Markov approximations and decay of correlations for Anosov flows

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Abstract

We develop Markov approximations for very general suspension flows. Based on this, we obtain a stretched exponential bound on time correlation functions for 3-D Anosov flows that verify 'uniform nonintegrability of foliations'. These include contact Anosov flows and geodesic flows on compact surfaces of variable negative curvature. Our bound on correlations is stable under small smooth perturbations.

1 Introduction

Let $\phi^t : M \to M$ be a measurable flow preserving a probability measure μ, and F, G two square integrable functions on M. The time correlation function is defined by

$$C_{F,G}(t) = \int_M F(\phi^t y) G(y)\, d\mu(y) - \left(\int_M F(y)\, d\mu(y)\right) \cdot \left(\int_M G(y)\, d\mu(y)\right) \quad (1.1)$$

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The flow ϕ^t is mixing if and only if the correlation function (1.1) converges to zero (decays) as $t \to \infty$. The asymptotics of $C_{F,G}(t)$, as $t \to \infty$, is of great interest in the theory of chaotic dynamics and statistical physics.

The basic classes of mixing flows are Anosov and Axiom A flows, including geodesic flows on manifolds with negative curvature. In 1975 Bowen and Ruelle [7, 30] raised a question:

Do the correlations for mixing Axiom A flows with Gibbs invariant measures and smooth functions F, G decay exponentially in t?

Since then it has been a major challenging problem to obtain upper bounds on correlations for mixing flows. Formally, a negative answer to the above question was given by Ruelle [31] and Pollicott [25] who found mixing Axiom A flows for which the correlations decay arbitrarily slowly. There are no examples of mixing Anosov flows or geodesic flows on negatively curved manifolds with a decay of correlations slower than exponential. There is a strong belief and excessive numerical evidence that, normally, the correlations for such flows decay exponentially fast in t.

Only a few rigorous results exist in this direction, however. Exponential upper bounds on correlations have been established for geodesic flows on manifolds of *constant* negative curvature in two dimensions (Moore [23], Ratner [29], Collet et al. [13]) and three dimensions (Pollicott [26]). For these flows representation group theory does the job, but it presumably cannot be adapted to Anosov or Axiom A flows.

Our poor knowledge of the asymptotics of correlation functions for mixing flows contrasts to the well established exponential bounds on correlations for all the basic classes of discrete-time mixing systems, including Anosov and Axiom A diffeomorphisms, expanding interval maps, etc. The reason why the correlations for flows are substantially harder to estimate than those for diffeomorphisms is that flows have a zero Lyapunov exponent in the direction of the flow. The time t map ϕ^t is then only partially hyperbolic [8] for any $t \in \mathbb{R}$. In other words, there is no exponential instability of trajectories in the flow direction. The mechanism of mixing in the phase space of a flow is, therefore, more subtle than that of a diffeomorphism.

The lack of rigorous bounds on the correlations for mixing flows is a real headache for physicists. For example, the Green-Kubo formulas for transport coefficients involve integrals of autocorrelation functions, $\int_{-\infty}^{\infty} C_{F,F}(t)\, dt$, for which nobody can rigorously prove integrability, let alone exponential decay.

In some cases bounds on time correlations are the only missing components in rigorous proofs of Green-Kubo formulas and related transport laws [10].

In this paper we give a partial solution to this old problem, as described below.

Let ϕ^t be a C^2 Anosov flow on a 3-D compact Riemannian manifold M. We assume that it is topologically mixing and its stable and unstable foliations satisfy a condition that we call 'uniform nonintegrability of foliations' (the exact meaning is given by the assumption (A5) stated in Sections 13). Let μ be the Sinai-Bowen-Ruelle (SBR) measure for ϕ^t. Let F, G be two so called generalized Hölder continuous functions (defined in Section 2).

Theorem 1.1 *Under the above conditions we have*

$$|C_{F,G}(t)| \leq v(F,G) \cdot c_\phi e^{-a_\phi \sqrt{t}} \tag{1.2}$$

Here $c_\phi > 0$ and $a_\phi > 0$ depend on the flow ϕ^t alone. On the contrary, the factor $v(F,G)$ is independent of the flow.

The exact expression for $v(F,G)$ is given in (7.5).

The function of t on the right-hand side of (1.2) is often called a *stretched* exponential. It decays slower than any exponential, but fast enough for virtually all physical applications.

We also show that this theorem covers all contact 3-D Anosov flows and all geodesic flows on surfaces of variable negative curvature.

For physical applications mentioned before, it is important that the bound on correlations be uniform for all flows close enough to the given one [10]. Responding to these needs, we study small perturbations of Anosov flows in C^1 metric. Let ϕ^t and ψ^t be two flows on M satisfying the assumptions[1] of Theorem 1.1.

Theorem 1.2 *For any $\varepsilon > 0$ there is a $\delta > 0$ such that if ψ^t is δ-close to ϕ^t in C^1 metric, then $|c_\psi - c_\phi| < \varepsilon$ and $|a_\psi - a_\phi| < \varepsilon$.*

In other words, the values of a_ϕ and c_ϕ depend continuously on the flow ϕ^t in C^1 metric.

[1] In (A5) stated in Section 13 two positive numbers $\underline{d}$ and $\overline{d}$ and a ball $B_0 \subset M$ are involved. Those must be the same for both flows ϕ^t and ψ^t.

The proofs of Theorems 1.1 and 1.2 rely on purely dynamical arguments. We use Markov partitions and symbolic dynamics created by Sinai, Bowen and Ruelle, and develop special approximations of mixing flows by Markov chains.

The paper consists of two parts. In the first, spanning Sections 2-7, we develop the techniques of Markov approximations to flows in a very general setup – we work with suspension flows built under generalized Hölder continuous functions over measurable transformations of metric spaces. These are 'bare bones' of our techniques, readily applicable to wide classes of mixing flows, including billiards, nonuniformly hyperbolic flows, etc. The results of this part are summarized in Section 7, were we give sufficient conditions for a stretched exponential bound on correlations.

In the second, principal part of the paper, presented in Sections 8-18, we study Anosov flows and prove the above two theorems. The main Theorem 1.1 is proved in Sections 8-16. Theorem 1.2 is proved in Section 17. Contact and geodesic flows are discussed in Section 18.

Our method presently does not produce an exponential bound on correlations. But we conjecture that under the assumptions of Theorem 1.1 the correlations do decay exponentially fast, and our method can be refined to produce an exponential bound. We also believe that the present results and proofs can be extended to physically interesting billiard models, and to multi-dimensional Anosov flows.

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2 Suspension flows

Let Ω be a metric space with a metric ρ and a nonatomic Borel probability measure ν. Let $T : \Omega \to \Omega$ be an invertible measure-preserving transformation (i.e. $\nu(A) = \nu(TA) = \nu(T^{-1}A)$ for every measurable $A \subset \Omega$). We assume that $\operatorname{diam}\Omega < \infty$.

Now let $l(x)$ be a positive bounded measurable function on Ω. A suspension flow build under the function $l(x)$ is defined on the measurable space $\mathcal{M} = \{(x,s) : x \in \Omega,\, 0 \le s < l(x)\}$ by the rule

$$\Phi^t(x,s) = \begin{cases} (x, s+t) & \text{for } \ 0 \le t < l(x) - s \\ (Tx, s+t-l(x)) & \text{for } \ l(x) - s \le t < l(Tx) + l(x) - s \end{cases} \tag{2.1}$$

This flow is measurable and preserves the probability measure μ on $\mathcal{M}$ defined by $d\mu = c_\mu \cdot d\nu \times ds$, where $c_\mu^{-1} = \int_\Omega l(x) d\nu(x)$. The map T is often called the base transformation and $l(x)$ the ceiling function. For any point $y = (x,s) \in \mathcal{M}$ we denote its 'coordinate' projections by $\pi_1(y) = x$ and $S(y) = s$.

We intentionally do not assume any ergodic or mixing property of the map T or the flow Φ^t, nor the existence of local coordinates in Ω or $\mathcal{M}$. All we need for the machinery developed in Sections 2-7 is a metric on Ω. The metric ρ is extended to $\mathcal{M}$ in a natural way: for any $y = (x,s)$ and $y' = (x',s')$ we put $\rho(y,y') = (\rho^2(x,x') + |s-s'|^2)^{1/2}$. Notice that $\operatorname{diam}\mathcal{M} < \infty$.

Here we do not follow a tradition to identify points $(x, l(x))$ and $(Tx, 0)$ of $\mathcal{M}$. In our metric the distance between these points is positive, but this will not be essential for our results.

We will study functions that satisfy the following generalized Hölder continuity. Its definition is similar to the one introduced in [21]. Let M' be a metric space with a probability measure, μ'. For any measurable function $f : M' \to \mathbb{R}$ and a subset $B \subset M'$ we define the oscillation of $f(x)$ on B by $\operatorname{osc}(f,B) = \sup_B f(x) - \inf_B f(x)$. Let $B_r(x)$ be the ball in M' with radius r centered at $x \in M'$. We define the *oscillations* of $f(x)$ by $\operatorname{osc}_r(f,x) = \operatorname{osc}(f, B_r(x))$.

Definition (motivated by [21]). A function $f(x)$ on a metric space M' with a measure μ' is said to be generalized Hölder continuous if there are $\alpha > 0$ and $C > 0$ such that

$$\int_{M'} \operatorname{osc}_r(f,x)\, d\mu'(x) \le C r^\alpha \tag{2.2}$$

for all $r > 0$. We call α the (generalized) Hölder exponent of f and denote the class of functions satisfying (2.2) with some $C = C_f < \infty$ by $GH_\alpha(M')$.

Any (ordinary) Hölder continuous function on M' with an exponent α belongs in $GH_\alpha(M')$. If M' is a compact manifold, the measure μ' has a bounded density and the function f is Hölder continuous on a finite number of domains in M' with piecewise smooth boundary that make a partition of M', then f is also generalized Hölder continuous. In particular, these include characteristic functions of piecewise smooth domains. Yet, the class $GH_\alpha(M')$ is far larger than all the classes just mentioned.

For any $f \in GH_\alpha(M')$ we define the α-variation of f by

$$\mathrm{var}_\alpha(f) = \mathrm{osc}(f, M') + \sup_r r^{-\alpha} \int_{M'} \mathrm{osc}_r(f, x)\, d\mu'(x) \tag{2.3}$$

We now make two assumptions on the suspension flow Φ^t.

Assumption L1 (Generalized Hölder continuity of the ceiling function). Let $l(x) \in GH_{\alpha_l}(\Omega)$ for some $\alpha_l \in (0, 1]$.

Assumption L2 (Lower bound on the density of returns to Ω). There are $t_0 > 0$ and $m_0 \geq 1$ such that for any point $y = (x, s) \in \mathcal{M}$ its trajectory $\{\Phi^t y\}$ for $0 \leq t < t_0$ intersects the base Ω no more than m_0 times, i.e.,

$$l(x) + l(Tx) + \cdots + l(T^{m_0}x) > t_0.$$

for any $x \in \Omega$. Without loss of generality we assume that $t_0 \leq 1$, so that $m_0/t_0 \geq 1$.

In particular, L2 holds if the ceiling function $l(x)$ has a positive lower bound, $l_{\min} > 0$.

Next, we specify which correlation functions (1.1) will be studied. Let $F, G \in GH_\alpha(\mathcal{M})$ be two generalized Hölder continuous functions on $\mathcal{M}$ for some $\alpha > 0$. We are interested in the asymptotics of $C_{F,G}(t)$, as $t \to \infty$, and so we always assume that $t > 1$. Since $C_{F+c,G+d}(t) \equiv C_{F,G}(t)$ for any constants c and d, we can (and will) assume that both functions F and G have zero means:

$$\int_{\mathcal{M}} F(y)\, d\mu(y) = \int_{\mathcal{M}} G(y)\, d\mu(y) = 0.$$

Note that in this case $||F||_\infty \leq \mathrm{osc}_F(\mathcal{M}) \leq \mathrm{var}_\alpha(F)$ and $||G||_\infty \leq \mathrm{osc}_G(\mathcal{M}) \leq \mathrm{var}_\alpha(G)$.

We will assume that $\alpha \leq \alpha_l$. This simply means that the functions F and G are not supposed to be any 'better' (smoother) than the function l.

This is not a restrictive assumption: if $F, G \in GH_\alpha(\mathcal{M})$ and $\alpha_l < \alpha$, then $F, G \in GH_{\alpha_l}(\mathcal{M})$ as well. In any case, one cannot gain much by making F and G 'better' (smoother) than $l(x)$, since the smoothness of $F(\Phi^t(y))$ in (1.1) will depend on how smooth both F and l are, not just F.

3 Markov approximations for the base map T

We will employ the definition of Markov approximations to discrete time dynamical systems introduced in [11].

Let $\mathcal{A} = \{A_i\}$ be a finite or countable measurable partition of the space Ω into subsets of positive measure. By a Markov approximation to the map T we mean a probabilistic stationary Markov chain with transition probabilities

$$\pi_{ij} = \nu(A_j/TA_i) = \nu(A_j \cap TA_i)/\nu(A_i) \tag{3.1}$$

and the stationary distribution

$$p_i = \nu(A_i). \tag{3.2}$$

The ideas behind this approximation, its properties and existing applications are discussed in [11, 12]. If (Ω, T, ν) is a hyperbolic dynamical system with a smooth invariant measure and $\{A_i\}$ is a Markov partition or a Markov sieve [9, 11], then the above Markov approximation leads to an effective estimation of the decay of correlations and gives a proof of the certain limit theorem.

We now briefly recall the necessary properties of the above Markov chain. The 'discrepancy' of the Markov approximation defined by (3.1) and (3.2), within N iterates of the map T, is measured by the following quantity:

$$\chi_N := \sup_{n \le N} \sum_{i_0, i_{-1}, \ldots, i_{-n}} |\nu(A_{i_0}/TA_{i_{-1}} \cap \cdots \cap T^n A_{i_{-n}}) - \nu(A_{i_0}/TA_{i_{-1}})|$$
$$\times \nu(TA_{i_{-1}} \cap \cdots \cap T^n A_{i_{-n}}) \tag{3.3}$$

Here and further on $\nu(A/B)$ means the conditional measure, $= \nu(A \cap B)/\nu(B)$, and we always set it to zero whenever $\nu(B) = 0$. The quantity χ_N measures how distant the 'long-memory' and 'short-memory' conditional distributions are within the first N iterates.

Recall that given two probability distributions $P = \{p_i\}$ and $Q = \{q_i\}$ on the same index set $\{i\}$, the distance in variation between P and Q is defined to be

$$\mathrm{Var}\,(P, Q) = \frac{1}{2} \sum_i |p_i - q_i|. \tag{3.4}$$

Now (3.3) estimates twice the mean distance in variation between the long- and short-memory conditional distributions on $\{A_i\}$.

By means of (3.3) one can estimate how the finite dimensional distributions of the Markov chain,

$$p_{i_{-n}i_{-n+1}\cdots i_{-1}i_0} = p_{i_{-n}} \pi_{i_{-n}i_{-n+1}} \cdots \pi_{i_{-1}i_0} \tag{3.5}$$

are close to those of the dynamical system in the variational metric (3.4). It is shown in [11] that

$$\sum_{i_0, i_{-1}, \ldots, i_{-n}} |\nu(A_{i_0} \cap T A_{i_{-1}} \cap \cdots \cap T^n A_{i_{-n}}) - p_{i_{-n}\cdots i_{-1}i_0}| \le (n-1)\chi_N \tag{3.6}$$

for any $n \le N$.

For any $x \in \Omega$, let $A(x)$ be the atom of the partition $\mathcal{A}$ that contains x. Let $d(x) = \mathrm{diam}\, A(x)$, measured in the metric ρ, and

$$D = D(\mathcal{A}) = \int_\Omega d(x)\, d\nu(x). \tag{3.7}$$

The techniques of Markov approximations work well when both χ_N and $D(\mathcal{A})$ are small enough.

4 Discretization of the suspension flow

We will also define Markov approximations to the suspension flow Φ^t, but first we need to discretize this flow. We start with discretizing the ceiling function $l(x)$, which will be done in two steps. Let $\bar{l}(x) = E(l/\mathcal{A})$ be l conditioned on the partition $\mathcal{A}$, i.e.

$$\bar{l}(x) = [\nu(A(x))]^{-1} \cdot \int_{A(x)} l(x')\, d\nu(x')$$

for any $x \in \Omega$. The function $\bar{l}(x)$ is constant on every atom of $\mathcal{A}$.

Next, let $\delta > 0$ be a small parameter, a 'quantum' of time. Consider another discrete function on Ω defined by

$$\hat{l}(x) = \hat{l}_{\delta,\mathcal{A}}(x) := ([\bar{l}(x)/\delta] + 2)\delta \tag{4.1}$$

where $[a]$ stands for the integral part of a real number a. Clearly, $|\hat{l}(x) - \bar{l}(x)| \leq 2\delta$. The function $\hat{l}(x)$ is not only discrete, but its values are integral multiples of δ. Its minimum value on Ω is not less than 2δ.

Denote the suspension flow build under the function $\hat{l}(x)$ by $\hat{\Phi}^t$. We call it a discrete flow. This flow acts on $\hat{\mathcal{M}} = \{(x, s) : x \in \Omega, 0 \leq s < \hat{l}(x)\}$. The metric ρ is defined on $\hat{\mathcal{M}}$ in the same way as on $\mathcal{M}$, so that we now have a unique metric on $\mathcal{M} \cup \hat{\mathcal{M}}$. The invariant measure of the flow $\hat{\Phi}^t$ is

$$d\hat{\mu} := c_{\hat{\mu}} \cdot d\nu \times ds \quad \text{with} \quad c_{\hat{\mu}}^{-1} = \int_{\Omega} \hat{l}(x) d\nu(x)$$

Note that on the set $\mathcal{M} \cap \hat{\mathcal{M}}$ we have $d\hat{\mu}/d\mu = c_{\hat{\mu}}/c_{\mu}$. Since $\bar{l}(x) \leq \hat{l}(x) \leq \bar{l}(x) + 2\delta$, we have

$$1 \leq \frac{c_{\mu}}{c_{\hat{\mu}}} \leq 1 + 2\delta c_{\mu} \tag{4.2}$$

We assume that δ is small enough, so that at least $c_{\mu}/c_{\hat{\mu}} \leq 2$.

We call $\hat{\Phi}^t$ the discrete version of the original flow Φ^t generated by the partition $\mathcal{A}$ and $\delta > 0$. The smaller the values of δ and D, the more similar are the flows $\hat{\Phi}^t$ and Φ^t. Next, we specify how similar they are in terms of the correlation function (1.1).

First, we extend the functions F and G to $\mathcal{M} \cup \hat{\mathcal{M}}$ by setting them to zero on $\hat{\mathcal{M}} \setminus \mathcal{M}$. This extension is not harmless, however, since it can increase the oscillations of these functions. We thus have to prove that F and G, after their extension to $\hat{\mathcal{M}} \setminus \mathcal{M}$, will be still generalized Hölder continuous, i.e. belong to both $GH_{\alpha}(\mathcal{M})$ and $GH_{\alpha}(\hat{\mathcal{M}})$. Let us denote, for a moment, the 'extended' functions by $\hat{F}$ and $\hat{G}$, respectively.

Let $y = (x, s) \in \mathcal{M} \cup \hat{\mathcal{M}}$ and $r > 0$. Then either $B_r(y) \cap (\hat{\mathcal{M}} \setminus \mathcal{M}) = \emptyset$ and then $\text{osc}_r(\hat{F}, y) = \text{osc}_r(F, y)$, or $B_r(y) \cap \mathcal{M} = \emptyset$ and then $\text{osc}_r(\hat{F}, y) = 0$, or else there is a point $y' = (x', s') \in B_r(y)$ for which $(s - l(x)) \cdot (s' - l(x')) < 0$. In the last case $\rho(x, x') < r$, $s' \geq l(x')$, and $|s - s'| < r$, so that

$$|s - l(x)| \leq |s - s'| + |l(x) - l(x')| \leq r + \text{osc}_r(l, x)$$

In the last case we use the bound $\text{osc}_r(\hat{F},y) \le \text{osc}\,(F,\mathcal{M})$ and then get

$$\begin{aligned}\int_{\mathcal{M}} \text{osc}_r(\hat{F},y)\,d\mu(y) &\le \int_{\mathcal{M}} \text{osc}_r(F,y)\,d\mu(y) + c_\mu \int_\Omega (\text{osc}_r(l,x)+r)\cdot \text{osc}\,(F,\mathcal{M})\,d\nu(x)\\ &\le \text{var}_\alpha(F)\cdot [r^\alpha + c_\mu \cdot (\text{var}_{\alpha_l}(l)\cdot r^{\alpha_l} + r)]\end{aligned}$$

Due to our assumption $\alpha \le \alpha_l$ we conclude that $\hat{F} \in GH_\alpha(\mathcal{M})$. Similarly, one can integrate over $\hat{\mathcal{M}}$ and verify that $\hat{F} \in GH_\alpha(\hat{\mathcal{M}})$. In all that follows, we will denote the new, extended functions by the same symbols, F and G, and this will cause no confusion.

We now consider the correlation function of the discrete flow $\hat{\Phi}^t$ on $\hat{\mathcal{M}}$:

$$\hat{C}_{F,G}(t) = \int_{\hat{\mathcal{M}}} F(\hat{\Phi}^t y)G(y)\,d\hat{\mu}(y) - \Big(\int_{\hat{\mathcal{M}}} F(y)\,d\hat{\mu}(y)\Big)\cdot\Big(\int_{\hat{\mathcal{M}}} G(y)\,d\hat{\mu}(y)\Big) \tag{4.3}$$

Theorem 4.1 *Let Φ^t be a suspension flow and $\hat{\Phi}^t$ its discrete version generated by a partition $\mathcal{A}$ and $\delta > 0$. For any $t > 1$ we set $N = [t/\delta]$. Then*

$$|C_{F,G}(t) - \hat{C}_{F,G}(N\delta)| \le \theta_1 \cdot \text{var}_\alpha(F)\text{var}_\alpha(G)\cdot \Big[t(\delta + D^{\alpha_l/(\alpha_l+1)})\Big]^{\alpha/(2\alpha+2)} \tag{4.4}$$

where the constant $\theta_1 > 0$ is determined by the flow Φ^t, and θ_1 is a continuous function of the following six parameters of this flow: $m_0, t_0, c_\mu, \alpha_l, \text{var}_{\alpha_l}(l)$ and $\text{diam}\,\mathcal{M}$.

Convention. We will denote by θ_i, $i \ge 1$, various positive constants determined by the suspension flow Φ^t and depending continuously on its parameters listed above.

Note that we have 'discretized' the time also, by substituting $N\delta$ for t in (4.3). Theorem 4.1 shows that the difference between the correlation functions of the original flow and that of its discrete version is bounded by an algebraic function in δ and D. Intuitively, this is clear because the smaller δ and D, the finer is the partition $\mathcal{A}$ and the closer the two flows, after that the (generalized) Hölder continuity of F, G and l provides the above power-law estimates.

A complete proof of Theorem 4.1 is provided in Appendix.

5 Markov approximation to the suspension flow

We assume that the partition $\mathcal{A}$ of the base space Ω is fixed and the value of δ is chosen. We will study the map $\hat{T} := \hat{\Phi}^\delta$ that acts on the space $\hat{\mathcal{M}}$ and preserves the measure $\hat{\mu}$. Then the quantity $\hat{C}_{F,G}(N\delta)$ entering (4.4) is the correlation coefficient for the transformation $\hat{T}$ and its Nth iterate.

Denote by $\hat{\mathcal{A}}$ the partition of the space $\hat{\mathcal{M}}$ into atoms $A_i \times [s\delta, (s+1)\delta)$, where $A_i \in \mathcal{A}$ and $s = 0, 1, \ldots, \hat{l}(A_i)/\delta - 1$. We denote the atoms of $\hat{\mathcal{A}}$ by X_i, numbered in an arbitrary order. For any $X_i = A_j \times [s\delta, (s+1)\delta)$ we denote $A(X_i) = \pi_1(X_i) = A_j \in \mathcal{A}$ and $s(X_i) = s$. We call the atoms X_i with $s(X_i) = 0$ the *bottom* atoms and those with $s(X_i) = \hat{l}(A(X_i))/\delta - 1$ the *top* atoms. Over every atom $A \in \mathcal{A}$, there is a column of atoms of $\hat{\mathcal{A}}$ starting with a bottom atom and terminating with a top atom. Every atom X except the top ones is shifted (elevated) by the map $\hat{T}$ one level up, so that $\hat{T}X$ is another atom with $A(\hat{T}X) = A(X)$ and $s(\hat{T}X) = s(X)+1$. Every top atom breaks into pieces under $\hat{T}$ which 'fall' down into some bottom atoms according to the action of T on Ω.

We now define the Markov approximation to the map $\hat{T}$ generated by the partition $\hat{\mathcal{A}}$ of the space $\hat{\mathcal{M}}$ according to the rules similar to (3.1)-(3.2): we take a Markov chain with transition probabilities

$$\hat{\pi}_{ij} = \hat{\mu}(X_j/\hat{T}X_i) \tag{5.1}$$

and stationary distribution

$$\hat{p}_i = \hat{\mu}(X_i). \tag{5.2}$$

The quality of this approximation is given by the value similar to (3.3):

$$\hat{\chi}_N := \sup_{n\le N} \sum_{i_0,i_{-1},\ldots,i_{-n}} |\hat{\mu}(X_{i_0}/\hat{T}X_{i_{-1}} \cap \cdots \cap \hat{T}^n X_{i_{-n}}) - \hat{\mu}(X_{i_0}/\hat{T}X_{i_{-1}})| \times \hat{\mu}(\hat{T}X_{i_{-1}} \cap \cdots \cap \hat{T}^n X_{i_{-n}}) \tag{5.3}$$

Theorem 5.1 *There are constants θ_2, θ_3 and $\hat{\kappa}$ determined by the flow $\{\Phi^t\}$ and depending continuously on its parameters, such that*

$$\hat{\chi}_N \le \delta\left(\theta_2\chi_{[\hat{\kappa}t]} + \theta_3 D^{\alpha_l/(\alpha_l+1)}\right) \tag{5.4}$$

Furthermore, if the ceiling function $l(x)$ has a positive lower bound, we can set $\theta_3 = 0$.

Theorem 5.1 shows that if a partition $\mathcal{A}$ generates a good Markov approximation for the base transformation T and the atoms of $\mathcal{A}$ are small enough, then $\hat{\mathcal{A}}$ will generate a good Markov approximation for the map $\hat{T}$. The rather technical proof of Theorem 5.1 is provided in Appendix.

Let $\hat{\Pi} = ||\hat{\pi}_{ij}||$ be the matrix of transition probabilities and $\hat{P} = ||\hat{p}_i||$ its stationary row vector. We also denote by $\hat{\pi}_{ij}^{(N)}$ the N-step transition probabilities for this Markov chain.

To bound the correlation coefficient $\hat{C}_{F,G}(N\delta)$, we will discretize the functions F and G. Let $\bar{F}$ and $\bar{G}$ be the functions F and G conditioned on the partition $\hat{\mathcal{A}}$ of $\hat{\mathcal{M}}$. That is, on every $X_i \in \hat{\mathcal{A}}$ the function $\bar{F}$ takes the value

$$\bar{F}(X_i) = \bar{F}_i := (\hat{\mu}(X_i))^{-1} \int_{X_i} F(y)\, d\hat{\mu}(y)$$

Similarly, we denote by $\bar{G}(X_i) = \bar{G}_i$ the values of the function $\bar{G}$. Let $\Delta F(y) = F(y) - \bar{F}(y)$ and $\Delta G(y) = G(y) - \bar{G}(y)$ for any $y \in \hat{\mathcal{M}}$.

For brevity, we denote by $\langle f(y)\rangle$ the average of a function f on $\hat{\mathcal{M}}$ with respect to the measure $\hat{\mu}$. The following expansion of the correlation function is immediate:

$$\hat{C}_{F,G}(N\delta) = \langle \bar{F}(\hat{T}^N y)\bar{G}(y)\rangle - \langle \bar{F}(y)\rangle \cdot \langle \bar{G}(y)\rangle$$

$$+\langle \Delta F(\hat{T}^N y)\bar{G}(y)\rangle + \langle \bar{F}(\hat{T}^N y)\Delta G(y)\rangle + \langle \Delta F(\hat{T}^N y)\Delta G(y)\rangle \qquad (5.5)$$

Theorem 5.2 *The last three terms in the expansion (5.5), combined, do not exceed* $\theta_4 \cdot \mathrm{var}_\alpha(F)\mathrm{var}_\alpha(G) \cdot (\delta + D)^{\alpha/(\alpha+1)}$.

The proof of Theorem 5.2 is provided in Appendix.

The main part of the expansion (5.5) is

$$\begin{aligned} \hat{C}_{F,G}^{(\mathrm{main})}(N) &:= \langle \bar{F}(y)\bar{G}(\hat{T}^{-N} y)\rangle - \langle \bar{F}(y)\rangle \cdot \langle \bar{G}(y)\rangle \\ &= \sum_{i,j} \bar{G}_i \bar{F}_j \cdot \left[\hat{\mu}(\hat{T}^N X_i \cap X_j) - \hat{\mu}(X_i)\hat{\mu}(X_j)\right] \qquad (5.6) \end{aligned}$$

This last sum can be approximated in terms of the Markov chain (5.1)-(5.2) as follows:

$$\hat{C}_{F,G}^{(\mathrm{chain})}(N) := \sum_{i,j} \bar{G}_i \bar{F}_j \cdot \left[\hat{p}_i \hat{\pi}_{ij}^{(N)} - \hat{p}_i \hat{p}_j\right] \qquad (5.7)$$

with the discrepancy estimated by

$$
\begin{aligned}
|\hat{C}^{(\mathrm{main})}_{F,G}(N) - \hat{C}^{(\mathrm{chain})}_{F,G}(N)| &\leq ||F||_\infty ||G||_\infty \sum_{i,j} \left|\hat{\mu}(\hat{T}^N X_i \cap X_j) - \hat{p}_i \hat{\pi}^{(N)}_{ij}\right| \\
&\leq ||F||_\infty ||G||_\infty N \hat{\chi}_N \\
&\leq t \cdot ||F||_\infty ||G||_\infty \left(\theta_2 \chi_{[\hat{\kappa}t]} + \theta_3 D^{\alpha_l/(\alpha_l+1)}\right) \quad (5.8)
\end{aligned}
$$

Here the middle inequality follows from the bound (3.6) applied to the Markov chain (5.1)-(5.2), and the last inequality follows from Theorem 5.1. The quantity (5.7) measures 'intrinsic' correlations in the Markov chain (5.1)-(5.2).

The next construction is, at first sight, artificial. Its purpose will become clear only in Section 16.

We perturb the chain (5.1)-(5.2) slightly, 'smoothing it out' in the direction of the flow. The degree of 'smoothing' is specified by an integer parameter η, $1 < \eta < \delta^{-1}$.

First, we define an auxiliary Markov chain whose states are still $X_i \in \hat{\mathcal{A}}$, as follows. Its transition probabilities, $\Pi^* = ||\pi^*_{ij}||$, are defined for all $i \neq j$ by

$$
\pi^*_{ij} = \begin{cases} 1/(2\eta+1) & \text{if } A(X_i) = A(X_j) \text{ and } |s(X_i) - s(X_j)| \leq \eta \\ 0 & \text{otherwise} \end{cases} \qquad (5.9)
$$

and then we set $\pi^*_{ii} = 1 - \sum_{j \neq i} \pi^*_{ij}$ for every $X_i \in \hat{\mathcal{A}}$. Roughly speaking, every atom $X \in \hat{\mathcal{A}}$ under the action of Π^* is 'dispersed' uniformly into $(2\eta + 1)$ neighboring atoms that line up around X in the same column of $\hat{\mathcal{A}}$.

Since the distribution (5.2) is uniform within every column of atoms of $\hat{\mathcal{A}}$, it is invariant for the auxiliary chain (5.9).

We now pick a point of time $t_1 \in (1, t)$ and set $K = [t_1/\delta]$. Let $L = [N/K]$, so that $N = KL + L_0$ for some $L_0 < K$. We also pick K_1, $1 < K_1 < K$, and put $K_2 = K - K_1$. Denote the K-step transition probabilities of the Markov chain (5.1)-(5.2) by $\tilde{\pi}_{ij} = \hat{\pi}^{(K)}_{ij}$, and let $\tilde{\Pi} = ||\tilde{\pi}_{ij}||$ be the corresponding matrix. Also, let $\tilde{\Pi}' = ||\tilde{\pi}'_{ij}|| = ||\hat{\pi}^{(K_1)}_{ij}||$ and $\tilde{\Pi}'' = ||\tilde{\pi}''_{ij}|| = ||\hat{\pi}^{(K_2)}_{ij}||$. Then obviously,

$$
\hat{\pi}^{(N)}_{ij} = \sum_{i_1,\ldots,i_L} \tilde{\pi}'_{ii_1} \cdot \tilde{\pi}_{i_1 i_2} \cdot \tilde{\pi}_{i_2 i_3} \cdots \tilde{\pi}_{i_{L-1} i_L} \hat{\pi}^{(K_2+L_0)}_{i_L i_{L+1}} \qquad (5.10)
$$

where $i_{L+1} = j$. One can think of the action of a nonhomogeneous Markov chain whose transition probabilities are taken firstly from $\tilde{\Pi}'$, then $L-1$ times from $\tilde{\Pi}$, and lastly, at the $(L+1)$-st step, from $\hat{\Pi}^{K_2+L_0}$.

We now consider another nonhomogeneous Markov chain, whose transition probabilities L times are taken from the matrix $\tilde{\Pi}' \cdot \Pi^* \cdot \tilde{\Pi}''$, and at the last, $(L+1)$-st step, they are taken from $\hat{\Pi}^{L_0}$. The resulting transition probability, that of the transition from X_i to X_j, is then

$$\pi_{ij}^{(\mathrm{per})} := \sum_{j_1,l_1,j_2,l_2,\ldots,j_L,l_L} \tilde{\pi}'_{ij_1} \cdot \pi^*_{j_1 l_1} \tilde{\pi}_{l_1 j_2} \pi^*_{j_2 l_2} \cdots \tilde{\pi}_{l_{L-1} j_L} \pi^*_{j_L l_L} \hat{\pi}^{(K_2+L_0)}_{l_L j_{L+1}} \qquad (5.11)$$

where $j_{L+1} = j$. In other words, we perturb the homogeneous Markov chain (5.1)-(5.2) by applying (5.9) first at the K_1-th step and then after every $K_1 + K_2 = K$ steps. Based on obvious similarity of (5.10) and (5.11) we will compare the quantity

$$C_{F,G}^{(\mathrm{per})}(N) := \sum_{i,j} \bar{G}_i \bar{F}_j \cdot \left[\hat{p}_i \pi_{ij}^{(\mathrm{per})} - \hat{p}_i \hat{p}_j\right] \qquad (5.12)$$

with $\hat{C}_{F,G}^{(\mathrm{chain})}(N)$ defined by (5.7). If the perturbation parameter η is small enough ($\ll \delta^{-1}$), the Markov chains (5.10) and (5.11) are close. Precisely:

Theorem 5.3 *We have*

$$|\hat{C}_{F,G}^{(\mathrm{chain})}(N) - C_{F,G}^{(\mathrm{per})}(N)| \leq \theta_5 \cdot \mathrm{var}_\alpha(F) \mathrm{var}_\alpha(G) \left(D^{\alpha/(\alpha+1)} + (t/t_1)^2 (\eta\delta)^{\alpha/(\alpha+1)}\right) \qquad (5.13)$$

This theorem is proved in Appendix.

The quantity (5.12) is bounded as follows:

$$|C_{F,G}^{(\mathrm{per})}(N)| \leq ||F||_\infty ||G||_\infty \cdot \sum_{i,j} \hat{p}_i \left|\pi_{ij}^{(\mathrm{per})} - \hat{p}_j\right| \qquad (5.14)$$

Consider the Markov chain with the transition matrix

$$\bar{\Pi} = ||\bar{\pi}_{ij}|| := \tilde{\Pi}' \cdot \Pi^* \cdot \tilde{\Pi}'' \qquad (5.15)$$

whose stationary distribution is still $\hat{P} = ||\hat{p}_i||$. For any $n \geq 1$ denote by $\bar{\pi}_{ij}^{(n)}$ the n-step transition probabilities of this chain, i.e. $||\bar{\pi}_{ij}^{(n)}|| = \bar{\Pi}^n$. In virtue of (5.11) and (5.14) we have $||\pi_{ij}^{(\text{per})}|| = \bar{\Pi}^L \cdot \hat{\Pi}^{L_0}$. Therefore,

$$|C_{F,G}^{(\text{per})}(N)| \leq ||F||_\infty ||G||_\infty \cdot \sum_{i,j} \hat{p}_i \left| \bar{\pi}_{ij}^{(L)} - \hat{p}_j \right| \tag{5.16}$$

since the correlations in Markov chains decay monotonically. We call the quantity

$$\bar{V}^{(L)} := \frac{1}{2} \sum_{i,j} \hat{p}_i \left| \bar{\pi}_{ij}^{(L)} - \hat{p}_j \right| \tag{5.17}$$

the *mixing coefficient* of the Markov chain $(\bar{\Pi}, \hat{P})$. As it turns out, it is this coefficient that is ultimately responsible for the mixing rates of the flow Φ^t.

We summarize the results of Sections 4-5 in the following corollary:

Corollary 5.4 *For all $t > 1$ we have*

$$|C_{F,G}(t)| \leq \theta_6 \cdot \text{var}_\alpha(F) \text{var}_\alpha(G) \cdot \left[t^{\theta_7} \cdot \left((\eta\delta)^{\theta_8} + D^{\theta_9} + \chi_{[\hat{\kappa} t]} \right) + \bar{V}^{(L)} \right]$$

According to our convention, the constants $\theta_i > 0$ and $\hat{\kappa}$ depend continuously on the parameters of the flow Φ^t. In particular, they do not depend on the choice of t_1 or K_1.

6 Mixing coefficients in Markov chains

From now on we assume that the partitions $\mathcal{A}$ and $\hat{\mathcal{A}}$ are *finite*, so that all the above Markov chains are finite.

In the previous section we ended up with the Markov chain $(\bar{\Pi}, \hat{P})$, whose mixing coefficient $\bar{V}^{(L)}$ is yet to be estimated. It measures the mean distance in variation between the L-step transition probabilities and the stationary distribution $\hat{P}$. In this section we find sufficient conditions under which the mixing coefficients of Markov chains can be effectively bounded.

In order to display the results of this section in a general form, we will work with an abstract homogeneous Markov chain with a finite number of states. We denote the sates by $1, 2, \ldots, I$, the matrix of transition probabilities by $\Pi = ||\pi_{ij}||$ with $1 \leq i, j \leq I$ and the stationary distribution by $P = ||p_i||$. We denote by $\pi_{ij}^{(m)}$ the m-step transition probabilities, i.e. $||\pi_{ij}^{(m)}|| = \Pi^m$. We denote by $\mathcal{J}$ the set of indices $\{1, 2, \ldots, I\}$. Let

$$p_{\min} = \min p_i > 0$$

For any $m \geq 1$ let

$$V_i^{(m)} = \frac{1}{2} \sum_{j=1}^{I} |\pi_{ij}^{(m)} - p_j| \quad \text{and} \quad V^{(m)} = \sum_{i=1}^{I} p_i V_i^{(m)}$$

$V^{(m)}$ is the mixing coefficient of the Markov chain. For every $i, j \in \mathcal{J}$ let

$$\gamma_{i,j} = \sum_{k=1}^{I} \frac{\pi_{ik}\pi_{jk}}{p_k}$$

Theorem 6.1 *Suppose that the Markov chain satisfies the following regularity condition:*

$$\gamma = \min_{i,j} \gamma_{i,j} > 0 \tag{6.1}$$

Then for any $m \geq 1$ and all $i \in \mathcal{J}$ we have

$$V^{(m)} \leq \sup_i V_i^{(m)} \leq 50\gamma^{-1/2} p_{\min}^{-1} \cdot (1 - \gamma/2)^{m/3}$$

It is possible to relax the assumption (6.1) of this theorem: we will now assume a positive lower bound on $\gamma_{i,j}$ for an 'overwhelming majority' of pairs (i, j) rather than for every single pair (i, j). In that case the bound on $V^{(m)}$ is weaker, but still sufficient for us. Let

$$Q(\gamma) := \sum_{(i,j):\, \gamma_{i,j} < \gamma} p_i p_j$$

Theorem 6.2 *For any $\gamma > 0$ and any $m \geq 1$ we have*

$$V^{(m)} \leq \text{const} \cdot \left[\gamma^{-1/2} p_{\min}^{-2} (1 - \gamma/40)^{m/3} + m(p_{\min} + Q(\gamma)) \right] \tag{6.2}$$

where const *is an absolute constant (one can set* const $= 50$*).*

Note that Theorem 6.2 formally does not guarantee any convergence to equilibrium. It can be used effectively only when $m(p_{\min} + Q(\gamma)) \ll 1$. We will demonstrate how it works in the next section.

The last theorem in this section shows that the hypotheses of Theorem 6.2 are stable under certain perturbations. Consider two Markov chains with matrices of transition probabilities $\Pi = ||\pi_{ij}||$ and $\Pi' = ||\pi'_{ij}||$ and with a common stationary distribution $P = ||p_i||$. Let

$$\gamma'_{i,j} = \sum_{k=1}^{I} \frac{\pi'_{ik}\pi'_{jk}}{p_k} \quad \text{and} \quad Q'(\gamma) := \sum_{(i,j):\, \gamma'_{i,j} < \gamma} p_i p_j$$

Theorem 6.3 *Let*

$$\chi' := \frac{1}{2} \sum_{i,j=1}^{I} p_i |\pi_{ij} - \pi'_{ij}| < 1.$$

Then for any $\gamma > 0$ *we have*

$$Q'(\gamma/2) \leq Q(\gamma) + 50\gamma^{-1}\chi'$$

Theorems 6.1, 6.2 and 6.3 are purely probabilistic. They are introduced and proved in [12].

Remark. Theorem 6.1 alone would suffice for Anosov flows treated below in Sections 8-18. Theorems 6.2 and 6.3 are designed for flows with singularities (hence, with countable Markov partitions) and we include them here for future use.

7 Sufficient conditions for a stretched exponential bound on correlations

We now turn back to the Markov chain $(\bar{\Pi}, \hat{P})$ defined in Section 5. Consider yet another chain, with the following matrix of transition probabilities

$$\bar{\bar{\Pi}} = ||\bar{\bar{\pi}}_{ij}|| := \Pi_{K_1} \cdot \Pi^* \cdot \Pi_{K_2} \tag{7.1}$$

where for any $k \geq 1$ we set

$$\Pi_k = ||(\pi_k)_{ij}|| := ||\hat{\mu}(X_j/\hat{T}^k X_i)||$$

The matrices $\bar{\Pi}$ and $\bar{\bar{\Pi}}$ have a common stationary vector, $\hat{P}$. Furthermore,

$$\begin{aligned}
\sum_{i,j} \hat{p}_i |\bar{\pi}_{ij} - \bar{\bar{\pi}}_{ij}| &= \sum_{i,j} \hat{p}_i \left| \sum_{k,l} \left(\tilde{\pi}'_{ik} \pi^*_{kl} \tilde{\pi}''_{lj} - (\pi_{K_1})_{ik} \pi^*_{kl} (\pi_{K_2})_{lj} \right) \right| \\
&\leq \sum_{i,j} \hat{p}_i \sum_{k,l} \tilde{\pi}'_{ik} \pi^*_{kl} |\tilde{\pi}''_{lj} - (\pi_{K_2})_{lj}| \\
&+ \sum_{i,j} \hat{p}_i \sum_{k,l} |\tilde{\pi}'_{ik} - (\pi_{K_1})_{ik}| \pi^*_{kl} (\pi_{K_2})_{lj} \\
&\leq K_1 \hat{\chi}_N + K_2 \hat{\chi}_N = K \hat{\chi}_N \qquad (7.2)
\end{aligned}$$

where we have applied the bound (3.6) to the Markov chain $(\hat{\Pi}, \hat{P})$.

We now apply Theorem 6.2 to the Markov chains $(\bar{\Pi}, \hat{P})$ and $(\bar{\bar{\Pi}}, \hat{P})$. We put

$$\begin{aligned}
\hat{\gamma}_{i,j} : &= \sum_k \frac{\bar{\bar{\pi}}_{ik} \bar{\bar{\pi}}_{jk}}{\hat{p}_k} \qquad (7.3) \\
&\geq \sum_k \sum_{l_1 l_2 l_3 l_4}{}^{(c)} \frac{\hat{\mu}(\hat{T}^{K_1} X_i \cap X_{l_1}) \hat{\mu}(X_{l_3} \cap \hat{T}^{-K_2} X_k) \hat{\mu}(\hat{T}^{K_1} X_j \cap X_{l_2}) \hat{\mu}(X_{l_4} \cap \hat{T}^{-K_2} X_k)}{(2\eta + 1)^2 \hat{\mu}(X_k) \hat{\mu}(X_i) \hat{\mu}(X_{l_3}) \hat{\mu}(X_j) \hat{\mu}(X_{l_4})}
\end{aligned}$$

where the summation in $\sum^{(c)}$ is taken over the quadruples (l_1, l_2, l_3, l_4) satisfying a "coupling" condition: the atoms X_{l_1} and X_{l_3} must be in the same column in $\hat{\mathcal{A}}$ separated by no more than $\eta - 1$ other atoms, and the same must hold for the atoms X_{l_2} and X_{l_4}. For any $\gamma > 0$ let

$$\hat{Q}(\gamma) := \sum_{(i,j):\, \hat{\gamma}_{i,j} < \gamma} \hat{\mu}(X_i) \hat{\mu}(X_j) \qquad (7.4)$$

Then, by virtue of Theorems 6.2 and 6.3 and due to (7.2) we can bound the mixing coefficient $\bar{V}^{(L)}$ of the the Markov chain $(\bar{\Pi}, \hat{P})$ by

$$\bar{V}^{(L)} \leq \text{const} \cdot \left[\gamma^{-1} \hat{p}_{\min}^{-2} (1 - \gamma/80)^{L/3} + L(\hat{p}_{\min} + \hat{Q}(\gamma) + 25\gamma^{-1} K \hat{\chi}_N) \right]$$

for any $\gamma > 0$, where, say, const$= 2500$. Here $\hat{p}_{\min} = \min \hat{p}_i > 0$. Recall that $LK \leq N$ and $L = [t/\delta]/[t_1/\delta] < t$. By Theorem 5.1 we can bound $\hat{\chi}_N$ and obtain the following corollary.

Corollary 7.1 *For any $\gamma > 0$ we have*

$$\bar{V}^{(L)} \le \theta_{10} \cdot \gamma^{-1} \left[\hat{p}_{\min}^{-2} (1 - \gamma/80)^{L/3} + t \left(\hat{p}_{\min} + \hat{Q}(\gamma) + \chi_{[\hat{\kappa} t]} + D^{\alpha_l/(\alpha_l+1)} \right) \right]$$

We are now in a position to formulate all the assumptions under which a stretched exponential bound on $C_{F,G}(t)$ holds. The first assumption involves only the base transformation T:

Assumption T. For any $H > 1$ there is an $m_H > 0$ such that for all $m \ge m_H$ there is a partition $\mathcal{A} = \mathcal{A}^{(m,H)}$ of the base space Ω such that
(i) $D = D(\mathcal{A}) \le c_1 e^{-a_1 m}$;
(ii) $c_2 e^{-a_2 m} \le \nu(A) \le c_3 e^{-a_3 m}$ for every $A \in \mathcal{A}$;
(iii) $\chi_{[m^H]} \le c_4 e^{-a_4 m}$.
Here a_i and c_i, $i = 1, \ldots, 4$, are positive constants (independent of m and H).

Note that the only nontrivial condition here is (iii), which requires a 'good' Markov approximation to the map T by the chain (3.1)-(3.2). Assumption T does not require any ergodic or mixing properties of the transformation T. In fact, it is satisfied for plenty of non-ergodic transformations, for example, rational circle rotations: for a rotation $z \to e^{2\pi i k/n} z$ it is enough to partition the circle $|z| = 1$ into mn equal arcs for any $m \ge 1$.

The second assumption involves the flow $\{\hat{\Phi}^t\}$.

Assumption F. For any $H > 1$ there is $m'_H \ge m_H$ such that for all $m \ge m'_H$ there are
(i) $\delta = c_5 e^{-a_5 m}$;
(ii) $\eta \le c_6 e^{a_6 m}$ with $a_6 < a_5$, so that $\eta\delta$ decays exponentially in m;
(iii) $K_1 = [\beta_1 m/\delta]$, $K_2 = [\beta_2 m/\delta]$, and $K = K_1 + K_2$;
such that the partition $\mathcal{A} = \mathcal{A}^{(m,H)}$ entering Assumption T can be chosen so that in the above notations we have
(iv) $\hat{Q}(\gamma_0) \le c_7 e^{-a_7 m}$.
where $\hat{Q}(\gamma)$ is given by (7.3)-(7.4). Here γ_0, β_1, β_2, and a_i, c_i, $i = 5, 6, 7$, are positive constants (independent of m and H).

The core of this assumption is, of course, the item (iv): this is a sufficient condition for bounding the mixing coefficient of the approximating Markov chain $(\bar{\bar{\Pi}}, \hat{P})$ in Corollary 7.1.

Theorem 7.2 *Under Assumptions T and F, there are constants $c > 0$ and $a > 0$ such that the correlation function for the suspension flow $\{\Phi^t\}$ obeys the bound*

$$|C_{F,G}(t)| \le \mathrm{var}_\alpha(F)\mathrm{var}_\alpha(G) \cdot c \cdot e^{-a\sqrt{t}} \tag{7.5}$$

for all $t > 0$. The constants $c > 0$ and $a > 0$ depend continuously on γ_0, β_1, β_2, m'_H, c_i, a_i, $1 \le i \le 7$, and on the parameters of the flow: $m_0, t_0, c_\mu, \alpha_l, \mathrm{var}_{\alpha_l}(l)$ and $\mathrm{diam}\,\mathcal{M}$.

Proof. Let $m = [z\sqrt{t}]$ with a sufficiently small $z > 0$ that will be chosen below. We put $H = 3$ and take the partition $\mathcal{A} = \mathcal{A}^{(m,3)}$ satisfying Assumptions T and F (one exists for every $m \ge m'_3$). The partition $\mathcal{A}$ and the value δ given by Assumption F then specify the 'discretized' space $(\hat{\mathcal{M}}, \hat{\mu})$, the map $\hat{T}$ on it and the partition $\hat{\mathcal{A}}$. The value $\hat{p}_{\min} = \min\{\hat{\mu}(X);\, X \in \hat{\mathcal{A}}\}$ satisfies the bounds

$$\hat{c}_\mu c_2 c_5 e^{-(a_2+a_5)m} \le \hat{p}_{\min} \le \hat{c}_\mu c_3 c_5 e^{-(a_3+a_5)m}$$

Next, for any $z > 0$ and all $t > (\hat{\kappa}/z^3 + 3/z)^2$ (the constant $\hat{\kappa}$ enters Corollary 5.4) we have $m^3 \ge \hat{\kappa}t$, and so

$$\chi_{[\hat{\kappa}t]} \le \chi_{m^3} \le c_4 e^{-a_4 m}$$

We then substitute the two above bounds into the inequalities in Corollaries 5.4 and 7.1 (with $\gamma = \gamma_0$) and we make use of Assumptions T and F immediately obtaining exponential (in m) bounds on all the terms but one. That one, 'naughty' term is $\hat{p}_{\min}^{-2}(1-\gamma_0/80)^{L/3}$. Recall that

$$L = [N/K] \ge \frac{[t/\delta]}{2[\beta_1 m/\delta] + 2[\beta_2 m/\delta]} \ge \frac{t-\delta}{2(\beta_1+\beta_2)m} \ge \frac{m}{4z^2(\beta_1+\beta_2)}$$

Hence

$$\hat{p}_{\min}^{-2}(1-\hat{\gamma}/80)^{L/3} \le 4(c_\mu c_2 c_5)^{-2} \exp\left(\left[2(a_2+a_5) + \frac{\ln(1-\gamma_0/80)}{12z^2(\beta_1+\beta_2)}\right] m\right).$$

If z is small enough, the exponent will be negative, and this term will decay exponentially in m, too. The above estimates are valid for all

$$t \ge t_* = \max\{(m'_3/z)^2, (\hat{\kappa}/z^3 + 3/z)^2\}$$

and so we obtain (7.5) for all these values of t. On the finite interval $t \in [0, t_*]$ nothing can go wrong, since the function $C_{F,G}(t)$ is always bounded by $||F||_\infty \cdot ||G||_\infty$. Theorem 7.2 is proved.

Remark. Anosov flows discussed below satisfy some stronger assumptions than our T and F. In particular, we will prove that $\chi_n \leq c_4 e^{-a_4 m}$ for all $n \geq 1$ in T and $\hat{Q}(\gamma_0) = 0$ in F.

8 Anosov flows

In Sections 8-18 we work with transitive Anosov flows on 3-D manifolds. We will first invoke the techniques of Markov partitions and symbolic dynamics and prove Assumption T. Then, under an extra condition stated in Section 13, we prove Assumption F, which implies a stretched exponential bound on correlations.

Here we use the terminology and results of classical works by Anosov, Sinai, Bowen and Ruelle [1, 34, 5, 6, 7].

Let M be a compact connected C^∞ Riemannian manifold, $\dim M = 3$. Let $\phi^t : M \to M$ be a flow of class at least C^2. This means that its trajectories are defined by ordinary differential equations $d\phi^t/dt = v(y)$ with a C^2 vector field $v(y)$ on M. We assume that ϕ^t is an Anosov flow, i.e.

(A1) M does not have fixed points, i.e. $v(y) \neq 0$ for every $y \in M$;

(A2) for every $y \in M$ there is a $D\phi^t$-invariant splitting

$$\mathcal{T}_y M = \mathcal{E}_y^\phi \oplus \mathcal{E}_y^u \oplus \mathcal{E}_y^s \tag{8.1}$$

into three one-dimensional subspaces, such that $\mathcal{E}_y^\phi$ is tangent to the flow, and there are constants $C_\phi > 0$ and $\lambda_\phi \in (0, 1)$ such that

$$||D\phi^t(v)|| \leq C_\phi \lambda_\phi^t ||v|| \quad \text{for } \ v \in \mathcal{E}_y^s,\ t \geq 0$$

$$||D\phi^{-t}(v)|| \leq C_\phi \lambda_\phi^t ||v|| \quad \text{for } \ v \in \mathcal{E}_y^u,\ t \geq 0.$$

The Anosov splitting (8.1) depends on y continuously [1]. It is even Hölder continuous, as it was proved in [2], see the proof in [20].

For any $y \in M$ there are local (strongly) stable and unstable fibers $\mathcal{W}_y^s$ and $\mathcal{W}_y^u$, respectively [1, 5]. These fibers are as smooth as the flow itself [3], i.e., they are at least C^2. However, the foliations of M by local stable and unstable fibers may not be even C^1, they are typically only Hölder continuous [24] (even in 3-D, despite Anosov's original claim in [1]).

For small $\varepsilon > 0$, we call the C^2 surfaces

$$\mathcal{W}^{wu}_y := \phi^{[-\varepsilon,\varepsilon]}(\mathcal{W}^u_y) \quad \text{and} \quad \mathcal{W}^{ws}_y := \phi^{[-\varepsilon,\varepsilon]}(\mathcal{W}^s_y)$$

local weakly unstable and stable manifolds (or leaves), respectively. Here and further on we adopt Bowen's notation

$$\phi^{[a,b]}A := \cup_{a\le t\le b}\phi^t A$$

for any $A \subset M$. For every two close points, $y', y'' \in M$ the sets

$$[y', y'']_u := \mathcal{W}^{ws}_{y'} \cap \mathcal{W}^u_{y''} \quad \text{and} \quad [y', y'']_s := \mathcal{W}^s_{y'} \cap \mathcal{W}^{wu}_{y''}$$

are not empty and each consists of one point.

Remark. The above points $[y', y'']_u$ and $[y', y'']_s$ obviously lie on one orbit of the flow, so that $[y', y'']_s = \phi^\tau[y', y'']_u$ for some $|\tau| \le \varepsilon$. We call this $\tau = \tau(y', y'')$ the *temporal distance* between the fibers $\mathcal{W}^u_{y''}$ and $\mathcal{W}^s_{y'}$. This distance will play a key role in Section 13.

Our next assumption is

(A3) Anosov flow ϕ^t is topologically transitive, i.e. it has a dense orbit[2];

Equivalently, we can assume that periodic orbits of the Anosov flow ϕ^t are dense in M, or that it has no wandering points in M. Under any of these assumptions the Anosov flow ϕ^t satisfies Smale's Axiom A for flows [35, 5] and its only basic set is the whole M.

A transitive Anosov flow is said to be *topologically mixing* if for some (and then for all) points $x \in M$ their (full) stable and unstable fibers are dense in M.

Fact (Alternative for Anosov flows [1, 7, 24]). For every transitive Anosov flow ϕ^t there are exactly two possibilities:
(i) the flow ϕ^t is topologically mixing, then all its Gibbs invariant measures (see below) are mixing and Bernoulli [28];

[2]Examples of nontransitive Anosov flows were constructed in [15].

(ii) the flow ϕ^t has continuous nonconstant eigenfunctions (= a discrete component in its spectrum), it is then a suspension flow under a constant ceiling function, and it has no mixing invariant measures.

Naturally, we assume that

(A4) the flow ϕ^t is topologically mixing.

In the case of Axiom A flows, even this assumption does not guarantee any bound on correlations, as it was demonstrated by Pollicott [25]. It is unknown if (A4) guarantees any bound on correlations for Anosov flows. We will prove Theorem 1.1 under an extra assumption, (A5), stated in Section 13. On the other hand, many strong statistical properties, like the central limit theorem and its invariance principle, have been established for both types of Anosov (and Axiom A) flows [14].

We equip the flow ϕ^t with the so called Sinai-Bowen-Ruelle (SBR) measure μ. It is defined [34] to be a ϕ^t-invariant probability measure whose conditional measures on local unstable fibers are absolutely continuous with respect to the Riemannian volume on those fibers (the volume, or the length, is induced on the fibers by the Riemannian metric in M). Every topologically mixing Anosov flow has a unique SBR measure μ. Likewise, there is a unique invariant measure μ_- absolutely continuous on stable fibers. These two measures are generally different. They coincide iff the flow preserves an absolutely continuous measure on M, and in this case $\mu = \mu_-$ is that measure. Note, however, that typical Anosov flows do not enjoy this property, they form an open dense subset in the space of C^2 flows [34]. Our main results hold for both measures μ and μ_-, but in view of the time symmetry it is enough to study one.

The SBR measure μ is a Gibbs measure, see [34] and Section 10 below. Its other remarkable property is that for any absolutely continuous probability measure ρ_0 on M its image ρ_t defined by $\rho_t(A) = \rho_0(\phi^{-t}A)$ weakly converges, as $t \to \infty$, to μ, see, e.g., [7]. In other words, any initial smooth probability distribution on M converges under the action of ϕ^t to the SBR measure. In nonequilibrium statistical physics, SBR measures describe stationary states for nonhamiltonian systems [10, 16].

Denote by $\Lambda_t^u(y)$ the Jacobian of the linear map $D\phi^t : \mathcal{E}_y^u \to \mathcal{E}_{\phi^t y}^u$, where the Euclidean structure in $\mathcal{E}_y^u$ is induced by the Riemannian metric in M. Let ξ be any measurable partition of M whose atoms are local unstable fibers. For any $y \in M$ denote by $\xi(y)$ the atom of ξ containing y and by

$f_\xi(y)$ the density of the SBR measure μ conditioned on $\xi(y)$ with respect to the Riemannian length on $\xi(y)$. Then for every $y_1, y_2 \in \xi(y)$ we have a remarkable formula [32, 20]

$$\frac{f_\xi(y_1)}{f_\xi(y_2)} = \lim_{t\to-\infty} \frac{\Lambda_t^u(y_1)}{\Lambda_t^u(y_2)} \tag{8.2}$$

The function f_ξ is at least Lipschitz continuous [32] on every fiber $\xi(y)$, and for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $y_1, y_2 \in \xi(y)$ we have [7]

$$\text{dist}\,(y_1, y_2) < \delta \quad \Longrightarrow \quad e^{-\varepsilon} < f_\xi(y_1)/f_\xi(y_2) < e^{\varepsilon} \tag{8.3}$$

The foliations by local stable and unstable fibers and leaves are all absolutely continuous [1, 3, 8] in the following sense. For any two close local unstable leaves $\mathcal{W}_1^{wu}$, $\mathcal{W}_2^{wu}$ the map $H : \mathcal{W}_1^{wu} \to \mathcal{W}_2^{wu}$ defined by $H(y) = \mathcal{W}_y^s \cap \mathcal{W}_2^{wu}$ is called canonical isomorphism or holonomy map. Its Jacobian with respect to the Riemannian area on unstable leaves is bounded away from 0 and ∞. Moreover, it is close to one if the leaves are close enough to each other. The same property holds for stable leaves. Likewise, for two close local unstable fibers $\mathcal{W}_1^u$, $\mathcal{W}_2^u$ the map $H : \mathcal{W}_1^u \to \mathcal{W}_2^u$ defined by $H(y) = \mathcal{W}_y^{ws} \cap \mathcal{W}_2^u$ can also be termed a canonical isomorphism or holonomy map. Its Jacobian with respect to the Riemannian length has similar properties, see Sect. 13 for a detailed argument.

9 Markov families and special representations

A closed subset $R \subset M$ is called a *rectangle* if there is a small closed smooth disk $D \subset M$ transversal to the flow ϕ^t, such that $R \subset D$, and for any $y', y'' \in R$ the point

$$[y', y'']_R := D \cap \phi^{[-\varepsilon,\varepsilon]}\{[y', y'']_u\}$$

exists and also belongs to R. A rectangle R is said to be *proper* if $R = \overline{R^*}$, where R^* is the set of the interior points of R considered as a subset of D. For any rectangle R and $y \in R$ we put

$$W_y^u(R) := R \cap \mathcal{W}_y^{wu} \quad \text{and} \quad W_y^s(R) := R \cap \mathcal{W}_y^{ws}$$

Then R is a direct product of the sets $W^u_y(R)$ and $W^s_y(R)$ in the set-theoretic and topological senses.

Definition (cf. [5]). A finite collection of closed sets $\mathcal{R} = \{R_1, \dots, R_I\}$ is said to be a *proper family of size* $\alpha > 0$ if
(i) $M = \phi^{[-\alpha,0]}(\Omega)$, where $\Omega = R_1 \cup \cdots \cup R_I$;
and there are disks $D_1, \dots, D_I$ containing these sets such that for every i
(ii) $\operatorname{diam} D_i < \alpha$;
(iii) $R_i = \overline{R_i^*}$, where R_i^* is the interior of R_i considered as a subset of D_i;
(iv) for any $i \neq j$ at least one of the two sets $D_i \cap \phi^{[0,\alpha]} D_j$ and $D_j \cap \phi^{[0,\alpha]} D_i$ is empty; in particular, $D_i \cap D_j = \emptyset$.

It follows from (i) that for any $x \in \Omega$ there is a smallest positive $l(x)$ such that $\phi^{l(x)}(x) \in \Omega$. According to (i) and (iv), the function $l(x)$ is bounded from above and below: $0 < l_{\min} \leq l(x) \leq l_{\max} < \infty$. The set Ω is called a cross-section of the manifold M. It generates the first return map (Poincaré map), $T(x) = \phi^{l(x)}(x)$, which is a one-to-one map $T : \Omega \to \Omega$. It is easy to check that the functions $l(x)$ and $T(x)$ are locally as smooth as the flow ϕ^t, i.e. at least of class C^2. They are not continuous globally on Ω, but $l(x)$ and all iterations of T are continuous on the T-invariant set

$$\Omega^* = \Big\{ x \in \Omega : \, T^k x \in \cup_{i=1}^I R_i^* \quad \text{for all} \quad k \in \mathbb{Z} \Big\}$$

Definition. A proper family of sets $\mathcal{R} = \{R_1, \dots, R_I\}$ of a small size α is said to be a *Markov family*, if
(i) every $R_i \in \mathcal{R}$ is a (proper) rectangle;
(ii) if $x \in R_i^* \cap T^{-1}(R_j^*)$, then $W^s_x(R_i) \subset \overline{T^{-1}(W^s_{Tx}(R_j))}$ and $\overline{T(W^u_x(R_i))} \supset W^u_{Tx}(R_j)$.

Theorem 9.1 *Any transitive 3-D Anosov flow* $\phi^t : M \to M$ *has Markov families of arbitrary small sizes whose rectangles are connected.*

The existence of Markov families was proved by Bowen [5]. Marcus [22] showed how to establish the connectivity of rectangles of Markov partitions with one-dimensional stable and unstable fibers for Axiom A diffeomorphisms. He proved that every rectangle in Bowen's construction of Markov partitions [5, 6] consists of a finite number of connected subrectangles, and those form a (naturally, finer) Markov family. Alternatively, one can use Sinai's construction of Markov partitions for Anosov diffeomorphisms [33] to construct Markov families for 3-D Anosov flows, and then all the rectangles will be automatically connected.

We will only consider Markov families with connected rectangles. For every connected rectangle R and $x \in R$ the sets $W_x^{u,s}(R)$ are segments of C^2-smooth curves. The boundary ∂R consists of two parts: $\partial^u R$ and $\partial^s R$, where

$$\partial^u R = \cup_{x\in R}\partial W_x^s(R) \quad \text{and} \quad \partial^s R = \cup_{x\in R}\partial W_x^u(R)$$

Each part is the union of exactly two segments of smooth curves, $\partial^u R = W_{y'}^u(R) \cup W_{y''}^u(R)$ and $\partial^u R = W_{y'}^u(R) \cup W_{y''}^u(R)$ for some $y', y'' \in R$. Every connected rectangle R is then a curvilinear quadrilateral in the corresponding disk D.

The set Ω is now a manifold with I connected components, each of which is a smooth surface with boundary. The function $l(x)$ and the Poincaré map T are piecewise smooth (of class at least C^2) on Ω with a finite number of discontinuity lines, which are formed by $\partial\Omega \cup T^{-1}(\partial\Omega)$. Here $\partial\Omega = \cup_i \partial R_i$. We also put $\partial^{u,s}\Omega = \cup_i \partial R_i^{u,s}$.

For every $R_i \in \mathcal{R}$ and $x \in R_i$ let $E_x^{s,u}$ be the tangent lines to $W_x^{s,u}(R_i)$, respectively. The splitting

$$\mathcal{T}_x\Omega = E_x^s \oplus E_x^u \tag{9.1}$$

is DT-invariant and there are constants $C_T > 0$ and $\lambda_T \in (0,1)$ such that

$$\begin{aligned} ||DT^n(v)|| &\le C_T\lambda_T^n||v|| \quad \text{for } v \in E_x^s,\ n \ge 0 \\ ||DT^{-n}(v)|| &\le C_T\lambda_T^n||v|| \quad \text{for } v \in E_x^u,\ n \ge 0. \end{aligned} \tag{9.2}$$

(at singular points $x \in \partial\Omega \cup T^{-1}(\partial\Omega)$ a one-sided derivative DT can be defined and used in (9.2)). The Anosov splitting (9.1) depends on x continuously, it is even Hölder continuous, since such is the splitting (8.1). At every point $x \in R_i \in \mathcal{R}$ the curves $W_x^{s,u}(R_i)$ are stable and unstable fibers for T, respectively. We denote by $\Lambda^u(x)$ the Jacobian of the linear map $DT : E_x^u \to E_{Tx}^u$. This is also a Hölder continuous function on Ω.

Remark. The map $T : \Omega \to \Omega$ is not exactly an Anosov diffeomorphism, since Ω is disconnected and T is discontinuous on $\Omega \cap T^{-1}\partial\Omega$. Nontheless, all the main results by Sinai [32, 34] and Bowen [6] apply to T, because the discontinuity lines of both T and T^{-1} coincide with some stable and unstable fibers.

The invariant measure μ of the flow ϕ^t induces an invariant measure ν for the Poincaré map $T : \Omega \to \Omega$. For any Borel $A \subset \Omega$ and a small $t > 0$ we define ν by $\mu(\phi^{[0,t]}A) = t\nu(A)$. The measure ν is a SBR measure for the map T, i.e. its conditional measures on unstable fibers $W^u_x(R_i)$, $x \in R_i$, are absolutely continuous with respect to the Riemannian length. The ergodicity of ν is obviously equivalent to that of μ. The measure ν need not be mixing, however, despite the mixing property of μ. If ν is not mixing, we have $\Omega = \Omega^{(1)} \cup \cdots \cup \Omega^{(r)}$, with $\Omega^{(i)} \cap \Omega^{(j)} = \emptyset$ for $i \neq j$, so that $T(\Omega^{(i)}) = \Omega^{(i+1)}$ (with $\Omega^{(r+1)} = \Omega^{(1)}$, of course), every $\Omega^{(i)}$ is a union of some rectangles of $\mathcal{R}$ and the map T^r is mixing on $\Omega^{(1)}$, see [7]. Then we replace Ω by $\Omega^{(1)}$, the function $l(x)$ by $l(x) + \cdots + l(T^{r-1}x)$ and T by T^r. Thus, we can (and will) always assume that the measure ν is mixing.

We now represent every point $y \in M$ by a unique pair

$$y = (x, s) : \quad x \in \Omega,\ 0 \leq s < l(x), \quad y = \phi^s x$$

Then the flow $\phi^t : M \to M$ becomes isomorphic to a suspension flow built under $l(x)$ over the map $T : \Omega \to \Omega$. This is called a special representation of the flow ϕ^t. We will use the notations of Section 2 and denote this flow by $\Phi^t : \mathcal{M} \to \mathcal{M}$. The metric ρ on Ω is induced by the Riemannian metric on M, and its extension to $\mathcal{M}$ was defined in Sect. 2. The 'coordinate' projections $\pi_1 : \mathcal{M} \to \Omega$ and $S : \mathcal{M} \to [0, l_{\max})$ are defined by $\pi_1(y) = x$ and $S(y) = s$.

The isomorphism between (M, ϕ^t) and $(\mathcal{M}, \Phi^t)$ is understood in the measure theoretic sense: it is a one-to-one map $\psi : M \to \mathcal{M}$ that preserves the dynamics, $\psi \circ \phi^t = \Phi^t \circ \psi$, and the invariant measure (we even use here the same notations $\Omega, T, l(x), \mu, \nu$ as in Section 2, but there will be no confusion). We often denote ψ-isomorphic subsets of M and $\mathcal{M}$ by the same symbols, slightly abusing notations.

The isomorphism $\psi : M \to \mathcal{M}$ does not, however, preserve the metric or even the topology of M, because ψ is only a piecewise smooth map. It is locally a diffeomorphism with bounded derivatives, but it has a finite number of discontinuity surfaces in M. The union of those is $\Omega \cup \Omega_1$, where

$$\Omega_1 = \pi_1^{-1}(\partial\Omega) = \{(x, s) : x \in \partial\Omega,\ 0 \leq s < l(x)\} \tag{9.3}$$

The set $\Omega \cup \Omega_1$ is a finite union of smooth compact surfaces in M. It is the preimage of the boundary $\partial\mathcal{M}$ under the natural continuous extension of ψ^{-1} to the closure $\bar{\mathcal{M}}$. We call the smooth components of Ω_1 'side walls' of

$\mathcal{M}$, thinking of Ω as its 'floor'. We also put $\Omega_1^{u,s} = \pi_1^{-1}(\partial^{u,s}\Omega)$. Note that $\phi^t(\Omega_1^s) \subset \Omega_1^s$ and $\phi^{-t}(\Omega_1^u) \subset \Omega_1^u$ for all $t > 0$.

The discontinuities of ψ will not cause any problems for our main results. Indeed, if F is a generalized Hölder continuous function on M, then so is the function $\mathcal{F} = F \circ \psi^{-1}$ on $\mathcal{M}$. Moreover, the generalized Hölder exponent α for F, recall (2.2), will be also such for $\mathcal{F}$. The ratio of the α-variations of these two functions, defined by (2.3), is uniformly bounded:

$$0 < C^{-1} \le \mathrm{var}_\alpha(F)/\mathrm{var}_\alpha(\mathcal{F}) \le C < \infty$$

where C is determined by α and $\mathcal{R}$. Therefore, we obtain

Proposition 9.2 *If the stretched exponential bound (7.5) holds for the flow $\Phi^t : \mathcal{M} \to \mathcal{M}$, then it also holds for $\phi^t : M \to M$.*

The ceiling function $l(x)$ on $\mathrm{int}\,\Omega = \cup_i R_i^*$ is piecewise smooth with a finite number of discontinuity lines, $T^{-1}(\partial\Omega) \setminus \partial\Omega$, coinciding with some stable fibers of T (we can disregard the discontinuities of $l(x)$ on $\partial\Omega$ since $\nu(\partial\Omega) = 0$). The ε-neighborhood of these lines has ν-measure $<$ const $\cdot\, \varepsilon$ because ν is a SBR measure. Thus, $l(x)$ belongs in $GH_{\alpha_l}(\Omega)$ with $\alpha_l = 1$, and we get Assumption L1 in Section 2. Assumption L2 is also ensured by the lower bound on $l(x)$: $l(x) \ge l_{\min} > 0$. It will be enough, hence, to prove Assumptions T and F, and then Theorem 1.1 will follow from Theorem 7.2.

10 Symbolic dynamics and Gibbs measures

Here we invoke the symbolic dynamics generated by the Markov family $\mathcal{R}$ and study the necessary properties of Gibbs measures.

We recall the basic definitions of symbolic dynamics. A partition matrix $A = A(\mathcal{R})$ is defined by

$$A_{ij} = \begin{cases} 1 & \text{if } R_i^* \cap T^{-1}R_j^* \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{10.1}$$

Let $\mathcal{J} = \{1, \ldots, I\}$ and $\Sigma = \mathcal{J}^{\mathbb{Z}}$ denote the set of all doubly infinite sequences $\underline{\omega} = \{\omega_i\}_{-\infty}^{\infty}$. Fix a $d < 1$ and a metric ρ_d on Σ such that

$\rho_d(\underline{\omega}', \underline{\omega}'') = d^n$, where $n = \max\{n \geq 0 : \omega_i' = \omega_i'' \quad \forall |i| < n\}$. Consider the set

$$\Sigma_A = \{\underline{\omega} \in \Sigma : A_{\omega_i \omega_{i+1}} = 1 \text{ for all } -\infty < i < \infty\}$$

and a left shift homeomorphism $\sigma : \Sigma_A \to \Sigma_A$ defined by $(\sigma(\underline{\omega}))_i = \omega_{i+1}$. The system (Σ_A, σ) is called a subshift of finite type, or a topological Markov chain.

Fact [5]. There is a continuous onto map $\pi : \Sigma_A \to \Omega$ such that[3]
(i) $\pi(\underline{\omega}) \in R_{\omega_0}$;
(ii) the projection π is one-to-one on the σ-invariant subset $\Sigma_A^* := \pi^{-1}(\Omega^*)$, and on this subset we have $T \circ \pi = \pi \circ \sigma$

Remark. The identity $T \circ \pi = \pi \circ \sigma$ may fail on $\Sigma_A \setminus \Sigma_A^*$, but it is possible [5] to slightly redefine the function $l(x)$ (and hence the map $T(x) = \phi^{l(x)}(x)$) on $\partial\Omega$ so that this identity will hold on the entire Σ_A. This will not be important for us, however.

The subshift (Σ_A, σ) is topologically transitive and topologically mixing. Equivalently, the matrix A is irreducible and aperiodic, i.e. A^{K_0} contains no zeroes for some $K_0 > 0$.

For any Hölder continuous function h on Σ_A (with respect to the metric ρ_d on Σ_A) there is a (unique) Gibbs σ-invariant measure ν_Σ^h on Σ_A, see [6, 30] for definitions and basic properties of Gibbs measures. We only recall that there are $0 < C_1 < C_2 < \infty$ and $-\infty < P < \infty$ such that for any $m \geq 1$ and all $\underline{\omega}' \in \Sigma_A$ we have

$$C_1 \leq \frac{\nu_\Sigma^h\{\underline{\omega} : \omega_i = \omega_i' \quad \forall i = 0, \ldots, m-1\}}{\exp[h(\underline{\omega}') + h(\sigma\underline{\omega}') + \cdots + h(\sigma^{m-1}\underline{\omega}') - Pm]} \leq C_2 \tag{10.2}$$

Here $P = P(h)$ is the so called the topological pressure of h. For any Hölder continuous function f on Ω the function $h = f \circ \pi$ is [6] Hölder continuous on Σ_A. Then the Gibbs measure ν_Σ^h projected down to Ω is called the Gibbs measure ν_Ω^f on Ω generated by the function f (or the equilibrium state for the potential function f). It is T-invariant, and since the shift (Σ_A, σ) is topologically mixing, the measure ν_Ω^f is ergodic and mixing. It is also Bernoulli and enjoys strong statistical properties [6]. For any f we have $\nu_\Omega^f(\Omega \setminus \Omega^*) = 0$.

[3]The map π is Lipschitz continuous [5] with respect to the metric ρ_d for some $d > 0$.

Fact. The SBR measure ν for the map T is a Gibbs measure for a Hölder continuous potential function g on Ω. Moreover, the function g can be chosen so that
(i) for some $m_g \geq 1$ and $b_g > 0$ and for all $x \in \Omega$ we have

$$g(x) + g(Tx) + \cdots + g(T^{m_g-1}x) \leq -b_g < 0 \tag{10.3}$$

(ii) the topological pressure $P = P(g \circ \pi)$ is zero.

Remark. This fact follows from [34, 6] with the function $g_1(x) = -\ln \Lambda^u(x)$. The bound (10.3) follows from (9.2). Alternatively, one can invoke the results by Bowen and Ruelle for Axiom A flows [7, Propositions 3.1 and 4.4]. The above fact then readily follows for the function $g_2(x) = -\ln \Lambda^u_{l(x)}(x)$, where $\Lambda^u_t(y)$ was defined in Sect. 8. The bound (10.3) then follows from (A2). Note that the functions $g_1(x)$ and $g_2(x)$ are homologous, i.e., there exists another Hölder continuous function h on Ω, such that $g_2(x) = g_1(x) + h(Tx) - h(x)$. Due to this, $g_1(x)$ and $g_2(x)$ have the same equilibrium state $(= \nu)$ and the same pressure $(= 0)$.

Next, for any two integers $-\infty < p \leq 0 \leq q < \infty$ and a symbolic string $(\omega'_p, \omega'_{p+1}, \ldots, \omega'_q) \in \mathcal{J}^{|p|+q}$ the set

$$C_{p,q} = C_{p,q}(\omega'_p, \ldots, \omega'_q) = \{\underline{\omega} \in \Sigma_A : \omega_i = \omega'_i \text{ for all } p \leq i \leq q\}$$

is called a cylinder (of length $q - p + 1$, or a cylinder 'from p to q'). The projection $\pi(C_{p,q})$ of any cylinder down to Ω is a proper connected rectangle $R \subset \Omega$ (more precisely, $R \subset R_{\omega'_0} \in \mathcal{R}$).

Lemma 10.1 *There are positive constants b_1, b_2, b_3 and $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ determined by the Markov family $\mathcal{R}$ such that for every cylinder $C_{p,q}$ the rectangle $R = \pi(C_{p,q})$ has the diameter less than $b_1 \lambda_1^{\min\{|p|,q\}}$, and the measure*

$$b_2 \lambda_2^{|p|+q} \leq \nu(R) \leq b_3 \lambda_3^{|p|+q}$$

Proof. The bound on the diameter follows from (9.2). The bounds on the measure readily follow from (10.2), since the potential g of the measure ν satisfies (10.3) and its topological pressure P is zero. Hence the lemma.

Remark. This lemma can be easily proved for an arbitrary Gibbs measure ν^h_Ω, with constants b_i, λ_i depending continuously on the potential function h, but we will not need this.

11 Markov partitions for the map T

Let Υ be a finite partition of Σ_A into disjoint cylinders (generally, of different lengths). Its projection $\pi(\Upsilon) = \{\pi(C) : C \in \Upsilon\}$ down to Ω will be a finite covering of Ω by proper connected rectangles, which can intersect only in boundary points.

Definition [6]. A finite covering of Ω by closed proper rectangles is called a Markov partition for the map T if they intersect only in boundary points, and for any two rectangles R', R'' and any point $x \in (R')^* \cap T^{-1}(R'')^*$ we have $W_x^s(R') \subset \overline{T^{-1}(W_{Tx}^s(R''))}$ and $\overline{T(W_x^u(R'))} \supset W_{Tx}^u(R'')$.

The Markovian property of the family $\mathcal{R}$ implies that the partition of Ω into the rectangles $R_1, \ldots, R_I$ is a Markov partition for the map T. Let us note that there is a different definition and construction of Markov partitions for Anosov flows, due to Ratner [27], but we will not use those in this paper.

Definition. We say that a finite partition Υ of Σ_A into cylinders satisfies the Markov condition (MC) if for any two cylinders $C' = C_{p',q'}$ and $C'' = C_{p'',q''}$ in Υ such that $\sigma(C') \cap C'' \neq \emptyset$ we have $p' - 1 \leq p''$ and $q' - 1 \leq q''$.

Lemma 11.1 *Let Υ be a finite partition of Σ_A into cylinders. The covering $\pi(\Upsilon)$ of Ω is a Markov partition iff the partition Υ satisfies the Markov condition (MC).*

The proof is a direct inspection, and we omit it. We will only work with partitions Υ of Σ_A satisfying the Markov condition.

The 'longer' the cylinders $C \in \Upsilon$ are, the smaller are the rectangles of the Markov partition $\pi(\Upsilon)$. It is customary to refine Markov partitions by taking $\mathcal{R}_{-n,n} = T^{-n}\mathcal{R} \vee \cdots \vee T^{-n}\mathcal{R}$, which corresponds to the (canonical) partition Υ_n of Σ_A into cylinders of constant length, $C_{-n,n}$, i.e. all cylinders 'from $-n$ to n'. We will also need partitions into cylinders of variable length. For any finite partition Υ of Σ_A into cylinders we put

$$r_{\min}(\Upsilon) = \min_{C_{p,q} \in \Upsilon} \{|p|, q\} \quad \text{and} \quad r_{\max}(\Upsilon) = \max_{C_{p,q} \in \Upsilon} \{|p|, q\}$$

Lemma 11.2 *Let $C_{p,q} \subset \Sigma_A$ be a cylinder, $r = \min\{|p|, q\}$, and x an arbitrary point in the rectangle $R' = \pi(C_{p,q}) \subset \Omega$. Then there is a product* *measure* $d\nu^p_{R'} = d\nu^{p,u}_{R'} \times d\nu^{p,s}_{R'}$ *on* R'*, where the measures* $d\nu^{p,u}_{R'}$ *and* $d\nu^{p,s}_{R'}$ *are defined on some* $W^u_x(R')$ *and* $W^s_x(R')$*, respectively, such that* $\nu^p_{R'}$ *is equivalent to* ν *on* R' *and*

$$\exp\left(-b_4\lambda_4^r\right) \le \frac{d\nu^p_{R'}}{d\nu} \le \exp\left(b_4\lambda_4^r\right) \tag{11.1}$$

Here $b_4 > 0$ *and* $\lambda_4 \in (0,1)$ *are constants determined by the Markov family* $\mathcal{R}$*.*

Proof. Our proof of this lemma works for an arbitrary Gibbs measure ν^h_Σ, where h is a Hölder continuous function on Σ_A. For any $n \ge 1$ let

$$\mathrm{var}_n(h) = \sup\{|h(\underline{\omega}) - h(\underline{\omega}')| : \omega_i = \omega_i' \ \ \forall |i| \le n\}$$

We pick an arbitrary $\underline{\omega}^{(0)} \in C_{p,q}$. Let

$$C^u_{p,q} = \{\underline{\omega} \in \Sigma_A : \omega_i = \omega_i^{(0)} \ \ \forall i \le q\}$$

and

$$C^s_{p,q} = \{\underline{\omega} \in \Sigma_A : \omega_i = \omega_i^{(0)} \ \ \forall i \ge p\}$$

Note that $\pi(C^{u,s}_{p,q}) = W^{u,s}_x(R')$, where $x = \pi(\underline{\omega}^{(0)})$. We then define a product measure on the cylinder $C_{p,q}$ obtained by multiplying two conditional measures generated by ν^h_Σ on $C^u_{p,q}$ and $C^s_{p,q}$, respectively, and then multiplying the product by a constant factor, $\nu^h_\Sigma(C_{p,q})$.

Gibbs measures have locally a product structure, see, e.g., [17]. Haydn [17] obtained an exact formula for the Radon-Nikodym derivative involved in (11.1), from which it follows that

$$\exp\left[-2\sum_{n=r}^{\infty}\mathrm{var}_n(h)\right] \le \frac{d\nu^p_{R'}}{d\nu} \le \exp\left[2\sum_{n=r}^{\infty}\mathrm{var}_n(h)\right] \tag{11.2}$$

We also mention another way to prove (11.2), suggested by K. Khanin. A standard definition of Gibbs measures is based on its conditional distributions on finite cylinders with fixed boundary conditions, see, e.g., Ruelle's book [30]. We can specify the boundary conditions by the above element $\underline{\omega}^{(0)} \in \Sigma_A$. Then (11.2) can be obtained by a direct (but lengthy) calculation.

Now, since the function h is Hölder continuous on Σ_A, we have $\mathrm{var}_n(h) \le c_h\beta_h^n$ for some $c_h > 0$ and $\beta_h \in (0,1)$. Lemma 11.2 then follows from (11.2).

Note that the values of b_4 and λ_4 depend continuously on c_h and β_h.

The next lemma is specific for SBR measures. Recall [6] that $R'' \subset R'$ is called a u-subrectangle in a rectangle R' if $W^u_x(R') \subset R''$ for all $x \in R''$. Similarly, $R'' \subset R'$ is an s-subrectangle in R' if $W^s_x(R') \subset R''$ for all $x \in R''$.

Lemma 11.3 *There is a constant $\lambda_5 = \lambda_5(\mathcal{R}) \in (0,1)$ such that for any s-subrectangle $R'' \subset R'$ of any rectangle $R' \subset \Omega$ and any point $x \in R''$ we have*

$$\lambda_5 \frac{|W^u_x(R'')|}{|W^u_x(R')|} \le \frac{\nu(R'')}{\nu(R')} \le \lambda_5^{-1} \frac{|W^u_x(R'')|}{|W^u_x(R')|}$$

where $|\cdot|$ stands for the Riemannian length of a curve.

Proof. First, we show that for any rectangle $R' \subset \Omega$ and $x', x'' \in R'$ we have

$$\lambda_6 \le |W^u_{x'}(R')|/|W^u_{x''}(R')| \le \lambda_6^{-1} \tag{11.3}$$

for some constant $\lambda_6 = \lambda_6(\mathcal{R}) \in (0,1)$. Indeed, let $H : W^u_{x'}(R') \to W^u_{x''}(R')$ be the holonomy map defined by $H(z) = W^s_z(R') \cap W^u_{x''}(R')$. Let $DH(z)$ for $z \in W^u_{x'}(R')$ be the Jacobian of H with respect to the Riemannian length on fibers. Then Anosov-Sinai's formula [3, Equation (5.3)] says that

$$DH(z) = \lim_{n\to\infty} \prod_{i=0}^{n} \frac{\Lambda^u(T^i z)}{\Lambda^u(T^i H(z))} \tag{11.4}$$

Since $\Lambda^u(\cdot)$ is a Hölder continuous function on Ω and the points z and $H(z)$ belong to the same stable fiber, we get (11.3). Now, Lemma 11.3 follows from (8.3).

Proposition 11.4 *Let Υ be a finite partition of Σ_A into cylinders that satisfy the Markov condition (MC). Then the partition $\mathcal{A} = \pi(\Upsilon)$ of Ω enjoys the following properties (in notations of Section 3):*

(i) $D = D(\mathcal{A}) \le b_1 \lambda_1^{r_{\min}(\Upsilon)}$;

(ii) $b_2 \lambda_2^{2r_{\max}(\Upsilon)} \le \nu(A) \le b_3 \lambda_3^{2r_{\min}(\Upsilon)}$ *for every* $A \in \mathcal{A}$;

(iii) $\chi_n \le \exp\left(4 b_4 \lambda_4^{r_{\min}(\Upsilon)}\right) - 1$ *for all* $n \ge 1$.

Proof. The parts (i) and (ii) follow from Lemma 10.1, and for these we do not need the Markov condition (MC). The proof of (iii) is based on the fact that the atoms $A_{i_0}, \ldots, A_{i_{-n}}$ in (3.3) are rectangles of the Markov partition $\mathcal{A}$. The set $A_{i_{-1}} \cap T^{-1}A_{i_0}$ is then an s-subrectangle in $A_{i_{-1}}$ and $A_{i_{-1}} \cap TA_{i_{-2}} \cap \cdots \cap T^{n-1}A_{i_{-n}}$ is a u-subrectangle in $A_{i_{-1}}$. The existence of the product measure $\nu^p_{A_{i_{-1}}}$ stated in Lemma 11.2 then readily gives

$$\nu^p_{A_{i_{-1}}}(T^{-1}A_{i_0}/A_{i_{-1}} \cap \cdots \cap T^{n-1}A_{i_{-n}}) = \nu^p_{A_{i_{-1}}}(T^{-1}A_{i_0}/A_{i_{-1}})$$

and so

$$\exp\left(-4b_4\lambda_4^{r_{\min}(\Upsilon)}\right) \le \frac{\nu(A_{i_0}/TA_{i_{-1}} \cap \cdots \cap T^n A_{i_{-n}})}{\nu(A_{i_0}/TA_{i_{-1}})} \le \exp\left(4b_4\lambda_4^{r_{\min}(\Upsilon)}\right)$$

The part (iii) now easily follows. Proposition 11.4 is proved.

Corollary 11.5 *Let $0 < d_1 < d_2 < \infty$. Then for any $m \ge 1$ any partition Υ of Σ_A into cylinders that satisfies the Markov condition (MC) and such that*

$$d_1 m \le r_{\min}(\Upsilon) \le r_{\max}(\Upsilon) \le d_2 m \tag{11.5}$$

will generate a partition $\mathcal{A} = \pi(\Upsilon)$ of Ω which satisfies Assumption T for any $H \ge 1$ with some a_i, c_i depending on the Markov family $\mathcal{R}$ and d_1, d_2.

Remarks. We actually proved something more than Assumption T, because $\chi_n \le c_4 e^{-a_4 m}$ for *all* $n \ge 1$. Assumption T is weak enough, it can be easily proved for arbitrary Anosov or Axiom A flows with Gibbs measures.

12 Uniform transitivity of unstable fibers

Here we prove an auxiliary property of Anosov flows with mixing Gibbs measures.

Let $\mathcal{W}^u \subset M$ be an unstable fiber. Its image, $\phi^t\mathcal{W}^u$, as t grows, gets longer, and it will fill the space M more and more densely. We will estimate, roughly speaking, how much of the fiber $\phi^t\mathcal{W}^u$ ends up in a given domain $U \subset M$.

Without loss of generality, we assume that there is a rectangle $R' = \pi(C_{p',q'})$, for some cylinder $C_{p',q'} \subset \Sigma_A$, such that $\pi_1(\mathcal{W}^u) = W^u_{\pi_1(y)}(R')$ for any $y \in \mathcal{W}^u$. Let ν be an arbitrary Gibbs measure on Ω and μ the corresponding measure on $\mathcal{M}$, and thus on M. The measure μ is then a mixing Gibbs measure for the suspension flow $\Phi^t : \mathcal{M} \to \mathcal{M}$, see [7]. The

Gibbs measure ν, restricted on R', induces a one-dimensional probability measure on the unstable fiber $\pi_1(\mathcal{W}^u)$. We lift this measure up onto $\mathcal{W}^u$ (under π_1^{-1}) and denote the obtained measure on $\mathcal{W}^u$ by ν_0^u. For every $t > 0$ we denote by ν_t^u the image of ν_0^u on the fiber $\phi^t \mathcal{W}^u$.

Lemma 12.1 *For any open domain $U \subset M$ there are constants $\beta_U > 0$ and $t_U > 0$ such that for all $t > t_U$ we have*

$$\nu_{t_1+t}^u(U) \geq \beta_U > 0$$

where $t_1 = q' \cdot l_{\max}$. The constants β_U and t_U are independent of the fiber $\mathcal{W}^u$ (but they depend on the Gibbs measure ν, see the end of this section).

Remark. In particular, ν can be the SBR measure for T. In that case, in view of (8.3), we can simply replace ν_0^u by the (normalized) Riemannian length on $\mathcal{W}^u$, and the statement of Lemma 12.1 will hold true. For this case, a discrete-time version of the lemma was proved in [9], and it was called there the uniform transitivity of unstable fibers.

Before we start the proof, we introduce some convenient terminology. Let $R \subset \Omega$ be a proper connected rectangle. For small $0 < s_1 < s_2$ we call the set

$$X = \phi^{[s_1,s_2]} R = R \times [s_1, s_2] \subset \mathcal{M}$$

a box. For all $y \in X$ we denote by

$$\mathcal{W}_y^{wu,ws}(X) = \mathcal{W}_y^{wu,ws} \cap X \quad \text{and} \quad \mathcal{W}_y^{u,s}(X) = \mathcal{W}_y^{u,s} \cap X$$

the unstable and stable leaves and unstable and stable fibers in the box X, respectively. Every box X is a closed connected domain in $\mathcal{M}$ with piecewise smooth boundary consisting of six faces. These include two stable leaves, two unstable leaves and two surfaces parallel to $R \subset \Omega$, which we call the top and bottom of X. Every box X is foliated by both stable and unstable leaves, which are canonically isomorphic in the following natural sense: the map $H : \mathcal{W}_{y'}^{wu}(X) \to \mathcal{W}_{y''}^{wu}(X)$ defined by $H(x,s) = (W_x^s \times \{s\}) \cap \mathcal{W}_{y''}^{wu}$ is one-to-one.

On the contrary, the unstable and stable fibers $\mathcal{W}_y^{u,s}(X)$ are not all canonically isomorphic (in the sense of Sect. 8). Some of them may cross all stable and unstable leaves $\mathcal{W}_y^{ws,wu}(X)$, respectively, and we call them *full-size* fibers.

Some other fibers in X may terminate on its top or bottom, and will not be then full-size.

Proof of Lemma 12.1. Pick a ball $B_{2r}(y_0) \subset U$ of some radius $2r > 0$ centered at some $y_0 \in U$. Let $v_1 = \sup_{y \in U} ||d\phi^t(y)/dt||$ be the maximum speed of the flow in U. We put

$$\tau = \min\{l_{\min}/10,\, r/(10v_1),\, r/10\}$$

For any $q \geq 1$ we take the partition Υ_q into cylinders $C_{-q,q}$, see Section 11. It generates a Markov partition $\mathcal{A}_q = \pi(\Upsilon_q) = \{A_1^{(q)}, \dots, A_J^{(q)}\}$ of the space Ω. Consider closed boxes

$$Y_{k,j} = \phi^{[k\tau,(k+2)\tau]} A_j^{(q)} \subset M$$

for $1 \leq j \leq J$ and $k = 0, 1, \dots k_{\max} = l_{\max}/\tau + 1$. There are $K = Jk_{\max}$ of those boxes, and they cover M. Moreover, they overlap, so that every point of M is covered at least twice.

We now pick $q \geq 1$ large enough (thus making the rectangles $A_j^{(q)} \in \mathcal{A}_q$ small enough compared to τ), so that for every $y \in Y_{k,j}$ the projection $S(\mathcal{W}_y^u(Y_{k,j}))$ of the curve $\mathcal{W}_y^u(Y_{k,j})$ on the s-axis in $\mathcal{M}$ will be a segment of length $< \tau$. This is clearly possible due to the transversality of Ω and the flow ϕ^t. Then it is easy to check that any point $y \in M$ belongs to a full-size unstable fiber in some box $Y_{k,j}$. Consequently, if an unstable fiber $\mathcal{W}_1^u \subset M$ terminates on stable 'side walls' Ω_1^s, then it can be covered by full-size unstable fibers

$$\mathcal{W}_1^u = \mathcal{W}_{y_1}^u(Y_{k_1,j_1}) \cup \dots \cup \mathcal{W}_{y_L}^u(Y_{k_L,j_L}) \tag{12.1}$$

i.e. every $\mathcal{W}_{y_i}^u(Y_{k_i,j_i})$ crosses all stable leaves in the box Y_{k_i,j_i}.

By increasing q, if necessary, we can ensure that $|\mathcal{W}_y^s(Y_{k,j})| \leq C_\phi^{-1} r/2$ for all $y \in Y_{k,j}$. It is then clear that for any $Y_{k,j}$, any $t > 0$ and any point $y \in Y_{k,j} \cap \phi^{-t} B_r(y_0)$ we have $\mathcal{W}_y^{ws}(Y_{k,j}) \subset \phi^{-t} B_{2r}(y_0)$. The mixing property of the measure μ on M implies that there is a $t_U > 0$ such that for all $Y_{k,j}$ we have

$$\mu(Y_{k,j} \cap \phi^{-t_U} B_r(y_0)) \geq 0.5\mu(Y_{k,j})\mu(B_r(y_0))$$

(t_U also depends on the domains $Y_{k,j}$, i.e. on q and τ, but these are determined by U). Hence, for every k, j there is a subset $Y_{k,j}^s \subset Y_{k,j}$ which is a union of stable leaves in $Y_{k,j}$ and

$$Y_{k,j}^s \subset \phi^{-t_U} U \quad \text{and} \quad \mu(Y_{k,j}^s) > 0.5\mu(Y_{k,j})\mu(B_r(y_0)) \tag{12.2}$$

The set

$$A_{k,j}^s = \phi^{[-l_{\max}-3\tau,0]} Y_{k,j}^s \cap A_j^{(q)}$$

is an s-subrectangle in the rectangle $A_j^{(q)}$. Then the bound in (12.2) can be rewritten as

$$\nu(A_{k,j}^s) > 0.5\nu(A_j^{(q)})\mu(B_r(y_0))$$

Note that all our constructions were so far independent of the fiber $\mathcal{W}^u$.

We now take the unstable fiber $\mathcal{W}^u$. It is clear that for every $t > t_1$ the fiber $\phi^t \mathcal{W}^u$ terminates on Ω_1^s. Due to (12.1) there is a box Y_{k_t,j_t} in which the full-size unstable fibers $\mathcal{W}_{l,t}^u \subset \phi^t \mathcal{W}^u \cap Y_{k_t,j_t}$, $l \geq 1$, satisfy

$$\nu_t^u(\cup_l \mathcal{W}_{l,t}^u) \geq 1/K \tag{12.3}$$

The curves $\pi_1(\mathcal{W}_{l,t}^u)$, $l \geq 1$, are unstable fibers in the rectangle $A_{j_t}^{(q)}$, which we denote by $W_{l,t}^u$. Let $\nu_{l,t}^u$ be the one-dimensional probability measure on $W_{l,t}^u$ induced by the Gibbs measure ν on $A_{j_t}^{(q)}$.

Let $R' = A_{j_t}^{(q)}$ and $d\nu_{R'}^p = d\nu_{R'}^{p,u} \times d\nu_{R'}^{p,s}$ be the product measure on R' involved in Lemma 11.2. For every l we can certainly pick $d\nu_{R'}^{p,u} = \nu_{l,t}^u$, cf. the proof of Lemma 11.2. It is then immediate that

$$\begin{aligned} \nu_{l,t}^u(A_{k_t,j_t}^s) &= \nu_{R'}^p(A_{k_t,j_t}^s)/\nu_{R'}^p(R') \\ &\geq \exp(2b_4\lambda_4^q) \cdot \nu(A_{k_t,j_t}^s)/\nu(R') \\ &\geq 0.5 \cdot \exp(2b_4\lambda_4^q) \cdot \mu(B_r(y_0)) \end{aligned}$$

Summing up over l and combining with (12.3) gives $\nu_t^u(Y_{k_t,j_t}^s) \geq K^{-1} \cdot 0.5\exp(2b_4\lambda_4^q)\mu(B_r(y_0))$. Now, Lemma 12.1 follows with $\beta_U = (2K)^{-1}\exp(2b_4\lambda_4^q)\mu(B_r(y_0))$.

Remark. For any $U \subset M$ the values of t_U and β_U depend on the Gibbs measure μ continuously (in the weak topology of measures), but certainly not uniformly in U.

13 Uniform nonintegrability of foliations

We now turn to the proof of Assumption F. For this, we need an extra assumption on the Anosov flow ϕ^t.

Denote by $B_r(y) \subset M$ an open ball of radius $r > 0$ centered at $y \in M$. If r is small enough, then both families of stable and unstable fibers are orientable inside $B_r(y)$. Pick a $\bar{y}_0 \in M$, a small $r_0 > 0$ and fix some orientations of the families of stable and unstable fibers in $B_0 = B_{r_0}(\bar{y}_0)$. Let $y \in B_0$ and $\delta < r_0$ a small number. On the fibers $\mathcal{W}_y^u$ and $\mathcal{W}_y^s$ we take two positively oriented segments of length δ, starting at y and ending at some $y_1 \in \mathcal{W}_y^u$ and $y_2 \in \mathcal{W}_y^s$, respectively. We denote by $\tau_y(\delta) = \tau(y_1, y_2)$ the temporal distance between the fibers $\mathcal{W}_{y_2}^u$ and $\mathcal{W}_{y_1}^s$ defined in Section 8.

The foliations by local stable and unstable fibers are said to be *jointly integrable* [24] in B_0 if $\tau_y(\delta) = 0$ for all $y \in B_0$ and small $\delta > 0$. In that case those are subfoliations of the same C^1 foliation of B_0 by surfaces. Plante's [24] results imply that the flow ϕ^t is topologically mixing iff its stable and unstable foliations are not jointly integrable in some ball B_0. Our next assumption is a sort of 'uniform nonitegrability' for stable and unstable foliations:

(A5) there is an open ball $B_0 = B_{r_0}(\bar{y}_0) \subset M$ where both families of stable and unstable fibers are orientable, and for some orientation we have, at every $y \in B_0$,

$$0 < \underline{d} < \underline{\lim}_{\delta\to 0} \frac{\tau_y(\delta)}{\delta^2} \leq \overline{\lim}_{\delta\to 0} \frac{\tau_y(\delta)}{\delta^2} < \overline{d} < \infty \tag{13.1}$$

where $\underline{d}$ and $\overline{d}$ do not depend on y.

Remarks. Obviously, (A5) implies the previous assumption (A4). The lower bound ($\underline{d} > 0$) is a principal one here. The upper bound ($\overline{d} < \infty$) is assumed for mere convenience. We could relax it, but this would cause unpleasant complications in our proofs.

Remark. One can also measure the length of fibers in the metric of $\mathcal{M}$ rather than M. This will not alter (A5), only the values of $\underline{d}, \overline{d}$ may change. All the subsequent arguments in this section are valid for both metrics.

We can assume that the radius r_0 of the ball B_0 is so small that there are local coordinates in B_0 in which for any points $y, y' \in B_0$ the angles between $\mathcal{E}_y^u$ and $\mathcal{E}_{y'}^u$, and also between $\mathcal{E}_y^s$ and $\mathcal{E}_{y'}^s$ do not exceed $\gamma/100$, where γ is the smallest angle between $\mathcal{E}_y^\phi$, $\mathcal{E}_y^u$ and $\mathcal{E}_y^s$. This means that all the stable fibers in B_0 are almost parallel, and so are all unstable fibers. We also can assume that the speed of the flow, $|d\phi^t/dt|$, is almost constant in B_0, i.e. for some $\bar{v}_0 > 0$ we have

$$0.99\bar{v}_0 \leq |d\phi^t/dt| \leq 1.01\bar{v}_0$$

Without loss of generality, we can assume that $\bar{v}_0 = 1$ (this amounts to just a rescaling of time or length). In the metric of $\mathcal{M}$ we always have $d\Phi^t/dt = 1$.

For two points y, y' on the same local unstable (or stable) fiber we will denote by $|y - y'|_u$ (resp., $|y - y'|_s$) the length of that fiber between y and y'.

For any two unstable fibers $\mathcal{W}_1^u, \mathcal{W}_2^u \subset B_0$ denote by $H : \mathcal{W}_1^u \to \mathcal{W}_2^u$ the holonomy map, see Sect. 8. Let $DH(y)$ be the Jacobian of H with respect to the length on unstable fibers.

Lemma 13.1 *For any two unstable fibers $\mathcal{W}_1^u, \mathcal{W}_2^u \in B_0$ the Jacobian DH is uniformly bounded away from 0 and ∞. Moreover, by making the ball B_0 smaller if necessary, we can ensure that* $0.99 \le DH(y) \le 1.01$ *for all $y \in \mathcal{W}_1^u$ at which $H(y)$ is defined.*

Proof. Let $\mathcal{W}_2^{wu} = \phi^{[-r_0, r_0]}\mathcal{W}_2^u$. Put $y_* = \mathcal{W}_y^s \cap \mathcal{W}_2^{wu}$. Denote by H_* the holonomy map $\mathcal{W}_1^u \to \mathcal{W}_{y_*}^u$. (Note that H_* depends on y.) There is a $|\tau_*| < r_0$ such that $\phi^{\tau_*}\mathcal{W}_{y_*}^u = \mathcal{W}_2^u$. Then we have $DH(y) = DH_*(y) \cdot \Lambda_{\tau_*}^u(y_*)$. For the Jacobian $DH_*(y)$ an analog of Anosov-Sinai formula (11.4) holds, which says that $DH_*(y) = \lim_{t\to\infty} \Lambda_t^u(y)/\Lambda_t^u(y_*)$. The existence of this limit and its closeness to one follows from the fact that $y_* \in \mathcal{W}_y^s$, and the function $\Lambda_t^u(\cdot)$ is Hölder continuous on M for any t. Lemma 13.1 is proved.

Remark. Similarly, the Jacobian of the holonomy map between any two stable fibers in B_0 will be in the interval $[0.99, 1.01]$.

Let $0 < r_1 < r_0/1000$, so that $r_1 \ll r_0$.

Definition. We call an H-frame any triple of fibers $\{\mathcal{W}_{y_0}^s, \mathcal{W}_{y_1}^u, \mathcal{W}_{y_2}^u\}$ lying in the ball B_0, all terminating on ∂B_0 such that
(i) $y_0 \in B_{r_1}(\bar{y}_0)$ and $y_1, y_2 \in \mathcal{W}_{y_0}^s$;
(ii) the point y_0 lies between y_1 and y_2 on the fiber $\mathcal{W}_{y_0}^s$, and this fiber is positively oriented from y_1 to y_2;
(iii) $3r_1 < |y_l - y_0|_s < 5r_1$ for $l = 1, 2$.
One can visualize an H-frame as a letter H, it consists of two unstable fibers joined by a stable one.

For any H-frame $\{\mathcal{W}_{y_0}^s, \mathcal{W}_{y_1}^u, \mathcal{W}_{y_2}^u\}$ and any $y \in \mathcal{W}_{y_1}^u$ we denote by $\tau_*(y) = \tau(y, y_2)$ the temporal distance between the fibers $\mathcal{W}_{y_2}^u$ and $\mathcal{W}_y^s$. Our assumption (A5), along with Lemma 13.1 and the subsequent remark, implies that the function $\tau_*(y)$ is monotone increasing as y moves along $\mathcal{W}^u_{y_1}$ in the positive direction, and

$$4r_1\underline{d} \le \frac{|\tau_*(y') - \tau_*(y'')|}{|y' - y''|_u} \le 12r_1\overline{d} \qquad (13.2)$$

for all $y', y'' \in \mathcal{W}^u_{y_1}$ such that $\max\{|y' - y_1|_u, |y'' - y_1|_u\} < 100r_1$.

We now fix an H-frame $\{\mathcal{W}^s_{z_0}, \mathcal{W}^u_{z_1}, \mathcal{W}^u_{z_2}\}$ with $z_0 = \bar{y}_0$, the center of the ball B_0, and $|z_1 - z_0|_s = |z_2 - z_0|_s = 4r_1$. Let $D^s_r(z_1)$ and $D^s_r(z_2)$ be two open discs on the surface $\mathcal{W}^{ws}_{z_0}$ with a small radius $0 < r < r_1$ centered at z_1 and z_2, respectively. Denote by U^1_r and U^2_r the unions of all local unstable fibers in B_0 terminating on ∂B_0 and crossing the discs $D^s_r(z_1)$ and $D^s_r(z_2)$, respectively. For small r, the set U^1_r and U^2_r are kind of tubular neighborhoods of the fibers $\mathcal{W}^u_{z_1}$ and $\mathcal{W}^u_{z_2}$, respectively. They are 'distorted' cylinders (of radius r) with axes $\mathcal{W}^u_{z_l}$, $l = 1, 2$. We put $r = r_2 = r_1^2\underline{d}$, so that $r_2 \ll r_1$. Then (13.2) immediately implies

Proposition 13.2 *Let $y'_1 \in D^s_{r_2}(z_1)$ and $y'_2 \in D^s_{r_2}(z_2)$. Then there is a local stable fiber $\mathcal{W}^s_{y_0}$ for some $y_0 \in B_{r_1}(\bar{y}_0)$ that crosses the unstable fibers $\mathcal{W}^u_{y'_1}$ and $\mathcal{W}^u_{y'_2}$ at some points y_1 and y_2, respectively. The points y_1 and y_2 lie in the r_1-neighborhoods of the discs $D^s_{r_2}(z_1)$ and $D^s_{r_2}(z_2)$, respectively.*

Remark. All triples of fibers $\{\mathcal{W}^s_{y_0}, \mathcal{W}^u_{y_1}, \mathcal{W}^u_{y_2}\}$ involved in this proposition are H-frames. In other words, we have plenty of H-frames around: one can take *any* pair of unstable fiber $\mathcal{W}^u_1 \in U^1_{r_2}$ and $\mathcal{W}^u_2 \in U^2_{r_2}$ and join them by a stable fiber $\mathcal{W}^s$ making an H-frame.

Note that all the points z_l, y'_l, y_l, $l = 1, 2$, in the above constructions, and their r_1-neighborhoods, are in the ball $B_{20r_1}(\bar{y}_0)$. We put $B_1 = B_{50r_1}(\bar{y}_0)$, so that $B_1 \subset B_0$. Without loss of generality we can assume that there is a cylinder $C_{p_0,q_0} \subset \Sigma_A$ such that the rectangle $R_0 = \pi(C_{p_0,q_0})$ satisfies $\pi_1(B_1) \subset R_0 \subset \pi_1(B_0)$ and there is a box

$$Y = \phi^{[s_1,s_2]} R_0 = R_0 \times [s_1, s_2]$$

such that $B_1 \subset Y \subset B_0$.

Remark. In Sections 14-15 three extra conditions, (B1)-(B3), will be imposed on domains Y, B_1, B_0 and the values of r_0, r_1, r_2. After that all of them will be fixed.

We now construct special tubular neighborhoods of the middle bar (stable fiber) of an H-frame.

For any H-frame $\{\mathcal{W}^s_{y_0}, \mathcal{W}^u_{y_1}, \mathcal{W}^u_{y_2}\}$ and $l = 1,2$ let $D^u_\varepsilon(y_l)$ be a disc on the surface $\mathcal{W}^{wu}_{y_l}$ centered at y_l with radius $\varepsilon \in (0, r_1)$. Due to (A5) and Lemma 13.1, for any point $y \in D^u_\varepsilon(y_l)$ we have $|\tau(y, y_{3-l})| \le d_0\varepsilon$, where $d_0 = 15r_1\bar{d} + 2$. For r_1 small enough, we have $d_0 \le 3$. Let $V_\varepsilon \subset Y$ be the union of stable fibers terminating on ∂Y that cross both discs $D^u_\varepsilon(y_1)$ and $D^u_\varepsilon(y_2)$. Then V_ε is a sort of tubular neighborhood of the stable fiber $\mathcal{W}^s_{y_0}$, it is a 'distorted' cylinder of radius ε stretching across the box Y. We call V_ε a stable ε-tube corresponding to the given H-frame. Our previous estimates imply that

$$V_\varepsilon \cap \mathcal{W}^{wu}_{y_l} \supset D^u_{d_0^{-1}\varepsilon}(y_l) \supset D^u_{\varepsilon/3}(y_l) \qquad (13.3)$$

for $l = 1,2$.

Lemma 13.3 *Let $\{\mathcal{W}^s_{y_0}, \mathcal{W}^u_{y_1}, \mathcal{W}^u_{y_2}\}$ be an H-frame. For all $\varepsilon \in (0, r_1)$ its stable ε-tube V_ε satisfies $\mu(V_\varepsilon) > \kappa_1\varepsilon^2$, where $\kappa_1 = \kappa_1(Y) > 0$ does not depend on the frame.*

Proof. Due to (13.3), it is enough to show that $\nu(\pi_1(V_\varepsilon)) > \kappa_1'\varepsilon$ with some $\kappa_1' = \kappa_1'(Y) > 0$. Put $x_1 = \pi_1(y_1) \in R_0$. The set $\pi_1(V_\varepsilon)$ is an s-subrectangle in R_0 such that $|W^u_{x_1}(\pi_1(V_\varepsilon))| > \kappa_1''\varepsilon$ with some $\kappa_1'' = \kappa_1''(Y) > 0$, due to (13.3) and the transversality of Ω and the flow ϕ^t. Now the result follows from Lemma 11.3. Lemma 13.3 is proved.

Remark. The proof of this lemma is the only place where we rely on the fact that μ is the SBR measure (otherwise Lemma 11.3 could not be applied). Lemma 13.3 will be only used at the very end of Section 16. All the other arguments in Sections 8–16 hold true for arbitrary Gibbs measures. It would be desirable to extend Theorem 1.1 to arbitrary Gibbs measures, but for its present proof the use of Lemma 11.3 here and that of Lemma 13.3 in Section 16 seem to be indispensable.

14 Markov partitions for the flow $\hat{\Phi}^t$

Let Υ be a finite partition of the symbolic space Σ_A into cylinders, which satisfy the Markov condition (MC). Its projection $\mathcal{A} = \pi(\Upsilon)$ is then a Markov partition $\{A_i\}$ of the space Ω, whose atoms are connected rectangles. We consider $\mathcal{A}$ as a (mod 0) partition of the space Ω, neglecting possible intersections of its atoms in boundary points.

For any such $\mathcal{A}$ and $\delta > 0$ we can use the notations and constructions of Sections 4-5 thinking of δ as a quantum of time. In particular, $\hat{\Phi}^t : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ is a discrete approximation to the flow $\Phi^t : \mathcal{M} \to \mathcal{M}$, and $\hat{T} = \hat{\Phi}^\delta$, and $\hat{\mathcal{A}}$ is a partition of $\hat{\mathcal{M}}$. Since the base Ω is a manifold with boundary, the space $\hat{\mathcal{M}}$ is also a manifold with boundary. Every atom $X \in \hat{\mathcal{A}}$ is a box in the terminology of Section 12. It is a direct product of the rectangle $A(X) = \pi_1(X) \in \mathcal{A}$ and the segment $S(X)$ of length δ on the s axis. One can think of $\hat{\mathcal{M}}$ as made of boxes, as of building blocks. They are piled up over Ω lining up in nice columns, each consisting of identical boxes (in the metric of $\mathcal{M}$).

We call $X' \in X$ a u-subbox (s-subbox) in a box $X \in \hat{\mathcal{A}}$ if $X' = \overline{\operatorname{int} X'}$ and X' is a connected union of some unstable (stable) leaves of X. Obviously, in this case $\pi_1(X')$ is a u-subrectangle (s-subrectangle) of the rectangle $A(X)$.

For any boxes $X', X'' \in \hat{\mathcal{A}}$ and $n \geq 1$ the intersection $\hat{T}^n X' \cap X''$ (with a nonempty interior) consists of one or more u-subboxes in X'', and $\hat{T}^{-n} X'' \cap X'$ will consist of s-subboxes in X'. We refer to this as the Markov property of the partition $\hat{\mathcal{A}}$ of $\hat{\mathcal{M}}$. One can restate the Markov property in this way: for any stable leaf $\mathcal{W}^{ws}_y(X)$ of any box X its image $\hat{T}^n(\mathcal{W}^{ws}_y(X))$, $n \geq 1$, lies entirely in one atom of $\hat{\mathcal{A}}$, and so does every preimage of any unstable leaf of any box.

The ceiling function $l(x)$ is C^2 up to the singularity set $\partial\Omega \cup T^{-1}(\partial\Omega)$. Due to Lemma 10.1 there is a constant $b_5 = b_5(\mathcal{R}) > 0$ such that

$$\operatorname{osc}(l, A) \leq b_5 \lambda_1^{r_{\min}(\Upsilon)} \quad \text{for any } A \in \mathcal{A}. \tag{14.1}$$

Lemma 14.1 *There is a constant $b_6 = b_6(\mathcal{R}) > 0$ such that if $\delta \geq b_6 \lambda_1^{r_{\min}(\Upsilon)}$, then every unstable leaf $\mathcal{W}^{wu}_y(X)$ of every box $X \in \hat{\mathcal{A}}$ will contain a full-size unstable fiber. Likewise, every stable leaf in X must contain a full-size stable fiber.*

Proof. It is enough to establish the lemma for the bottom boxes $X \in \hat{\mathcal{A}}$ only, i.e. those with $S(X) = [0, \delta]$, because the boxes in every column are identical. Then the lemma readily follows from the transversality of the cross-section Ω to the flow ϕ^t and from Lemma 10.1, which provides a bound on $\operatorname{diam} A(X)$.

Given a partition Υ, we fix

$$\delta = \max\{b_5, b_6\} \cdot \lambda_1^{r_{\min}(\Upsilon)} \tag{14.2}$$

so that Lemma 14.1 will always apply. It also follows from (14.1) that

$$\sup\{|l(x) - \hat{l}(x)| : x \in \Omega\} \leq 3\delta \tag{14.3}$$

For δ small enough ($\ll l_{\min}$) it is clear that the maps Φ^t and $\hat{\Phi}^t$ are close to each other as long as $|t| \ll \delta^{-1}$. Precisely,

Lemma 14.2 *Let $y = (x, s) \in \mathcal{M} \cap \hat{\mathcal{M}}$ and $y_t = (x_t, s_t) = \Phi^t y$. Then either*

$$\min\{s_t, \hat{l}(x_t) - s_t\} \leq 3\delta(|t|/l_{\min} + 2) \tag{14.4}$$

or[4]

$$\rho(\Phi^t y, \hat{\Phi}^t y) \leq 3\delta(|t|/l_{\min} + 2) \tag{14.5}$$

Proof. It is enough to do this for $t > 0$. The trajectory $\Phi^s y$, $0 \leq s \leq t$, can cross the surface Ω not more than $t/l_{\min} + 1$ times. Every time the 'asynchronism' between the two flows can grow by at most 3δ due to (14.3). It may happen that the trajectories $\Phi^s y$ and $\hat{\Phi}^s y$, $0 < s < t$, cross Ω a different number of times, and then we have (14.4). Otherwise we have (14.5), and Lemma 14.2 is proved.

We now turn back to the constructions of Section 13, including the balls B_1, B_0, the box Y and the quantities $r_2 \ll r_1 \ll r_0$. By making all of these smaller, if necessary, we can ensure two extra properties:

(B1) for any point $y = (x, s) \in B_0$ we have $s > r_0$ and $\hat{l}(x) - s > r_0$, so that B_0 is a way from the bottom and top of the manifold $\hat{\mathcal{M}}$.
(B2) for every $y \in B_1$ we have $S(\mathcal{W}^{u,s}_y \cap Y) \subset [s_1 + 2r_1, s_2 - 2r_1]$.

Note that, in virtue of (B2), for every $y \in B_1$ the fibers $\mathcal{W}^{u,s}_y(Y) = \mathcal{W}^{u,s}_y \cap Y$ will be full-size in the box Y (crossing all the leaves of the opposite foliation).

[4] the metric ρ defined in Sect. 2 is here simply the 'distance in time' along an orbit of the flow.

We will then consider only partitions Υ of Σ_A into cylinders such that

$$r_{\min}(\Upsilon) \geq \max\{|p_0|, q_0\} \quad \text{and} \quad \delta < r_2/100 \tag{14.6}$$

where δ is defined by (14.2). Let

$$0 < t \leq l_{\min} r_2/(20\delta) \tag{14.7}$$

and put $n = [t/\delta]$. Let $X_i \in \hat{\mathcal{A}}$. Consider closely the intersection $\hat{T}^n X_i \cap Y$. Pick a point $y' \in \operatorname{int} X_i$ with a full-size unstable fiber $\mathcal{W}^u_{y'}(X_i)$ (one exists due to Lemma 14.1). Its image, $\Phi^t(\mathcal{W}^u_{y'}(X_i))$, may intersect the box Y in a finite number of smooth curves. Denote by $\tilde{\mathcal{W}}_\zeta$, $\zeta = 1, \dots, Z_i$, all of those curves which (i) also intersect the smaller ball B_1 and (ii) do not terminate at the image of either endpoint of the original fiber $\mathcal{W}^u_{y'}(X_i)$. Due to (B2), every curve $\tilde{\mathcal{W}}_\zeta$ in this collection crosses all the stable leaves of the box Y.

Next, for any ζ the set

$$X_i^\zeta = \cup_{y \in \Phi^{-t}\tilde{\mathcal{W}}_\zeta} \mathcal{W}^{ws}_y(X_i)$$

is an s-subbox in X_i. We also put $\tilde{X}_i^\zeta = \hat{T}^n(X_i^\zeta) = \hat{\Phi}^{n\delta}(X_i^\zeta)$. Due to (B1), (14.6), (14.7) and Lemma 14.2, the set $\tilde{X}_i^\zeta$ lies in the ε_t-neighborhood of the curve $\tilde{\mathcal{W}}_\zeta$, with

$$\varepsilon_t = 3\delta t/l_{\min} + 8\delta < r_2/4 \tag{14.8}$$

In virtue of (B2), this set lies wholly in the box Y. According to the Markov property of the partition $\hat{\mathcal{A}}$, for every box $X_l \subset Y$ the intersection $\tilde{X}_i^\zeta \cap X_l$ (if it has a nonempty interior) will be a u-subbox in X_l, which we denote by $\tilde{X}_i^\zeta[l]$. Its projection $\pi_1(\tilde{X}_i^\zeta[l]) \subset \Omega$ will be a u-subrectangle in the rectangle $A(X_l)$ that covers the curve $\pi_1(\tilde{\mathcal{W}}_\zeta) \cap A(X_l)$. The projection $\pi_1(\tilde{X}_i^\zeta)$ will be then a long narrow u-subrectangle in the rectangle $R_0 = \pi_1(Y)$, completely covering the curve $\pi_1(\tilde{\mathcal{W}}_\zeta)$.

Summarizing, we may think of $\tilde{X}_i^\zeta$ as a chain of u-subboxes in some boxes $X_l \subset Y$. This chain stretches along the curve $\tilde{\mathcal{W}}_\zeta$ and stays in its ε_t-neighborhood. For every rectangle $A \in \mathcal{A}$ in R_0 that intersects the curve $\pi_1(\tilde{\mathcal{W}}_\zeta)$ there is exactly one box X_l in the column of boxes over A (i.e., one with $A(X_l) = A$) which contains a nonempty u-subbox $\tilde{X}_i^\zeta[l]$ of this chain.

A dual property, obviously, holds true for the set $\hat{T}^{-n} X_k \cap Y$ for any box $X_k \in \hat{\mathcal{A}}$, under again the condition (14.7). That set contains the chains of s-subboxes, $\tilde{X}_k^\zeta$, $\zeta = 1, \dots, Z_k$, lining up along stable fibers stretching across the box Y. We also put $\tilde{X}_k^\zeta[l] = \tilde{X}_k^\zeta \cap X_l$.

15 Synchronization

Here we construct a rich family of finite partitions of Σ_A into cylinders that enjoy a special property, which we call synchronization.

For any $y \in M$ and $t > 0$ denote by $J(y,t)$ the number of times the reverse trajectory $\phi^\tau y$, $-t < \tau < 0$, crosses the surface Ω. For any subset $B \subset M$ we put $J^+(B,t) = \max\{J(y,t) : y \in B\}$ and $J^-(B,t) = \min\{J(y,t) : y \in B\}$. For any cylinder $C_{p,q} \subset \Sigma_A$ we put

$$M(C_{p,q}) = \psi^{-1}(\pi_1^{-1}(\pi(C_{p,q}))) \subset M$$

Next, for every large integer $\bar{q} \geq 1$ we define an increasing sequence of finite partitions, $\Upsilon_{n,\bar{q}}$, $n \geq 1$, of Σ_A into cylinders, such that for any atom $C_{p,q} \in \Upsilon_{n,\bar{q}}$ we will have $p \leq -\bar{q}$ and $q = \bar{q}$. Note first, that under these conditions we will have $r_{\min}(\Upsilon_{n,\bar{q}}) = \bar{q}$ for all $n \geq 1$, and so the value of $\delta = \delta_{\bar{q}}$ defined by (14.2) will not depend on n. We will also assume that $\bar{q}$ is large enough, so that (14.6) holds. In particular, $\delta < l_{\min}$, so that we will have $J^+(B,(n+1)\delta) \leq J^+(B,n\delta) + 1$ for all $n \geq 1$ and any $B \subset M$.

Our definition of the sequence of partitions $\Upsilon_{n,\bar{q}}$ is recurrent in n. We put $\Upsilon_{1,\bar{q}} = \Upsilon_{\bar{q}}$, a partition into cylinders $C_{-\bar{q},\bar{q}}$. For any $n \geq 2$ and any atom $C_{p,q}$ of the partition $\Upsilon_{n-1,\bar{q}}$ we declare it an atom of $\Upsilon_{n,\bar{q}}$ if $J^+(M(C_{p,q}), n\delta) \leq |p|$, otherwise we subdivide it into 'longer' cylinders $C_{p-1,q} \subset C_{p,q}$ declaring them all atoms of $\Upsilon_{n,\bar{q}}$. Clearly, for every $n \geq 2$ the cylinders $C_{p,q} \in \Upsilon_{n,\bar{q}}$ may have different 'lengths' (different p's but always $q = \bar{q}$).

Note that for any $n \geq 1$ and any $C_{p,q} \in \Upsilon_{n,\bar{q}}$ we have

$$J^+(M(C_{p,q}), n\delta) \leq |p| \tag{15.1}$$

and also

$$\bar{q} = r_{\min}(\Upsilon_{n,\bar{q}}) \leq r_{\max}(\Upsilon_{n,\bar{q}}) \leq \max\{\bar{q}, J^+(M, n\delta)\} \tag{15.2}$$

Lemma 15.1 *The partition $\Upsilon_{n,\bar{q}}$ satisfies the Markov condition (MC) for all $\bar{q}$ and $n \geq 1$.*

Proof. The proof is inductive on n. Obviously, $\Upsilon_{1,\bar{q}} = \Upsilon_{\bar{q}}$ satisfies (MC). Suppose that (MC) is violated for $\Upsilon_{n,\bar{q}}$ but holds for $\Upsilon_{n-1,\bar{q}}$. This can only happen if there are two cylinders $C'_{p,q}, C''_{p-1,q} \in \Upsilon_{n-1,\bar{q}}$, $\sigma(C'_{p,q}) \supset C''_{p-1,q}$, such that $J^+(M(C'_{p,q}), n\delta) \leq |p|$ and $J^+(M(C''_{p-1,q}), n\delta) > |p-1| = |p| + 1$. In that case the cylinder $C'_{p,\bar{q}}$ is an atom of $\Upsilon_{n,\bar{q}}$, while $C''_{p-1,\bar{q}}$ is not (it has to be subdivided into $C''''_{p-2,q} \subset C''_{p-1,q}$). But this is impossible, since the inclusion $\sigma(C'_{p,q}) \supset C''_{p-1,q}$ implies $\overline{T(\pi(C'_{p,q}))} \supset \pi(C''_{p-1,q})$, so that $J^+(M(C''_{p-1,q}), n\delta) \le J^+(M(C'_{p,q}), n\delta) + 1$. Lemma 15.1 is proved.

Now, for any $n \ge 1$ we have a Markov partition $\mathcal{A} = \pi(\Upsilon_{n,\bar{q}})$ of Ω and $\delta = \delta_{\bar{q}}$ defined by (14.2). They generate the space $\hat{\mathcal{M}}$, its partition $\hat{\mathcal{A}}$ into boxes, and a map $\hat{T} = \hat{\Phi}^\delta : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$.

We will always assume that n is bounded by

$$l_{\max}\bar{q}/\delta < n < l_{\min} r_2/(20\delta^2) \tag{15.3}$$

Note that the lower bound implies $J^-(M, n\delta) > \bar{q}$, and so $p < -\bar{q}$ for every atom $C_{p,q} \in \Upsilon_{n,\bar{q}}$. Consider an arbitrary $C_{p,q} \in \Upsilon_{n,\bar{q}}$ and $A = \pi(C_{p,q}) \in \mathcal{A}$. Let $X \in \hat{\mathcal{A}}$ be a box 'over' A, i.e. $\pi_1(X) = A$. Pick a point $y \in X$ with a full-size stable fiber $\mathcal{W}^s_y(X)$ (one exists due to Lemma 14.1). The curve $\tilde{\mathcal{W}} = \phi^{-n\delta}(\mathcal{W}^s_y(X))$ is a stable fiber in M. It may be cut by surfaces of $\Omega \cup \Omega_1$ into pieces, which correspond to the smooth components of $\psi(\tilde{\mathcal{W}}) \subset \mathcal{M}$.

The crucial observation is that, since $J^+(\mathcal{W}^s_y(X), n\delta) \le |p|$, which follows from (15.1), then the above curve $\tilde{\mathcal{W}}$ cannot cross any 'side wall' in Ω_1. It can only cross one or more times the 'floor' Ω. Hence, due to the transversality of Ω and the flow ϕ^t, the length of the curve $\tilde{\mathcal{W}}$ (in both metrics of M and $\mathcal{M}$) will be bounded by a constant $C_1 = C_1(\mathcal{R}) < \infty$. In particular, $\tilde{\mathcal{W}}$ can cross Ω not more than J_1 times, where $J_1 = J_1(\mathcal{R}) \in \mathbb{Z}_+$ is another constant. Therefore, $J^+(\mathcal{W}^s_y(X), n\delta) - J^-(\mathcal{W}^s_y(X), n\delta) \le J_1$, and so

$$J^+(M(C_{p,q}), n\delta) - J^-(M(C_{p,q}), n\delta) \le J_2 = J_1 + [l_{\max}/l_{\min}] + 3 \tag{15.4}$$

Next, recall that $|p| > \bar{q}$, so that $C_{p,q} \notin \Upsilon_{1,\bar{q}} = \Upsilon_{\bar{q}}$. Put $n_1 = \min\{n' \le n : C_{p,q} \in \Upsilon_{n',\bar{q}}\}$. Let $C_{p+1,q}$ be the atom of $\Upsilon_{n_1-1,\bar{q}}$ that contains $C_{p,q}$. Then $J^+(M(C_{p+1,q}), n_1\delta) > |p+1| = |p| - 1$. Since (15.4) also holds for $M(C_{p+1,q})$ (with $n_1 - 1$ instead of n), we get $J^-(M(C_{p+1,q}), n_1\delta) \ge |p| - J_2 - 1$. Since $M(C_{p,q}) \subset M(C_{p+1,q})$ and $n_1 \le n$, we get

$$|p| - J_2 - 1 \le J^-(M(C_{p,q}), n\delta) \le J^+(M(C_{p,q}), n\delta) \le |p| \tag{15.5}$$

Now we impose the last condition on the box Y constructed in Section 13:

(B3) The rectangle $\pi_1(Y) = R_0 = \pi(C_{p_0,q_0})$ is small enough, so that $|p_0| > J_2 + 1$.

We now turn back to the stable fiber $\tilde{\mathcal{W}} = \phi^{-n\delta}(\mathcal{W}^s_y(X))$. Let y_1 and y_2 be its endpoints. Each of $\pi_1(y_1)$ and $\pi_1(y_2)$ belongs to $T^j(\partial^u\Omega)$ for some $j \geq 1$. Due to (15.5) we have $|p| - J_2 - 1 \leq j \leq |p|$. Since $T^{-|p|}(\partial^u A) \subset \partial^u\Omega$, we have

$$\pi_1(y_1), \pi_1(y_2) \in T^{J_2+1}(\partial^u\Omega) \tag{15.6}$$

Assume now that $\tilde{\mathcal{W}} \cap B_1 \neq \emptyset$. Then (B3) and (15.6) imply that $y_1, y_2 \notin Y$. By virtue of (B2), the fiber $\tilde{\mathcal{W}} \cap Y$ is full-size in Y.

Next, the set $\hat{T}^{-n}X$ consists of s-subboxes in some boxes of $\hat{\mathcal{A}}$. Those subboxes lie in the $\varepsilon_{n,\bar{q}}$-neighborhood of the curve $\tilde{\mathcal{W}}$, where

$$\varepsilon_{n,\bar{q}} = 3\delta^2 n/l_{\min} + 8\delta \leq r_2/4$$

in view of (14.5), (14.6) and (15.3), unless they are separated from $\tilde{\mathcal{W}}$ by Ω, in which case (14.4) applies instead of (14.5). Therefore, the set $\hat{T}^{-n}X \cap Y$ is a chain of s-subboxes lining up along the curve $\tilde{\mathcal{W}} \cap Y$. We denote by $\tilde{X}$ the union of s-subboxes in this chain.

Lemma 15.2 *In the above notations we have*

$$\hat{\mu}(\tilde{X})/\hat{\mu}(X) \geq \kappa_2 > 0$$

for some constant $\kappa_2 = \kappa_2(Y, \mathcal{R}) > 0$, independent of $X, n, \bar{q}$.

Proof. Let $R_0 = \pi_1(Y) \subset R_{i_0} \in \mathcal{R}$. Since $J^+(\mathcal{W}^s_y(X), n\delta) \leq |p|$ and $\tilde{\mathcal{W}}$ does not cross any side walls $\Omega_1 \subset M$, there is a continuous function, $s(y)$, on $\tilde{\mathcal{W}}$ such that $\phi^{s(y)}y \in R_{i_0}$ for all $y \in \tilde{\mathcal{W}}$ and $\phi^{s(y)}y = \pi_1(y)$ for all $y \in \tilde{\mathcal{W}} \cap Y$. The curve $\tilde{W} = \{\phi^{s(y)}y : \, y \in \tilde{\mathcal{W}}\} \subset R_{i_0}$ is then a stable fiber for the map T.

Now, for any $y \in \hat{T}^{-n}X$ we pick a $z_y \in \tilde{\mathcal{W}}$ such that $\operatorname{dist}(y, z_y) \leq \varepsilon_{n,\bar{q}}$ (if such a z_y does not exist, we denote by z_y the closest point on $\tilde{\mathcal{W}}$ to y in the metric of M). Let $\pi_1^* : \hat{T}^{-n}X \to R_{i_0}$ be a map defined by $\pi_1^*(y) = \pi_1(\phi^{s(z_y)+l_{\min}/2}y)$. The image $\pi_1^*(\hat{T}^{-n}X)$ is a rectangle $R_X \subset R_{i_0}$, which covers the curve $\tilde{W}$. Clearly, $\pi_1(\hat{T}^{-n}X \cap Y) = R_X \cap \pi_1(Y)$ is a u-subrectangle in R_X. The statement of Lemma 15.2 is now equivalent to

$$\nu[R_X \cap \pi_1(Y)]/\nu(R_X) \geq \kappa_2 \tag{15.7}$$

Let $\nu^p_0 = \nu^{p,u}_0 \times \nu^{p,s}_0$ be the product measure involved in Lemma 11.2, for the rectangle R_{i_0}, with $\nu^{p,u}_0$ and $\nu^{p,s}_0$ defined on fibers $W^u_{x_0}(R_{i_0})$ and $W^s_{x_0}(R_{i_0})$, respectively, for some $x_0 \in \pi_1(Y)$, which we assume to be fixed (independent of $X, n, \bar{q}$). Then we have

$$\frac{\nu_0^p(R_X \cap \pi_1(Y))}{\nu_0^p(R_X)} = \frac{\nu_0^{p,s}(W_{x_0}^s(R_{i_0}) \cap \pi_1(Y))}{\nu_0^{p,s}([W_{x_0}^s(R_{i_0}), R_X])} \geq \frac{\nu_0^{p,s}(W_{x_0}^s(R_{i_0}) \cap \pi_1(Y))}{\nu_0^{p,s}(W_{x_0}^s(R_{i_0}))}$$

where $[W_{x_0}^s(R_{i_0}), R_X] = \{W_{x_0}^s(R_{i_0}) \cap W_{x'}^u : x' \in R_X\}$. Denote the last ratio by $\kappa_2' = \kappa_2'(Y, R) > 0$. Now, (15.7) readily follows from (11.1), with $\kappa_2 = e^{2b_4}\kappa_2'$. Lemma 15.2 is proved.

Proposition 15.3 (Synchronization) *There is $m_0(\mathcal{R}) > 0$ such that for every $m \geq m_0(R)$ there is a finite partition $\Upsilon^{(m)}$ of Σ_A into cylinders satisfying both the Markov condition (MC) and the bound (11.5) with some $0 < d_1(\mathcal{R}) < d_2(\mathcal{R}) < \infty$ and enjoying the following property:*

For every box $X_k \in \hat{\mathcal{A}}$ such that $\hat{T}^{-[m/\delta]}X_k \cap B_1 \neq \emptyset$, there is exactly one chain[5] $\tilde{X}_k^1 \subset \hat{T}^{-[m/\delta]}X_k \cap Y$ and

$$\hat{\mu}(\tilde{X}_k^1) \geq \kappa_2 \hat{\mu}(X_k) \tag{15.8}$$

Proof. We put $\Upsilon^{(m)} = \Upsilon_{n,\bar{q}}$ with $\bar{q} = [m/2l_{\max}] + 1$ and $n = [m/\delta_{\bar{q}}]$. Our assumptions (14.6) on $\bar{q} = r_{\min}(\Upsilon^{(m)})$ and $\delta = \delta_{\bar{q}}$ and (15.3) on n will hold for all $m \geq m_0(\mathcal{R})$ with some $m_0(\mathcal{R}) > 0$. The inequalities (15.2) imply the bounds (11.5) with

$$d_1 = (2l_{\max})^{-1} \quad \text{and} \quad d_2 = 2/l_{\min}$$

The main property (15.8) follows from Lemma 15.2. Proposition 15.3 is proved.

Remark. The key idea of this proposition is that all the boxes $X \in \hat{\mathcal{A}}$ are stretched under the map $\hat{T}^{-[m/\delta]}$ substantially, so that they are transformed into long chains of s-subboxes, but at the same time 'not too much' (every box is transformed into a few chains of finite total 'length'). Broadly speaking, the map $\hat{T}^{-[m/\delta]}$ stretches all the boxes 'synchronically'.

[5]Here we again use the terminology and notations of the previous section.

16 A proof of Assumption F

In this section we prove Assumption F stated in Section 7, and thus complete the proof of Theorem 1.1.

For any $m \geq m_0(\mathcal{R})$ we take the partition $\Upsilon^{(m)}$ of the symbolic space Σ_A into cylinders, which is provided by Proposition 15.3. It generates a Markov partition $\mathcal{A}^{(m)} = \pi(\Upsilon^{(m)})$ of the base Ω, which verifies Assumption T due to Corollary 11.5. The value of δ defined by (14.2) verifies (i) of Assumption F. We put

$$K_1 = [\beta_1 m/\delta] \quad \text{with} \quad \beta_1 = 4l_{\max}/l_{\min} \quad \text{and} \quad K_2 = [m/\delta]$$

so that $\beta_2 = 1$ in Assumption F. We also put

$$\eta = 100\beta_1 l_{\min}^{-1} m$$

Clearly, the parts (i)-(iii) of Assumption F will hold for all $m \geq m_0(\mathcal{R})$.

In view of our choice of δ the bounds (14.6) and (14.7) with $t = K_1\delta$ will hold for all $m \geq m_1(\mathcal{R})$ with some $m_1(\mathcal{R}) > 0$. Thus, for all $t \leq K_1\delta$ we get (14.8) and a stronger bound

$$\varepsilon_t \leq 3\delta^2 K_1/l_{\min} + 8\delta < \eta\delta/10 \tag{16.1}$$

We now turn back to the equation (7.3). We consider only such quadruples (l_1, l_2, l_3, l_4) that $X_{l_r} \subset Y$ for all $r = 1, 2, 3, 4$. The sets $\hat{T}^{K_1}X_i$, $\hat{T}^{K_1}X_j$ and $\hat{T}^{-K_2}X_k$ admit the description in terms of chains of subboxes developed in Sections 14-15. Let $\tilde{X}_i^{\zeta_1}$, $\zeta_1 = 1, \dots, Z_i$, be all the chains of u-subboxes in $(\hat{T}^{K_1}X_i) \cap Y$, also $\tilde{X}_j^{\zeta_2}$, $\zeta_2 = 1, \dots, Z_j$, be all the chains of u-subboxes in $(\hat{T}^{K_1}X_j) \cap Y$, and $\tilde{X}_k^1$ be the chain of s-subboxes in $(\hat{T}^{-K_2}X_k) \cap Y$, this one is unique for a given k due to Proposition 15.3.

Consider any pair of chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_k^1$. The sets $\pi_1(\tilde{X}_i^{\zeta_1})$ and $\pi_1(\tilde{X}_k^1)$ are a u- and s-subrectangles in $R_0 = \pi_1(Y)$, respectively. Therefore, they intersect each other inside some rectangle $A' \in \mathcal{A}$. Hence, there is exactly one column of boxes in $\hat{\mathcal{A}}$ (the one over A') in which both chains have 'representatives', i.e. a u-subbox $\tilde{X}_i^{\zeta_1}[l_1] \subset X_{l_1}$ and an s-subbox $\tilde{X}_k^1[l_3] \subset X_{l_3}$. We put $\Gamma_{i,k}^{\zeta_1} = 1$ if $|s(X_{l_1}) - s(X_{l_3})| < \eta$ and $\Gamma_{i,k}^{\zeta_1} = 0$ otherwise[6]. Thus, every

[6] The difference $|s(X_{l_1}) - s(X_{l_3})| - 1$ is the number of boxes between X_{l_1} and X_{l_3} in the column of boxes over A'.

pair of chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_k^1$ has at most one representative in (7.3), and it does iff $\Gamma_{i,k}^{\zeta_1} = 1$, according to the "coupling" condition on l_1, l_3 in the setup of equation (7.3). A similar conclusion is, of course, true for every pair of chains $\tilde{X}_j^{\zeta_2}$ and $\tilde{X}_k^1$.

Lemma 16.1 *Consider an arbitrary pair of chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_k^1$, and the corresponding boxes X_{l_1}, X_{l_3} described above. We have*

$$\hat{\mu}(\hat{T}^{K_1} X_i \cap X_{l_1}) \cdot \hat{\mu}(\hat{T}^{-K_2} X_k \cap X_{l_3}) \geq e^{-6b_4} \frac{\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_k^1)\hat{\mu}(X_{l_3})}{c_{\hat{\mu}}\nu(R_0)\delta}$$

Proof. First, it is enough to substitute $\hat{\mu}(\tilde{X}_i^{\zeta_1}[l_1])$ for $\hat{\mu}(\hat{T}^{K_1} X_i \cap X_{l_1})$ and $\hat{\mu}(\tilde{X}_k^1[l_3])$ for $\hat{\mu}(\hat{T}^{-K_2} X_k \cap X_{l_3})$. Then the lemma is equivalent to the inequality

$$\nu(\pi_1(\tilde{X}_i^{\zeta_1}[l_1])) \cdot \nu(\pi_1(\tilde{X}_k^1[l_3])) \geq e^{-6b_4} \frac{\nu(\pi_1(\tilde{X}_i^{\zeta_1}))\nu(\pi_1(\tilde{X}_k^1))\nu(A(X_{l_3}))}{\nu(R_0)}$$

Note that $\pi_1(\tilde{X}_i^{\zeta_1}[l_1])$ and $\pi_1(\tilde{X}_k^1[l_3])$ are u- and s-subrectangles of $A(X_{l_3})$, respectively. It is then a simple calculation that for the product measure $\nu_{R_0}^p$ on R_0 involved in Lemma 11.2 we have

$$\nu_{R_0}^p(\pi_1(\tilde{X}_i^{\zeta_1}[l_1])) \cdot \nu_{R_0}^p(\pi_1(\tilde{X}_k^1[l_3])) = \frac{\nu_{R_0}^p(\pi_1(\tilde{X}_i^{\zeta_1}))\nu_{R_0}^p(\pi_1(\tilde{X}_k^1))\nu_{R_0}^p(A(X_{l_3}))}{\nu_{R_0}^p(R_0)}$$

The bound (11.1) now implies the lemma.

A similar statement holds for any pair of chains $\tilde{X}_j^{\zeta_2}$ and $\tilde{X}_k^1$. This lemma will allow us to 'uncouple' the indices i, j, k from l_1, l_2, l_3, l_4 in (7.3). Combining Lemma 16.1 and Proposition 15.3 gives

$$\begin{aligned} \hat{b}_{i,j} &\geq \sum_k \sum_{\zeta_1,\zeta_2} \Gamma_{i,k}^{\zeta_1} \Gamma_{j,k}^{\zeta_2} \frac{e^{-12b_4}\kappa_2}{c_{\hat{\mu}}^2 \nu^2(R_0)} \frac{\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_j^{\zeta_2})\hat{\mu}(\tilde{X}_k^1)}{[(2\eta+1)\delta]^2 \hat{\mu}(X_i)\hat{\mu}(X_j)} \\ &\geq \frac{e^{-12b_4}\kappa_2}{c_{\hat{\mu}}^2 \nu^2(R_0)} \sum_{\zeta_1,\zeta_2} \left(\sum_k \Gamma_{i,k}^{\zeta_1} \Gamma_{j,k}^{\zeta_2} \frac{\hat{\mu}(\tilde{X}_k^1)}{[(2\eta+1)\delta]^2} \right) \frac{\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_j^{\zeta_2})}{\hat{\mu}(X_i)\hat{\mu}(X_j)} \end{aligned} \quad (16.2)$$

We now invoke the constructions of Sect. 13. It is enough to sum in (16.2) over such ζ_1, ζ_2 and k that

(i) the chain $\tilde{X}_i^{\zeta_1}$ intersects the tube $U^1_{r_2/2}$ and stretches completely across the box Y;
(ii) the chain $\tilde{X}_j^{\zeta_2}$ intersects the tube $U^2_{r_2/2}$ and stretches completely across the box Y.

For any pair of ζ_1, ζ_2 just specified, there are two unstable fibers $\tilde{\mathcal{W}}^u_{\zeta_1} \subset \Phi^{K_1\delta} X_i$ and $\tilde{\mathcal{W}}^u_{\zeta_2} \subset \Phi^{K_1\delta} X_j$ whose $\varepsilon_{K_1\delta}$-neighborhoods contain the chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_j^{\zeta_2}$, respectively. These fibers lie, respectively, in the tubes $U^1_{r_2}$ and $U^2_{r_2}$, due to (14.8). Then Proposition 13.2 says that there is a stable fiber $\tilde{\mathcal{W}}^s_{\zeta_1\zeta_2} \subset Y$ intersecting both $\tilde{\mathcal{W}}^u_{\zeta_1}$ and $\tilde{\mathcal{W}}^u_{\zeta_2}$ at some points y_1 and y_2, respectively, so that these three fibers make an H-frame. Then every chain $\tilde{X}_k^{(1)}$ lying in the vicinity of the fiber $\tilde{\mathcal{W}}^s_{\zeta_1\zeta_2} \cap Y$ must be 'coupled' with $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_j^{\zeta_2}$ in Eq. (16.2). Precisely, the union of all the chains $\tilde{X}_k^1$ with $\Gamma^{\zeta_1}_{i,k} = \Gamma^{\zeta_2}_{j,k} = 1$ covers the stable ε-tube $V_\varepsilon = V_\varepsilon^{\zeta_1\zeta_2}$ of the above H-frame with $\varepsilon = \eta\delta/4$. This follows from Proposition 15.3 and the bound (16.1).

Remark. The necessity to take into account the asynchronism between the flows Φ^t and $\hat{\Phi}^t$, which is manifested in ε_t estimated by (14.8) and (16.1), was the only reason why we introduced the parameter η and the corresponding perturbed Markov chains in Section 5.

Now, Lemma 13.3 implies that

$$\sum_k \Gamma^{\zeta_1}_{i,k}\Gamma^{\zeta_2}_{j,k}\frac{\hat{\mu}(\tilde{X}_k^1)}{[(2\eta+1)\delta]^2} \geq \frac{\hat{\mu}(V^{\zeta_1\zeta_2}_{\eta\delta/4})}{[(2\eta+1)\delta]^2} \geq \kappa_1/100$$

This reduces (16.2) to

$$\hat{b}_{i,j} \geq \frac{e^{-12b_4}\kappa_1\kappa_2}{100c^2_{\hat{\mu}}\nu^2(R_0)} \cdot \frac{\sum_{\zeta_1}\hat{\mu}(\tilde{X}_i^{\zeta_1})}{\hat{\mu}(X_i)} \cdot \frac{\sum_{\zeta_2}\hat{\mu}(\tilde{X}_j^{\zeta_2})}{\hat{\mu}(X_j)} \tag{16.3}$$

Next, Lemma 12.1 can be applied to any full-size unstable fiber in the boxes X_i, X_j and the open sets $U_1 = \mathrm{int}\,(U^1_{r_2/4} \cap Y)$ and $U_2 = \mathrm{int}\,(U^2_{r_2/4} \cap Y)$, respectively. In terms of Lemma 12.1, for any such fiber $\mathcal{W}^u$ we have

$$\nu^u_t(U_1) > \beta_* \quad \text{and} \quad \nu^u_t(U_2) > \beta_*$$

for all $t > l_{\max} d_2 m + t_*$, where $t_* = \max\{t_{U_1}, t_{U_2}\}$ and $\beta_* = \min\{\beta_{U_1}, \beta_{U_2}\}$. Note that $K_1\delta > l_{\max} d_2 m + t_*$ for all $m \geq m_2$ for some $m_2 = m_2(\mathcal{R}, U_1, U_2) >$

0. Once again, invoking (14.8), we get

$$\sum_{\zeta_1} \hat{\mu}(\tilde{X}_i^{\zeta_1}) \geq \beta_* \hat{\mu}(X_i) \quad \text{and} \quad \sum_{\zeta_2} \hat{\mu}(\tilde{X}_j^{\zeta_2}) \geq \beta_* \hat{\mu}(X_j)$$

Finally, we obtain

$$\hat{b}_{i,j} \geq \frac{e^{-12b_4}\kappa_1\kappa_2\beta_*^2}{100c_{\hat{\mu}}^2\nu^2(R_0)}$$

The right hand side of this bound is the constant γ_0 for Assumption F.

Thus, we obtain $\hat{b}_{i,j} \geq \gamma_0$ for *all* pairs i, j. For all $H > 1$ we set $m'_H = \max\{m_0, m_1, m_2\}$ (independently of H). Then Assumption F is proved, and, moreover, we get $\hat{Q}(\gamma_0) = 0$. As it was promised, we did prove something more than Assumption F.

17 Smooth perturbations of Anosov flows

Here we sketch a proof of Theorem 1.2. The following theorem [1] establishes the so called structural stability of Anosov flows:

Theorem 17.1 *Let $\phi^t : M \to M$ be an Anosov flow. Then for any other flow $\phi_1^t : M \to M$ close to ϕ^t in C^1 metric, there is a homeomorphism $\Psi : M \to M$, close to identity in C^0 metric, which takes (directed) orbits of ϕ^t to (directed) orbits of ϕ_1^t.*

Note that Ψ is not a conjugacy of the two flows, because it need not preserve the parametrization of trajectories. The homeomorphism Ψ can be chosen to be Hölder continuous [20].

Any flow ϕ_1^t close to an Anosov flow ϕ^t in C^1 metric[7] will be also an Anosov flow [1], for which the constants λ_ϕ and C_ϕ in (A2) can be chosen the same as for ϕ^t, see also [8]. Based on these general theorems and mere definitions (A1)-(A2), it is easy to show that the Anosov splitting (8.1) and all the local stable and unstable leaves and fibers depend uniformly continuously on the flow ϕ^t (in C^0 metric for fibers and C^1 metric for flows).

[7] The distance between two flows is defined to be the distance in C^1 metric between their velocity vector fields.

It is then easy to verify that all our constructions and parameters in Sections 9-16 depend continuously on the flow ϕ^t. We only mention a few crucial points.

The Markov family $\mathcal{R}$ can be built on the same disks $D_1, \dots, D_I$ for all flows C^1 close to ϕ^t. The rectangles $R_i \in \mathcal{R}$ depend on the flow ϕ^t continuously (in, say, the Hausdorff metric), and then the symbolic dynamics $\sigma : \Sigma_A \to \Sigma_A$ will be the same for all flows close to ϕ^t, see also [7]. The space $\mathcal{M}$ will then change continuously (in, again, the Hausdorff metric) with the flow ϕ^t. The parameters of the suspension flow $\Phi^t : \mathcal{M} \to \mathcal{M}$, listed in Theorem 4.1, will then depend continuously on ϕ^t.

The Gibbs measure $\nu_\Sigma = \pi^{-1} \circ \nu$ on Σ_A depends continuously on ϕ^t in the weak topology of measures [7]. The potential function $g \circ \pi$ and the ceiling function $l \circ \pi$ on Σ_A are uniformly continuous in ϕ^t, see [7]. Hence, our parameters b_i, λ_i, d_i and m_g, b_g in Sect. 10-15 depend continuously on ϕ^t.

Our constructions in Section 13 are performed in a small ball $B_0 \subset M$, which is supposed to be independent of ϕ^t, see a footnote in Introduction. The parameters r_0, r_1, r_2 are also independent of ϕ^t, and the domains $Y, U^1_{r_2}, U^2_{r_2}$ depend on ϕ^t continuously. Then κ_1, κ_2 will depend on ϕ^t continuously.

The parameters β_U and t_U is Sect. 12 depend continuously on ϕ^t, but not uniformly in U. However, Lemma 12.1 is only applied in Section 16 to just two specific domains, U_1 and U_2, both depending on ϕ^t continuously. Thus, the values of β_* and t_* in Sect. 16 will depend on ϕ^t continuously as well.

Summarizing, we conclude that all the parameters affecting the values of a and c in Theorem 7.2 depend continuously on the flow ϕ^t. Theorem 1.2 is proved.

Another interesting question is whether our assumptions (A1)-(A5) will hold for any flow ϕ^t_1 close to ϕ^t in C^1 metric. As we already mentioned, ϕ^t_1 has to be an Anosov flow, so that (A1) and (A2) will hold. If ϕ^t is topologically transitive, then so is ϕ^t_1 due to Theorem 17.1.

Proposition 17.2 *If the Anosov flow ϕ^t is of codimension one (i.e., $\dim \mathcal{E}^u_y = 1$ or $\dim \mathcal{E}^s_y = 1$) and topologically mixing, then so is any flow ϕ^t_1 close to ϕ^t in C^1 metric.*

Proof. It follows from [24] that the (strongly) stable and unstable foliations for the flow ϕ^t are not jointly integrable[8], cf. Section 13. Hence, $\tau_y(\delta) \neq 0$ for some $y \in M$ and small $\delta > 0$. Since local stable and unstable fibers depend continuously on the flow ϕ^t (in C^0 metric), so does the value of $\tau_y(\delta)$. Hence, $\tau_y(\delta) \neq 0$ for any flow ϕ_1^t close to ϕ^t, so that the stable and unstable foliations of ϕ_1^t are not jointly integrable either. (If $\dim M > 3$, the argument is essentially the same.) It now follows from [24] that ϕ_1^t is topologically mixing. Proposition 17.2 is proved.

Thus, our assumptions (A1)-(A4) are stable under smooth perturbations of flows (for (A4), this is proved for codimension one flows). Assumption (A5) apparently is not stable under perturbations.

18 Contact Anosov flows

Here we discuss an important class of Anosov flows, which includes geodesic flows on compact surfaces of negative curvature.

Let $\phi^t : M \to M$ be a C^2 Anosov flow, $\dim M = 3$. Assume that the Anosov splitting (8.1) is C^1 smooth. Then the family of tangent 2-planes $\mathcal{E}_y^u \oplus \mathcal{E}_y^s$ in $\mathcal{T}M$ is C^1 smooth (the converse is also true, see [20]).

In this case let ω be a C^1 smooth ϕ^t-invariant 1-form in $\mathcal{T}M$, such that its kernel is $\mathcal{E}_y^u \oplus \mathcal{E}_y^s$ and $\omega(v_y) = 1$, where $v_y = d\phi^t/dt$. Denote by $d\omega$ the exterior derivative of ω, it is a continuous ϕ^t-invariant 2-form.

Definition. The flow ϕ^t is called an Anosov contact flow if the 3-form $\omega \wedge d\omega$ is not degenerate. The form ω is then called the contact form of ϕ^t. The bundle of planes $\mathcal{E}_y^u \oplus \mathcal{E}_y^s$ is then called a contact structure [4].

Theorem 18.1 *Let ϕ^t be a C^2 Anosov flow on a 3-D compact manifold with C^1 smooth Anosov splitting. Then*
(i) *ϕ^t is topologically transitive;*
(ii) *ϕ^t is topologically mixing iff it is contact.*
In the second case the 3-form $\omega \wedge d\omega$ gives an absolutely continuous ϕ^t-invariant measure, and our assumption (A5) holds true.

[8] This was also conjectured in [24] for all transitive Anosov flows. If this is true, Proposition 17.2 will cover all those flows as well.

Proof. The part (i) was proved by Plante [24]. He also proved that ϕ^t is topologically mixing iff the family of the tangent planes $\mathcal{E}^u_y \oplus \mathcal{E}^s_y$ is not integrable, i.e. it is not a tangent bundle to a C^1 foliation of M by surfaces. By Hartman's theorem [24], this is equivalent to $\omega \wedge d\omega \neq 0$, thus we get (ii). Plante also proved [24] that in this case $\omega \wedge d\omega$ determines a ϕ^t-invariant measure equivalent to the Riemannian volume on M.

It remains to verify (A5). It is proved in [19] that for contact Anosov flows we have

$$\lim_{\delta\to 0} \tau_y(\delta)/\delta^2 = \pm d\omega(v^u_y, v^s_y) \neq 0 \tag{18.1}$$

where $v^u_y \in \mathcal{E}^u_y$ and $v^s_y \in \mathcal{E}^s_y$ are unit vectors ($\pm$ corresponds to the choice of orientation). The value of $d\omega(v^u_y, v^s_y)$ depends on y continuously, thus (A5) follows. Theorem 18.1 is proved.

Therefore, in the case of C^1 Anosov splitting, our (A3) holds automatically, and (A4) is equivalent to (A5) and both are equivalent to the contact property. Thus, for the flows with C^1 Anosov splitting the correlations either decay fast, as in (1.2), or do not decay at all.

For generic C^2 Anosov flows, the limit in (18.1) need not exist. In that case our (A5) generalizes contactness. It is perhaps reasonable to call Anosov flows satisfying (A1)-(A5) *generalized* contact Anosov flows.

It is well known that geodesic flows on C^∞ compact surfaces with negative curvature are contact Anosov flows. Indeed, they are Anosov [1], their Anosov splittings are C^1 [18], and they are topologically mixing by Arnold's theorem, see an extensive discussion in [1]. Thus, these flows satisfy our assumptions (A1)-(A5), and Theorem 1.1 applies. Theorem 1.2 then covers small perturbations of geodesic flows, which preserve their contact structure.

Appendix

Here we provide the proofs of Theorems 4.1, 5.1, 5.2 and 5.3.

A.1. We start with a lemma on generalized Hölder continuous functions.

Lemma A.1 *Let M' be a metric space with a Borel probability measure μ' and $f(x) \in GH_\alpha(M')$. Let $r(x) \geq 0$ be an integrable function on M' and $R = \int_{M'} r(x)\, d\mu'(x)$. Then*

$$\int_{M'} \mathrm{osc}_{r(x)}(f,x)\, d\mu'(x) \leq \mathrm{var}_\alpha(f) \cdot R^{\alpha/(\alpha+1)}$$

Proof. For any $\varepsilon > 0$ we obviously have $\mu'\{x \in M' : r(x) \geq R/\varepsilon\} \leq \varepsilon$. Therefore,

$$\int_{M'} \operatorname{osc}_{r(x)}(f,x)\, d\mu'(x) \leq \int_{M'} \operatorname{osc}_{R/\varepsilon}(f,x)\, d\mu'(x) + \varepsilon \cdot \operatorname{osc}(f, M').$$

Setting $\varepsilon = R^{\alpha/(\alpha+1)}$ and utilizing the definitions (2.2) and (2.3) accomplishes the proof.

Since $\operatorname{diam} A(x) = d(x)$, we have $|l(x) - \bar{l}(x)| \leq \operatorname{osc}_{d(x)}(l,x)$. Lemma A.1 and the definition (3.7) now imply that

$$\int_\Omega |l(x) - \bar{l}(x)|\, d\nu(x) \leq \operatorname{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)}. \tag{A.1}$$

Recall that $|\hat{l}(x) - \bar{l}(x)| \leq 2\delta$, and so

$$\int_\Omega |l(x) - \hat{l}(x)|\, d\nu(x) \leq 2\delta + \operatorname{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)}. \tag{A.2}$$

As a result,

$$\begin{aligned} \mu(\mathcal{M} \setminus \hat{\mathcal{M}}) + \hat{\mu}(\hat{\mathcal{M}} \setminus \mathcal{M}) &\leq (c_\mu + c_{\hat{\mu}}) \cdot \int_\Omega |l(x) - \hat{l}(x)|\, d\nu(x) \\ &\leq 3c_\mu \left(2\delta + \operatorname{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)}\right). \end{aligned} \tag{A.3}$$

Fix a $t > 0$ and set $N = [t/\delta]$. We will compare the maps Φ^t on $\mathcal{M}$ and $\hat{\Phi}^{\delta N}$ on $\hat{\mathcal{M}}$. For any $y = (x,s) \in \mathcal{M}$ let $J_t(y)$ be the number of times the trajectory $\Phi^s y$, $0 \leq s \leq t$, crosses the base Ω. Assumption L2 ensures that

$$J_t(y) \leq ([t/t_0] + 1)m_0 \tag{A.4}$$

For any $y = (x,s) \in \mathcal{M}$ let $\Delta l(y) = |l(x) - \hat{l}(x)|$ and

$$\Delta_t l(y) = \Delta l(x) + \Delta l(Tx) + \cdots + \Delta l(T^{J_t(y)-1}x) + \delta. \tag{A.5}$$

Now denote the point $\Phi^t y$ by $y_t = (x_t, s_t)$ and define

$$\hat{\Delta}_t l(y) = \begin{cases} \Delta_t l(y) & \text{if } \Delta_t l(y) < \min\{s_t, \hat{l}(x_t) - s_t\} \\ \operatorname{diam} \mathcal{M} & \text{otherwise} \end{cases} \tag{A.6}$$

A direct inspection shows that for any point $y \in \mathcal{M} \cap \hat{\mathcal{M}}$ we have $\rho(\Phi^t y, \hat{\Phi}^{\delta N} y) \le \hat{\Delta}_t l(y)$, and so

$$|F(\Phi^t y) - F(\hat{\Phi}^{\delta N} y)| \le \mathrm{osc}_{\hat{\Delta}_t l(y)}(F, \Phi^t y)$$

Notice that the point $\hat{\Phi}^{\delta N} y$ may lie in $\hat{\mathcal{M}} \setminus \mathcal{M}$, where the function F is set to zero.

A.2. We are now in a position to prove Theorem 4.1. First, we have

$$\begin{aligned} \int_{\mathcal{M}} F(\Phi^t y) G(y)\, d\mu(y) &= \int_{\mathcal{M} \cap \hat{\mathcal{M}}} F(\Phi^t y) G(y)\, d\mu(y) \\ &+ \int_{\mathcal{M} \setminus \hat{\mathcal{M}}} F(\Phi^t y) G(y)\, d\mu(y) \end{aligned}$$

The latter integral does not exceed const$\cdot ||F||_\infty ||G||_\infty \cdot (\delta + D^{\alpha_l/(\alpha_l+1)})$ with const$= 3c_\mu(2 + \mathrm{var}_{\alpha_l}(l))$, according to (A.3). Next,

$$\begin{aligned} \int_{\mathcal{M} \cap \hat{\mathcal{M}}} F(\Phi^t y) G(y)\, d\mu(y) &= \int_{\mathcal{M} \cap \hat{\mathcal{M}}} F(\hat{\Phi}^{\delta N} y) G(y)\, d\mu(y) \\ &+ \int_{\mathcal{M} \cap \hat{\mathcal{M}}} (F(\Phi^t y) - F(\hat{\Phi}^{\delta N} y)) G(y)\, d\mu(y) \end{aligned} \quad \text{(A.7)}$$

The latter integral does not exceed

$$||G||_\infty \int_{\mathcal{M} \cap \hat{\mathcal{M}}} |F(\Phi^t y) - F(\hat{\Phi}^{\delta N} y)|\, d\mu(y) \le ||G||_\infty \int_{\mathcal{M}} \mathrm{osc}_{\hat{\Delta}_t l(y)}(F, \Phi^t y)\, d\mu(y)$$

$$= ||G||_\infty \int_{\mathcal{M}} \mathrm{osc}_{\hat{\Delta}_t l(\Phi^{-t} y)}(F, y)\, d\mu(y) \le ||G||_\infty \mathrm{var}_\alpha(F) \left(\int_{\mathcal{M}} \hat{\Delta}_t l(y)\, d\mu(y) \right)^{\frac{\alpha}{\alpha+1}}$$

(at the last step we use Lemma A.1 and the invariance of the measure μ under the flow Φ^t). In order to estimate the integral in the last bound, we will integrate the equation (A.5). First, for any $i \ge 0$

$$\begin{aligned} \int_{\mathcal{M}} \Delta l(T^i x)\, d\mu(y) &= c_\mu \int_\Omega l(x) \cdot \Delta l(T^i x)\, d\nu(x) \\ &\le c_\mu l_{\max} \int_\Omega \Delta l(T^i x)\, d\nu(x) \end{aligned}$$

where $l_{\max} = \max_\Omega l(x)$. Notice that $c_\mu l_{\max} \le c_\mu (c_\mu^{-1} + \mathrm{osc}\,(l, \Omega)) = 1 + c_\mu \cdot \mathrm{osc}\,(l, \Omega)$. By virtue of (A.2) we get

$$\int_{\mathcal{M}} \Delta l(T^i x)\, d\mu(y) \le [1 + c_\mu \cdot \mathrm{var}_{\alpha_l}(l)] \cdot \left[2\delta + \mathrm{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)} \right].$$

We now integrate (A.5) by using the bound (A.4) and get

$$\int_{\mathcal{M}} \Delta_t l(y)\, d\mu(y) \le \text{const} \cdot t(\delta + D^{\alpha_l/(\alpha_l+1)})$$

where const$= 2m_0 t_0^{-1}(2 + (1 + c_\mu)\text{var}_{\alpha_l}(l))^2$.

The definition of $\hat{\Delta}_t l(y)$ by (A.6) involves the function $u_t(y) := \min\{s_t, \hat{l}(x_t) - s_t\}$. Let $u_t^+(y) := \max\{u_t(y), 0\}$. Notice that $u_t(y) < 0$ iff $\Phi^t(y) \in \mathcal{M} \setminus \hat{\mathcal{M}}$. It is an easy calculation that $\mu\{y \in \mathcal{M} :\, u_t^+(y) < r\} \le 2c_\mu r + \mu(\mathcal{M} \setminus \hat{\mathcal{M}})$ for all $r > 0$. It then follows that for any subset $B \subset \mathcal{M}$ such that $\mu(B) > \mu(\mathcal{M} \setminus \hat{\mathcal{M}})$ we have

$$\int_B u_t^+(y)\, d\mu(y) \ge (4c_\mu)^{-1}[\mu(B) - \mu(\mathcal{M} \setminus \hat{\mathcal{M}})]^2$$

Therefore, for any subset $B \subset \mathcal{M}$ we have

$$\mu(B) \le 2\left[c_\mu \int_B u_t^+(y)\, d\mu(y)\right]^{1/2} + \mu(\mathcal{M} \setminus \hat{\mathcal{M}})$$

We now obtain that

$$\begin{aligned} \mu\{y \in \mathcal{M} :\, \hat{\Delta}_t l(y) \ne \Delta_t l(y)\} &= \mu\{y \in \mathcal{M} :\, \Delta_t l(y) \ge u_t^+(y)\} \\ &\le 2\left(c_\mu \int_{\mathcal{M}} \Delta_t l(y)\, d\mu(y)\right)^{1/2} + \mu(\mathcal{M} \setminus \hat{\mathcal{M}}) \end{aligned}$$

and so

$$\begin{aligned} \int_{\mathcal{M}} \hat{\Delta}_t l(y)\, d\mu(y) &\le \int_{\mathcal{M}} \Delta_t l(y)\, d\mu(y) + \operatorname{diam} \mathcal{M} \cdot \mu\{y :\, \hat{\Delta}_t l(y) \ne \Delta_t l(y)\} \\ &\le \text{const} \cdot t(\delta + D^{\alpha_l/(\alpha_l+1)})^{1/2} \end{aligned}$$

with const$= 10m_0 t_0^{-1}(1 + c_\mu)(1 + \operatorname{diam} \mathcal{M})(2 + (1 + c_\mu)\text{var}_{\alpha_l}(l))^2$. Thus, the last integral in (A.7) is properly bounded. The first integral in the RHS of (A.7) equals

$$\frac{c_\mu}{c_{\hat{\mu}}} \int_{\hat{\mathcal{M}}} F(\hat{\Phi}^{\delta N} y) G(y)\, d\hat{\mu}(y)$$

It differs from

$$\int_{\hat{\mathcal{M}}} F(\hat{\Phi}^{\delta N} y) G(y)\, d\hat{\mu}(y)$$

by less than $2c_\mu ||F||_\infty ||G||_\infty \delta$ due to (4.2).

Finally, observe that, since $\int_{\mathcal{M}} F(y)\,d\mu(y) = 0$, we have

$$\left|\int_{\hat{\mathcal{M}}} F(y)\,d\hat{\mu}(y)\right| \leq 2||F||_\infty \cdot \mu(\mathcal{M} \setminus \hat{\mathcal{M}})$$

A similar bound holds for G, and then (A.3) gives

$$\left|\int_{\hat{\mathcal{M}}} F(y)\,d\hat{\mu}(y) \cdot \int_{\hat{\mathcal{M}}} G(y)\,d\hat{\mu}(y)\right| \leq ||F||_\infty ||G||_\infty \cdot 36 c_\mu^2 (2+\mathrm{var}_{\alpha_l}(l))^2 \cdot \left(\delta + D^{\alpha_l/(\alpha_l+1)}\right)^2$$

Theorem 4.1 is now proved.

A.3. We now turn to the proof of Theorem 5.1 and consider the quantity (5.3). First, notice that if X_{i_0} is not a bottom atom, then both conditional measures in (5.3) are either equal to one or not defined (in which case we set them to zero, recall the remark after equation (3.3)). Hence, non-bottom atoms X_{i_0} do not contribute to the value of $\hat{\chi}_N$. Now consider an arbitrary bottom atom, $X_{i_0} = A_{j_0} \times [0, \delta)$. Both conditional measures in (5.3) are now defined only if $X_{i_{-1}}$ is a top atom. Let $A_{j_{-1}} = A(X_{i_{-1}})$. Observe that now $\hat{\mu}(X_{i_0}/\hat{T}X_{i_{-1}}) = \nu(A_{j_0}/TA_{j_{-1}})$. Furthermore, the intersection $\hat{T}X_{i_{-1}} \cap \cdots \cap \hat{T}^n X_{i_{-n}}$ in (5.3) has a nonzero measure only if the chain of atoms $X_{i_{-1}}, \dots, X_{i_{-n}}$ has the following structure. The top atom $X_{i_{-1}}$ goes first, it is followed by the whole column of atoms down to $A(X_{i_{-1}}) \times [0, \delta)$, then goes another column of atoms over some $A_{j_{-2}}$ (listed from top to bottom), etc. The last atom in the chain, $X_{i_{-n}}$, is not necessarily a top or bottom one, but it must terminate a subcolumn of atoms going down from a top one. We denote by $A_{j_{-1}}, \dots, A_{j_{-k}}$ the atoms of $\mathcal{A}$ over which the above columns (including the last subcolumn) are situated, listed in the above order. Now, a direct inspection shows that

$$\hat{\mu}(X_{i_0}/\hat{T}X_{i_{-1}} \cap \cdots \cap \hat{T}^n X_{i_{-n}}) = \nu(A_{j_0}/TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}})$$

and

$$\hat{\mu}(\hat{T}X_{i_{-1}} \cap \cdots \cap \hat{T}^n X_{i_{-n}}) = c_{\hat{\mu}} \delta \nu(TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}})$$

Here the value of k depends on n and on atoms $A_{j_{-1}}, A_{j_{-2}}, \dots$, and it is uniquely defined by the inequalities

$$\sum_{m=1}^{k-1} \hat{l}(A_{j_{-m}}) < n\delta \leq \sum_{m=1}^{k} \hat{l}(A_{j_{-m}}) \tag{A.8}$$

We now turn back to the equation (5.3) and rewrite it as follows:

$$\hat{\chi}_N = \sup_{n\le N} \sum_{k,j_0,j_{-1},\dots,j_{-k}} |\nu(A_{j_0}/TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}) - \nu(A_{j_0}/TA_{j_{-1}})|$$

$$\times c_{\hat{\mu}} \delta \nu(TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}) \qquad \text{(A.9)}$$

where the summation is taken over all the $k, A_{j_0}, A_{j_{-1}}, \dots, A_{j_{-k}}$ that satisfy the constraints (A.8). The summation in the RHS of (A.9) is performed in a different way compared to that of (3.3), because k is variable in the former and n is constant in the latter. We will eliminate this difference and reduce (A.9) to (3.3). Observe that, in virtue of Assumption L2, for any $x \in \Omega$ and all $k \ge 1$ we have

$$\sum_{m=1}^{k-1} l(T^{-m}x) \ge \left[\frac{k-1}{m_0+1}\right] t_0 \ge \left[\frac{k-1}{2m_0}\right] t_0 \qquad \text{(A.10)}$$

The same bound with the function $\hat{l}$ instead of l is not necessarily true. However, the following weaker bound involving the function $\hat{l}$ will be established on an 'ample' set of points $x \in \Omega$:

Lemma A.2 *For any $k \ge 1$ we have*

$$\nu\left\{x \in \Omega : \sum_{m=1}^{k-1} \hat{l}(T^{-m}x) \ge \left[\frac{k-1}{4m_0}\right] t_0\right\} \ge 1 - 8m_0 t_0^{-1} \mathrm{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)} \qquad \text{(A.11)}$$

Proof. It is enough to prove (A.11) for the function $\bar{l}$ instead of $\hat{l}$ since $\hat{l}(x) \ge \bar{l}(x)$ for every point $x \in \Omega$. Due to (A.1) we have

$$\int_\Omega \left| \sum_{m=1}^{k-1} \bar{l}(T^{-m}x) - \sum_{m=1}^{k-1} l(T^{-m}x) \right| d\nu(x) \le (k-1) \cdot \mathrm{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)}$$

Together with (A.10) this completes the proof of Lemma A.2.

Since the function $\hat{l}$ is constant on the atoms of $\mathcal{A}$, the sum $\sum_{m=1}^{k-1} \hat{l}(T^{-m}x)$ is constant on any intersection $TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}$. According to (A.11), for an ample collection of such intersections we have

$$\sum_{m=1}^{k-1} \hat{l}(A_{j_{-m}}) \ge \left[\frac{k-1}{4m_0}\right] t_0$$

Along with (A.8), this implies $k \leq [4m_0 t_0^{-1} n\delta] + 4m_0$, and thus

$$k \leq K := [8m_0 t] \tag{A.12}$$

(recall that $t_0 \leq 1$). Then (A.11) implies the following 'tail bound':

$$\sum_{k>K, j_0, j_{-1}, \ldots, j_{-k}} \nu(TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}) \leq 8m_0 t_0^{-1} \mathrm{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)} \tag{A.13}$$

where the summation is taken, again, over $k, A_{j_{-1}}, \ldots, A_{j_{-k}}$ that satisfy (A.8), with an additional restriction $k > K$. Therefore, the total contribution to the value of $\hat{\chi}_N$ in (A.9) of all the 'tail' terms involved in (A.13), i.e. those for which $k > K$, does not exceed the value of the RHS of (A.13) times $c_{\hat{\mu}}\delta$. All the terms in (A.9) with $k \leq K$ make the following contribution to $\hat{\chi}_N$:

$$\begin{aligned}
\hat{\chi}_N^{(\text{main})} &:= \sup_{n \leq N} \sum_{k \leq K, j_0, j_{-1}, \ldots, j_{-k}} |\nu(A_{j_0}/TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}) - \nu(A_{j_0}/TA_{j_{-1}})| \\
&\quad \times c_{\hat{\mu}}\delta\nu(TA_{j_{-1}} \cap \cdots \cap T^k A_{j_{-k}}) \\
&\leq \sup_{n \leq N} \sum_{j_0, j_{-1}, \ldots, j_{-K}} |\nu(A_{j_0}/TA_{j_{-1}} \cap \cdots \cap T^K A_{j_{-K}}) - \nu(A_{j_0}/TA_{j_{-1}})| \\
&\quad \times c_{\hat{\mu}}\delta\nu(TA_{j_{-1}} \cap \cdots \cap T^K A_{j_{-K}}) \\
&\leq c_{\hat{\mu}}\delta\chi_K
\end{aligned} \tag{A.14}$$

Here the first sum is taken over all $k \leq K, j_0, j_{-1}, \ldots, j_{-k}$ that satisfy (A.8), and the second sum is taken over all $j_0, j_{-1}, \ldots, j_{-K}$ with the value of K specified by (A.12). The first inequality allowing the transition from the first sum, with a variable k, to the second one with a bigger but constant K, is based on the following simple trick (valid for any Borel subsets $B, C, D \subset \Omega$):

$$\begin{aligned}
|\nu(B/C \cap D) - \nu(B/C)| \cdot \nu(C \cap D) &= |\nu(B \cap C \cap D) - \nu(B/C)\nu(C \cap D)| \\
&= \left| \sum_i \nu(B \cap C \cap D \cap A_i) - \nu(B/C) \sum_i \nu(C \cap D \cap A_i) \right| \\
&\leq \sum_i |\nu(B \cap C \cap D \cap A_i) - \nu(B/C)\nu(C \cap D \cap A_i)| \\
&= \sum_i |\nu(B/C \cap D \cap A_i) - \nu(B/C)| \cdot \nu(C \cap D \cap A_i)
\end{aligned}$$

This trick allows us to 'extend' the conditions of the conditional measures in the first sum of (A.14). Combining (A.13) with (A.14) gives the following bound:

$$\hat{\chi}_N \le c_{\hat{\mu}}\delta\chi_{[8m_0t]} + 8m_0t_0^{-1}c_{\hat{\mu}}\delta \cdot \mathrm{var}_{\alpha_l}(l) \cdot D^{\alpha_l/(\alpha_l+1)} \tag{A.15}$$

The main statement of Theorem 5.1 is now proved.

In the case when the function $l(x)$ has a positive lower bound, $l_{\min} > 0$, we obviously have $\hat{l}(x) \ge \bar{l}(x) \ge l_{\min}$ for every $x \in \Omega$. Therefore, we can set $m_0 = 1$, $t_0 = l_{\min}$, and the bound (A.10) will also hold for the function $\bar{l}(x)$ instead of $l(x)$. Then (A.12) will be true uniformly (the 'tail' involved in (A.13) will be empty). Thus, we get the second statement of Theorem 5.1 and complete its proof.

A.4. Next, we prove Theorem 5.2. The two middle integrals in the expansion (5.5), combined, are bounded by

$$||G||_\infty \cdot \int_{\hat{\mathcal{M}}} |\Delta F(y)|\, d\hat{\mu}(y) + ||F||_\infty \cdot \int_{\hat{\mathcal{M}}} |\Delta G(y)|\, d\hat{\mu}(y)$$

The last integral in (5.5) does not exceed either of the above two summands. Adopting again the notation $y = (x, s)$ for points of $\hat{\mathcal{M}}$ and invoking Lemma A.1, we obtain

$$\int_{\hat{\mathcal{M}}} |\Delta F(y)|\, d\hat{\mu}(y) \le \int_{\hat{\mathcal{M}}} \mathrm{osc}_{d(x)+\delta}(F, y)\, d\hat{\mu}(y)$$

$$\le \mathrm{var}_\alpha(F) \cdot \left(\delta + \int_{\hat{\mathcal{M}}} d(x)\, d\hat{\mu}(y)\right)^{\alpha/(\alpha+1)} \le \mathrm{var}_\alpha(F) \cdot (\delta + 2l_{\max}D)^{\alpha/(\alpha+1)}$$

The same bound is true for the function G. Theorem 5.2 is proved.

A.5. We now turn to the proof of Theorem 5.3. Obviously,

$$\hat{C}^{(\mathrm{chain})}_{F,G}(N) - C^{(\mathrm{per})}_{F,G}(N) = \sum_i \bar{G}_i\hat{p}_i \sum_j \left[\bar{F}_j\hat{\pi}^{(N)}_{ij} - \bar{F}_j\pi^{(\mathrm{per})}_{ij}\right] \tag{A.16}$$

We now compare the expansions (5.10) and (5.11) carefully. We will call the string $i_1, i_2, \dots, i_L, i_{L+1} = j$ entering (5.10) an *interior* one if

$$L\eta < s(X_{i_r}) < \hat{l}(A(X_{i_r}))/\delta - L\eta$$

for all $r = 1, \dots, L+1$. This means that every atom X_{i_r} in some column of atoms in $\hat{\mathcal{A}}$ is at least $L\eta$ atoms away from both top and bottom of that

column. For any interior string we collect all the strings $j_1, l_1, \dots, j_L, l_L, j_{L+1}$ entering (5.11) such that
(i) $j_1 = i_1$;
(ii) $A(X_{l_r}) = A(X_{j_r})$ and $|s(X_{l_r}) - s(X_{j_r})| \le \eta$ for all $r = 1, \dots, L$;
(iii) $A(X_{j_r}) = A(X_{i_r})$ and

$$s(X_{j_r}) = s(X_{i_r}) + \sum_{u=1}^{r-1} \Big(s(X_{l_u}) - s(X_{j_u}) \Big)$$

for all $r = 2, \dots, L+1$.

For any string satisfying (i)-(iii) we have $\tilde{\pi}_{i_{r-1}i_r} = \tilde{\pi}_{l_{r-1}j_r}$ for all $r = 2, \dots, L$ and $\hat{\pi}^{(K_2+L_0)}_{i_L i_{L+1}} = \hat{\pi}^{(K_2+L_0)}_{l_L j_{L+1}}$, and also $\pi^*_{j_r l_r} = (2\eta+1)^{-1}$ for all $r = 1, \dots, L$. The number of strings satisfying (i)-(iii) is equal to $(2\eta+1)^L$. Besides, the last atoms, $X_{i_{L+1}}$ and $X_{j_{L+1}}$, are in the same column of $\hat{\mathcal{A}}$, and $\operatorname{dist}(X_{i_{L+1}}, X_{j_{L+1}}) \le (L\eta - 1)\delta$. Hence,

$$|\bar{F}_{i_{L+1}} - \bar{F}_{j_{L+1}}| \le \operatorname{osc}_{d(x')+(L\eta+1)\delta}(F, y') \tag{A.17}$$

for any point $y' = (x', s') \in X_{i_{L+1}}$.

We then substitute the expansions (5.10) and (5.11) into (A.16) and take first the sum over the interior strings in (5.10) and their counterparts in (5.11). Due to (A.17), the resulting sum will not exceed

$$||G||_\infty \int_{\hat{\mathcal{M}}} \operatorname{osc}_{d(x)+(L\eta+1)\delta}(F, y)\, d\hat{\mu}(y)$$

By virtue of Lemma A.1 this quantity is bounded by

$$||G||_\infty \operatorname{var}_\alpha(F) \cdot \left(\int_{\hat{M}} [d(x) + (L\eta+1)\delta]\ d\hat{\mu}(y) \right)^{\alpha/(\alpha+1)}$$

$$\le ||G||_\infty \operatorname{var}_\alpha(F) \cdot [2c_\mu l_{\max}(D + (L\eta+1)\delta)]^{\alpha/(\alpha+1)}$$

Lastly, the contribution to (A.16) of all the strings in (5.10) other than interior and the remaining strings in (5.11), combined, does not exceed $4L^2 c_\mu ||F||_\infty ||G||_\infty \eta\delta$.

Theorem 5.3 is proved.

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