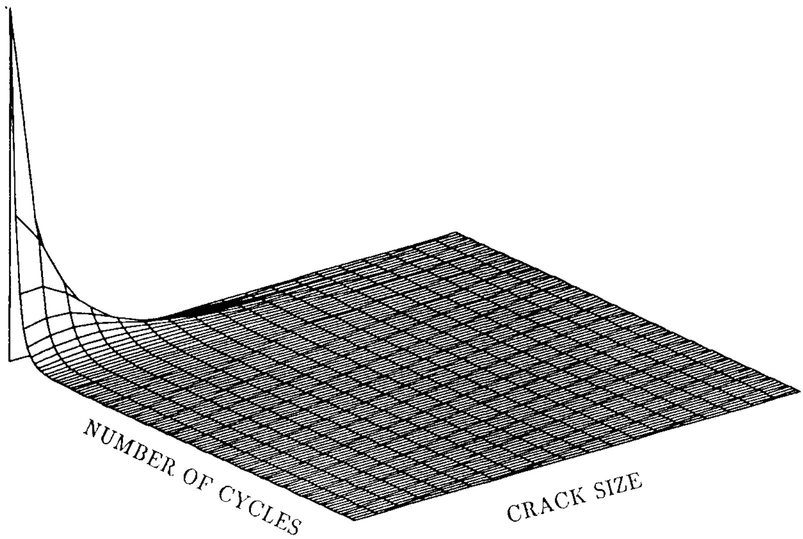

Fig. 1. Joint probability distribution function of the crack length at the end of the retardation and post retardation phase and the number of cycle within the block (β = 40 Hz.).

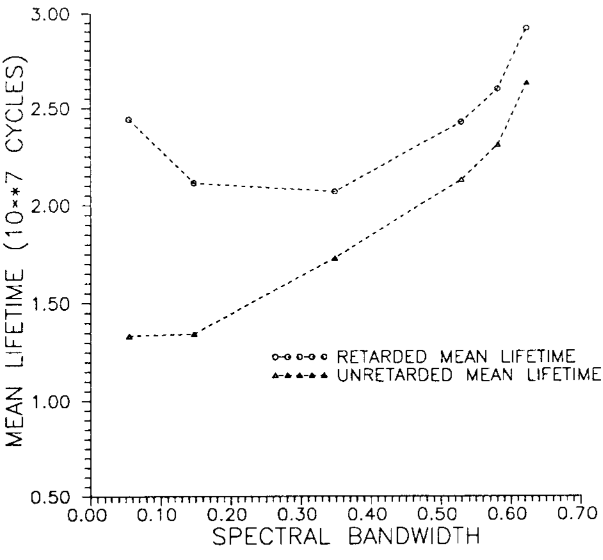

Fig. 2. Mean value of the lifetime as function of the bandwidth parameter ϵ for the retardated and unretarded fatigue crack growth.