MARKOV EXTENSIONS, ZETA FUNCTIONS, AND FREDHOLM THEORY FOR PIECEWISE INVERTIBLE DYNAMICAL SYSTEMS

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ABSTRACT. Transfer operators and zeta functions of piecewise monotonic and of more general piecewise invertible dynamical systems are studied. To this end we construct Markov extensions of given systems, develop a kind of Fredholm theory for them, and carry the results back to the original systems. This yields e.g. bounds on the number of ergodic maximal measures or equilibrium states.

1. INTRODUCTION

A. Background.

Piecewise invertible dynamical systems have attracted considerable attention during the last two decades. They occur in the literature under various names, the most prominent are perhaps: piecewise monotonic transformations [Lasota/Yorke, 1973], f-expansions or number-theoretical algorithms, see [Watermann, 1970] or [Schweiger, 1973] for references, and one-sided subshifts, in particular subshifts of finite type or topological Markov chains, cf. [Denker et al., 1976]. They all can be described as triples $(X, T, \mathscr{Z})$ where X is a topological space, $\mathscr{Z}$ is a finite or countable partition of X, and $T: X \to X$ is a mapping whose restrictions T_Z to elements of $\mathscr{Z}$ are injective.

[Hofbauer, 1986] investigates topological properties of (a restricted class of) such systems. He constructs a topological Markov chain on a countable state space in such a way that the system under consideration is a continuous factor of this chain (in fact, it is a one-block factor). Although the factor map is generally not finite-to-one, it is nice enough to carry over a great part of the chain's topological structure to the underlying system yielding in this way a description of the set of nonwandering points and of the measures of maximal entropy. Such results ignore, however, a possible smooth structure of the space X and a priori ideas about what subsets of X are small in a measure-theoretic sense.

To be definite, consider the family of systems $([0,1), T_p, \mathscr{Z}_p)$, $0 < p < 1$, where $\mathscr{Z}_p = \{[0,p), [p,1)\}$, $T_p(x) = p^{-1}x \ (0 \le x < p)$, $T_p(x) = (1-p)^{-1}(x-p)$ $(p \le x < 1)$. All T_p leave the Lebesgue measure on $[0,1)$ invariant and are

measurably isomorphic to a one-sided Bernoulli-shift with stationary probability vector $(p, 1-p)$. Their metric entropy $p \log p + (1-p) \log(1-p)$ is maximized for $p = \frac{1}{2}$ such that the Lebesgue measure is singular to the measure of maximal entropy if $p \neq \frac{1}{2}$. Nevertheless we are mostly interested in properties of orbits $(T_p^n x)_{n=1,2,\ldots}$ valid for Lebesgue-almost all $x \in [0, 1)$.

This motivates the attempt to investigate extensions $(\hat{X}, \hat{T}, \hat{\mathscr{X}})$ of "smooth" systems which at the same time inherit the "smooth" local structure of the system and show a global Markov structure similar to that exploited by Hofbauer. As a consequence of the Markov structure, "smooth" probability densities on $\hat{X}$ are transformed by the action of $\hat{T}$ into "smooth" densities, provided that $\hat{T}$ itself is "smooth" enough. This allows a systematic study of the spectral properties of the *Perron-Frobenius operator* of $\hat{T}$ acting on spaces of "smooth" functions on $\hat{X}$. To number-theorists this operator is better known as *Kuzmin operator*, and in statistical mechanics generalized versions of it are called *transfer operators*. (I shall adopt this last name.) In a second step these results must be interpreted for the system $(X, T, \mathscr{X})$.

During my work on these problems I learned that the same techniques can be used for exploring equilibrium states on various subshifts. For subshifts of finite type [Mayer, 1980] has done interesting work in this direction (in the context of one-dimensional spin systems with long-range interactions). From his work I learned how to use the theory of nuclear operators for investigating dynamical systems. The results presented here apply to a much broader class of subshifts including sofic systems (see [Coven/Paul, 1975], [Weiss, 1973]) and many coded systems (see [Blanchard/Hansel, 1986]).

B. Organization and results.

§2 begins with a quick review of the spectral theory for quasicompact operators and of Fredholm-determinants for nuclear operators on Banach-spaces. Both concepts are combined in the definition of *quasinuclear operators* (Definition 2.3. Some technical aspects of this definition are motivated by later applications.) The main result for quasinuclear operators P (Theorem 2.5) describes that part of their spectrum that can be characterized by the zeroes of an analytic function associated with P and called the *Fredholm-determinant* of $(\mathrm{Id} - zP)$.

The next three sections are independent of these functional analytic preparations, and the reader might prefer to begin his reading of this paper with definitions, basic facts, and simple examples for *piecewise invertible dynamical systems* and with the construction of *Markov extensions* for given systems (§3), followed by an introduction to *transfer operators* and *(Ruelle-)zeta functions* for *weighted* piecewise invertible systems and their Markov extensions (§4)[1] and

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[1] Very briefly: If $T: X \to X$ is an at most countable-to-one transformation and if $g: X \to \mathbb{C}$ is a "weight function", then $(P_g f)(x) = \sum_{y \in T^{-1}x} f(y) g(y)$ is the associated transfer operator and $\zeta_g(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} g(x) g(Tx) \cdot \ldots \cdot g(T^{n-1}x)\right)$ is the corresponding zeta function.

some considerations on *analytic structures* for such weighted systems (§5) including notions of (generalized) holomorphic functions and expanding systems. §6 is the core of this paper. We investigate transfer operators P_g acting on a space $\mathscr{H}^\infty$ of bounded holomorphic functions: *Theorem 6.1 gives conditions under which transfer operators of Markov extensions of expanding systems are quasicompact on $\mathscr{H}^\infty$ and describes in some detail the isolated eigenvalues of P_g and their associated eigenspaces. Our main result, Theorem 6.2, gives additional conditions under which P_g is quasinuclear and under which (some of) the isolated eigenvalues of P_g are just the inverses of the poles of $\zeta_g(z)$.* This uses heavily Theorem 2.5. Finally, *Proposition 6.10 and Theorem 6.11 relate the peripheral spectrum of P_g to invariant measures on irreducible components of the Markov extension.*

While the zeta function of a weighted piecewise invertible system and that of its Markov extension are often equivalent in the sense that they have the same poles and zeroes in the domain referred to by Theorem 6.2, the relations between the spectral properties of the corresponding transfer operators are more delicate: Proposition 7.1 provides some general criteria for an eigenvalue of the original system to be also an eigenvalue of the extended system. Unfortunately I was not able to obtain a useful, general result in the other direction, except for peripheral eigenvalues (Lemma 7.3).[2] Nevertheless the results from §§6 and 7 show that *for many transfer operators P_g associated to a weighted piecewise invertible system there is $r < r(P_g)$ (= spectral radius of P_g) such that the inverse of each eigenvalue λ of P_g with $|\lambda| > r$ is a pole of the corresponding zeta function with multiplicity equal to the algebraic multiplicity of λ.*

In §8 some of the previous results are translated into the "language" of equilibrium states. For example, *if the greatest positive eigenvalue of the transfer operator $\hat{P}_g$ of the Markov extension of a weighted system is simple, then there is a unique equilibrium state μ for $\log g$ (Theorem 8.3.b), and if (T, μ) is weakly mixing, then the partition $\mathscr{Z}$ is weak Bernoulli under T with an exponential mixing rate (Theorem 8.5).* (Both theorems say a bit more than indicated here.)

In §9 we touch upon several applications of Theorem 6.1: It is pointed out, how the spectral theory for P_g can be used to approach various probabilistic limit theorems via the routes (a) L^2- approximation, (b) strongly mixing stationary processes and functionals of them, (c) Fourier- or Laplace-transforms. *We also relate exponential extinction rates to the pressure of T restricted to a certain "repelling" subsystem (Proposition 9.7) and give a rigorous proof of the fact that long transients behave stochastically just like typical trajectories from the "repelling" subsystem.*

I have chosen three classes of examples in order to illustrate the sometimes rather general definitions and results:

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[2] For a rather special case the reverse implication has been proved in [Hofbauer/Keller, 1984].

(I) *Topological Markov shifts, sofic systems* [3]

These are among the simplest piecewise invertible systems, and they are included only, because everything reduces to ideas well-known to most readers.

(II) *Up-and-down-counters* [4]

This is a class of subshifts which are neither of finite type nor sofic (i.e. homomorphic images of subshifts of finite type). Using a rather explicit formula for their zeta function $\zeta_1(z)$ we show e.g. *that they have unique maximal measures* (8.7).

(III) *Piecewise analytic and expanding interval transformations* [5]

These are maps $\tilde{T} : [0,1] \to [0,1]$ for which there is a partition $0 = a_0 < a_1 < \cdots < a_N = 1$ such that each $\tilde{T}_{|(a_{i-1},a_i)}$ is monotone and extends holomorphically to a map $\tilde{T}_i$ defined on a complex neighbourhood of $[a_{i-1}, a_i]$ and satisfying $|\tilde{T}'_i| \geq \alpha > 1$. If $\phi : [0,1] \to \mathbb{C}$ is such that each $\phi_{|(a_{i-1},a_i)}$ extends holomorphically to the same neighbourhood as the corresponding $\tilde{T}_i$ and if $g = \exp \phi$,

$$g_\infty = \lim_{n \to \infty} \left(\sup_x |g(x) \cdot g(Tx) \cdot \ldots \cdot g(T^{n-1}x)| \right)^{1/n},$$

then $\zeta_g(z)$ extends to a meromorphic function on $\{|z| < g_\infty^{-1}\}$, and λ^{-1} is a pole of ζ_g if and only if λ is an eigenvalue of $\hat{P}_g$, the transfer operator of the Markov extension of $\tilde{T}$. In this case, the multiplicities of λ^{-1} as a pole and of λ as an eigenvalue coincide. If $g = 1/|\tilde{T}'|$, i.e. if P_g is the classical Perron-Frobenius operator, then each eigenvalue λ of P_g acting on $BV[0,1]$ [6] that satisfies $|\lambda| > g_\infty^{1/2}$ is the inverse of a pole of ζ_g. The multiplicity of λ as an eigenvalue is less than or equal to the multiplicity of λ^{-1} as a pole (see (7.7)). In this case it is also true that all equilibrium states for $\log g$ are absolutely continuous with respect to Lebesgue measure, and the number of ergodic equilibrium states equals the multiplicity of the eigenvalue 1 of P_g acting on $BV[0,1]$ (cf. 7.7). This number, in turn, coincides with the multiplicity of the pole at 1 of $\zeta_g(z)$ (cf. 6.14 and 7.7). An informal discussion of the Fredholm theory for transfer operators of interval maps can be found in [Oono/Takahashi, 1980].

C. Discussion.

I feel that the rather heavy machinery used for the description of the general setting and the proof of the main result needs some additional justification:

(1) Analyticity results for Fredholm determinants and for the inverses of zeta functions have been obtained so far only for very simple dynamics (topological Markov shifts over a finite alphabet) but quite general g, see e.g. [Ruelle,

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[3] See 3.12, 4.8, 5.9, 6.12, 7.5.
[4] See 3.13, 4.9, 5.9, 6.13, 7.6, 8.7.
[5] See 3.15, 4.10, 5.10, 6.14, 7.7, 8.8.
[6] The space of functions of bounded variation on $[0,1]$.

1976] or [Pollicott, 1986], and, on the other hand, for more general dynamics (piecewise monotonic interval maps) but rather simple g (piecewise constant), see e.g. [Takahashi, 1981], [Hofbauer/Keller,1984], or [Mori, 1985]. The combination of more complex dynamics and more general g necessitates a theorem like 2.5 and justifies the use of holomorphic functions in 6.2. It seems possible, however, that a clever combination of Pollicott's arguments with Theorem 2.5 could yield similar results for P_g acting on spaces of Lipschitz-continuous functions or on spaces of functions of bounded variation.

(2) The general framework introduced in §§3 and 4 allows to treat a class of systems much broader than just interval transformations (which have a very special Markov extension). This is illustrated by the subshifts in Example II.

(3) Even for interval transformations, and even if one is not interested in zeta functions, the use of Markov extensions and of holomorphic functions has its merits: It seems quite difficult to obtain rigorous results about extinction rates and transient behaviour as presented in §9.C by studying transfer operators acting on spaces of functions of bounded variation, cf. [Keller, 1984].

(4) All results apply to higher-dimensional systems, too, but it seems to be nontrivial to verify the assumptions of Theorems 6.1 and 6.2 for fairly general classes of such systems. The complex β-transformation that maps the unit square (regarded as a subset of the complex plane) to itself by $z \mapsto \beta z \bmod \mathbb{Z}^2$ ($|\beta| > 1$) should be a good candidate to test ones ideas for more general two-dimensional piecewise analytic and expanding maps. It already turned out to be tractable from a related point of view in [Keller, 1979].

2. Quasicompact and quasinuclear operators

A. Quasicompact operators.

Throughout this section P is a linear operator on some complex Banach-space $(B, \|.\|)$. Generalizing Definition V-3-1 and Lemma V-3-1 in [Neveu, 1964], we say that P is *quasicompact* if there is $k \in \mathbb{N}$ and a compact operator $K: B \to B$ such that $\|P^k - K\| < r(P)^k$ where $r(P)$ is the spectral radius of P. The existence of the limit

$$(2.1) \qquad r_K = \lim_{n \to \infty} (\inf\{\|P^n - K\| : K \text{ a compact operator on } B\})^{1/n}$$

is trivial. Obviously $r_K < r(P)$, and it follows from Lemma VIII.8.2 in [Dunford/Schwartz, 1958] that every spectral point λ of P with $|\lambda| > r_K$ is isolated in the spectrum $\sigma(P)$ of P and that the corresponding eigenprojection E_λ has a finite-dimensional range. Thus we obtain the following restricted spectral representation for a quasicompact operator P (cf. [Kato, 1976], III.6.5):

For any $r > r_K$ there are finitely many spectral values $\lambda_1, \dots, \lambda_N$ with $|\lambda_i| > r$. All of them are eigenvalues, and P can be decomposed as

$$(2.2) \qquad P = \sum_{i=1}^{N} \lambda_i (E_i + N_i) + Q$$

where E_i is the finite-dimensional eigenprojection corresponding to λ_i, $N_i = (P - \lambda_i \,\mathrm{Id})E_i$ its eigennilpotent[7], and Q is a linear operator with $r(Q) \leq r$. Furthermore $E_i E_j = 0$ (for $i \neq j$), $E_i N_i = N_i E_i = E_i$ (for all i), and $E_i Q = Q E_i = 0$ (for all i).

In particular, all λ_i are poles of the resolvent of P, and λ_i is a pole of order ν_i if and only if $(P - \lambda_i \,\mathrm{Id})^{\nu_i} E_i = 0$ but $(P - \lambda_i \,\mathrm{Id})^{\nu_i - 1} E_i \neq 0$ (Theorem VII.3.18 in [Dunford/Schwartz, 1958]).

With this notation we obtain a representation for P^n:

$$(2.3) \qquad P^n = \sum_{i=1}^{N} \lambda_i^n E_i \left(\sum_{k=0}^{\nu_i - 1} \binom{n}{k} N_i^k \right) + Q^n.$$

For $T: B \to B$ let $\ker(T) = \{x \in B: Tx = 0\}$ and $\mathrm{range}(T) = \{Tx: x \in B\}$. Theorem VII.3.24 in [Dunford/Schwartz, 1958] implies

$$\ker(P - \lambda_i \,\mathrm{Id})^{\nu_i} = \mathrm{range}(E_i) \simeq B/\mathrm{range}(\mathrm{Id} - E_i),$$

$$\mathrm{range}(P - \lambda_i \,\mathrm{Id})^{\nu_i} = \mathrm{range}(\mathrm{Id} - E_i)$$

for all λ_i. In particular $(P - \lambda_i \,\mathrm{Id})$ is a Fredholm operator.

B. The essential spectrum.

The above spectral decomposition motivates the following definition: For a bounded linear operator P on B the *nonessential spectrum* is the set of all isolated spectral points λ of P for which $\mathrm{range}(P - \lambda \,\mathrm{Id})$ is closed and $\bigcup_{n \geq 0} \ker((P - \lambda \,\mathrm{Id})^n)$ is finite-dimensional. Its complement in $\sigma(P)$ is just the *essential spectrum* $\mathrm{ess}(P)$ of P in the sense of [Browder, 1961, Definition 11]. The essential spectral radius is defined as

$$(2.4) \qquad r_{\mathrm{ess}} = \sup\{|\lambda|: \lambda \in \mathrm{ess}(P)\}.$$

From the results listed in part A it is obvious that for quasicompact operators $r_{\mathrm{ess}} \leq r_K$, and, using "measures of noncompactness", [Nussbaum, 1970] proved that for arbitrary bounded linear operators

$$(2.5) \qquad r_{\mathrm{ess}} = r_K$$

holds.

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[7] Id denotes the identity mapping on B.

C. Nuclear operators.

Denote by B' the space of bounded linear functionals on B. According to [Grothendieck, 1956, Chapter II.1], a linear operator P on B is *nuclear* (or of *trace class*), if it can be represented as

$$(2.6) \qquad P = \sum_{i=1}^{\infty} \alpha_i\, y_i' \otimes x_i$$

where $x_i \in B$, $y_i' \in B'$, $\|x_i\|, \|y_i'\| \le 1$, and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$. $y_i' \otimes x_i$ represents the operator on B mapping x to $y_i'(x) \cdot x_i$. The *trace norm* $\|P\|_T$ is the infimum of all $\sum_{i=1}^{\infty} |\alpha_i|$, the α_i arising from a representation (2.6) of P.

Observe that rigorously the right-hand side of (2.6) is an element of the norm-completion $B' \hat{\otimes} B$ of $B' \otimes B$ (ibid.), and in general the same operator P can be represented by different elements from $B' \hat{\otimes} B$. However, if P is p-summing for some $0 < p \le 2/3$, i.e. if P has a representation (2.6) with $\sum_{i=1}^{\infty} |\alpha_i|^p < \infty$ for some $0 < p \le 2/3$, then there is only one element in $B' \hat{\otimes} B$ representing P.[8] Since we are only working with $\frac{2}{3}$-summing operators, we call P *nuclear*, if it is nuclear in the sense of Grothendieck and has a unique representation in $B' \hat{\otimes} B$. This allows us to define the *trace*[9] of a nuclear P unambiguously as

$$(2.7) \qquad \mathrm{tr}(P) = \sum_{i=1}^{\infty} \alpha_i \cdot y_i'(x_i)$$

where α_i, x_i, y_i' are taken from any representation (2.6), and the *Fredholm determinant*[10]

$$(2.8) \qquad \det(\mathrm{Id} - zP) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\, \mathrm{tr}(P^n) \right)$$

for $|z| < 1/\overline{\lim}_{n\to\infty}(\mathrm{tr}(P^n))^{1/n}$. Theorem 4 and its corollaries in [Grothendieck, 1955] tell us that $\det(\mathrm{Id} - zP)$ is an entire function and

$$(2.9) \qquad \det(\mathrm{Id} - zP) = \prod_i (1 - z\lambda_i)$$

where the product extends over all eigenvalues λ_i of P counted according to their multiplicities. The sequence $(\lambda_i)_{i=1}^{\infty}$ is r-summable if P is p-summing and $r^{-1} = p^{-1} - \frac{1}{2}$. Finally (ibid.)

$$(2.10) \qquad \mathrm{tr}(P) = \sum_i \lambda_i .$$

D. Quasinuclear operators.

Remember that the defining property of a quasicompact operator P is its approximability (in the sense of part A) by compact operators. The resulting

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[8] See Corollary 3 of Theorem 4 in Chapter II,§1 of [Grothendieck, 1955].

[9] See Chapter I, §3, no. 2 of [Grothendieck, 1955] or Chapter II, Proposition 1 of [Grothendieck, 1956].

[10] See Chapter II.3 of [Grothendieck, 1956].

decomposition (2.2) asserts a much nicer approximation: For any $r > r_{\text{ess}}$ there is a finite-rank (and a fortiori nuclear) operator F such that spectral radius $(P - F) < r$ and $(P - F)F = F(P - F) = 0$. So there is nothing special, from this point of view, about approximability by nuclear operators. However, the operator F is not explicitly known in general, and the Fredholm determinant of F, which might serve as an approximate Fredholm determinant for P, cannot be described in known terms. Therefore we shall define quasinuclear operators in terms of "effective" approximation schemes of nuclear operators.

2.1. Definition. A family $\mathscr{F}$ of continuous projections on a Banach space B is a *projection net* if

(a) $\alpha, \beta \in \mathscr{F}$ implies $\alpha\beta = \beta\alpha \in \mathscr{F}$ and there is $\gamma \in \mathscr{F}$ with $\alpha\gamma = \alpha$, $\beta\gamma = \beta$.

(b) $\|\alpha\| \leq C$ for some $C > 0$ and all $\alpha \in \mathscr{F}$.

The projection net $\mathscr{F}$ is *continuous* if

(c) $\lim_{\alpha \in \mathscr{F}} \alpha(x) = x$ for all $x \in B$ in the norm topology of B.[11]

If $\mathscr{F}_0 = \mathscr{F} \cup \{\text{Id}\}$, then $\mathscr{F}_0$ is also a projection net.

Given a linear operator P and a projection net $\mathscr{F}$ we fix some further notation: For $\alpha \in \mathscr{F}$ denote by r_α the spectral radius of

$$P_\alpha := (\text{Id} - \alpha)P(\text{Id} - \alpha)$$

and let $r_{\text{as}} = \varliminf_{\alpha \in \mathscr{F}} r_\alpha$. Observe that $r_0 = r(P)$.

2.2. Proposition. *Let P be a linear operator on B and let $\mathscr{F}$ be a projection net. If $P\alpha$ is compact*[12] *for all $\alpha \in \mathscr{F}$ and if $r_{\text{as}} < r(P)$, then P is quasicompact and $r_{\text{ess}} \leq r_{\text{as}}$.*

If additionally $\mathscr{F}$ is continuous and if there is $A > 0$ such that for all $m \in \mathbb{N}$ and $0 < \varepsilon < r_{\text{as}}$ there is $n \geq m$ with

$$(2.11) \qquad \varlimsup_{\beta \in \mathscr{F}} \|(\text{Id} - \beta)P^n(\text{Id} - \beta)\| \geq A \cdot (r_{\text{as}} - \varepsilon)^n,$$

then $r_{\text{ess}} = r_{\text{as}}$. In particular r_{as} is independent of the special choice of $\mathscr{F}$ in this case.

Proof. Fix $\alpha \in \mathscr{F}$. By induction on n we show

$$(2.12) \qquad P^n - PP_\alpha^{n-1} \text{ is compact for all } n \geq 1.$$

For $n = 1$ this is trivial. For $(n + 1)$ we have

$$P^{n+1} - PP_\alpha^n = P^n P\alpha + P^n P(\text{Id} - \alpha) - PP_\alpha^{n-1}P(\text{Id} - \alpha) + PP_\alpha^{n-1}\alpha P(\text{Id} - \alpha)$$

$$= P^n P\alpha + PP_\alpha^{n-1}\alpha P(\text{Id} - \alpha) + (P^n - PP_\alpha^{n-1})P(\text{Id} - \alpha).$$

The first term in this sum is compact as $P\alpha$ is, the third one by the inductive hypothesis, and the second term is different from 0 only for $n = 1$, in which case its compacity follows from that of $P\alpha$.

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[11] $\lim_{\alpha \in \mathscr{F}} \alpha(x) = x$ means that for each neighbourhood U of x there is $\beta \in \mathscr{F}$ such that for all $\alpha \geq \beta$ (i.e. $\alpha\beta = \beta$) holds: $\alpha(x) \in U$.

[12] A similar proof yields the same result if the αP are compact.

Now (2.5) and (2.12) imply $r_{\text{ess}} \leq \lim_{n \to \infty}(\|P\| \cdot \|P_\alpha^{n-1}\|)^{1/n} = r_\alpha$, and as $\alpha \in \mathscr{F}$ is arbitrary, $r_{\text{ess}} \leq r_{\text{as}} < r(P)$, i.e. P is quasicompact.

For the reverse inequality let K be any compact linear operator on B. Then

$$(2.13) \qquad \lim_{\beta \in \mathscr{F}} \|(\mathrm{Id} - \beta)K(\mathrm{Id} - \beta)\| \leq (1 + C) \cdot \lim_{\beta \in \mathscr{F}} \|(\mathrm{Id} - \beta)K\| = 0$$

by continuity of $\mathscr{F}$ and the fact that $\{Kx : \|x\| \leq 1\}$ is relatively compact in B. Fix $n \in \mathbb{N}$ such that (2.11) holds. Then

$$\|P^n - K\| \geq (1 + C)^{-2}\|(\mathrm{Id} - \beta)P^n(\mathrm{Id} - \beta) - (\mathrm{Id} - \beta)K(\mathrm{Id} - \beta)\|$$

for all $\beta \in \mathscr{F}$, and in view of (2.11) and (2.13)

$$\|P^n - K\| \geq (1 + C)^{-2}\, \overline{\lim_{\beta \in \mathscr{F}}} \, \|(\mathrm{Id} - \beta)P^n(\mathrm{Id} - \beta)\| \geq A \cdot (1 + C)^{-2} \cdot (r_{\text{as}} - \varepsilon)^n.$$

Now $r_{\text{ess}} \geq r_{\text{as}}$ follows from Nussbaums's result (2.5) and from (2.1). $\qquad \square$

2.3. Definition. (a) The linear bounded operator $P \colon B \to B$ is $\mathscr{F}$-*quasinuclear*, if $\mathscr{F}$ is a projection net for which all $P\alpha$ $(\alpha \in \mathscr{F})$ are nuclear and which satisfies:

(i) For $\beta \in \mathscr{F}$ and $n \in \mathbb{N}$ there exists $\lim_{\alpha \in \mathscr{F}} \mathrm{tr}((\alpha P_\beta \alpha)^n)$. This limit is denoted by $\mathrm{tr}_{\mathscr{F}}(P_\beta^n)$, and we set $t_\beta = \overline{\lim}_{n \to \infty} |\mathrm{tr}_{\mathscr{F}}(P_\beta^n)|^{1/n}$. For $\beta = 0$ we have $P_0 = P$ and write $t(P)$ instead of t_0. We also let $t_{\text{as}} := \underline{\lim}_{\beta \in \mathscr{F}} t_\beta$.

(ii) There is $S > 0$ such that $|\mathrm{tr}((\alpha P\alpha)^n)| \leq S^n$ $(n \in \mathbb{N}, \alpha \in \mathscr{F})$.

(iii) For each $\varepsilon > 0$ there are $C_\varepsilon > 0$ and $\beta = \beta_\varepsilon \in \mathscr{F}$ such that

$$\|(\alpha P_\beta \alpha)^n\| \leq C_\varepsilon(r_{\text{as}} + \varepsilon)^n, \quad |\mathrm{tr}((\alpha P_\beta \alpha)^n)| \leq C_\varepsilon(t_{\text{as}} + \varepsilon)^n$$

for $n \in \mathbb{N}$ and $\alpha \in \mathscr{F}_0$.

(iv)

$$\lim_{\alpha \in \mathscr{F}} \|\alpha(P_\beta \alpha)^j P\beta - P_\beta^j P\beta\|_T = 0$$

for all $\beta \in \mathscr{F}$ and $j \in \mathbb{N}$.

(b) Having defined a kind of trace for $\mathscr{F}$-quasinuclear operators in (i) we can extend the definition of a *Fredholm determinant* (cf. (2.8)) to this class of operators: For $\beta \in \mathscr{F}$ let

$$(2.14) \qquad \det{}_{\mathscr{F}}(\mathrm{Id} - zP_\beta) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}_{\mathscr{F}}(P_\beta^n)\right).$$

2.4. Remark. (a) The definition of t_β and t_{as} depends on the particular choice of $\mathscr{F}$. This should be kept in mind although the above notation ignores this dependence.

(b) Condition (iv) means that P and P_β map "small" subspaces "close" to ones that are not much "bigger" (in the sense of the filtration $\mathscr{F}$). Its special form is motivated by later applications, where it may be satisfied, as in Example 6.14, not only in the limit but also for "finite" α.

(c) If one does not assume nuclearity of the $P\alpha$ but instead of the αP ($\alpha \in \mathscr{F}$) and if one replaces the inequality in (iv) by

$$\lim_{\alpha \in \mathscr{F}} \|\alpha (P_\beta \alpha)^j P\beta - P_\beta^j P\beta\| = 0,$$

then one arrives at a slightly different definition of quasinuclearity that will not be referred to in this paper but might be useful in other situations. The following theorem and its corollary remain valid (see also the footnote to Proposition 2.2).

(d) We did not require $r_{as} < r(P)$ for $\mathscr{F}$-quasinuclearity. Hence $\mathscr{F}$-quasinuclearity does not imply quasicompactness.

2.5. Theorem. *Suppose P is $\mathscr{F}$-quasinuclear. Let $R = \max\{r_{as}, t_{as}\}$. Then*

(a) $\det_{\mathscr{F}}(\mathrm{Id} - zP)$ *is analytic in $\{z \in \mathbb{C} : |z| < R^{-1}\}$,*

(b) $\det_{\mathscr{F}}(\mathrm{Id} - z_0 P) = 0$ *if and only if z_0^{-1} is an eigenvalue of P. The multiplicity of z_0 as a zero of $\det_{\mathscr{F}}(\mathrm{Id} - zP)$ is finite and equals the algebraic multiplicity of z_0^{-1} as an eigenvalue of P.*

2.6. Corollary. *Under the assumptions of the theorem the following holds:*

(a) *All spectral points λ of P with $|\lambda| > R$ are eigenvalues of P.*

(b) $r(P) = t(P)$, *if $\mathrm{tr}_{\mathscr{F}}(P^n) \geq 0$ for all $n \in \mathbb{N}$ and if*

(1) $t_{as} < t(P)$ *or* $t_{as} \leq r(P)$ *and*   (2) $r_{as} < r(P)$ *or* $r_{as} \leq t(P)$.

Proof of the corollary. By Proposition 2.2 P is quasicompact and $r_{ess} \leq r_{as} \leq R$. This proves (a). If $r(P) > r_{as}$, we have

$$r(P) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P\}.$$

Hence, if $t(P) > R$ or $r(P) > R$, we obtain from (2.14), Pringsheim's theorem, and Theorem 2.5

$$t(P) = \varlimsup_{n \to \infty} \mathrm{tr}_{\mathscr{F}}(P^n)^{1/n} = \max\{|z|^{-1} : \det_{\mathscr{F}}(\mathrm{Id} - zP) = 0\} = r(P).$$

Otherwise we have $r(P) \leq R$ and $t(P) \leq R$, and the additional assumption implies $r(P) = t(P)$. $\square$

Proof of the theorem. Fix $\varepsilon > 0$ and $\beta = \beta_\varepsilon \in \mathscr{F}$ as in Definition 2.3(aiii). For each $\alpha \in \mathscr{F}_0$ $(\mathrm{Id} - z\alpha P_\beta \alpha)^{-1}$ exists and is an analytic function of z in $G_\varepsilon = \{|z| < (R + \varepsilon)^{-1}\}$. Let

$$A_\alpha(z) = -z(\mathrm{Id} - \beta)(\mathrm{Id} - z\alpha P_\beta \alpha)^{-1}(\mathrm{Id} - \beta)\alpha P\alpha\beta,$$
$$B_\alpha(z) = -z\beta\alpha P\alpha(\mathrm{Id} - A_\alpha(z))\beta.$$

Then
(2.15)
$$(\mathrm{Id} - z\alpha P\alpha) = \left(\beta + (\mathrm{Id} - z\alpha P\alpha)(\mathrm{Id} - \beta)\right)\left(A_\alpha(z) + B_\alpha(z) + \mathrm{Id}\right)$$
$$= \left(\mathrm{Id} - z\alpha P_\beta \alpha - z\beta\alpha P\alpha(\mathrm{Id} - \beta)\right)\left(\mathrm{Id} + B_\alpha(z)\right)\left(\mathrm{Id} + A_\alpha(z)\right)$$

Although the first identity is not mysterious (its matrix version is Schur's equality, see also Lemma 2 of [Hofbauer/Keller, 1984]), the proof is a bit tricky:

Let $A = A_\alpha(z)$, $B = B_\alpha(z)$, and make use of $\alpha\beta = \beta\alpha$. Then

$$(\beta + (\mathrm{Id} - z\alpha P\alpha)(\mathrm{Id} - \beta))(A + B + \mathrm{Id}) - (\mathrm{Id} - z\alpha P\alpha)$$

$$= B + (\mathrm{Id} - z\alpha P\alpha)A + z\alpha P\alpha\beta$$

$$= z(\mathrm{Id} - \beta)\alpha P\alpha\beta + (\mathrm{Id} - z(\mathrm{Id} - \beta)\alpha P\alpha)A$$

$$= z(\mathrm{Id} - \beta)\alpha P\alpha\beta + (\mathrm{Id} - z\alpha P_\beta\alpha)A$$

$$= z(\mathrm{Id} - \beta)\alpha P\alpha\beta - z(\mathrm{Id} - \beta)\alpha P\alpha\beta + z\beta(\mathrm{Id} - z\alpha P_\beta\alpha)^{-1}(\mathrm{Id} - \beta)\alpha P\alpha\beta$$

$$= z\beta(\mathrm{Id} - \beta)\alpha P\alpha\beta + z^2\beta\alpha P_\beta\alpha(\mathrm{Id} - z\alpha P_\beta\alpha)^{-1}(\mathrm{Id} - \beta)\alpha P\alpha\beta$$

$$= 0$$

Suppose now that $\alpha \in \mathscr{F}$. As P is $\mathscr{F}$-quasinuclear, all factors occurring in (2.15) are of the type "Id + nuclear operator". Hence

$$(2.16) \qquad \det(\mathrm{Id} - z\alpha P\alpha) = \det(\mathrm{Id} - z\alpha P_\beta\alpha) \cdot \det(\mathrm{Id} + B_\alpha(z)),$$

because $\beta\alpha P\alpha(\mathrm{Id} - \beta)$ and $A_\alpha(z)$ are "transient" off-diagonal terms which do not contribute to the determinants (see Chapter II.3, Proposition 2 of [Grothendieck, 1956]). We take the $\lim_{\alpha \in \mathscr{F}}$ on both sides of (2.16): As

$$\det(\mathrm{Id} - z\alpha P\alpha) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr}((\alpha P\alpha)^n)\right) \quad \text{for } |z| < S^{-1}$$

by (2.8) and Definition 2.3(aii), we have in view of Definition 2.3(ai, ii) and (b)

$$(2.17) \quad \lim_{\alpha \in \mathscr{F}} \det(\mathrm{Id} - z\alpha P\alpha) = \det_{\mathscr{F}}(\mathrm{Id} - zP) \quad \text{uniformly for } |z| < (S + \varepsilon)^{-1}$$

Similarly, but using (iii) instead of (ii) of Definition 2.3(a) one obtains

$$\lim_{\alpha \in \mathscr{F}} \det(\mathrm{Id} - z\alpha P_\beta\alpha) = \det_{\mathscr{F}}(\mathrm{Id} - zP_\beta)$$

$$(2.18)$$

$$= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr}_{\mathscr{F}}(P_\beta^n)\right) \quad \text{uniformly in } G_\varepsilon.$$

As $B_\alpha(z)$ is nuclear for all $\alpha \in \mathscr{F}_0$ (i.e. also in the limit $\alpha = \mathrm{Id}$), we obtain

$$(2.19) \qquad \lim_{\alpha \in \mathscr{F}} \det(\mathrm{Id} + B_\alpha(z)) = \det(\mathrm{Id} + B_{\mathrm{Id}}(z)) \qquad (z \in G_\varepsilon)$$

if we can show that $\lim_{\alpha \in \mathscr{F}} B_\alpha(z) = B_{\mathrm{Id}}(z)$ in trace-norm $\|.\|_T$, for the determinant is continuous with respect to this norm (see Chapter II.2, Proposition 1 of [Grothendieck, 1956]). To this end fix $l \in \mathbb{N}$, and let $\alpha \geq \beta$ such that for $0 \leq j < l$

$$\|\alpha(P_\beta\alpha)^j P\beta - (P_\beta)^j P\beta\|_T \leq (r_{\mathrm{as}} + \varepsilon)^j \cdot l^{-1}$$

(cf. Definition 2.3(aiv)). [13]   Then (cf. Chapter II, §1, no. 2 of [Grothendieck, 1955])

$$\|B_{\mathrm{Id}}(z) - B_\alpha(z)\|_T = \|z\beta P(A_{\mathrm{Id}}(z) - \alpha A_\alpha(z))\|_T$$

$$\leq |z|^2 \cdot \|\beta P\| \sum_{j=0}^\infty |z|^j \|(\mathrm{Id} - \beta)(P_\beta^j - \alpha(P_\beta \alpha)^j)P\beta\|_T$$

$$\leq |z|^2 \cdot \mathrm{const} \cdot \left(\frac{1}{l(1 - |z|(r_{\mathrm{as}} + \varepsilon))} + \sum_{j=l}^\infty \left(|z|(r_{\mathrm{as}} + \varepsilon)\right)^j \cdot \|P\beta\|_T \right)$$

by Definition 2.3(aiii), and as l was arbitrary, we obtain

$$(2.20) \qquad \lim_{\alpha \in \mathscr{F}} \|B_{\mathrm{Id}}(z) - B_\alpha(z)\|_T = 0 \quad \text{for all } z \in G_\varepsilon.$$

Now (2.16)–(2.20) imply

$$(2.21) \qquad \det{}_{\mathscr{F}}(\mathrm{Id} - zP) = \det{}_{\mathscr{F}}(\mathrm{Id} - zP_\beta) \cdot \det(\mathrm{Id} + B_{\mathrm{Id}}(z)) \quad \text{for } |z| < S^{-1},$$

and the right-hand side of this equation is analytic in G_ε. This proves (a) of the theorem.

Let $B(z) = B_{\mathrm{Id}}(z)$, and suppose that $\det(\mathrm{Id} + B(z_0)) = 0$ for some $z_0 \in G_\varepsilon$. The following identity is a generalization of Theorem 5.1 in [Gohberg/Sigal, 1971]:

$$(2.22) \qquad \begin{aligned} &\mathrm{tr} \int_{|z - z_0| = r} B'(z)(\mathrm{Id} + B(z))^{-1} dz \\ &= \int_{|z - z_0| = r} \frac{d}{dz} \det(\mathrm{Id} + B(z)) \cdot \det(\mathrm{Id} + B(z))^{-1} dz \\ &= 2\pi i \cdot \left(\text{multiplicity of the zero } z_0 \text{ of } \det(\mathrm{Id} + B(z))\right) \end{aligned}$$

for small r. We prove it: Recall that

$$B(z) = -z\beta P\beta - z^2 \beta P(\mathrm{Id} - \beta)P\beta - z^2 \beta P \left(\sum_{j=1}^\infty z^j P_\beta^j \right) P\beta.$$

Since $\|SP\beta\|_T \leq \|S\| \cdot \|P\beta\|_T$ for each bounded linear operator S on B, $B(z)$ is an analytic function of $z \in G_\varepsilon$ with respect to the trace-norm, and $\|B(z)\|_T < \frac{1}{2}$ for $|z| < z_0$, say. Hence, observing the fact that $\mathrm{tr}(UV) = \mathrm{tr}(VU)$ for nuclear operators U and V on B,

$$\mathrm{tr}(B'(z)(\mathrm{Id} + B(z))^{-1}) = \mathrm{tr}\left(\frac{d}{dz} \log(\mathrm{Id} + B(z)) \right)$$

$$= \frac{d}{dz} \mathrm{tr}(\log(\mathrm{Id} + B(z))) = \frac{d}{dz} \log \det(\mathrm{Id} + B(z)) \quad \text{for } |z| < \varepsilon_0$$

---

[13] This is the only instant of the proof where Definition 2.3(aiv) is needed. If the definition of quasinuclearity is varied as proposed in Remark 2.4(c), the proof of the subsequent identity changes only slightly.

i.e.

$$(2.23) \qquad \mathrm{tr}(B'(z)(\mathrm{Id}+B(z))^{-1}) \cdot \det(\mathrm{Id}+B(z)) = \frac{d}{dz}\det(\mathrm{Id}+B(z))$$

for $|z| < \varepsilon_0$, and by analytic continuation (2.23) continues to hold on G_ε (where $(\mathrm{Id}+B(z))$ is defined and analytic). Using once again the continuity of the trace functional with respect to the trace-norm one finally arrives at (2.22).

As all $B(z)$, $z \in G_\varepsilon$, are nuclear and a fortiori compact, $(\mathrm{Id}+B(z))$ is an analytic operator-valued function, normal with respect to $\mathrm{bd}(G_\varepsilon)$ in the sense of [Gohberg/Sigal, 1971, §2]. In particular, Theorem 2.1 (ibid.) says

$$(2.24) \qquad M(\mathrm{Id}+B(z_0)) = \frac{1}{2\pi i}\,\mathrm{tr}\int_{|z-z_0|=r} B'(z)(\mathrm{Id}+B(z))^{-1}dz$$

where $M(\mathrm{Id}+B(z_0))$ is the multiplicity of the singular point z_0 of $(\mathrm{Id}+B(z))$ (cf. §1 (ibid.)).The multiplicity M is invariant under equivalence, i.e. if $E(z)$, $F(z)$ are equivalent holomorphic families of invertible operators defined in a neighbourhood of z_0, then $M\big(E(z_0)(\mathrm{Id}+B(z_0))F(z_0)\big) = M(\mathrm{Id}+B(z_0))$.

Recall (2.15) for $\alpha = \mathrm{Id}$:

$$(\mathrm{Id} - zP) = (\mathrm{Id} - zP_\beta - z\beta P(\mathrm{Id} - \beta))\,(\mathrm{Id}+B(z))\,(\mathrm{Id}+A(z))$$

The first and the last factor on the right-hand side are analytic and invertible in G_ε, i.e. $(\mathrm{Id}-zP)$ and $(\mathrm{Id}+B(z))$ are equivalent at $z_0 \in G_\varepsilon$. Hence

$$(2.25) \qquad\qquad M(\mathrm{Id}+B(z_0)) = M(\mathrm{Id}-z_0 P)$$

and the right-hand side of this equation coincides with the usual notion of (algebraic) multiplicity of the eigenvalue z_0^{-1} of P (ibid., §1,1). As $\det_\mathscr{F}(\mathrm{Id}-zP_\beta) \neq 0$ for $z \in G_\varepsilon$ by (2.18), assertion (b) of the theorem follows now from (2.21), (2.22), (2.24), and (2.25). □

2.7. *Remark.* The theorem allows us to define $\det(\mathrm{Id}-zP)$ as the equivalence class of $\det_\mathscr{F}(\mathrm{Id}-zP)$ of analytic functions on $\{|z| < R\}$ under the equivalence $f \sim g$ if and only if f/g and g/f are analytic on $\{|z| < R\}$. This makes particular sense if there is some $\mathscr{F}$ such that $R = r_{\mathrm{as}} = t_{\mathrm{as}} = r_{\mathrm{ess}}$.

3. Piecewise invertible systems and their Markov extensions

A. Piecewise invertible systems.

Having accomplished the functional analytic prerequisites we now define the class of piecewise invertible dynamical systems to be investigated in the rest of this paper. For similar definitions see [Schweiger, 1975] and [Hofbauer, 1986].

3.1. **Definition.** Let X be a topological space.

(a) The triple $(X, T, \mathscr{Z})$ is a *piecewise invertible system*, if $\mathscr{Z}$ is a finite or countable partition of X such that for each $Z \in \mathscr{Z}$ $\mathrm{cl}(Z)$ is compact and if the restrictions $T_{|Z}$ can be extended to continuous injective maps $T_Z: \mathrm{cl}(Z) \to X$. (cl denotes the topological closure, int the interior, and (for later use)

bd the boundary.) Often we shall just say *system* instead of piecewise invertible system.

(b) $(X, T, \mathscr{Z})$ is a *piecewise compact system* if all $Z \in \mathscr{Z}$ are compact open subsets of X.

(c) The system $(X, T, \mathscr{Z})$ is a *Markov system* if $TZ_1 \cap Z_2 \neq \varnothing$ implies $TZ_1 \supseteq Z_2$ for all $Z_1, Z_2 \in \mathscr{Z}$.

3.2. *Remark.* If $(X, T, \mathscr{Z})$ is piecewise compact, then $T_Z = T_{|Z}$ is a homeomorphism from Z onto TZ $(Z \in \mathscr{Z})$.

3.3. **Definition** (Cylinders). For $n \geq 0$ let

$$\mathscr{Z}_n = \bigvee_{i=0}^{n} T^{-i} \mathscr{Z} = \{Z_0 \cap T^{-1} Z_1 \cap \cdots \cap T^{-n} Z_n \neq \varnothing : Z_i \in \mathscr{Z}\}$$

be the *cylinders of length* n of the system $(X, T, \mathscr{Z})$. Denote by $V_n(x)$ the cylinder of length n containing x, $(x \in X)$ and by $\mathrm{reg}(X, T, \mathscr{Z}) = \{x \in X : T^k x \in \mathrm{int}\, V_n(T^k x)$ for all $k \geq 0, n \geq 0\}$ the set of all *regular points* of $(X, T, \mathscr{Z})$. Observe that if $(X, T, \mathscr{Z})$ is piecewise compact, then each $Z \in \mathscr{Z}_n$ is compact and $\mathrm{reg}(X, T, \mathscr{Z}) = X$.

$(X, T, \mathscr{Z})$ is *generating*, if $[x] := \bigcap_{n \geq 0} V_n(x) = \{x\}$ for all $x \in X$.

3.4. **Definition** (Factors, extensions). $(X, T, \mathscr{Z})$ is a *factor* of $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$, and $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ is an *extension* of $(X, T, \mathscr{Z})$, if there is a continuous map π from $\hat{X}$ onto X such that $T \circ \pi = \pi \circ \hat{T}$ on the set $\pi^{-1}(\mathrm{reg}(X, T, \mathscr{Z}))$. Then $T^k \circ \pi = \pi \circ \hat{T}^k$ for all k on the same set, because this set is $\hat{T}$-invariant. We also say *π-factor* or *π-extension* if we want to specify the factoring map. Observe that $T \circ \pi = \pi \circ \hat{T}$ on all of $\hat{X}$ if $(X, T, \mathscr{Z})$ is piecewise compact.

B. Canonical extensions.

Our main goal in this section is to construct in a canonical way a Markov extension for a given piecewise compact system. At the end we also show how to find, in a canonical way, a piecewise compact extension for an arbitrary system.

From now on let $(X, T, \mathscr{Z})$ be a piecewise compact system. The ideas behind the following construction of a Markov extension are borrowed from [Hofbauer, 1986] and some of his earlier works, e.g. [Hofbauer, 1979]. References can be found in his 1986 paper. Analogous constructions for (two-sided) sofic systems are carried out in [Krieger, 1984].

3.5. **Definition** (Successors). Suppose C is a compact subset of X. The nonempty sets among $TC \cap Z$ with $Z \in \mathscr{Z}$ are called the *successors* of C. We write $C \to D$ if $D = TC \cap Z$. As all $Z \in \mathscr{Z}$ are assumed to be compact, successors of compact sets are again compact, and one can iterate the formation of successors. Let $\mathscr{D}$ be the smallest set of (compact) subsets of X that contains $\mathscr{Z}$ and is closed under forming successors.

The following lemma relates the above construction to cylinders. Its proof is simple and can be found in [Hofbauer, 1986, Lemma 1].

3.6. Lemma. *Suppose that* $Z_i \in \mathscr{Z}$ $(i = 0, \ldots, k)$ *and that* $D \subseteq Z_0$. *Let* $A_k = D \cap \bigcap_{i=1}^{k} T^{-i} Z_i$ *and* $D_0 = D$, $D_i = T(D_{i-1}) \cap Z_i$ $(i = 1, \ldots, k)$. *Then* $D_k = T^k A_k$ *and* $A_k = \bigcap_{i=0}^{k} T^{-i} D_i$. *If* $D = Z_0$, *then* $A_k \in \mathscr{Z}_k$.

Now let

$$\hat{X} = \bigcup_{D \in \mathscr{D}} \hat{D}$$

where the $\hat{D}$ are pairwise disjoint copies of the sets $D \in \mathscr{D}$. Elements of $\hat{X}$ are denoted by $\hat{x} = (x, D)$ with $x \in D \in \mathscr{D}$. Define $\hat{T} : \hat{X} \to \hat{X}$ by

$$\hat{T}(x, C) = (Tx, D) \quad \text{if } C \to D \text{ and } Tx \in D$$

and let $\hat{\mathscr{Z}} = \{\hat{D} : D \in \mathscr{D}\}$. Then $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ is, by construction, a piecewise compact Markov system. It is a π-extension of $(X, T, \mathscr{Z})$ where $\pi : \hat{X} \to X$ is defined by $\pi(x, C) = x$. By $\pi_{\mathscr{D}}$ we denote the projection from $\hat{X}$ to $\mathscr{D}$, $\pi_{\mathscr{D}}(x, C) = C$.

3.7. Definition. (a) $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ is the *canonical Markov extension* of $(X, T, \mathscr{Z})$.

(b) The *Markov diagram* of $(X, T, \mathscr{Z})$ is the directed graph $(\mathscr{D}, \to)$ with vertex-set $\mathscr{D}$ and edges defined by the successor relation (cf. [Hofbauer, 1986]). As usual it can be represented by a $\{0, 1\}$-valued $\mathscr{D} \times \mathscr{D}$-matrix.[14]

(c) The Markov-diagram is *uniformly forward finite* if there is a uniform bound on the number of successors of a single vertex.

3.8. Remark. If $(X, T, \mathscr{Z})$ is a piecewise compact Markov system, it coincides with its canonical Markov extension, and the Markov diagram describes the Markov structure of $(X, T, \mathscr{Z})$ itself.

3.9. Definition (Orbits and paths). An *orbit of length* n (in X) is an ordered n-tuple $x, Tx, \ldots, T^{n-1} x$. It is *periodic* if $T^n x = x$. Orbits in $\hat{X}$ are defined analogously. A *path of length* n (in $\mathscr{D}$) is an ordered $(n+1)$-tuple $\omega = D_0 D_1 \ldots D_n$ with $D_{i-1} \to D_i$ $(i = 1, \ldots, n)$. It is closed, if $D_n = D_0$. With ω we associate the cylinder of length n $Z(\omega) = \bigcap_{i=0}^{n} T^{-i} D_i$. Infinite orbits and paths are defined analogously.

Note that Lemma 3.6 establishes a 1-1 correspondence $\omega \leftrightarrow \mathscr{Z}(\omega)$ between paths and cylinders of length n. The following relation between orbits in $\hat{X}$ and paths in $\mathscr{D}$ are implicit in Theorem 1 of [Hofbauer, 1986]. The proof is easy.

3.10. Lemma. *Suppose* $(X, T, \mathscr{Z})$ *is a piecewise compact system.*

(a) *Each* $\hat{x} \in \hat{X}$ *induces a unique infinite path* $\omega(\hat{x}) = D_0 D_1 D_2 \cdots$ *by* $D_j = \pi_{\mathscr{D}}(\hat{T}^j \hat{x})$ $(j = 0, 1, 2, \ldots)$. *Notation:* $\omega_n(\hat{x}) = D_0 \cdots D_n$.

---

[14] Hofbauer investigates the topological Markov chain defined by this matrix. It is again a piecewise compact system and can be obtained as an ω-factor of $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ by $\omega(\hat{x}) = (\pi_{\mathscr{D}}(\hat{T}^n \hat{x}))_{n \geq 0}$. If $(X, T, \mathscr{Z})$ is generating, then ω is a homeomorphism between $\hat{X}$ and the chain, cf. Lemma 3.6.

(b) *Each infinite path $D_0 D_1 D_2 \cdots$ in the Markov diagram represents a point $\hat{x} \in \hat{X}$, i.e. there is $\hat{x} \in \hat{X}$ with $\omega(\hat{x}) = D_0 D_1 D_2 \cdots$. This $\hat{x}$ is unique if $[x] = \{x\}$, in particular if $(X, T, \mathscr{Z})$ is generating.*

(c) *If $\omega(\hat{x}) = C_0 C_1 C_2 \cdots$, $\omega(\hat{y}) = D_0 D_1 D_2 \cdots$, $\pi(\hat{x}) = \pi(\hat{y}) = x$ for $\hat{x}, \hat{y} \in \hat{X}$, and if $C_0 \cap V_k(x) = D_0 \cap V_k(x)$ for some $k \geq 0$, then $C_i = D_i$ $(i \geq k)$.*

The following lemma, which relates periodic orbits in X and $\hat{X}$ to closed paths in $\mathscr{D}$, is a slight variation of Theorem 8 in [Hofbauer, 1986]:

3.11. Lemma. *Suppose $(X, T, \mathscr{Z})$ is a generating, piecewise compact system.*

(a) *If $D_0 D_1 \dots D_{n-1} D_0$ is a closed path, then there is exactly one $x \in D_0$ such that $\hat{T}^i(x, D_0) = (T^{i \bmod n} x, D_{i \bmod n})$ for $i = 0, 1, 2, \dots$.*

(b) *If $T^n x = x$ for some $x \in X$, and if $x \in \mathrm{int}(\pi_{\mathscr{D}} \circ \hat{T}^n(x, V_0(x)))$, then there is $\hat{x} \in \pi^{-1}\{x\}$ with $\hat{T}^n \hat{x} = \hat{x}$. If x is in the interior of each $D \in \mathscr{D}$ it belongs to, then $\hat{x}$ is unique.*

Proof. (a) By Lemma 3.6, $A_k = \bigcap_{i=0}^{k} T^{-i} D_{i \bmod n}$ is nonempty and compact for all $k \in \mathbb{N}$. Hence $A_\infty = \bigcap_{i=0}^{\infty} T^{-i} D_{i \bmod n} \subseteq D_0$ is nonempty. Let $x \in A_\infty$. By Lemma 3.6, $A_\infty \subseteq [x] = \{x\}$, and we obtain inductively $\hat{T}(T^i x, D_i) = (T^{i+1} x, D_{(i+1) \bmod n})$ $(i = 0, \dots, n-1)$. Since $\{T^n x\} = T^n A_\infty \subseteq A_\infty = \{x\}$, this yields

$$\hat{T}^i(x, D_0) = (T^{i \bmod n} x, D_{i \bmod n}) \qquad (i \geq 0).$$

As each point satisfying this relation belongs to $A_\infty = \{x\}$, x is uniquely determined.

(b) Let $Z = V_0(x)$. Then $\pi(x, Z) = x = \pi(\hat{T}^n(x, Z))$, and for some k big enough holds

$$\pi_{\mathscr{D}}(x, Z) \cap V_k(x) = V_k(x) = \pi_{\mathscr{D}}(\hat{T}^n(x, Z)) \cap V_k(x)$$

because $(X, T, \mathscr{Z})$ generates. Hence $\pi_{\mathscr{D}}(\hat{T}^i(x, Z)) = \pi_{\mathscr{D}}(\hat{T}^{n+i}(x, Z))$ for $i \geq k$ by Lemma 3.10(c), and for a suitable multiple ln of n and $\hat{x} := \hat{T}^{ln}(x, Z)$ holds

$$
\begin{aligned}
\hat{T}^n \hat{x} &= (T^{(l+1)n} x, \pi_{\mathscr{D}}(\hat{T}^{(l+1)n}(x, Z))) \\
&= (T^{ln} x, \pi_{\mathscr{D}}(\hat{T}^{ln}(x, Z))) \\
&= \hat{x} \in \pi^{-1}\{x\}
\end{aligned}
$$

and $\omega_n(\hat{x})$ is closed, i.e. $\omega(\hat{x})$ is periodic.

If $\hat{y}$ is another point with $\pi(\hat{y}) = x$, $\hat{T}^n \hat{y} = \hat{y}$, then also $\omega(\hat{y})$ is periodic, and by Lemma 3.10(c) the two paths coincide if x is in the interior of each $D \in \mathscr{D}$ it belongs to. $\hat{x} = \hat{y}$ follows now from part (a). $\quad\square$

3.12. **Example I** (topological Markov shifts, sofic systems).

Let Σ be a finite or countable set of symbols, A a $\{0,1\}$-valued $\Sigma \times \Sigma$-matrix which is row-finite (i.e. $\sum_{\tau \in \Sigma} A(\sigma, \tau) < \infty$ for all $\sigma \in \Sigma$), and let

$$\Omega_A = \{(\sigma_1 \sigma_2 \sigma_3 \cdots) \in \Sigma^{\mathbb{N}} \colon A(\sigma_i, \sigma_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\}$$

With the product topology of the discrete topology on Σ the shift space Ω_A is a topological space. If $S \colon \Omega_A \to \Omega_A$, $S(\sigma_1 \sigma_2 \sigma_3 \cdots) = (\sigma_2 \sigma_3 \sigma_4 \cdots)$ denotes the shift transformation, and if $\mathscr{Z} = \{[\sigma]_1 \colon \sigma \in \Sigma\}$ (where $[\sigma]_1 = \{(\sigma_1 \sigma_2 \sigma_3 \cdots) \in \Omega_A \colon \sigma_1 = \sigma\}$) is the partition into cylinders of length 1, Then $(\Omega_A, S, \mathscr{Z})$ is a generating, piecewise compact system. (For the compactness of the fibers see Lemma 1.5(b) in [Wagoner, 1988].) We have $\mathscr{D} = \mathscr{Z}$ because of the Markov structure of Ω_A, and it is easily seen that $(\Omega_A, S, \mathscr{Z})$ is its own canonical Markov extension.

If Σ is finite, if $\tilde{\Sigma}$ is another alphabet, and if $\theta \colon \Sigma \to \tilde{\Sigma}$, then θ extends coordinate-wise to a map from Ω_A to $\tilde{\Sigma}^{\mathbb{N}}$, and

$$\Omega_{A,\theta} = \{\theta(\omega) \colon \omega \in \Omega_A\}$$

is a sofic system (see [Coven/Paul, 1975]). As in [Krieger, 1984] it is easy to show that $\mathscr{D}$ is finite.

3.13. **Example II** (up-and-down-counter).

This is an example of a family of closed subshifts of $\{0,1\}^{\mathbb{N}}$ which are not sofic. Before I describe these systems, some notational conventions are introduced which will be used later, too.

For a set Σ of symbols let $\Sigma^+ = \{\sigma_1 \cdots \sigma_n \colon n \geq 1, \sigma_i \in \Sigma\}$ and $\Sigma^* = \Sigma^+ \cup \{\text{empty word}\}$. Similarly, for $W \subseteq \Sigma^*$ let $W^+ = \{w_1 \cdots w_n \colon n \geq 1, w_i \in W\}$ and $W^* = W^+ \cup \{\text{empty word}\}$. Furthermore let $W^\infty = \{w_1 w_2 w_3 \cdots \colon w_i \in W\}$. W^∞ is considered as a subset of $\Sigma^{\mathbb{N}}$. By $\Omega_0(W)$ we denote the smallest shift-invariant subset of $\Sigma^{\mathbb{N}}$ containing W^∞ and by $\Omega(W)$ its topological closure in $\Sigma^{\mathbb{N}}$.

If $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ or $B \subseteq \Sigma^{\mathbb{N}}$, we let $AB = \{uv \colon u \in A, v \in B\}$. $AB \subseteq \Sigma^*$ or $\Sigma^{\mathbb{N}}$, respectively.

If $\sigma \in \Sigma$ then $\sigma^n = \sigma \cdots \sigma$ (n-times).

Here comes the up-and-down-counter: Let $\Sigma = \{0,1\}$, fix $M \subseteq \mathbb{N}$ (infinite, otherwise we would come out with a sofic system), and let

$$W_M = \{0^i 1^i \colon i \in M\}$$
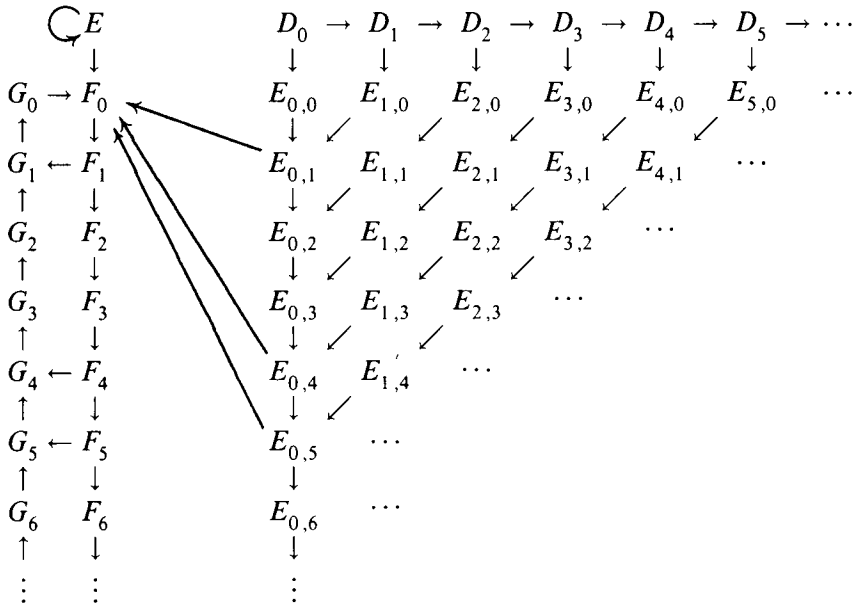
and

$$V_M = \{0^i 1^j \colon 0 \leq i \leq j < \infty \text{ and } j \in M \text{ if } i > 0\}.$$

It is quite obvious that $\Omega_0(W_M) = V_M W_M^\infty$, and it is also not hard to see that $\Omega(W_M) = V_M R_M$ where

$$R_M = W_M^\infty \cup W_M^* \{0^\infty\} \cup \{1^\infty\}$$

If $S: \Omega(W_M) \to \Omega(W_M)$ denotes again the shift-transformation and if $\mathscr{Z} = \{[0]_1, [1]_1\}$, then $(\Omega(W_M), S, \mathscr{Z})$ is a generating, piecewise compact system, and its Markov diagram has the following structure, if, for example, the three smallest members of M are $2, 5$, and 6:

$$
\begin{array}{l}
\circlearrowright E \qquad\qquad\qquad D_0 \to D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to \cdots \\
\quad \downarrow \qquad\qquad\qquad\quad \downarrow \quad\ \downarrow \quad\ \downarrow \quad\ \downarrow \quad\ \downarrow \quad\ \downarrow \\
G_0 \to F_0 \leftarrow\quad E_{0,0} \quad E_{1,0} \quad E_{2,0} \quad E_{3,0} \quad E_{4,0} \quad E_{5,0} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \nearrow \quad\ \nearrow \quad\ \nearrow \quad\ \nearrow \quad\ \nearrow \\
G_1 \leftarrow F_1 \qquad\quad E_{0,1} \quad E_{1,1} \quad E_{2,1} \quad E_{3,1} \quad E_{4,1} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \nearrow \quad\ \nearrow \quad\ \nearrow \quad\ \nearrow \\
G_2 \quad\ F_2 \qquad\quad E_{0,2} \quad E_{1,2} \quad E_{2,2} \quad E_{3,2} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \nearrow \quad\ \nearrow \quad\ \nearrow \\
G_3 \quad\ F_3 \qquad\quad E_{0,3} \quad E_{1,3} \quad E_{2,3} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \nearrow \quad\ \nearrow \\
G_4 \leftarrow F_4 \qquad\quad E_{0,4} \quad E_{1,4} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \nearrow \\
G_5 \leftarrow F_5 \qquad\quad E_{0,5} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \\
G_6 \quad\ F_6 \qquad\quad E_{0,6} \quad \cdots \\
\uparrow \quad\ \downarrow \qquad\qquad \downarrow \\
\vdots \quad\ \vdots \qquad\qquad \vdots
\end{array}
$$

Here

$$D_n = \{0^i 1^j : 1 \le i \le j - n, j \in M\} R_M \cup \{0^\infty\}, \quad \text{i.e.} \quad D_0 = [0]_1,$$

$$E_{n,m} = \{1^j : j \ge n+1, j+m \in M\} R_M,$$

$$E = \{1^j : 1 \le j < \infty\} R_M = [1]_1,$$

$$F_0 = R_M \setminus \{1^\infty\},$$

$$F_n = \{0^{j-n} 1^j : n < j < \infty, j \in M\} F_0 \cup \{0^\infty\} \quad \text{for } n \ge 1,$$

$$G_n = \{1^{n+1}\} F_0.$$

For special choices of M the right-hand side of the diagram may collapse. For $M = \mathbb{N}$, for example, $E_{n,m} = E_{n,0}$ for all m and $E_{0,0} = E$.

If M is finite, the same construction yields a finite diagram, and $(\Omega(W_M), S)$ is a one-block factor of the topological Markov chain defined by $(\mathscr{D}, \to)$, i.e. it is sofic. For later use we note some further facts:

(a) All closed paths are of the form $F_0 \cdots F_j G_j \cdots G_0 F_0$ $(j \in M)$ (or translates of these cycles), except for the trivial cycle EE.

(b)
$$D_0 = \text{int}(D_0), \quad D_n \setminus \text{int}(D_n) = \{0^\infty\} \quad (n \ge 1),$$

$$E = \text{int}(E), \quad E_{n,m} \setminus \text{int}(E_{n,m}) = \{1^\infty\} \quad (\text{if } M \ne \mathbb{N}),$$

$$F_n \setminus \text{int}(F_n) = \{0^\infty\}, \quad G_n = \text{int}(G_n).$$

3.14. *Extensions for nonpiecewise compact systems.*

The relations between $(X, T, \tilde{\mathscr{Z}})$ and its canonical Markov extension as stated in Lemmas 3.10 and 3.11 are no longer true, if $(X, T, \tilde{\mathscr{Z}})$ is not piecewise compact. In order to investigate a nonpiecewise compact system $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Z}})$ by means of a Markov extension it is therefore useful to construct first a (usually non-Markovian) piecewise compact extension $(X, T, \mathscr{Z})$ of that system. This can be done as follows:

Suppose that $\tilde{X}$ is compact and that $V \in \tilde{\mathscr{Z}}$. Let

$$X_0 = \tilde{X}, \qquad X_n = \sum_{Z \in \tilde{\mathscr{Z}}_{n-1}} \mathrm{cl}(Z), \qquad V_n = \sum_{\substack{Z \in \tilde{\mathscr{Z}}_{n-1} \\ Z \subseteq V}} \mathrm{cl}(Z) \quad (n \geq 1),$$

(disjoint topological sum), and let $T_n \colon X_n \to X_{n-1}$ be the continuous extension of $\tilde{T}$ to X_n $(n \geq 1)$. Denote by $\Phi_{k,n}$ the canonical projection $X_n \to X_k$ $(n \geq k)$ and by $(X_\infty, \Phi_{k,\infty})$ the inverse limit of the inverse spectrum $(X_n, \Phi_{k,n})$ (see Appendix 2, Definition 2.2 of [Dugundji, 1966]). Similarly $(V_\infty, \Phi_{k,\infty|V_\infty})$ is the inverse limit of $(V_n, \Phi_{k,n|V_n})$. The following diagram commutes:

$$
\begin{array}{ccccccccccc}
X_0 & \overset{\Phi}{\longleftarrow} & X_1 & \overset{\Phi}{\longleftarrow} & X_1 & \overset{\Phi}{\longleftarrow} & \cdots & \longleftarrow & X_{k-1} & \overset{\Phi}{\longleftarrow} & X_\infty \\
\big\uparrow{\scriptstyle T_{1|V_1}} & & \big\uparrow{\scriptstyle T_{2|V_2}} & & \big\uparrow{\scriptstyle T_{3|V_3}} & & & & \big\uparrow{\scriptstyle T_{k|V_k}} & & \big\uparrow{\scriptstyle T_{\infty|V_\infty}} \\
V_1 & \overset{\Phi}{\longleftarrow} & V_2 & \overset{\Phi}{\longleftarrow} & V_3 & \overset{\Phi}{\longleftarrow} & \cdots & \longleftarrow & V_k & \overset{\Phi}{\longleftarrow} & V_\infty
\end{array}
$$

The map T_∞ is uniquely defined by the T_k $(1 \leq k \leq \infty)$, and as all $T_{k|V_k}$ $(1 \leq k < \infty)$ are continuous and injective, the same holds for $T_{\infty|V_\infty}$.

Hence $(X_\infty, T_\infty, \Phi_{0,\infty}^{-1} \tilde{\mathscr{Z}})$ is a piecewise compact $\Phi_{0,\infty}$-extension of $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Z}})$. Properties analogous to Lemmas 3.10 and 3.11 must be deduced separately in particular cases from the above construction. They depend strongly on how far $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Z}})$ is from being piecewise compact.

For later use we note that

$$\Phi_{0,\infty}(T_\infty^k x) = \tilde{T}^k(\Phi_{0,\infty} x) \quad \text{for } x \in \Phi_{0,\infty}^{-1}(\mathrm{reg}(\tilde{X}, \tilde{T}, \tilde{\mathscr{Z}}))$$

(see Definition 3.3) and for all x if $\tilde{T}$ is continuous.

We close this section with a class of examples for nonpiecewise compact systems:

3.15. **Example III** (piecewise monotonic transformations).

Let $\tilde{X} \subseteq \mathbf{R} \cup \{+\infty, -\infty\}$ be a bounded or unbounded interval, $\tilde{\mathscr{Z}}$ a finite or countable partition of $\tilde{X}$ into intervals, and let $\tilde{T} \colon \tilde{X} \to \tilde{X}$ be a map for which $\tilde{T}_{|Z}$ is strictly monotone and continuous for each $Z \in \tilde{\mathscr{Z}}$.

(a) The β-transformation, $\beta > 1$: $\tilde{X} = [0,1]$, $\tilde{T}x = \beta x \bmod 1$, $\tilde{\mathscr{Z}} = \{[0, \beta^{-1}), [\beta^{-1}, 2\beta^{-1}), \ldots, [k\beta^{-1}, 1]\}$ where k is the integer part of β. In

this case

$$X_1 = \sum_{j=0}^{k-1}[j\beta^{-1},(j+1)\beta^{-1}] + [k\beta^{-1},1],$$

$$X_2 = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}[i\beta^{-1}+j\beta^{-2},i\beta^{-1}+(j+1)\beta^{-2}]$$

$$+ \sum_{i=0}^{k-1}[i\beta^{-1}+k\beta^{-2},(i+1)\beta^{-1}]$$

$$+ \sum_{j=0}^{l-1}[k\beta^{-1}+j\beta^{-2},k\beta^{-1}+(j+1)\beta^{-2}]$$

$$+[k\beta^{-1}+l\beta^{-2},1],$$

where l is the integer part of $\beta(\beta-k)$, and so on. The Markov diagram looks like
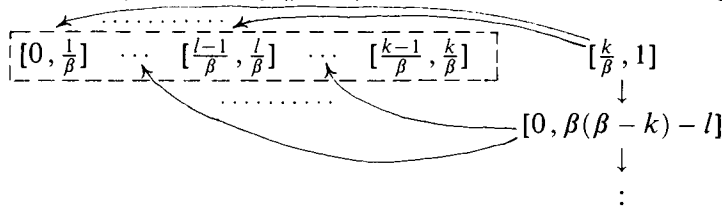
where also all intervals in the dashed box are pairwise joined by arrows.

(b) The continued fraction transformation:
$\hat{X} = [0,1]$, $\hat{T}x = \frac{1}{x} \bmod 1$ $(x>0)$ and $\hat{T}(0) = 0$. Obviously $\mathscr{D} = \{I_n = [\frac{1}{n+1},\frac{1}{n}]: n \in \mathbb{N}\} \cup \{\{0\}\}$, and there are arrows $I_n \to I_m$, $I_n \to \{0\}$, and $\{0\} \to \{0\}$ for all $m,n \in \mathbb{N}$. We shall not pursue further such systems whose Markov diagrams are not uniformly forward finite. [15]

4. TRANSFER OPERATORS AND ZETA FUNCTIONS

A. Transfer operators.

In this section we take a first step from the purely topological concepts of the last section towards a consideration of "smooth" and measure-theoretical structures of our systems:

Let $(X,T,\mathscr{Z})$ be a piecewise invertible system. By $\mathrm{mb}(X)$ (or simply mb if no confusion can occur) we denote the space of complex valued, bounded, Borel-measurable functions on X. Pick some $g \in \mathrm{mb}(X)$, and fix the following notation:

$$g_n(x) = g(x) \cdot g(Tx) \cdot \ldots \cdot g(T^{n-1}x).$$

---

[15] Cf. however the last paragraph of Example 4.10 and Remark 6.3.

4.1. Definition. (a) $(X, T, \mathcal{Z}, g)$ is a *weighted system* with *weight function* g, if

$$\sum_{Tx=y} |g(y)| \in mb(X)$$

as a function of x.

(b)

$$g_\infty = \lim_{n\to\infty} (\|g_n\|_\infty)^{1/n},$$

where $\|\cdot\|_\infty$ is the supremum-norm on $mb(X)$.

With a weighted system $(X, T, \mathcal{Z}, g)$ we associate the linear operator $P_g : mb \to mb$ (sometimes abbreviated as P)

$$(4.1) \qquad P_g f(x) = \sum_{Ty=x} g(y)f(y).$$

Our goal is to restrict P_g to suitable subspaces of smooth functions and to show that under certain assumptions on T and g this restriction is quasicompact or even quasinuclear.

For later use we note

$$(4.2) \qquad P_g^n f(x) = \sum_{T^n y=x} g_n(y)f(y) \qquad (n \in \mathbb{N})$$

4.2. Definition. P_g is the *transfer operator* of $(X, T, \mathcal{Z}, g)$.

4.3. Remark. These operators are frequently called *Perron-Frobenius operators*, in particular if the following setting is considered:

There is a nonatomic measure m on X with respect to which T is nonsingular and such that g^{-1} is its m-derivative. [16] Then P_g is dual to T in the sense that

$$(4.3) \qquad \int_X P_g f_1 \cdot f_2 \, dm = \int_X f_1 \cdot Tf_2 \, dm$$

whenever both sides of this equation are well defined ($Tf := f \circ T$). Obviously $m(\{g = 0\}) = 0$.

It is easily checked that for $0 \le f \in mb$ and $\mu = f \cdot m$

$$(4.4) \qquad P_g f = f \qquad \text{if and only if} \qquad T\mu = \mu,$$

where $(T\mu)(A) = \mu(T^{-1}A)$. But even if such an absolutely continuous, T-invariant measure exists, this is not the only choice of g for which the operator P_g describes relevant aspects of the dynamics of $(X, T, \mathcal{Z})$ (cf. §9.C).

We note that (4.3) implies

$$(4.5) \qquad \int_X |P_g f| \, dm \le \int_X |f| \, dm \qquad (f \in L_m^1).$$

---

[16] I.e. $\int_A g^{-1} dm = m(TA)$ for all measurable A contained in some $Z \in \mathcal{Z}$.

4.4. *Remark.* One might ask if, with g as in the preceding remark, $P_g : L_m^1 \to L_m^1$ has already interesting spectral properties. This is not the case:

Assume for simplicity that $P1 = 1$, i.e. $Tm = m$ and $P(Tf) = f$. For $\lambda \in \mathbb{C}$, $|\lambda| < 1$, define $Q_\lambda : L_m^1 \to L_m^1$ by

$$Q_\lambda = \sum_{k=0}^{\infty} \lambda^k T^k (\mathrm{Id} - TP).$$

Then $PQ_\lambda = \lambda Q_\lambda$, $Q_\lambda^2 = Q_\lambda$, and $Q_\lambda f = f$ if and only if $Pf = \lambda f$. Hence Q_λ is a projection onto $\{f \in L_m^1 : Pf = \lambda f\}$, and λ is a L_m^1-eigenvalue of P if and only if $Q_\lambda \neq 0$. Since T is not a bijection, there is $0 \neq f \in L_m^1$ with $0 \neq f - TPf = P^k T^k (f - TPf)$, i.e. $T^k(f - TPf) \neq 0$ for all k, and there are arbitrarily large n such that

$$Q_\lambda^{(n)} f = \sum_{k=0}^{n-1} \lambda^k T^k (\mathrm{Id} - TP) f \neq 0.$$

Now $Q_\lambda f \neq 0$ follows from

$$Q_\lambda f = Q_\lambda^{(n)} f + \sum_{k=1}^{\infty} \lambda^{kn} T^{kn} Q_\lambda^{(n)} f \quad \text{and}$$

$$\left\| \sum_{k=1}^{\infty} \lambda^{kn} T^{kn} Q_\lambda^{(n)} f \right\|_1 \leq \frac{|\lambda|^n}{1 - |\lambda|^n} \left\| Q_\lambda^{(n)} f \right\|_1 .$$

Hence λ is a L_m^1-eigenvalue of P, and the L_m^1-spectrum of P is the whole unit-disc. (This argument is taken from §IV in [Keller, 1984]. See also §1 of [Takahashi, 1981].)

B. Zeta-functions.

Another object we associate with $(X, T, \mathscr{Z}, g)$ is the *zeta function* $\zeta_g(z)$ (or simply $\zeta(z)$):

4.5. **Definition.** For $Y \subseteq X$ and $\mathscr{Y} \subseteq \mathscr{Z}$ let

$$(4.6) \qquad \gamma(Y, n) = \sum_{\substack{x: \, T^n x = x \\ T^i x \in Y, 0 \leq i \leq n}} g_n(x) \qquad \text{and} \qquad \gamma(\mathscr{Y}, n) = \gamma \left(\bigcup_{Z \in \mathscr{Y}} Z, n \right)$$

and define

$$\zeta_g(Y, z) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \gamma(Y, n) \right), \qquad \zeta_g(z) = \zeta_g(X, z).$$

Let

$$(4.7) \qquad\qquad\qquad \gamma(\mathscr{Y}) = \varlimsup_{n \to \infty} |\gamma(\mathscr{Y}, n)|^{1/n}.$$

Then $\zeta_g(z)$ is analytic in $\{|z| < \gamma(\mathscr{Z})^{-1}\}$, and using results of §2, we shall see that, under the same conditions we need in order to show that P_g is quasinuclear on some suitable function space, $\zeta_g(z)$ can be extended to a meromorphic function on a bigger domain, where its inverse is closely related to the Fredholm determinant of P_g.

C. Transfer operators and zeta functions for Markov extensions.

Suppose $(X, T, \mathscr{Z})$ is a piecewise compact system and $(\hat{X}, \hat{T}, \hat{\mathscr{Z}})$ its canonical Markov extension. A weight function g is lifted by the canonical projection $\pi: \hat{X} \to X$ to a weight function $\hat{g} \in \mathrm{mb}(\hat{X})$:

$$(4.8) \qquad \hat{g}(\hat{x}) = g(\pi(\hat{x})).$$

By $\hat{P} = \hat{P}_{\hat{g}}$ and $\hat{\zeta}(z) = \hat{\zeta}_{\hat{g}}(z)$ we denote the associated transfer operator and zeta function on $\hat{X}$.

A function $\hat{f} \in \mathrm{mb}(\hat{X})$ can also be regarded as a family $(f_D)_{D \in \mathscr{D}}$ of functions $f_D: D \to \mathbb{C}$ where $f_D(x) = \hat{f}(x, D)$. With this notation $\hat{P}_{\hat{g}}$ can be written in terms of the Markov diagram:

$$(4.9) \qquad (\hat{P}_{\hat{g}}^n \hat{f})_D = \sum_{\substack{C_0, \dots, C_{n-1} \in \mathscr{D} \\ C_0 \to \cdots \to C_{n-1} \to D}} (f_{C_0} \cdot g_{C_0}) \circ (T_{|C_0}^{-1} \circ \cdots \circ T_{|C_{n-1}}^{-1})_{|D}$$

Similarly, the zeta function $\hat{\zeta}$ can be expressed in terms of closed paths of the Markov diagram, provided $(X, T, \mathscr{Z})$ generates:

Suppose $\omega = D_0 \cdots D_{n-1} D_0$ is a closed path of length n in $\mathscr{D}$. By Lemma 3.11(a) there is exactly one $x \in D_0$ such that $\hat{T}^i(x, D_0) = (T^{i \bmod n} x, D_{i \bmod n})$ $(i \geq 0)$, and we can define $\hat{g}_n(\omega) = \hat{g}_n(x, D_0)$. Observing Lemmas 3.10(a) and 3.11(a) we thus obtain

$$(4.10) \qquad \hat{\zeta}_{\hat{g}}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\omega \text{ closed path of length } n} \hat{g}_n(\omega)\right).$$

The linear systems $(\mathrm{mb}(X), P_g)$ and $(\mathrm{mb}(\hat{X}), \hat{P}_{\hat{g}})$ are related by the maps

$$e_*: \mathrm{mb}(X) \to \mathrm{mb}(\hat{X}),$$
$$(4.11) \qquad (e_* f)(x, C) = \begin{cases} f(x), & \text{if } C \in \mathscr{Z}, \\ 0, & \text{if } C \in \mathscr{D} \setminus \mathscr{Z} \end{cases}$$

and

$$(4.12) \qquad \pi_*: \mathrm{mb}(\hat{X}) \to \mathrm{mb}(X), \quad (\pi_* \hat{f})(x) = \sum_{\pi(\hat{x}) = x} \hat{f}(\hat{x}).$$

Observe that

$$(4.13) \qquad \pi_*(e_* f) = f.$$

The basic (but not the only) device to "project" spectral properties from $\hat{P}$ down to P is

4.6. Lemma. *The linear system* $(\mathrm{mb}(X), P_g)$ *is a factor of the linear system* $(\mathrm{mb}(\hat{X}), \hat{P}_{\hat{g}})$ *via the onto-homomorphism* π_*, *i.e.*

$$\pi_* \circ \hat{P}_{\hat{g}} = P_g \circ \pi_*.$$

In particular

$$P_g^n f = \pi_* \hat{P}_{\hat{g}}^n (e_* f).$$

The proof is left to the reader. For similar statements cf. Lemma 10 in [Hofbauer/Keller, 1984] and Lemma 1 in [Mayer, 1984].

Denote by π_n the restriction of π to the set $\{\hat{x} \in \hat{X} : \hat{T}^n \hat{x} = \hat{x}\}$. The next lemma relates ζ_g and $\hat{\zeta}_{\hat{g}}$:

4.7. Lemma.

$$\hat{\zeta}_{\hat{g}}(z) = \zeta_g(z) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{z^{n'}}{n} \sum_{T^n x = x} g_n(x)(\mathrm{card}\ \pi_n^{-1}\{x\} - 1)\right).$$

Proof. Observe that $\hat{T}^n \hat{x} = \hat{x}$ implies $T^n \pi_n(\hat{x}) = \pi_n(\hat{x})$ and that $\hat{g}_n(\hat{x}) = g_n(\pi_n(\hat{x}))$. $\square$

D. Examples.

We evaluate the quotient $\hat{\zeta}/\zeta$ for the examples introduced in §3.

4.8. Example I (topological Markov shifts, sofic systems, cf. 3.12).

As $(\Omega_A, S, \mathscr{Z})$ is its own canonical Markov extension, $\hat{\zeta}_{\hat{g}}(z) = \zeta_g(z)$. We note by the way that if $g \equiv 1$ and Σ is finite, then $\zeta_g(z) = \det(\mathrm{Id} - zA)^{-1}$, see [Bowen/Lanford, 1970].

The zeta function of a sofic system is rational, and its inverse need not be a polynomial (see [Coven/Paul, 1975]).

4.9. Example II (up-and-down-counter, cf. 3.13).

As 0^∞ and 1^∞ are the only points which may occur as noninterior points of some $C \in \mathscr{D}$, we see from Lemma 3.11(b) that card $\pi_n^{-1}\{x\} = 1$ if $T^n x = x$ and $x \neq 0^\infty, 1^\infty$. From the diagram in 3.13 one infers that $\pi_n^{-1}\{0^\infty\} = \varnothing$ and $\pi_n^{-1}\{1^\infty\} = \{(1^\infty, E)\}$ for all n. Hence

$$\frac{\hat{\zeta}_{\hat{g}}(z)}{\zeta_g(z)} = \exp\left(-\sum_{n=1}^{\infty} \frac{(z \cdot g(0^\infty))^n}{n}\right) = 1 - z \cdot g(0^\infty).$$

As in Example I we can give a simpler expression for $\zeta_g(z)$ if $g \equiv 1$: W_M is a circular code in the sense of [Keller, 1987]. Hence

$$\exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \gamma(W_M^\infty, n)\right) = \left(1 - \sum_{n=1}^{\infty} z^n \cdot \mathrm{card}(W_M \cap \Sigma^n)\right)^{-1}$$

$$= \left(1 - \sum_{n \in M} z^{2n}\right)^{-1}$$

(ibid., Lemma 1 [17]) and

$$\zeta(z) = (1 - z)^{-2} \left(1 - \sum_{n \in M} z^{2n} \right)^{-1}$$

(ibid., Lemma 2). In the particular case $M = \mathbb{N}$ we thus have

$$\zeta(z) = \frac{(1 + z)}{(1 - z)(1 - 2z^2)}.$$

One should note that, although $\zeta(z)$ is meromorphic, neither ζ nor ζ^{-1} can be analytically extended to $\{|z| < r\}$ for $r > 1$ in contrast to the situation in Example I. If M is finite, then $\zeta(z) = 1/\text{polynomial}(z)$. (This is not necessarily true for general sofic systems, see e.g. the "even" system of [Weiss, 1973].

4.10. Example III (piecewise monotonic transformations, cf. 3.15).

Suppose that $\tilde{\mathcal{Z}}$ is finite. If $\tilde{x} \in \tilde{X}$ is regular [18] then $\Phi_{0,\infty}^{-1}\{\tilde{x}\}$ contains exactly one element, and $T^n x = x$ if and only if $\tilde{T}^n \tilde{x} = \tilde{x}$. On the other hand, if $\tilde{x}$ is not regular, then $\tilde{T}^k \tilde{x} \in \text{bd}(Z)$ for some $k \in \mathbb{N}$ and $Z \in \tilde{\mathcal{Z}}$, and there are at most finitely many nonregular points periodic under $\tilde{T}$, each of which has at most two preimages under $\Phi_{0,\infty}$. Hence

$$(4.14) \qquad \zeta_g(z) = \tilde{\zeta}_{\tilde{g}}(z) \hat{H}(z)$$

where $\hat{H}(z)$ is a finite product of expressions of the type

$$(4.15) \qquad \exp\left(\pm \sum_{n=1}^{\infty} \frac{z^{kn}}{n} (g_k(x))^n \right) = 1 - z^k g_k(x)$$

with $k \in \mathbb{N}$ and $x \in \text{bd}(Z), Z \in \tilde{\mathcal{Z}}$. (We have tacitly assumed that $\tilde{g} : \tilde{X} \to \mathbb{C}$ is continuous on each $Z \in \tilde{\mathcal{Z}}$ such that it can be lifted by $\Phi_{0,\infty}$ to a continuous weight function $g = \tilde{g} \circ \Phi_{0,\infty}$ on X.)

Similarly, if $T^n x = x$ and if x is interior to each $D \in \mathcal{D}$ it belongs to, then there is a unique $\hat{x} \in \pi_n^{-1}\{x\}$ with $\hat{T}^n \hat{x} = \hat{x}$ (see Lemma 3.11(b)). But if $x \in \text{bd}(D)$ for some $D = T^k Z, Z \in \mathcal{Z}_k$ (see Lemma 3.6), then $x \in T^j(\text{bd}(Z))$ for some $Z \in \mathcal{Z}$ and $0 \leq j \leq k$, i.e. one of the finitely many boundary points of the sets $Z \in \mathcal{Z}$ is ultimately periodic and its orbit contains x. Hence $\hat{x} \in \pi_n^{-1}\{x\}$ is unique for all but finitely many x from above, and for these exceptional x [Hofbauer, 1986] has shown that there are at most four $\hat{T}$-periodic $\hat{x} \in \pi^{-1}\{x\}$, each of prime period p or $2p$ if p is the prime period of x. Therefore

$$(4.16) \qquad \zeta_g(z) = \hat{\zeta}_{\hat{g}}(z) \hat{H}(z)$$

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[17] The mathematical essence of this lemma is similar to Corollary 1 of Theorem 1 in [Takahashi, 1983]. The language used there is very different, however.

[18] I.e. if $\tilde{T}^k \tilde{x} \in \text{int } V_n(\tilde{T}^k \tilde{x})$ for all $k, n \geq 0$.

where $\hat{H}(z)$ is again a finite product of expressions of the type (4.15). It will turn out later that the factors $\tilde{H}(z)$ and $\hat{H}(z)$ are unessential for our purposes.

A rather advanced treatment of the continued fraction transformation can be found in [Mayer, 1976]. There it is shown that $\zeta_g(z)$ is meromorphic in $\mathbb{C}$ for a wide class of functions g and that P_g acts as a nuclear operator on a suitable space of holomorphic functions. The proof relies heavily on the fact that each monotonicity interval is mapped by this transformation onto the whole unit interval.

5. ANALYTIC STRUCTURES

The phase space X of a piecewise compact system is in general highly discontinuous; indeed, if the system generates, it is completely disconnected, because each point x possesses a base of closed open neighbourhoods $V_n(x)$. Therefore we cannot expect X to carry a differentiable structure in a rigorous sense. On the other hand, if we are primarily interested in interval maps e.g., we have a clear idea of what differentiability of T means, although $(X, T, \mathscr{Z})$ is the system obtained from the "smooth" system $(\tilde{X}, \tilde{T}, \tilde{\mathscr{Z}})$ by doubling all preimages of critical points (cf. 3.14–3.15): We simply forget about the doubling (that was necessary in order to avoid troubles in constructing the Markov extension) and re-identify the split up points. In this way some subspace of $\mathrm{mb}(X)$ can be identified with a space of holomorphic functions on a complex neighbourhood of $[0, 1]$, and the transfer operator behaves as if it were acting on this space of holomorphic functions. From here it is only a small step to apply such an identification-procedure to systems which we never thought of being smooth, e.g. to subshifts.

We start recalling some facts about spaces of bounded holomorphic functions

A. Bounded holomorphic functions.

Suppose that Ω is a region in $\mathbb{C}^d$. By $\mathscr{H}^\infty(\Omega)$ we denote the space of bounded holomorphic functions on Ω. Endowed with the supremum-norm $\|\cdot\|_\infty$, $\mathscr{H}^\infty(\Omega)$ is a Banach space.

The following important theorem will turn out to be responsible for the nuclearity of certain "partial" transfer operators on $\mathscr{H}^\infty(\Omega)$ (cf. Theorem 5.8):

5.1. Theorem (of Montel as extended by Grothendieck[19]). *Let Ω_1, Ω_2 be regions in $\mathbb{C}^d$, $\mathrm{cl}(\Omega_1) \subseteq \Omega_2$. Then the natural embedding $\mathscr{H}^\infty(\Omega_2) \hookrightarrow \mathscr{H}^\infty(\Omega_1)$ is not only compact (Montel) but also nuclear (Grothendieck).*

B. Piecewise analytic systems.

Suppose $(X, T, \mathscr{Z}, g)$ is a weighted system and $(\mathscr{D}, \rightarrow)$ is its Markov diagram.

5.2. Definition. A pair $\mathscr{A} = (s, (U_D)_{D \in \mathscr{D}})$ is an *analytic structure* for $(X, T, \mathscr{Z}, g)$ if there is $d \geq 1$ with:

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[19] [Grothendieck, 1955], §1, no. 3

(i) U_D $(D \in \mathscr{D})$ are regions in $\mathbb{C}^d$, and

$$\sup\{\operatorname{diam}(U_D): D \in \mathscr{D}, U_D \neq \mathbb{C}^d\} < \infty.$$

(ii) s is a map from X to $\mathbb{C}^d$ with $\operatorname{cl}(s(D)) \subseteq U_D \cap \mathbb{R}^d$, and $s(D)$ is a uniqueness set for bounded holomorphic functions on U_D.

(iii) for $C, D \in \mathscr{D}$ with $C \to D$ there is a holomorphic map $\tau_{CD}: U_D \to U_C$ such that $s \circ (T_Z^{-1})_{|D} = \tau_{CD} \circ s_{|D}$ if $C \subseteq Z$.

Denote by $D\tau_{CD}$ the total derivative of τ_{CD} and by $\|D\tau_{CD}\|$ its Euclidean operator norm on $\mathbb{C}^d$. Since $\tau_{CD}(U_D \cap \mathbb{R}^d) \subseteq U_C \cap \mathbb{R}^d$, $D\tau_{CD}$ has real entries.

(iv) For each $D \in \mathscr{D}$ either $g_D \equiv 0$, or there is $\tilde{\psi}_D \in \mathscr{H}^\infty(U_D)$ with $g_D = \exp(\tilde{\psi}_D) \circ s_{|D} =: \tilde{g}_D \circ s_{|D}$. The $\tilde{\psi}_D$ and their derivatives are uniformly bounded. Here g_D stands for $g_{Z|D}$ when $C \subseteq Z$.

For a path $\omega = D_0 \cdots D_n$ let

$$\tau_\omega = \tau_{D_0,D_1} \circ \cdots \circ \tau_{D_{n-1},D_n}, \qquad U_\omega = U_{D_n},$$

and

$$T_\omega^{-1} = (T_{D_0}^{-1} \circ \cdots \circ T_{D_{n-1}}^{-1})_{|D_n}.$$

Then $T_\omega^{-1} = (T^n_{|Z(\omega)})^{-1}$ (cf. Lemma 3.6) and

$$s \circ T_\omega^{-1} = \tau_\omega \circ s_{|D_n}.$$

The meaning of $D\tau_\omega$ and $\|D\tau_\omega\|$ is obvious.

5.3. Definition. $(X, T, \mathscr{Z}, g)$ is $\mathscr{A}$-expanding if

(i) there are $n_0 \in \mathbb{N}$ and $\alpha < 1$ such that $\|D\tau_\omega\| \leq \alpha$ on U_ω for all paths ω in $\mathscr{D}$ of length $n \geq n_0$, for which $g_{n|Z(\omega)} \not\equiv 0$.

(ii) $\operatorname{cl}(\tau_{CD}(U_D)) \subseteq U_C$ for all $C, D \in \mathscr{D}$ with $C \to D$ and $g_D \not\equiv 0$.

5.4. Remark. At first sight it might look sufficient to define analytic structures only on the cylinders $Z \in \mathscr{Z}$ and not on all components D of the Markov extension. However, the notion of an $\mathscr{A}$-expanding system would become more complicated in this case.

C. Piecewise analytic Markov systems.

Throughout this subsection suppose that $(X, T, \mathscr{Z}, g)$ is Markovian. Remember that $\mathscr{Z} = \mathscr{D}$ in this case.

5.5. Remark. If $(X, T, \mathscr{Z}, g)$ is $\mathscr{A}$-expanding and if ω is a closed path of length n in $\mathscr{D}$ such that $g_{n|Z(\omega)} \not\equiv 0$, then τ_ω has a unique fix-point x_ω and $\|D\tau_\omega(x_\omega)\| \leq \alpha^{n/n_0}$ (see Lemma 1 of [Ruelle, 1976]). We denote

$$(5.1) \qquad \kappa(n) = \sum_\omega \|D\tau_\omega(x_\omega)\| \cdot |g_n(\omega)|$$

where the sum extends over all closed paths ω of length n with $g_{n|Z(\omega)} \not\equiv 0$, and

$$(5.2) \qquad \kappa := \varlimsup_{n \to \infty} \kappa(n)^{1/n} \leq \alpha^{1/n_0} \gamma(\mathscr{Z}).$$

Observe that $\kappa < \gamma(\mathscr{Z})$ if $\gamma(\mathscr{Z}) > 0$.

5.6. Definition. For an analytic structure $\mathscr{A}$ of $(X, T, \mathscr{Z}, g)$ let

$$\mathscr{H}^\infty(\mathscr{A}) = \left\{ f \in \mathrm{mb}(X) \colon \forall D \in \mathscr{D} \, \exists \tilde{f}_D \in \mathscr{H}^\infty(U_D) \text{ such that } f_D = \tilde{f}_D \circ s_{|D} \right.$$

$$\left. \text{and } \| f \|_{\mathscr{A}} := \sum_{D \in \mathscr{D}} \| \tilde{f}_D \|_\infty < \infty \right\}$$

be the space of $\mathscr{A}$-holomorphic functions on X.

Notation: $\| f_D \|_{\mathscr{A}} := \| \tilde{f}_D \|_\infty$.

5.7. Remark. $(\mathscr{H}^\infty(\mathscr{A}), \| \cdot \|_{\mathscr{A}})$ is a Banach space. (Observe that the map $\tilde{f}_D \mapsto \tilde{f}_D \circ s_{|D}$ is 1-1 by Definition 5.2(ii).)

5.8. Theorem. *Suppose that* $(X, T, \mathscr{Z}, g)$ *is* $\mathscr{A}$-*expanding. For a path* $\omega = D_0 \cdots D_n$ *define* $L_\omega \colon \mathscr{H}^\infty(\mathscr{A}) \to \mathscr{H}^\infty(\mathscr{A})$ *by*

$$(L_\omega f)(x) = \begin{cases} (g_n \cdot f) \circ T_\omega^{-1}(x), & \text{if } x \in D_n, \\ 0, & \text{otherwise}. \end{cases}$$

Then L_ω *is nuclear* (*in the sense of* §2.C), *and*

$$(5.3) \qquad \mathrm{tr}(L_\omega) = g_n(x_\omega) \cdot \det(\mathrm{Id} - D\tau_\omega(x_\omega))^{-1} \quad \text{if } \omega \text{ is closed},$$

$$(5.4) \qquad \mathrm{tr}(L_\omega) = 0 \quad \text{otherwise, i.e. if } D_0 \neq D_n.$$

Proof. The nuclearity is an immediate consequence of 5.1. (5.3) is contained in the proof of Lemma 2 of [Ruelle, 1976], and (5.4) follows from Chapter II.3, Proposition 2 of [Grothendieck, 1956].

D. Examples.

5.9. Examples I, II (closed subshifts, cf. 3.12–3.13).

For $j = 1, \ldots, d$ let $\phi_j \colon \Sigma \to \mathbb{C}$ be nonconstant bounded maps, let $\rho_j \in \mathbb{C}$, $|\rho_j| \leq \rho < 1$, and let $R = \max\{|\phi_j(\sigma)|/(1 - |\rho_j|) \colon j = 1, \ldots, d \,; \, \sigma \in \Sigma\}$. Define $s \colon \Omega \to U := \{z \in \mathbb{C}^d \colon |z_j| < R \ (j = 1, \ldots, d)\}$ by

$$s_j(\sigma_1, \sigma_2, \sigma_3, \ldots) = \sum_{i=1}^\infty \phi_j(\sigma_i) \cdot \rho_j^i.$$

For $D \in \mathscr{D}$ let $U_D = U$, and if $C \to D$ in $\mathscr{D}$ then define $\tau_{CD} \colon U \to U$ by

$$(\tau_{CD}(z))_j = (\phi_j(\sigma) + z_j) \cdot \rho_j \quad \text{if } C \subseteq [\sigma]_1.$$

We check that $(s, (U_D)_{D \in \mathscr{D}})$ is an analytic structure:

(a) $|s_j(w)| \leq \rho R \ (w \in \Omega)$, hence $\mathrm{cl}(s(D)) \subseteq U_D \ (D \in \mathscr{D})$.

(b) Let $C \subseteq [\sigma]_1$, $C \to D$, $w \in D$. Then

$$(s(T_C^{-1} w))_j = s_j(\sigma w) = \rho_j(\phi_j(\sigma) + s_j(w)) = (\tau_{CD}(s(w)))_j,$$

i.e. $s \circ (T_C^{-1})_{|D} = \tau_{CD} \circ s_{|D}$.

(c) For $d = 1$ it is an easy excercise to show that $s(D)$ is a uniqueness set $(D \in \mathscr{D})$ in case of Examples I and II, if one assumes for Example I that there are no trivial cylinders (i.e. consisting of finitely many points only). The corresponding assertion for $d > 1$ follows, because now $s(D)$ is the Cartesian product of one-dimensional uniqueness sets.

(d) Obviously the τ_{CD} are holomorphic on U and $\|D\tau_{CD}\| = \prod_{j=1}^{d} \rho_j \leq \rho^d < 1$. Finally, for $z \in U$

$$|(\tau_{CD}(z))_j| \leq \rho(R(1 - \rho) + R) = \rho(2 - \rho)R < R$$

such that $\mathrm{cl}(\tau_{CD}U) \subseteq U$ and $(\Omega, S, \mathscr{D})$ is $\mathscr{A}$-analytic. In particular $\kappa \leq \gamma(\mathscr{D}) \prod_{j=1}^{d} \rho_j$.

Similarly one may choose $U_D = \mathbb{C}$ $(D \in \mathscr{D})$ and $\tau_{CD}: \mathbb{C} \to \mathbb{C}, z \mapsto 0$ if $C \to D$. In this case $\kappa = 0$.

5.10. Example III (piecewise monotonic transformations, cf. 3.14–3.15).

Let $s = \Phi_{0,\infty}$, and for $D \in \mathscr{D}$ and $\varepsilon > 0$ let $U_D = \{z \in \mathbb{C}: \mathrm{dist}(z, s(D)) < \varepsilon\}$. U_D is a region in $\mathbb{C}$ and $\mathrm{cl}(s(D)) \subseteq U_D$. Suppose that $(\tilde{T}_{\tilde{Z}}^{-1})_{|\tilde{T}\tilde{Z}}$ has a bounded holomorphic extension to a complex neighbourhood of $\tilde{T}\tilde{Z}$ $(\tilde{Z} \in \tilde{\mathscr{Z}})$ and that $\inf_{\tilde{x} \in \tilde{X}} |\tilde{T}'(\tilde{x})| > 1$. Now, if $\tilde{\mathscr{Z}}$ is finite, $\varepsilon > 0$ can be chosen small enough such that there exist extensions τ_{CD} of $(\tilde{T}_{s(Z)})_{|s(D)}^{-1}$ to U_D (if $C \to D$ and $C \subseteq Z$), and by choosing ε even smaller (if necessary), one may assume

$$|\tau_{CD}'| \leq \alpha < 1 \qquad \text{for some } \alpha < 1.$$

In particular $\mathrm{cl}(\tau_{CD}(U_D)) \subseteq U_C$ for such C, D.

We still have to assure that $s(D)$ is a uniqueness set for bounded holomorphic functions on U_D, $(D \in \mathscr{D})$, i.e. $\mathrm{card}(s(D)) = \infty$ for all $D \in \mathscr{D}$. This requirement is fullfilled, however, since it is easily seen that the endpoints of each $D \in \mathscr{D}$ are non-isolated in D. Hence $\mathscr{A} = (s, (U_D)_{D \in \mathscr{D}})$ is an analytic structure for $(X, T, \mathscr{Z}, g)$ for suitable g, e.g. for $g = 1/|\tilde{T}' \circ s|$, (just the structure on X inherited from $\tilde{X} \subseteq \mathbb{R}$), and $(X, T, \mathscr{Z}, g)$ is $\mathscr{A}$-expanding.

6. Spectral theory for Markov systems

A. The main theorems.

Assume throughout this section that $(X, T, \mathscr{Z}, g)$ is a weighted Markov system, expanding with respect to the analytic structure $\mathscr{A} = (s, (U_D)_{D \in \mathscr{D}})$.

Fix an increasing sequence $(\mathscr{E}_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathscr{D}$ such that $\bigcup_{n \in \mathbb{N}} \mathscr{E}_n = \mathscr{D}$, and define projections $\beta[\mathscr{E}_n]: \mathscr{H}^{\infty}(\mathscr{A}) \to \mathscr{H}^{\infty}(\mathscr{A})$ by

$$(\beta[\mathscr{E}](f))_Z = \begin{cases} f_Z & \text{if } Z \in \mathscr{E}, \\ 0 & \text{otherwise}, \end{cases}$$

$\mathscr{F} = \{\beta[\mathscr{E}_n]: n \in \mathbb{N}\}$ is a continuous projection net (see Definition 2.1).

Finally suppose that $\phi \in \mathrm{mb}(X)$ is real-valued, $\phi_D = \tilde{\phi}_D \circ s$ for some $\tilde{\phi}_D \in \mathscr{H}^{\infty}(U_D)$ $(D \in \mathscr{D})$, and that the $\tilde{\phi}_D$ are uniformly bounded.

Under these assumptions and with the notational conventions of §2.D we can state

6.1. Theorem. *Suppose that the Markov diagram of* $(X, T, \mathscr{X})$ *is uniformly forward finite.* [20]

(a) *The transfer operator* $P = P_{g \cdot \exp(i\phi)}$ *leaves* $\mathscr{H}^\infty(\mathscr{A})$ *invariant.*

(b) *If* $r_{as} < r(P)$, *then* P *is quasicompact on* $\mathscr{H}^\infty(\mathscr{A})$ *and* $r_{ess} \leq r_{as}$. *If* $\phi \equiv 0$ *and* $g \geq 0$, *then* $r_{ess} = r_{as}$.

(c) *If* $r_{as} < r(P)$, *if there is some Borel measure* m *on* X *with* $\mathrm{supp}(m) = X$, $\sup_{D \in \mathscr{D}} m(D) < \infty$, *and such that* g^{-1} *is the derivative of* T *with respect to* m, [21] *and if* $\mathscr{H}^\infty(\mathscr{A})$ *is dense in* L_m^1, *then* P *admits a representation*

$$(6.1) \qquad P = \sum_{i=1}^{l} \lambda_i E_i + \sum_{i=l+1}^{N} \lambda_i (E_i + N_i) + Q$$

with E_i, N_i, *and* Q *as in* (2.2) *and where* $\Gamma = \{\lambda_1, \ldots, \lambda_l\} \neq \varnothing$ *is the set of eigenvalues of modulus* 1, *and* $|\lambda_j| < 1$ *for* $j = l+1, \ldots, N$. *For each* $\lambda_i \in \Gamma$

$$(6.2) \qquad E_i = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\lambda_i^{-1} P)^k$$

uniformly on the unit ball of $\mathscr{H}^\infty(\mathscr{A})$. E_i *extends to* L_m^1, $\|E_i\|_1 = 1$, *and* (6.2) *holds strongly on* L_m^1. $(P - \sum_{i=1}^{l} \lambda_i E_i)^n$ *tends to* 0 *as* $n \to \infty$ *strongly on* L_m^1. *If* $Pf = \lambda f$ *for some* $0 \neq f \in L_m^1$ *and* $|\lambda| = 1$, *then* $f \in \mathscr{H}^\infty(\mathscr{A})$ *and* $\lambda \in \Gamma$.

(If $\mathrm{supp}(m) \neq X$, *then* (6.1) *and* (6.2) *still hold in the* L_m^1-*sense.)*

(d) *If, in the situation of* (c), $\phi \equiv 0$, *i.e. if* P *is the classical Perron-Frobenius operator satisfying* (4.3), *then* $\lambda_1 = 1 \in \Gamma$, E_1 *is positive, and* Γ *is fully cyclic.* [22]

Let $d := \mathrm{rank}(E_1)$. *There are a* mod m-*partition* $(X_1, \ldots, X_d)$ *of* X *into* $T - T^{-1}$-*invariant measurable sets and a basis* $\{h_1, \ldots, h_d\}$ *of* $\mathrm{range}(E_1)$ *such that* $h_i \geq 0$ *and* $\int_{X_i} h_j \, dm = \delta_{ij}$ $(i, j = 1, \ldots, d)$. *If* h *is any convex combination of the* h_i *and* $d\mu = h \, dm$, *then* μ *is a* T-*invariant probability measure.*

(e) *If* P_g *satisfies the assumptions of* (c) *and* (d) *and if the eigenspace* $\mathrm{range}(E_1)$ *of* P_g *is one-dimensional, then (without any further assumptions on* $P = P_{g \cdot \exp(i\phi)}$) $P : L_m^1 \to L_m^1$ *has an eigenvalue of modulus* 1 *if and only if* $\exp(i\phi)$ *is equivalent to a constant, i.e. there are* $a \in \mathbf{R}$ *and* $\psi \in \mathrm{mb}(X)$ *such that* $\exp(i\phi) = \exp i(a + \psi \circ T - \psi)$ m-*a.e.*

6.2. Theorem. *Suppose* $(X, T, \mathscr{X})$ *generates, that the Markov diagram of* $(X, T, \mathscr{X}, g)$ *is uniformly forward finite and that* $g \geq 0$. *Let* $P = P_g$ *and*

$$\gamma_{as} = \lim_{n \to \infty} \gamma(\mathscr{D} \setminus \mathscr{E}_n) \qquad (cf. \ (4.7)).$$

---

[20] See Definition 3.7.

[21] Cf. Remark 4.3.

[22] I.e. $Pf = \lambda f$, $|\lambda| = 1$, implies $P(f^k / |f|^{k-1}) = \lambda^k f^k / |f|^{k-1}$ for all $k \in \mathbf{Z}$, see Definition V.4.5 in [Schaefer, 1974].

Assume

(i) $r(P) < \infty$, $\gamma(\mathscr{D}) < \infty$,

(ii) $[\gamma_{as} < \gamma(\mathscr{D})$ *or* $\gamma_{as} \leq r(P)]$ *and* $[r_{as} < r(P)$ *or* $r_{as} \leq \gamma(\mathscr{D})]$,

(iii) $\lim_{\alpha \in \mathscr{F}} \|\alpha(P_\beta \alpha)^j P\beta - P_\beta^j P\beta\|_T = 0$ *for all* $\beta \in \mathscr{F}$ *and* $j \in \mathbb{N}$.[23]

Then $P: \mathscr{H}^\infty(\mathscr{A}) \to \mathscr{H}^\infty(\mathscr{A})$ *is* $\mathscr{F}$-*quasinuclear and* $r(P) = \gamma(\mathscr{D})$. *The inverse of the zeta function* $\zeta_g(z)$ *can be analytically extended from* $\{|z| < \gamma(\mathscr{D})^{-1}\}$ *to*[24]

$$G_{\mathscr{F}} := \left\{ z \in \mathbb{C} : |z| < 1/\max\{\gamma_{as}, r_{as}, \kappa\} \right\},$$

and $\det_{\mathscr{F}}(\mathrm{Id} - zP_g) \cdot \zeta_g(z)$ *is analytic and nonzero on* $G_{\mathscr{F}}$. $z_0 \in G_{\mathscr{F}}$ *is a pole of* $\zeta_g(z)$ *if and only if* z_0^{-1} *is an eigenvalue of* P_g. *The order of the pole* z_0 *of* $\zeta_g(z)$ *is finite and equals the algebraic multiplicity of the eigenvalue* z_0^{-1} *of* P_g.

Even without assuming (ii) $1/\zeta_g(z)$ *can be analytically extended to* $G_{\mathscr{F}}$.

6.3. Remark. The assumption in both theorems that the Markov diagram of $(X, T, \mathscr{Z})$ is uniformly forward finite is needed only for showing that P is a bounded operator on $\mathscr{H}^\infty(\mathscr{A})$ and that $P\alpha$ is nuclear for $\alpha \in \mathscr{F}$. It can be replaced by these two assertions. Alternatively one could assume in view of Remark 2.4(c) that P is bounded on $\mathscr{H}^\infty(\mathscr{A})$ and αP is nuclear for $\alpha \in \mathscr{F}$. In this case the trace-norm in assumption (iii) of Theorem 6.2 may be replaced by the usual operator-norm. With these changes both theorems apply also to systems like the continued fraction transformation, but it may be quite hard to verify these assumptions (see 4.10 and [Mayer, 1976]).

6.4. Remark. Even if $G_{\mathscr{F}}$ is the greatest ball around 0 to which $\zeta_g(z)^{-1}$ can be analytically extended, it may still be possible to extend $\zeta_g(z)$ meromorphically to a bigger ball (see also 6.13, Example II).

The proof of the theorems needs the following monotonicity lemma:

6.5. Lemma. *Assume that* $g \geq 0$. *Let* $\Omega_1 \subseteq \Omega_2$ *be sets of paths of length* n *in* D, *and suppose that there are* $f_\omega \in \mathscr{H}^\infty(\mathscr{A})$ $(\omega \in \Omega_1)$. *If* a_D $(D \in \mathscr{D})$ *are positive constants with* $\|(f_\omega)_D\|_{\mathscr{A}} \leq a_D$ $(\omega \in \Omega_1)$ *and if* $f^* = \sum_{D \in \mathscr{D}} a_D \chi_D$, *then*

$$\sum_{\omega \in \Omega_1} \|(f_\omega \cdot g_n) \circ T_\omega^{-1}\|_{\mathscr{A}} \leq A \cdot \left\| \sum_{\omega \in \Omega_2} (f^* \cdot g_n) \circ T_\omega^{-1} \right\|_{\mathscr{A}}$$

with a constant A *independent of* $n, \Omega_1, \Omega_2, f_\omega$, *and* f^*.

Proof. Observing that $(X, T, \mathscr{Z}, g)$ is $\mathscr{A}$-expanding and that $\tilde{g}_D \equiv 0$ or $\tilde{g}_D = \exp \tilde{\psi}_D \in \exp \mathscr{H}^\infty(U_D)$ for $D \in \mathscr{D}$, where the derivatives of the $\tilde{\psi}_D$ are uniformly bounded, it needs a routine calculation to show that there is a constant $B > 0$ with

$$|\tilde{g}_n(\tau_\omega z_1)| \leq |\tilde{g}_n(\tau_\omega z_2)| \cdot \exp(B|z_1 - z_2|)$$

---

[23] $\|.\|_T$ is the trace-norm, see §2.C.
[24] For κ see Remark 5.5.

for $n \in \mathbb{N}$ and z_1, z_2 in the domain of τ_ω, $\omega = D_0 \cdots D_n$. In particular

$$(6.3) \qquad\qquad |\tilde{g}_n(\tau_\omega z_1)| \leq \text{const} \cdot |\tilde{g}_n(\tau_\omega z_2)|$$

for n, ω, z_1, z_2 as above. If $U_{D_n} \neq \mathbb{C}^d$ this follows from the uniform bound-edness of the sets $U_D \neq \mathbb{C}^d$ (see Definition 5.2(i)). Otherwise $\tilde{g}$ is constant on U_{D_n}, and it suffices to treat $\tilde{g}_{n-1} \circ \tau_{\omega'}$ where $\omega' = D_0 \cdots D_{n-1}$.

Now the positivity of g implies

$$\sum_{\omega \in \Omega_1} \|(f_\omega \cdot g_n) \circ T_\omega^{-1}\|_{\mathscr{A}} \leq \sum_{\substack{D \in \mathscr{L} \\ C \in \mathscr{L}}} \sum_{\substack{\omega \in \Omega_1 \\ \omega = C \cdots D}} \|(f_\omega)_C\|_{\mathscr{A}} \cdot \|\tilde{g}_n \circ \tau_\omega\|_\infty$$

$$\leq \sum_{\substack{D \in \mathscr{L} \\ C \in \mathscr{L}}} \sum_{\substack{\omega \in \Omega_2 \\ \omega = C \cdots D}} a_C \cdot \text{const} \cdot \min_{x \in D} \{\tilde{g}_n \circ \tau_\omega(s(x))\} \qquad \text{(by (6.3))}$$

$$\leq \text{const} \cdot \sum_{D \in \mathscr{L}} \left\| \sum_{\substack{\omega \in \Omega_2 \\ \omega = \dots D}} (f^* \cdot g_n) \circ T_\omega^{-1} \right\|_{\mathscr{A}}$$

$$= \text{const} \cdot \left\| \sum_{\omega \in \Omega_2} (f^* \cdot g_n) \circ T_\omega^{-1} \right\|_{\mathscr{A}} . \qquad \square$$

6.6. Remark. The need for such a lemma is the main reason why I restricted Theorem 6.2 to the case $g \geq 0$. If Lemma 6.5 holds because of other reasons, most of the theorem remains true for more general g.

Proof of Theorem 6.1. As the Markov diagram of $(X, T, \mathscr{Z})$ is uniformly forward finite, $P\beta$ $(\beta \in \mathscr{F})$ is a finite sum of operators of the type $f \mapsto (f g \exp(i\phi)) \circ T_{CD}^{-1}$ each of which is nuclear by Theorem 5.8. Hence $P\beta$ is nuclear and a fortiori compact for $\beta \in \mathscr{F}$. P leaves $\mathscr{H}^\infty(\mathscr{A})$ invariant, because for $f \in \mathscr{H}^\infty(\mathscr{A})$

$$\|| Pf \||_{\mathscr{A}} \leq \sum_{D \in \mathscr{L}} \sum_{\substack{C \in \mathscr{L} \\ C \to D}} \|(g \cdot \exp(i\phi) \cdot f) \circ (T_C^{-1})_{|D}\|_{\mathscr{A}}$$

$$(6.4) \qquad\qquad \leq \sum_{C \in \mathscr{L}} \sum_{\substack{D \in \mathscr{L} \\ C \to D}} \|g_C \exp(i\phi_C)\|_{\mathscr{A}} \|f_C\|_{\mathscr{A}}$$

$$\leq \text{const} \cdot \sup_{C \in \mathscr{L}} \|g_C \exp(i\phi_C)\|_{\mathscr{A}} \cdot \|| f \||_{\mathscr{A}}$$

The quasicompactness of P and $r_{\text{ess}} \leq r_{\text{as}}$ follow from Proposition 2.2.

Suppose now $\phi \equiv 0$ and $g \geq 0$. For $f \in \mathscr{H}^\infty(\mathscr{A})$ let $f^* = \sum_{D \in \mathscr{L}} \|f_D\|_{\mathscr{A}} \cdot \chi_D$. Then $f^* \in \mathscr{H}^\infty(\mathscr{A})$, $\||f^*\||_{\mathscr{A}} = \||f\||_{\mathscr{A}}$, and for $\alpha, \beta \in \mathscr{F}$ with $\alpha \geq \beta$ and $r_\alpha \geq r_{\text{as}} - \varepsilon$:

$$\||f\||_{\mathscr{A}} \cdot \|| (\text{Id} - \beta) P^n (\text{Id} - \beta) \||_{\mathscr{A}}$$

$$\geq \|| (\text{Id} - \beta) P^n (\text{Id} - \beta) f^* \||_{\mathscr{A}}$$

$$\geq A^{-1} \cdot \|| P_\alpha^n f \||_{\mathscr{A}} \qquad \text{(by Lemma 6.5)}.$$

Hence
$$A \cdot \||| (\mathrm{Id} - \beta) P^n (\mathrm{Id} - \beta) \|||_{\mathscr{A}} \geq \||| P_\alpha^n \|||_{\mathscr{A}} \geq r_\alpha^n \geq (r_{\mathrm{as}} - \varepsilon)^n,$$
and $r_{\mathrm{ess}} = r_{\mathrm{as}}$ follows from Proposition 2.2. This proves (a) and (b).

Suppose now that there is a measure m on X with $\mathrm{supp}(m) = X$ and with the properties from Remark 4.3. If $|\lambda_i| \geq 1$ and if $Pf = \lambda_i f$ for some $0 \neq f \in \mathscr{H}^\infty(\mathscr{A})$, then $|\lambda_i||f| = |Pf| \leq P_g|f|$ and $0 < \int |f| dm = \int P_g |f| dm$, whence $|\lambda_i| = 1$. We show that $N_i \equiv 0$: Suppose there is $f \in E_i(\mathscr{H}^\infty(\mathscr{A}))$ with $N_i f \neq 0$ (i.e. $\int |N_i f| dm > 0$ as $\mathrm{supp}(m) = X$) and $N_i^2 f = 0$. Then by (2.2) and (4.5)

$$0 < \int |N_i f| dm \leq \lim_{n \to \infty} \frac{1}{n} \left(\int P_g^n |f| dm + \int |E_i f| dm \right) \leq 0,$$

a contradiction. Hence $N_i \equiv 0$ if $|\lambda_i| = 1$. This yields the representation (6.1). (If $\mathrm{supp}(m) \neq X$, practically the same argument shows that (6.1) still holds in the L_m^1-sense.)

For $|\lambda| = 1$ and $n \in \mathbb{N}$ let $S_{\lambda,n} = \frac{1}{n} \sum_{k=0}^{n-1} (\lambda^{-1} P)^k$. Because of (6.1) $\lim_{n \to \infty} S_{\lambda,n} = E_i$ if $\lambda = \lambda_i \in \Gamma$ and $\lim_{n \to \infty} S_{\lambda,n} = 0$ if $\lambda \notin \Gamma$ (uniformly on bounded subsets of $\mathscr{H}^\infty(\mathscr{A})$). As $\int |Pf| dm \leq \int P_g |f| dm = \int |f| dm$, P and also $S_{\lambda,n}$ are L_m^1-contractions, and since $\mathscr{H}^\infty(\mathscr{A})$ is dense in L_m^1, $S_{\lambda,n}$ converges strongly on L_m^1 thereby extending E_i to a projection from L_m^1 onto $E_i(\mathscr{H}^\infty(\mathscr{A}))$ with $\|E_i\|_1 = 1$. (The strong convergence to 0 as $n \to \infty$ of $(P - \sum_{i=1}^l \lambda_i E_i)^n$ is an immediate consequence.) For $0 \neq f \in L_m^1$ and $|\lambda| = 1$ with $Pf = \lambda f$ we thus obtain $0 \neq f = \lim_{n \to \infty} S_{\lambda,n} f = E_i f \in \mathscr{H}^\infty(\mathscr{A})$ and $\lambda = \lambda_i \in \Gamma$. This finishes the proof of (c) and we turn to (d):

Let $\phi \equiv 0$. For $\lambda = 1$ and $f \in L_m^1$ with $\int f dm \neq 0$ identity (4.3) implies

$$0 \neq \int f dm = \lim_{n \to \infty} \int S_{1,n} f dm = \int E_1 f dm$$

and hence $1 = \lambda_1 \in \Gamma$. Obviously E_1 is positive in this case. Γ is fully cyclic by Proposition V.4.6 in [Schaefer, 1974] and the example thereafter. As $f \in \mathrm{range}(E_1)$ implies $\bar{f}, |f| \in \mathrm{range}(E_1),$[25] $\mathrm{range}(E_1)$ is a vector sublattice of L_m^1 (Riesz subspace).

Hence there are $0 \leq h_i \in \mathrm{range}(E_1)$, $\int h_i dm = 1$ $(i = 1, \dots, d)$ with mutually disjoint supports H_i.[26] As

$$\int_{T^{-n} H_i} h_j dm = \int_{H_i} P^n h_j dm = \int_{H_i} h_j dm = \delta_{ij},$$

we have for $X_i := \bigcup_{n \geq 0} T^{-n} H_i$

$$\int_{X_i} h_j dm = \delta_{ij}.$$

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[25] The latter because $Pf = f \Rightarrow |f| \leq P|f|$ and $\int |f| dm = \int P|f| dm$.

[26] See the preliminaries of §1 in [Schaefer, 1980].

In view of Remark 4.3 the assumption $\mathrm{supp}(m) = X$ implies $g > 0$. Therefore $TH_i = H_i$ and the X_i are disjoint and $T - T^{-1}$-invariant. Let $B = X \setminus \bigcup_{i=1}^{d} X_i$. Then

$$m(B) = \int_X E_1 \chi_B \, dm = \sum_{i=1}^{d} \int_{X_i} E_1 \chi_B \, dm$$

$$= \lim_{n\to\infty} \sum_{i=1}^{d} \frac{1}{n} \sum_{k=0}^{n-1} \int_{X_i} P^k \chi_B \, dm$$

$$= 0,$$

because $\int_{X_i} P^k \chi_B \, dm = \int_{T^{-k}X_i} \chi_B \, dm = 0$, i.e. $(X_1, \ldots, X_d)$ is a partition of X modulo m-null sets.

As the T-invariance of the measure $d\mu = h \, dm$ follows from (4.4), this finishes the proof of (d).

We prove (e): Let $d = 1$, $h = h_1$. Suppose there are $a \in \mathbb{R}$ and $\psi \in \mathrm{mb}(X)$ such that

(6.5) $\exp(i\phi) = \exp i(a + \psi \circ T - \psi)$ m-a.e.

Then

$$P(h \cdot \exp(i\psi)) = P_g(h \cdot \exp i(\phi + \psi))$$

$$= \exp(ia) P_g(h \cdot \exp(i\,\psi \circ T))$$

$$= \exp(ia) \cdot h \cdot \exp(i\psi)   m\text{-a.e.},$$

i.e. $\exp(ia)$ ia an eigenvalue of $P\colon L_m^1 \to L_m^1$.

Conversely, let $Pf = \lambda f$ for $0 \ne f \in L_m^1$ and $|\lambda| = 1$. Assume without loss of generality that $\int |f| \, dm = 1$, $f = |f| \cdot \exp(i\psi)$, and $\lambda = \exp(ia)$, $a \in \mathbb{R}$. Then

$$|f| \cdot \exp i(a + \psi) = Pf = P_g(f \cdot \exp(i\phi))   m\text{-a.e.},$$

i.e.

$$|f| = P_g(|f| \cdot \exp i(-a + \phi - \psi \circ T + \psi)) \le P_g|f| m\text{-a.e.}$$

as P_g is positive and, because of (4.3) $P_g|f| = |f|$, i.e. $|f| = h$, and

$$\exp i(-a + \phi - \psi \circ T + \psi) = 1   m\text{-a.e.}$$

on $\{h \ne 0\}$. This means that (6.5) holds μ-a.e. or, equivalently, m-a.e. on $H = \mathrm{supp}(h)$. Since the identity $f = h \cdot \exp(i\psi)$ determines ψ (mod 2π) only on H and since $X = \bigcup_{n\ge 0} T^{-n}H$ modulo m, the domain of validity of (6.5) can be iteratively extended to X (mod m).   □

6.7. *Remark.* Most ideas in the proof of (c)–(e) of Theorem 6.1 have already occured in one or another form in the literature, some of them several times. In a similar situation, [Pollicott, 1984] proved (e). His proof is slightly different.

Proof of Theorem 6.2. At the beginning of the proof of the last theorem we saw that $P \colon \mathscr{H}^\infty(\mathscr{A}) \to \mathscr{H}^\infty(\mathscr{A})$ is bounded (see (6.4)) and that all $P\beta$ ($\beta \in \mathscr{F}$) are nuclear. Let $\alpha = \beta[\mathscr{E}_k]$, $\beta = \beta[\mathscr{E}_l]$, $k \geq l$. Then

$$\mathrm{tr}((\alpha P_\beta \alpha)^n) = \sum_{\substack{\omega = D_0 \cdots D_n \\ D_i \in \mathscr{E}_k \backslash \mathscr{E}_l}} \mathrm{tr}\left(f \mapsto (g_n \cdot f) \circ \tau_\omega \right)$$

$$(6.6) \qquad\qquad = \sum_{\substack{\omega = D_0 \cdots D_{n-1} D_0 \\ D_i \in \mathscr{E}_k \backslash \mathscr{E}_l}} g_n(x_\omega) \cdot \det \left(\mathrm{Id} - D\tau_\omega(x_\omega) \right)^{-1}$$

by Theorem 5.8. As $(X, T, \mathscr{T}, g)$ is $\mathscr{A}$-expanding, $g_n(x_\omega) = 0$ or $\|D\tau_\omega(x_\omega)\| \leq \alpha^{n/n_0} < 1$ by Remark 5.5. Therefore, if d is the dimension of the analytic structure $\mathscr{A}$ and if $g_n(x_\omega) \neq 0$, then

$$(6.7) \qquad \left(\frac{1}{1 + \|D\tau_\omega(x_\omega)\|} \right)^d \leq \det \left(\mathrm{Id} - D\tau_\omega(x_\omega) \right)^{-1} \leq \left(\frac{1}{1 - \|D\tau_\omega(x_\omega)\|} \right)^d .$$

(Observe that $D\tau_\omega(x_\omega)$ has real entries, see Definition 5.2(iii).)

Hence all summands in (6.6) are positive, and the expression is monotone in k if l is fixed. Therefore

$$\lim_{\alpha \in \mathscr{F}} \mathrm{tr}((\alpha P_\beta \alpha)^n) = \mathrm{tr}_{\mathscr{F}}(P_\beta^n)$$

exists (possibly $= \infty$). Observing the positivity of g we thus obtain from (6.6), (6.7), and (4.6) for $\beta = \beta[\mathscr{E}_l]$ and $\mathscr{A} \subseteq \mathscr{E}_l \subseteq \mathscr{B} \subseteq \mathscr{D}$

$$(6.8) \qquad c_n \cdot \gamma(\mathscr{D} \backslash \mathscr{B}, n) \leq \mathrm{tr}_{\mathscr{F}}(P_\beta^n) \leq c_n^{-1} \cdot \gamma(\mathscr{D} \backslash \mathscr{A}, n)$$

for some constants $0 < c_n \nearrow 1$. In particular

$$(6.9) \qquad t_\beta = \varlimsup_{n \to \infty} |\mathrm{tr}_{\mathscr{F}}(P_\beta^n)|^{1/n} = \varlimsup_{n \to \infty} |\gamma(\mathscr{D} \backslash \mathscr{E}_l, n)|^{1/n} = \gamma(\mathscr{D} \backslash \mathscr{E}_l)$$

and

$$t_{\mathrm{as}} = \lim_{\beta \in \mathscr{F}} t_\beta = \lim_{l \to \infty} \gamma(\mathscr{D} \backslash \mathscr{E}_l) = \gamma_{\mathrm{as}}.$$

Now $t(P) = t_0 = \gamma(D) < \infty$ by (6.9), and all further assumptions on t in (ii) and (iii) of Definition 2.3 follow immediately from (6.8) and (6.9). The corresponding assumption on r in (iii) of Definition 2.3 is easily verified using Lemma 6.5 (i.e. the positivity of P_g). Finally, (iv) of Definition 2.3 is nothing but assumption (iii) of the theorem. Hence P is $\mathscr{F}$-quasinuclear.

Now the theorem will follow from Theorem 2.5 and Corollary 2.6, if we can show that $\det_{\mathscr{F}}(\mathrm{Id} - zP) \cdot \zeta(z)$ is analytic and nonzero on $G_{\mathscr{F}}$. (Note that $r(P)$ is then an eigenvalue of P, because $r(P)^{-1} = t(P)^{-1} = \gamma(\mathscr{D})^{-1}$ is a pole of $\zeta(z)$ and hence of $\det_{\mathscr{F}}(\mathrm{Id} - zP)$.)

Observing (2.14) and (4.10) we have

$$\det_{\mathscr{F}}(\mathrm{Id} - zP) \cdot \zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}(-\operatorname{tr}_{\mathscr{F}}(P^n) + \gamma(\mathscr{D}, n))\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\substack{\omega \text{ closed path of} \\ \text{length } n \text{ in } \mathscr{Y}}} g_n(x_\omega)(1 - \det(\mathrm{Id} - D\tau_\omega(x_\omega))^{-1})\right),$$

and by (6.7) $\det_{\mathscr{F}}(\mathrm{Id} - zP)\zeta(z)$ is analytic and nonzero for

$$|z| \leq 1 / \varlimsup_{n\to\infty} \left(\sum_\omega g_n(x_\omega)\left(\left(\frac{1}{1 - \|D\tau_\omega(x_\omega)\|}\right)^d - 1\right)\right)^{1/n}$$

$$= 1 / \varlimsup_{n\to\infty} \kappa(n)^{1/n} = \kappa^{-1}. \quad \square$$

6.8. *Remark.* We have seen that if $\phi \equiv 0$ and $g \geq 0$, then P behaves much like a positive operator, e.g. $r(P)$ is an eigenvalue of P. But since $\mathscr{H}^\infty(\mathscr{A})$ is far from being a Banach lattice, the general theory of positive operators does not apply. However, just as in the Banach lattice setting, one can show that if λ is an eigenvalue of P of modulus $r(P)$, then λ^{-1} is a pole of the resolvent $(\mathrm{Id} - zP)^{-1}$ whose order does not exceed the order of $r(P)^{-1}$, which we denote by σ. This follows easily from

$$G(\lambda, t, s) := t\left(\frac{1-t}{t}\right)^s (\mathrm{Id} - t\lambda^{-1}P)^{-1}$$

$$= t\left(\frac{1-t}{t}\right)^s \sum_{n=0}^{\infty} \left(\frac{t}{\lambda}\right)^n P^n$$

(6.10)

$$= \left(\frac{1-t}{t}\right)^s \left(\sum_{\lambda_i \in \Gamma} \frac{\lambda}{\lambda_i} \sum_{k=0}^{\infty} \left(\frac{t\lambda_i}{\lambda - t\lambda_i}\right)^{k+1} E_i N_i^k\right.$$

$$\left. + (\text{terms bounded as } t \to 1)\right)$$

in the limit $t \to 1$ for $s > \sigma$: Let $0 \leq f \in \mathscr{H}^\infty(\mathscr{A})$. Then

$$\lim_{t\to 1} |G(\lambda, t, s)(f)| \leq \lim_{t\to 1} G(r(P), t, s)(f) = 0,$$

and for general $f \in \mathscr{H}^\infty(\mathscr{A})$ this estimate can be applied separately to the positive functions $f^* = \sum_{D \in \mathscr{D}} \|f_D\|_{\mathscr{A}} \cdot \chi_D$ and $(f + f^*)$.

Also, if $0 \leq f \in \mathscr{H}^\infty(\mathscr{A})$, $D \in \mathscr{D}$, and $(E_1 N_1^{s+1} f)_{|D} = 0$ for some $s \geq 0$, then $(E_1 N_1^s f)_{|D} \geq 0$.

In Theorem 6.1(d) we have seen that, if P is a Perron-Frobenius operator with respect to a measure m with $\operatorname{supp}(m) = X$, then the positive cone of $\operatorname{range}(E_1)$ is just the space of densities of m-absolutely continuous T-invariant measures.

In general, the relation between $\mathrm{range}(E_1)$ and T-invariant measures is more complicated, even if $P = P_g$. For its investigation we need

6.9. Lemma. *Suppose that $g \geq 0$, that $P = P_g$ is quasicompact, that $(X, T, \mathscr{Z})$ generates, and that s is continuous. If σ is the order of the pole $r(P)^{-1}$ of the resolvent of P, then there is a nonnegative, $\mathrm{range}(E_1)$-valued Borel measure F on X such that*

$$E_1 N_1^{\sigma-1}(f) = \int f \, dF \quad \text{for all } f \in \mathscr{H}^\infty(\mathscr{A}),$$

$$\int Pf \, dF = r(P) \cdot \int f \, dF \quad \text{for all } f \in L_F^1, \text{ and}$$

$$\| F(Z) \|_{\mathscr{A}} \leq \| E_1 N_1^{\sigma-1} \|_{\mathscr{A}} \quad \text{for all } Z \in \mathscr{Z}.$$

Proof. Without loss of generality assume $r(P) = 1$. Since $P^k \chi_Z \in \mathscr{H}^\infty(\mathscr{A})$ for $Z \in \mathscr{Z}_k$,

$$F(Z) := E_1 N_1^{\sigma-1} P^k(\chi_Z)$$

is well defined for such Z, $F \geq 0$ by Remark 6.8. It is easy to check that $F(Z) = \sum F(Z')$ where the sum extends over all $Z' \in \mathscr{Z}_{k+1}$ with $Z' \subseteq Z$. Hence F extends uniquely to a finitely additive nonnegative set function on the algebra generated by $\bigcup_{k \geq 0} \mathscr{Z}_k$. Since all sets in this algebra are compact, F is countably additive, and since $\| F(Z) \|_{\mathscr{A}} \leq \| E_1 N_1^{\sigma-1} \|_{\mathscr{A}}$ for all $Z \in \mathscr{Z}$, F extends to a nonnegative $\mathrm{range}(E_1)$-valued measure on the σ-algebra generated by $\bigcup_{k \geq 0} \mathscr{Z}_k$, i.e. on the Borel sets.

Let $f \in \mathscr{H}^\infty(\mathscr{A})$, $D \in \mathscr{Z}$. For $k \geq 0$ and $Z \in \mathscr{Z}_k$ fix $\alpha_Z \in (f \cdot \chi_D)(Z)$ and consider $f_k := \sum_{Z \in \mathscr{Z}_k} \alpha_k \chi_Z$. $(f \cdot \chi_D)$ is continuous by continuity of s. Therefore $\|f \cdot \chi_D - f_k\|_\infty =: \delta_k \to 0$ as $k \to \infty$ and

$$\int_D f \, dF = \lim_{k \to \infty} \int f_k \, dF = \lim_{k \to \infty} E_1 N_1^{\sigma-1} P^k f_k$$

$$\leq \lim_{k \to \infty} E_1 N_1^{\sigma-1} P^k((f + \delta_k)\chi_D)$$

$$= E_1 N_1^{\sigma-1}(f \cdot \chi_D),$$

and for the reverse inequality consider $-f$ instead of f.

Since $\int P\chi_Z \, dF = E_1 N_1^{\sigma-1} P^{k+1}(\chi_Z) = E_1 N_1^{\sigma-1} P^k(\chi_Z) = \int \chi_Z \, dF$ for all $Z \in \mathscr{Z}_k$, $k \geq 0$, we also have $\int Pf \, dF = \int f \, dF$ for all $f \in L_F^1$. $\square$

Next we have to think about the structure of the graph $(\mathscr{D}, \to)$: A subset $\mathscr{I}$ of $\mathscr{D}$ is *irreducible*, if for all $C, D \in \mathscr{I}$ there is a path from C to D and if $\mathscr{I}$ is maximal with respect to this property. It is well known that there is a maximal positive integer p (the *period* of $\mathscr{I}$) such that the length of each closed path in $\mathscr{I}$ is a multiple of p (see e.g. [Seneta, 1980]).

6.10. Proposition. *Suppose that $P = P_g$ is quasicompact, that $(X, T, \mathscr{Z})$ generates, that s is continuous, that $(\mathscr{D}, \to)$ is irreducible with period p and that*

$g > 0$ *everywhere. Then*

(a) *The set* Γ *of eigenvalues of* P *of modulus* $r(P)$ *consists of the simple eigenvalues* $r(P) \cdot e^{2\pi i k/p}$ $(k = 0, \ldots, p - 1)$.

(b) *There are a Borel measure* m *on* X *with* $\mathrm{supp}(m) = X$ *and* $\sup_{D \in \mathscr{D}} m(D) < \infty$, *and a strictly positive function* $h \in \mathscr{H}^\infty(\mathscr{A})$ *such that*

$$E_1 f = \int f \, dm \cdot h \quad \text{for all } f \in \mathscr{H}^\infty(\mathscr{A}), \text{ and}$$

$$\int P f \, dm = r(P) \cdot \int f \, dm \quad \text{for all } f \in L^1_m.$$

The measure $\mu = h \cdot m$ *is an ergodic* T-*invariant probability measure.*

Proof. Let σ be the order of the pole $r(P)$ of the resolvent of P and denote $E = E_1$, $N = N_1$. Then $N^\sigma = 0$ but $N^{\sigma-1} \neq 0$. Without loss of generality assume $r(P) = 1$. Then $EN^{\sigma-1} = \lim_{t \to 1} G(1, t, \sigma) \geq 0$ by Remark 6.8, and $EN^{\sigma-1} P = EN^{\sigma-1}$.

Fix $C \in \mathscr{D}$ and let j be the maximal integer such that $EN^j \chi_C \neq 0$. Obviously $j < \sigma$. Suppose for a contradiction that $j \geq 1$. Then $EN^{\sigma-1}(EN^j \chi_C) = 0$, where $EN^j \chi_C \geq 0$ by Remark 6.8. As $(X, T, \mathscr{Z})$ generates, there are $\varepsilon > 0$ and $Z \in \mathscr{Z}_n$ with $(EN^j \chi_C)_{|Z} \geq \varepsilon$. Let ω be a path of length $k \geq n$ in $\mathscr{D}$ such that $Z(\omega) \subseteq Z$. Since $T^k Z(\omega) =: D \in \mathscr{D}$ and $\varepsilon_k := \varepsilon \cdot \inf g_{k | Z(\omega)} > 0$, we have

$$(P^k EN^j \chi_C)_{|D} \geq ((EN^j \chi_C) \cdot g_k) \circ \tau_\omega \geq \varepsilon_k \cdot \chi_D.$$

Hence

$$0 = EN^{\sigma-1}(EN^j \chi_C) = EN^{\sigma-1} P^k EN^j \chi_C \geq \varepsilon_k \cdot EN^{\sigma-1} \chi_D \geq 0,$$

i.e. $EN^{\sigma-1} \chi_D = 0$. This argument applies to all $D \in \mathscr{D}$, because $(\mathscr{D}, \to)$ is irreducible, in contradiction to $EN^{\sigma-1} \neq 0$. Hence $j = 0$, i.e. $N\chi_C = 0$, and as $C \in \mathscr{D}$ was arbitrary, $N = 0$ and $E \geq 0$. In view of Remark 6.8 all eigenvalues in Γ are semisimple.

We prove the simplicity of $r(P) = 1$: Let $Pf = f$ for some $f \in C(X, \mathbf{R})$, and assume that $f(x) > 0$ for some $x \in D \in \mathscr{D}$. As $(X, T, \mathscr{Z})$ generates, there are $n > 0$ and $Z \in \mathscr{Z}_n$ such that $f_{|Z} > 0$. Now the irreducibility of $(\mathscr{D}, \to)$ together with $g > 0$ implies $f > 0$ on all of X. Hence, if $Pf = f$ for $f \in C(X, \mathbf{R})$, then $f = 0$, $f < 0$ or $f > 0$ everywhere. If there are nonzero f_1 and f_2 as above, then $\psi_\alpha = f_1 + \alpha f_2$ satisfies the assumptions on f above, and since $\alpha^{-1} \psi_\alpha \to f_2$ as $\alpha \to \pm\infty$, there is some $\alpha \in \mathbf{R}$ with $\psi_\alpha = 0$, i.e. $f_1 = -\alpha f_2$. If $Ph = h$ for some $h \in \mathscr{H}^\infty(\mathscr{A})$, then $\Re h$ and $\Im h$ satisfy the assumptions on f, because s is continuous. Hence $\mathrm{rank}(E) = 1$, i.e. $r(P) = 1$ is a simple eigenvalue. The proof that $\Gamma = \{e^{2\pi i k/p} : k = 0, \ldots, p-1\}$ and that all $\lambda \in \Gamma$ are simple is postponed.

Now fix $0 < h \in \text{range}(E)$. By Lemma 6.9 there is a Borel measure m on X such that $Ef = h \cdot \int f \, dm$ and $\int Pf \, dm = r(P) \cdot \int f \, dm$ for all $f \in \mathscr{H}^\infty(\mathscr{A})$ and $m(D) \leq \||\, E \,\||_{\mathscr{A}}$ for all $D \in \mathscr{Z}$.

We show $\text{supp}(m) = X \colon m(D) > 0$ for all $D \in \mathscr{D}$, because $(\mathscr{D}, \rightarrow)$ is irreducible and $m(D) = \int P^n \chi_D \, dm$ for all $n \geq 0$. Hence, if $Z = Z(\omega) \in \mathscr{Z}_n$, then $m(Z) = \int P^n \chi_Z \, dm \geq \inf g_{n|Z} \cdot m(T^n Z) > 0$ since $T^n Z \in \mathscr{D}$. As $(X, T, \mathscr{Z})$ generates, this proves the claim.

Finally, if $f \in L_m^\infty$, then

$$\int f \circ T \, d\mu = \int f \circ T \cdot h \, dm = \int P(f \circ T \cdot h) \, dm$$

$$= \int f \cdot Ph \, dm = \int f \cdot h \, dm$$

$$= \int f \, d\mu,$$

and the ergodicity of μ follows immediately from $\text{rank}(E) = 1$. This proves (b).

We turn back to (a). Write $C \rightarrow_k D$ if there is a path of length k from C to D. $(\mathscr{D}, \rightarrow_k)$ is again a directed graph. Assume first that $p = 1$. Then $(\mathscr{D}, \rightarrow_k)$ is irreducible and of period 1 for each $k \geq 1$. Hence $r(P) = 1$ is a simple eigenvalue for all P^k. With the measure m constructed above we are in the situation of (c) and (d) of Theorem 6.1, in particular Γ is a finite group. If there were $1 \neq \lambda \in \Gamma$, one would have $\lambda^k = 1$ for some $k > 1$ contradicting the simplicity of the eigenvalue 1 of P^k. Hence $\Gamma = \{1\}$ in case $p = 1$.

If $p > 1$, then $\mathscr{D}$ can be decomposed uniquely into p disjoint subsets $\mathscr{D}_0, \ldots, \mathscr{D}_{p-1}$, each having the property that $C \in \mathscr{D}_i$ and $C \rightarrow_k D$ implies $D \in \mathscr{D}_{(i+k) \bmod p}$. The graphs $(\mathscr{D}_i, \rightarrow_p)$ are irreducible and of period 1 $(i = 0, \ldots, p-1)$. This is proved just as in the case of nonnegative matrices, see e.g. [Seneta, 1980]. Hence (a) and (b) of this proposition apply to each $P_{|V_i}^p$ where $V_i = \{f \in \mathscr{H}^\infty(\mathscr{A}) \colon f_D \equiv 0 \text{ if } D \notin \mathscr{D}_i\}$. In particular $P_{|V_i}^p$ has only 1 as an eigenvalue of modulus one, and its multiplicity is p. This implies immediately that $\sum_{\lambda \in \Gamma} \text{multiplicity}(\lambda) = p$, and since Γ is a group, we can finish the proof by showing that $e^{-2\pi i/p} \in \Gamma$: Let $\psi(x) = k$ if $x \in D \in \mathscr{D}_k$ and let $Ph = h$. Then $P(h e^{2\pi i \psi/p}) = e^{2\pi i (\psi - 1)/p} \cdot P(h) = e^{-2\pi i/p} \cdot h \cdot e^{2\pi i \psi/p}$, i.e. $e^{-2\pi i/p} \in \Gamma$. $\square$

Now we turn to the case of a general graph $(\mathscr{D}, \rightarrow)$. Let $\mathscr{I}_1, \mathscr{I}_2, \ldots$ be the irreducible subsets of $\mathscr{D}$, $X_i := \bigcup_{D \in \mathscr{I}_i} D$ for each i. Let $Y_i := \bigcap_{n \geq 0} T^{-n} X_i$. Since the $\mathscr{I}_i$ are irreducible, $TY_i = Y_i$, and it is easily checked that

$$\mathscr{Z}_i = \{D \cap Y_i \colon D \in \mathscr{I}_i, D \cap Y_i \neq \varnothing\}$$

is a Markov partition for $T_{|Y_i}$. $(Y_i, T, \mathscr{Z}_i)$ generates if $(X, T, \mathscr{Z})$ does. We make the following additional assumption:

(6.11)   If $(\mathscr{I}_i, \rightarrow)$ does not consist of a single loop only, and if $D \cap Y_i \neq \varnothing$, then $s(D \cap Y_i)$ is a uniqueness set for bounded holomorphic functions on U_D.

This strengthens requirement (ii) of Definition 5.2. One should note, however, that $D \cap Y_i$ is uncountable if $(\mathscr{I}_i, \rightarrow)$ does not consist of a single loop only, and hence (6.11) is not really restrictive.

Under this assumption $(Y_i, T, \mathscr{Z}_i, g)$ is $\mathscr{A}_i$-expanding if $(X, T, \mathscr{Z}, g)$ is $\mathscr{A}$-expanding where $\mathscr{A}_i = (s, (U_D)_{D \in \mathscr{I}_i})$. The transfer operator corresponding to $(Y_i, T, \mathscr{Z}_i, g)$ is denoted by P_i.

6.11. Theorem. *Suppose that $(X, T, \mathscr{Z}, g)$ satisfies (6.11) and the assumptions of Theorem 6.2, that $r_{\mathrm{as}} < r(P)$ and $\gamma_{\mathrm{as}} < \gamma(\mathscr{D})$, that $g > 0$ everywhere, and that s is continuous. Let $P = P_g$ and $d = \mathrm{rank}(E_1)$. Then $r(P_i) < r(P)$ except for d irreducible subsets of $\mathscr{D}$ (say $\mathscr{I}_1, \ldots, \mathscr{I}_d$) for which $r(P_i) = r(P)$ and for which holds:*

(a) The subsystem $(Y_i, T, \mathscr{Z}_i, g)$ satisfies all assumptions of Theorems 6.1 and 6.2 and Proposition 6.10, and it does not consist of a single loop only.

(b) There are a σ-finite Borel measure m_i on X with $\mathrm{supp}(m_i) = Y_i$ and a function $0 \leq h_i \in \mathscr{H}^\infty(\mathscr{A})$, zero outside X_i, satisfying

$$(E_1(f \cdot \chi_{X_i}))_{|X_i} = \int_{X_i} f \, dm_i \cdot h_{i|X_i} \quad \text{for all } f \in \mathscr{H}^\infty(\mathscr{A}).$$

The measure $\mu_i = h_i \cdot m_i$ is a T-invariant ergodic probability measure equivalent to m_i. h_i satisfies $(Ph_i)_{|X_i} = h_{i|X_i}$.

(c) There is a σ-finite Borel measure $\bar{m}_i$ on X such that $\bar{m}_{i|X_i} = m_{i|X_i}$,

$$\int Pf \, d\bar{m}_i = r(P) \cdot \int f \, d\bar{m}_i \quad \text{for all } f \in L^1_{\bar{m}_i}$$

and $\mathrm{supp}(\bar{m}_i) = \mathrm{cl}(\bigcup_{k \geq 0} T^{-k} Y_i)$.

Proof. Fix some $\mathscr{I}_i$. If $\gamma(\mathscr{I}_i) < \gamma(\mathscr{D}) = r(P)$, then $1/\zeta(Y_i, z)$ is analytic and nonzero in $\{|z| < (r(P) - \varepsilon)^{-1}\}$ for some $\varepsilon > 0$. In particular, $1/r(P)$ is not a zero of $1/\zeta(Y_i, z)$. Also $r(P_i) < r(P)$, since otherwise $r_{\mathrm{as}}(P_i) \leq r_{\mathrm{as}}(P) < r(P) = r(P_i)$ and $\gamma_{\mathrm{as}}(\mathscr{I}_i) \leq \gamma_{\mathrm{as}}(\mathscr{D}) \leq \gamma(\mathscr{D}) = r(P) = r(P_i)$ such that Theorem 6.2 would lead to the contradiction $r(P_i) = \gamma(\mathscr{I}_i) < \gamma(\mathscr{D}) = r(P)$.

If, on the other hand, $r(P) = \gamma(\mathscr{D}) = \gamma(\mathscr{I}_i)$, then $\gamma_{\mathrm{as}}(\mathscr{I}_i) \leq \gamma_{\mathrm{as}}(\mathscr{D}) < \gamma(\mathscr{D}) = \gamma(\mathscr{I}_i)$ and $r_{\mathrm{as}}(P_i) \leq r_{\mathrm{as}}(P) \leq r(P) = \gamma(\mathscr{I}_i)$, and Theorem 6.2 applies to P_i. Hence $r(P_i) = \gamma(\mathscr{I}_i) = r(P)$, and in view of Proposition 6.10 $r(P)$ is a simple eigenvalue of P_i, such that, by Theorem 6.2 again, $1/r(P)$ is a simple zero of $1/\zeta(Y_i, z)$.

As $\zeta(X, z) = \prod_i \zeta(Y_i, z)$, there are exactly d irreducible components $\mathscr{I}_i$ of $(\mathscr{D}, \rightarrow)$, say $\mathscr{I}_1, \ldots, \mathscr{I}_d$, for which $r(P_i) = r(P)$ is a simple eigenvalue of P_i.

Now Proposition 6.10 guarantees the existence of measures m_i and $\mu_i = h_i \cdot m_i$ with full topological support on Y_i. Since the Y_i are closed subsets of the X_i, these measures can be interpreted as measures on X_i with topological support Y_i. Since $TY_i = Y_i$, the μ_i are ergodic, T-invariant probability measures.

We use the operators $G(\lambda, t, s)$ defined in (6.10) and denote by $G_i(\lambda, t, s)$ the corresponding operators for the P_i. Observing (6.11) we see that D and $D \cap Y_i$ have actually the same analytic structure for all $D \in \mathscr{S}_i$. Hence $(P^n(f \cdot \chi_{X_i}))_{|Y_i} = P_i^n(f_{|Y_i})$, in particular $(Ph_i)_{Y_i} = P_i(h_{i|Y_i}) = h_{i|Y_i}$ and $(Ph_i)_{|X_i} = h_{i|X_i}$. Thus for all $0 < t < 1$ and $s \geq 1$

$$\big(G(r(P), t, s)(f \cdot \chi_{X_i})\big)_{|Y_i} = G_i(r(P), t, s)(f_{|Y_i}).$$

But $\lim_{t \to 1} G_i(r(P), t, s) = 0$ for $s \geq 2$, whence

$$\big(E_1(f \cdot \chi_{X_i})\big)_{|Y_i} = \lim_{t \to 1}\big(G(r(P), t, 1)(f \cdot \chi_{X_i})\big)_{|Y_i}$$

$$= \lim_{t \to 1} G_i(r(P), t, 1)(f_{|Y_i}) = E_{i,1}(f_{|Y_i})$$

$$= \int_{Y_i} f \, dm_i \cdot h_i = \int_{X_i} f \, dm_i \cdot h_i \quad \text{by Proposition 6.10.}$$

In view of (6.11), $\int_{X_i} f \, dm_i \cdot h_i$ extends uniquely to the nonnegative function $E_1(f \cdot \chi_{X_i})$ defined on X_i. This proves (a) and (b).

We turn to (c): For $k \geq 0$ let

$$X_{i,k} = \{x \in X : T^k x \in X_i, \, T^j x \notin X_i \, (j = 0, \dots, k-1)\},$$

and for a bounded measurable function f let

$$m_{i,k}(f) := m_i(P^k(f \cdot \chi_{i,k})), \quad \text{where } \chi_{i,k} := \chi_{X_{i,k}}.$$

Then $\bar{m}_i := \sum_{k \geq 0} r(P)^{-k} m_{i,k}$ is a σ-finite Borel measure, and using the relations

$$P f \cdot \chi_{i,0} = P(f \chi_{i,0} + f \chi_{i,1}),$$
$$P f \cdot \chi_{i,k} = P(f \chi_{i,k+1}) \quad (k \geq 1)$$

it is straightforward to check that $\bar{m}_i(Pf) = r(P) \cdot \bar{m}_i(f)$ for $f \in L^1_{\bar{m}_i}$. $\quad\square$

B. Examples.

6.12. Example I (topological Markov shifts, sofic systems).

For a finite state Markov shift whose transition matrix has a positive spectral radius Theorems 6.1 and 6.2 apply imediately, because $\gamma_{as} = r_{as} = 0 < r(P)$ and $\mathrm{Id} \in \mathscr{F}$. (The latter assures condition (iii) of Theorem 6.2.)

If $\rho_1, \dots, \rho_d$ are as in Example 5.9, then

$$G_{\mathscr{F}} = \left\{ z \in \mathbb{C}^d : |z| \cdot r(P) < \left(\prod_{j=1}^d \rho_j \right)^{-1} \right\}.$$

Hence, the smaller the ρ_j (i.e. the "smoother" the functions in $\mathscr{H}^\infty(\mathscr{A})$), the bigger is $G_\mathscr{F}$.

If $U_D = \mathbb{C}$ $(D \in \mathscr{D})$, i.e. if $\mathscr{H}^\infty(U_D) \cong \mathbb{C}^d$ and $\mathscr{H}^\infty(\mathscr{A}) \cong \mathbb{C}^{d \cdot \mathrm{card}(\mathscr{D})}$, then Theorems 6.1 and 6.2 reduce to statements about matrices (cf. 4.8).

These results are well known, of course. The analyticity part of Theorem 6.2 (and much more) for finite state Markov shifts is due to [Ruelle, 1976].

For general countable state Markov shifts the assumptions of the theorems must be checked in each case individually.

As the canonical Markov extensions of sofic systems are finite state topological Markov chains, the above remarks apply to these extensions as well.

6.13. Example II (up-and-down-counter, cf. 3.13).

We consider the canonical Markov extension of the counter $(\Omega(W_M), S, \mathscr{Z})$. Its Markov diagram is uniformly forward finite, since each $D \in \mathscr{D}$ has at most two successors. Fix an analytic structure $\mathscr{A}$ and a weight function g such that $(\Omega(W_M), S, \mathscr{Z}, g)$ is an $\mathscr{A}$-expanding weighted system (cf. 5.2, 5.3, 5.9). Let $\mathscr{E} = \{E, F_0\}$ (cf. 3.13). As there are no closed paths in $\mathscr{D} \setminus \mathscr{E}$, we have $\gamma_{\mathrm{as}} = \gamma(\mathscr{D} \setminus \mathscr{E}) = 0$. As the number of paths of length n in $\mathscr{D} \setminus \mathscr{E}$ starting at any $D \in \mathscr{D} \setminus \mathscr{E}$ is bounded by n,

$$r_{\mathrm{as}} \le \varlimsup_{n \to \infty} (n \cdot \|g_n\|_\infty)^{1/n} = g_\infty$$

for $\hat{P}$ acting on $\mathscr{H}^\infty(\mathscr{A})$. Hence Theorems 6.1 and 6.2 apply if $g_\infty < r(\hat{P})$ or $g_\infty \le \gamma(\mathscr{D})$.

If $g \equiv 1$, then $r(\hat{P}_{\hat{g}}) > 1 = g_\infty$ as the diagram $(\mathscr{D}, \to)$ contains finite subdiagrams describing topological Markov shifts of positive entropy. In particular Theorem 6.2 applies in this case and $G_\mathscr{F} \supseteq \{|z| < 1\}$ provided the ρ_j from 5.9 have been chosen such that $\prod_{j=1}^d \rho_j \cdot \gamma(\mathscr{D}) \le 1$ or $U_D = \mathbb{C}$ $(D \in \mathscr{D})$. From the explicit formula for the zeta function (see Example 4.9) or from Proposition 6.10 it follows immediately that $\gamma(\mathscr{D})^{-1} = r(\hat{P})^{-1}$ is a simple pole of $\hat{\zeta}(z)$. Hence $\lambda_1 = r(\hat{P})$ is a simple eigenvalue of $\hat{P}$.

If $M = \mathbb{N}$, then $\hat{\zeta}_1(z) = (1+z)/(1-2z^2)$ (see 4.10), and $\{|z| < 1\}$ is indeed the greatest ball around 0 to which $\hat{\zeta}_1^{-1}(z)$ can be analytically extended.

6.14. Example III (piecewise monotonic transformations with finitely many monotonicity intervals).

In order to check the assumptions made in Theorems 6.1 and 6.2 we need some informations from [Hofbauer, 1985, Theorem 9 and its Corollary 1] about the structure of the Markov diagram for piecewise monotonic transformations:

Let $\mathscr{D}_0 = \mathscr{Z}$ and $\mathscr{D}_{n+1} = \mathscr{D}_n \cup \{D \in \mathscr{D} : \exists C \in \mathscr{D}_n$ such that $C \to D\}$ $(n \ge 1)$. Then

$$(6.12) \qquad \mathscr{D}_n \subseteq \mathscr{D}_{n+1} \ (n \ge 1), \quad \bigcup_{n \ge 1} \mathscr{D}_n = \mathscr{D}, \quad \mathrm{card}(\mathscr{D}_n) = O(n),$$

$$(6.13) \qquad \lim_{n \to \infty} r(M_{|\mathscr{D} \setminus \mathscr{D}_n}) = 1,$$

where $M_{|\mathscr{D}\backslash\mathscr{D}_n}$ is the transition matrix of $(\mathscr{D}, \rightarrow)$ restricted to entries from $\mathscr{D} \backslash \mathscr{D}_n$ and acting by right-multiplication on row vectors of $l^1(\mathscr{D} \backslash \mathscr{D}_n)$. Now let $\omega = D_0 \cdots D_n$ be a path of length n in $\mathscr{D}$. Then

(6.14) $$D_n \in \mathscr{D}_{k+n} \quad \text{if } D_0 \in \mathscr{D}_k, \text{ and}$$

(6.15) $\quad$ ω is closed, then it is completely contained in $\mathscr{D}_{2n}$ provided $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

As $\mathscr{Z}$ is finite, each $D \in \mathscr{D}$ has at most $\text{card}(\mathscr{Z})$ successors, i.e. $(\mathscr{D}, \rightarrow)$ is uniformly forward finite.

Let $\mathscr{F} = \{\beta[\mathscr{D}_n] : n \in \mathbb{N}\}$. For $\beta = \beta[\mathscr{D}_l]$ and $\hat{f} \in \mathscr{H}^\infty(\mathscr{A})$ we have in view of (6.3)

$$\| \hat{P}_\beta^n \hat{f} \|_{\mathscr{A}} \leq \text{const} \cdot \| \hat{f} \|_{\mathscr{A}} \cdot \|g_n\|_\infty \cdot \|(M_{|\mathscr{D}\backslash\mathscr{D}_l})^n\|_1 \,,$$

i.e. $r_\beta \leq g_\infty \cdot r(M_{|\mathscr{D}\backslash\mathscr{D}_l})$, and therefore $r_{\text{as}} = \varliminf_{\beta \in \mathscr{F}} r_\beta \leq g_\infty$ by (6.13). Hence $r_{\text{as}} < r(\hat{P})$ provided that $g_\infty < r(\hat{P})$. Similarly it follows from (6.15) that $\gamma_{\text{as}} \leq g_\infty$, cf. the proof of Lemma 5 in [Hofbauer/Keller, 1984].

We turn to the particular case where $\tilde{T}$ is piecewise analytic as discussed in Example 5.10 and where $\tilde{g} = 1/|\tilde{T}'|$. Let g be the "lift" of $\tilde{g}$ to the associated piecewise compact space X (see 3.14, 3.15). Choosing $\varepsilon > 0$ in 5.10 small enough, $(X, T, \mathscr{Z}, g)$ becomes a weighted $\mathscr{A}$-expanding system. In particular $r(\hat{P}) \geq 1$ as $\int \hat{P}\hat{f}\, d\hat{m} = \int \hat{f}\, d\hat{m}$ (where $\hat{m}$ is the Lebesgue measure lifted to the Markov extension $\hat{X}$ of X). Hence Theorem 6.1 applies if $g_\infty < 1$.

Assumptions (i) and (ii) of Theorem 6.2 are now trivially sastisfied as

$$\text{card}(\{\hat{x}: \hat{T}^n \hat{x} = \hat{x}\}) \leq (\text{card}(\mathscr{Z}))^n \,.$$

We check assumption (iii): Let $\beta = \beta[\mathscr{D}_k]$, $n \in \mathbb{N}$, and $\alpha = \beta[\mathscr{D}_{k+n}]$. Then, in view of (6.14), $\alpha(P_\beta\alpha)^j P\beta = P_\beta^j P\beta$ for $0 \leq j < n$.

We summarize the results for the case $\tilde{g} = 1/|\tilde{T}'|$: $r_{\text{ess}} = r_{\text{as}} \leq g_\infty$, $\gamma_{\text{as}} \leq g_\infty$, $\gamma(\mathscr{D}) = r(\hat{P}) = 1$, and hence $\kappa \leq g_\infty \gamma(\mathscr{D}) = g_\infty$. In particular $G_{\mathscr{F}} \supseteq \{|z| < g_\infty^{-1}\}$. We also note that in this case

(6.16) $$1 = \gamma(\mathscr{D}) = \varlimsup_{n \to \infty} \left(\sum_\omega \|D\tau_\omega(x_\omega)\| \right)^{1/n}$$

where the sum extends over all closed paths ω of length n. If $\tilde{g} \neq 1/|\tilde{T}'|$, then (6.16) still allows the conclusion $\kappa \leq g_\infty \cdot \gamma(\mathscr{D}) = g_\infty$, and hence again $G_{\mathscr{F}} \supseteq \{|z| < g_\infty^{-1}\}$.

7. Eigenvalues and eigenfunctions for non-Markovian systems

A. A general result.

Suppose that $(X, T, \mathscr{Z}, g)$ is a weighted, $\mathscr{A}$-expanding, piecewise compact system (not necessarily Markovian) and that $(\hat{X}, \hat{T}, \hat{\mathscr{Z}}, \hat{g})$ is its canonical Markov extension. Our goal is to derive knowledge about eigenvalues and eigenfunctions of P_g and poles of $\zeta_g(z)$ from corresponding results for $\hat{P}_{\hat{g}}$ and $\hat{\zeta}_{\hat{g}}(z)$ proved in the last section.

$\zeta_g(z)$ and $\hat{\zeta}_{\hat{g}}(z)$ have been related in Lemma 4.7 and Examples 4.8–4.10, and it seems difficult to prove general results about this relation more precise than Lemma 4.7.

P_g and $\hat{P}_{\hat{g}}$ are related by

$$\pi_* \circ \hat{P}_{\hat{g}} = P_g \circ \pi_* \quad \text{(Lemma 4.6)}.$$

This identity, however, is of no use, unless one has some additional information about the range of $\pi_*: \mathscr{H}^\infty(\mathscr{A}) \to \text{mb}(X)$.

In order to state a rather general result, which applies to various examples, we introduce the following notation:

Let $P = P_g$, $\hat{P} = \hat{P}_{\hat{g}}$, and suppose

(A1) $(\hat{H}, \|\mkern-2mu| \,.\, \|\mkern-2mu|)$ is a Banach space of (equivalence classes of) functions on $\hat{X}$, $\hat{P}(\hat{H}) \subseteq \hat{H}$.

(A2) $(H, \| \,.\, \|)$ is a Banach space of (equivalence classes of) functions on X, $P(H) \subseteq H$.

(A3) $\pi_*(\hat{H}) \subseteq H$ and $\pi_*: \hat{H} \to H$ is continuous.

(A4) $\hat{P}: \hat{H} \to \hat{H}$ is quasicompact with spectral radius $r(\hat{P})$ and essential spectral radius $\hat{r}_{\text{ess}}$, and for $\hat{r} > \hat{r}_{\text{ess}}$ there is $N = N(\hat{r}) \in \mathbb{N}$ such that

$$\hat{P} = \sum_{i=1}^{N} \lambda_i(\hat{E}_i + \hat{N}_i) + \hat{Q}$$

as in (2.2). We write $\hat{E}_\lambda = \hat{E}_i$ if $\lambda = \lambda_i$ and $\hat{E}_\lambda = 0$ otherwise.

(A5) $P: H \to H$ is bounded with spectral radius $r(P)$.

Under these assumptions we can state

7.1. Proposition.

(a) *Assume that* $\|\mkern-2mu| \hat{P}^n \|\mkern-2mu| = O(r(\hat{P})^n)$, $\|P^n\| = O(r(P)^n)$, *and that* $\pi_*(\hat{H})$ *is dense in* H. *If* $(P - \lambda \operatorname{Id})^k f = 0$ *for some* $f \in H$, $k \geq 1$, $|\lambda| = r(P)$, *then* $Pf = \lambda f$, *and there is* $\hat{f} \in \hat{H}$ *with* $\pi_* \hat{f} = f$ *and* $\hat{P}\hat{f} = \lambda \hat{f}$. *Briefly*

$$\bigcup_{k \geq 1} \text{kernel}((P - \lambda \operatorname{Id})^k) = \text{kernel}(P - \lambda \operatorname{Id}) = \pi_*(\hat{E}_\lambda(\hat{H})).$$

(b) *If* $(P - \lambda \operatorname{Id})^k f = 0$ *for some* $f \in H$, $k \geq 1$, $|\lambda| > \hat{r}_{ess}$, *and if condition* (A6) *below is satisfied, then there is* $\hat{f} \in \hat{H}$ *with* $\pi_* \hat{f} = f$ *and* $\hat{E}_\lambda \hat{f} = \hat{f}$. *Briefly*

$$\bigcup_{k \geq 1} \operatorname{kernel}((P - \lambda \operatorname{Id})^k) = \pi_*(\hat{E}_\lambda(\hat{H})).$$

The condition mentioned above is:

(A6) *There are a sequence of functions* $\hat{f}_n \in \operatorname{mb}(\hat{X})$ *and constants* α, β, $\gamma_1, \gamma_2, C \geq 0$ *such that for* $m = [\beta n]$

$$\hat{P}^m \hat{f}_n \in \hat{H}, \quad \||\hat{P}^m \hat{f}_n\|| \leq C \cdot \gamma_1^n, \quad \|f - \pi_* \hat{f}_n\| \leq C \cdot \gamma_2^n,$$

$$\gamma_1 < \frac{|\lambda|^{\alpha+\beta}}{\hat{r}_{ess}^\alpha}, \quad \gamma_2 < \left|\frac{\lambda}{r(P)}\right|^{\alpha+\beta}.$$

7.2. *Remark.* If $f \in \pi_*(\hat{H})$, one may choose $\hat{f}_n = \hat{f}$, $\alpha = 1$, $\beta = 0$, $\gamma_1 = 1$, $\gamma_2 = 0$ in (A6), and the condition reduces to $|\lambda| > \hat{r}_{ess}$. This is always the case if $e_*(H) \subseteq \hat{H}$.

Proof of the proposition.

(a) Define $S_{\lambda,n} = \frac{1}{n} \sum_{j=0}^{n-1} (\lambda^{-1}P)^j$ and $\hat{S}_{\lambda,n}$ analogously. As $\||\hat{P}^n\|| = O(r(\hat{P})^n)$, $\lim_{n \to \infty} \hat{S}_{\lambda,n} = \hat{E}_\lambda$ strongly and $\hat{P}\hat{E}_\lambda = \lambda \hat{E}_\lambda$ (see Corollary VIII.5.2 in [Dunford/Schwartz, 1958]). Hence

$$\lim_{n \to \infty} S_{\lambda,n} \circ \pi_* = \lim_{n \to \infty} \pi_* \circ \hat{S}_{\lambda,n} = \pi_* \circ \hat{E}_\lambda$$

strongly. Since $\pi_*(\hat{H})$ is dense in H and $\|P^n\| = O(r(P)^n)$, we may assume $Pf = \lambda f$, and there are $\hat{f}_j \in \hat{H}$ with $\|f - S_{\lambda,n}(\pi_* \hat{f}_j)\| \leq j^{-1}$ for all n. Hence $\|f - \pi_*(\hat{E}_\lambda \hat{f}_j)\| \leq j^{-1}$, and as $\operatorname{range}(\pi_* \circ \hat{E}_\lambda)$ is finite-dimensional, $f \in \operatorname{range}(\pi_* \circ \hat{E}_\lambda)$. This proves (a).

(b) In view of (A6) we can fix $\hat{r} > \hat{r}_{ess}$ and $r > r(P)$ such that

(7.1) $$\hat{r} < |\lambda|, \quad \gamma_1 < \frac{|\lambda|^{\alpha+\beta}}{\hat{r}^\alpha}, \quad \gamma_2 < \left|\frac{\lambda}{r}\right|^{\alpha+\beta}.$$

We may assume that $\hat{Q}$ from (A4) has spectral radius less than $\hat{r}$.

Suppose now

(7.2) $$(P - \lambda \operatorname{Id})f = h = \pi_* \hat{h} \in \pi_*(\hat{E}_\lambda(\hat{H})).$$

We shall show

(7.3) $$f \in \pi_*(\hat{E}_\lambda(\hat{H})).$$

Observe first that (7.2) implies

(7.4) $$(\lambda^{-1}P)^n f - f = \lambda^{-1} \sum_{j=0}^{n-1} (\lambda^{-1}P)^j h \in \pi_*(\hat{E}_\lambda(\hat{H}))$$

because $(P^j \circ \pi_* \circ \hat{E}_\lambda)(\hat{H}) = (\pi_* \circ \hat{P}^j \hat{E}_\lambda)(\hat{H}) = (\pi_* \circ \hat{E}_\lambda \hat{P}^j)(\hat{H}) \subseteq \pi_*(\hat{E}_\lambda(\hat{H}))$.

Let $l = [\alpha n] + 1$ and define

$$\hat{u}_n = (\lambda^{-1}\hat{P} - \lambda^{-1}\hat{Q})^l (\lambda^{-1}\hat{P})^m \hat{f}_n.$$

$\hat{u}_n$ is an element of the finite-dimensional space $\hat{W} := \text{range}(\sum_{i=1}^N \hat{E}_i)$, $\hat{E}_i$ as in (A4).

Note that $\hat{E}_\lambda(\hat{H}) \subseteq \hat{W}$. Now

$$\left\| f - (f - (\lambda^{-1}P)^{l+m} f) - \pi_*(\hat{u}_n) \right\|$$

$$\leq \left\| (\lambda^{-1}P)^{l+m}(f - \pi_*\hat{f}_n) \right\| + \left\| \pi_* \left((\lambda^{-1}\hat{P})^{l+m}\hat{f}_n - \hat{u}_n \right) \right\|$$

$$\leq \text{const} \cdot \left| \frac{r}{\lambda} \right|^{l+m} \cdot \gamma_2^n + \text{const} \cdot \left\| | (\lambda^{-1}\hat{Q})^l (\lambda^{-1}\hat{P})^m \hat{f}_n \right\| |$$

by (A3), (A6), and the choice of r,

$$\leq \text{const} \cdot \left(\left| \frac{r}{\lambda} \right|^{\alpha+\beta} \cdot \gamma_2 \right)^n + \text{const} \cdot \hat{r}^l |\lambda|^{-(l+m)} \gamma_1^n$$

$$\leq \text{const} \cdot \left(\left(\left| \frac{r}{\lambda} \right|^{\alpha+\beta} \cdot \gamma_2 \right)^n + \left(\frac{\hat{r}^\alpha}{|\lambda|^{\alpha+\beta}} \gamma_1 \right)^n \right)$$

$$\longrightarrow 0 \qquad \text{as } n \to \infty \qquad \text{by (7.1)}.$$

Hence f can be approximated by elements from $\pi_*(\hat{W})$ (see (7.4)), and as $\dim(\pi_*(\hat{W})) \leq \dim(\hat{W}) < \infty$, we have $f \in \pi_*(\hat{W})$, say $f = \pi_*\hat{w}$, $\hat{w} \in \hat{W}$.

Let $\hat{K} = \{\hat{w} \in \hat{W} : \pi_*(\hat{w}) = 0\}$. As $\pi_* \circ \hat{P} = P \circ \pi_*$, we have

$$(7.5) \qquad\qquad\qquad (\hat{P} - \lambda\,\text{Id})\hat{K} \subseteq \hat{K}.$$

Let $\hat{U} = \hat{E}_\lambda(\hat{H})$ and $\hat{V} = (\sum_{\substack{i=1,\dots,N \\ \lambda_i \neq \lambda}} \hat{E}_{\lambda_i})(\hat{H})$. Then $\hat{U} + \hat{V} = \hat{W}$, $\hat{U} \cap \hat{V} = \{0\}$, and both spaces are stable under $(\hat{P} - \lambda\,\text{Id})$. $(\hat{P} - \lambda\,\text{Id})_{|\hat{V}}$ is an isomorphism of the finite-dimensional space $\hat{V}$ and hence also of its stable subspace $\hat{V} \cap \hat{K}$.

Let $\hat{f} = \hat{u} + \hat{v}$, $\hat{u} \in \hat{U}$, $\hat{v} \in \hat{V}$. For large s we have

$$\pi_*(\hat{P} - \lambda\,\text{Id})^s \hat{v} = \pi_*(\hat{P} - \lambda\,\text{Id})^s \hat{w} = (P - \lambda\,\text{Id})^s f$$

$$= (P - \lambda\,\text{Id})^{s-1}\pi_*\hat{h} = \pi_*(\hat{P} - \lambda\,\text{Id})^{s-1}\hat{h} = 0.$$

This shows that $(\hat{P} - \lambda\,\text{Id})^s \hat{v}$ and hence also $\hat{v}$ belong to $\hat{V} \cap \hat{K}$. Therefore $f = \pi_*\hat{w} = \pi_*\hat{u} \in \pi_*(\hat{E}_\lambda(\hat{H}))$, and (7.3) is proved.

With $h \equiv 0$ in (7.2) this proves (b) of the proposition for the case $k = 1$.

For $k \geq 1$ we proceed by induction:

Suppose we know that for some $k \geq 1$

$$(7.6) \qquad\qquad\qquad (P - \lambda\,\text{Id})^k f \in \pi_*(\hat{E}_\lambda(\hat{H}))$$

implies

$$(7.7) \qquad\qquad\qquad f \in \pi_*(\hat{E}_\lambda(\hat{H})).$$

For $k = 1$ this has just been proved $((7.2) \Rightarrow (7.3))$.

If $(P - \lambda \,\mathrm{Id})^{k+1} f \in \pi_*(\hat{E}_\lambda(\hat{H}))$, then $f_1 := (P - \lambda\,\mathrm{Id})f$ satisfies (7.6) and hence $f_1 \in \pi_*(\hat{E}_\lambda(\hat{H}))$ by (7.7). But this means that f satisfies (7.2) with h replaced by f_1, and (7.3) tells us that $f \in \pi_*(\hat{E}_\lambda(\hat{H}))$, i.e. (7.7). $\square$

Proposition 7.1 does not exclude the case that $\pi_* \circ \hat{E}_{\lambda_i} = 0$ for some eigenvalue λ_i of $\hat{P}$, such that λ_i is not an eigenvalue of P. It just says that under suitable assumptions each eigenvalue of P is also an eigenvalue for $\hat{P}$. It seems quite difficult indeed, to show the injectivity of $\pi_{*|\hat{E}_\lambda(\hat{H})}$ for general λ, even for particular examples. [27] (But see also the last paragraph of Example 7.6.) For $\lambda \in \Gamma$, however, we have

7.3. Lemma. *Suppose Theorem 6.1(c) applies to $\hat{P}$ together with a Borel measure $\hat{m}$ satisfying for all $C, D \in \mathscr{D}$*

$$\hat{m}\{\hat{x} \in \hat{C} \cap \pi^{-1} D : V_k(\pi\hat{x}) \nsubseteq D\} \to 0 \qquad as\ k \to \infty.$$

Then $\hat{E}_\lambda \hat{f} = 0$ for all $\lambda \in \Gamma$ and all $\hat{f} \in L^1_{\hat{m}}$ with $\pi_* \hat{f} = 0$ $\hat{m} \circ \pi^{-1}$-a.e. (For $V_k(x)$ see Definition 3.3.) In particular $\pi_*: \mathrm{range}(\hat{E}_\lambda) \to L^\infty_{\hat{m} \circ \pi^{-1}}$ is 1-1.

Proof. Let $N_0 = \{x \in X : \pi_* \hat{f}(x) \neq 0\}$. Fix $\delta > 0$ and choose a decomposition $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$ with a finite $\mathscr{D}_1$ and such that $\|\|\hat{f}_2\|\|_{\mathscr{A}} < \delta$ where $\hat{f}_2 := \sum_{D \in \mathscr{D}_2} \hat{f} \cdot \chi_{\hat{D}}$, $\hat{f}_1 := \hat{f} - \hat{f}_2$. For $k \in \mathbb{N}$ let

$$B_k = \{x \in X : \exists D \in \mathscr{D}_1 \text{ such that } x \in D \text{ but } V_k(x) \nsubseteq D\}.$$

Then

$$\pi_*|\hat{E}_\lambda \hat{f}|(x) \leq \|\| \hat{E}_\lambda \hat{f}_2 \|\|_{\mathscr{A}} + \|\| \hat{E}_\lambda(\hat{f}_1 \cdot (\chi_{N_0 \cup B_k} \circ \pi)) \|\|_{\mathscr{A}}$$
$$\text{(7.8)} \qquad\qquad + \pi_*|\hat{E}_\lambda(\hat{f}_1 \cdot (\chi_{X \setminus (N_0 \cup B_k)} \circ \pi))|(x).$$

Now

$$\|\| \hat{E}_\lambda \hat{f}_2 \|\|_{\mathscr{A}} \leq \delta \,\|\| \hat{E}_\lambda \|\|_{\mathscr{A}}$$

and

$$\|\| \hat{E}_\lambda(\hat{f}_1 \cdot (\chi_{N_0 \cup B_k} \circ \pi)) \|\|_{\mathscr{A}} \leq \mathrm{const} \cdot \|\hat{E}_\lambda\|_1 \int_{\pi^{-1}(N_0 \cup B_k)} |\hat{f}_1|\, d\hat{m}$$

$$\leq \mathrm{const} \cdot \|\hat{E}_\lambda\|_1 \,\|\| \hat{f} \|\|_{\mathscr{A}} \left(\hat{m}(\pi^{-1} N_0) + \sum_{C \in \mathscr{D}_1} \hat{m}(\hat{C} \cap \pi^{-1} B_k) \right)$$

where the constant is the norm of $\mathrm{Id}: (\mathrm{range}(\hat{E}_\lambda), \|\cdot\|_1) \to (\mathrm{range}(\hat{E}_\lambda), \|\| \cdot \|\|_{\mathscr{A}})$. Since for $j \geq k$ and $y \in X \setminus B_k$ there is just one $\hat{x} \in \hat{X}$ such that $\hat{T}^j \hat{y} = \hat{x}$

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[27] *Added in proof.* Recently V. Baladi and the author proved the injectivity of $\pi_{*|\hat{E}_\lambda(\hat{H})}$ for $|\lambda| > \hat{r}_{\mathrm{ess}}$ in the case of piecewise monotonic transformations. The proof can be adapted to quite general systems. It will appear in Comm. Math. Phys. as Lemma 4.3 of the paper *Zeta functions and transfer operators for piecewise monotone transformations.*

for all $\hat{y} \in \pi^{-1}\{y\}$ (see Lemma 3.10(c)), we have

$$\pi_*|(\lambda^{-1}\hat{P})^j(\hat{f}_1 \cdot (\chi_{X\setminus(N_0\cup B_k)} \circ \pi))|(x)$$

$$\leq \sum_{y\in T^{-j}x} |g_n(y)| \cdot \chi_{X\setminus(N_0\cup B_k)}(y) \cdot |\pi_*\hat{f}_1(y)|.$$

For $y \in X\setminus N_0$ we have $|\pi_*\hat{f}_1(y)| = |\pi_*\hat{f}_2(y)| \leq \||\hat{f}_2\||_{\mathscr{A}} < \delta$, such that the third term in (7.8) can be estimated by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=k}^{n-1} \pi_*|(\lambda^{-1}\hat{P})^j(\hat{f}_1 \cdot (\chi_{X\setminus(N_0\cup B_k)} \circ \pi))|(x)$$

$$\leq \delta \cdot \lim_{n\to\infty} \frac{1}{n}\sum_{j=k}^{n-1} P_g^n 1 \leq \delta \cdot \||\hat{E}e_*1\||_{\mathscr{A}}$$

where $\hat{E}$ is the projection onto $\{\hat{P}_{\hat{g}}\hat{\phi} = \hat{\phi}\}$.

Putting everything together we have in the limit $k \to \infty$: $\pi_*|\hat{E}_\lambda\hat{f}| \leq \delta C_1$ with a constant C_1 independent of δ, and as $\delta > 0$ was arbitrary, $\pi_*|\hat{E}_\lambda\hat{f}| = 0$, i.e. $\hat{E}_\lambda\hat{f} = 0$. □

7.4. *Remark.* Theorem 6.1(d) underlines the importance of measures $\hat{m}$ on $\hat{X}$ with

(7.9) $\text{supp}(\hat{m}) = \hat{X}$ and $\int \hat{P}\hat{f}\,d\hat{m} = \int \hat{f}\,d\hat{m}$ for $\hat{f} \in L^1_{\hat{m}}$.

Similarly there may be a measure m on X with

(7.10) $\text{supp}(m) = X$ and $\int Pf\,dm = \int f\,dm$ for $f \in L^1_m$.

The relation between such measures on X and $\hat{X}$ is as follows: If m satisfies (7.10), then $\hat{m} := m \circ \pi_*$ satisfies (7.9), observe only Lemma 4.6. On the other hand suppose that some $\hat{m}$ sastisfies (7.9) and that Lemma 7.3 applies to $\hat{P}$ and $\hat{m}$. We show that $m := \hat{m} \circ e_*$ satisfies (7.10):

$$\int Pf\,dm - \int f\,dm = \int (e_*P - \hat{P}e_*)f\,d\hat{m} = \int \hat{E}_1(e_*P - \hat{P}e_*)f\,d\hat{m} = 0$$

by Lemma 7.3, since $\pi_*(e_*P - \hat{P}e_*) = 0$ by (4.13) and Lemma 4.6.

Observe that in any case $\int \hat{f}\,d\hat{m} = 0$ if $\pi_*\hat{f} \equiv 0$.

B. Examples.

7.5. **Example I** (topological Markov shift).

Suppose that Theorem 6.1(b) applies to the system (cf. Example 6.12). Although P and $\hat{P}$ coincide as operators on $\text{mb}(X) = \text{mb}(\hat{X})$, we distinguish between $P: C(X) \to C(X)$ and $\hat{P}: \mathscr{H}^\infty(\mathscr{A}) \to \mathscr{H}^\infty(\mathscr{A})$, and Proposition 7.1(a) still tells us that if $|\lambda| = r(P)$ and $\|P^n\| = O(r(P)^n)$, then each continuous eigenfunction of P with eigenvalue λ belongs to $\mathscr{H}^\infty(\mathscr{A})$. For finite

state Markov shifts part (b) of that proposition implies that if $E_\lambda f = f$ and if f varies by at most $C \cdot \gamma_2^n$ on cylinders of length n (where $\gamma_2 < \frac{|\lambda|}{r(P)}$), then $f \in \mathscr{H}^\infty(\mathscr{A})$. As $\hat{r}_{\text{ess}} = 0$, it suffices to choose $\beta = 1$, $\gamma_1 > r(\hat{P})$, and α close to 0 and to use Lemma 6.5 in order to show that if $\hat{f}_n$ is constant on cylinders of length n, then $||| \hat{P}^n \hat{f}_n |||_{\mathscr{A}} \leq \text{const} \cdot ||\hat{f}_n||_\infty \cdot ||| \hat{P}^n 1 |||_{\mathscr{A}}$.

7.6. Example II (up-and-down-counter, cf. 3.13 and 4.9).

Although the same remarks as above apply, this does not really help in this case, because eigenfunctions may be (and will be in general) discontinuous and cannot be approximated as before. ($\mathscr{D}$ does not consist of cylinders only!) Hence Proposition 7.1 must be applied in a different way.

In Example 3.13 we saw that $\mathscr{I} = \{F_i, G_i : i \geq 0\}$ is the only irreducible component of $(\mathscr{D}, \to)$ not consisting of a single loop only, and Theorem 6.11.c) provides a σ-finite measure $\hat{m}$ on $\hat{X}$ such that (7.9) holds for $\hat{P}_{\lambda_1^{-1}}$. Since $D \setminus \text{int}(D) \subseteq \{0^\infty, 1^\infty\}$ for all $D \in \mathscr{D}$, and since for cylinders $\hat{Z} \in \hat{\mathscr{Z}}_n$

$$(7.11) \qquad \hat{m}(\hat{Z}) = \lambda_1^{-n} \int \hat{P}^n \chi_{\hat{Z}} \, d\hat{m} \leq \lambda_1^{-n} \sup_{D \in \mathscr{D}} m(D),$$

we can apply Lemma 7.3. (Observe that $\lambda_1 > 1$, cf. Example 6.14). Hence Remark 7.4 shows that the measure $m = \hat{m} \circ e_*$ on X satisfies (7.10) for the operator $P_{\lambda_1^{-1}}$. In particular $m(Z) = \hat{m}(e_* \chi_Z) \leq \text{const} \cdot \lambda_1^{-n}$ for $Z \in \mathscr{Z}_n$ by (7.11).

Hence, if $f \in \text{mb}(X)$ is of bounded variation with respect to the lexicographic ordering on X, and if $f_n = \sum_{Z \in \mathscr{Z}_n} \chi_Z \cdot \frac{1}{m(Z)} \int_Z f \, dm$, then

$$\int_X |f - f_n| \, dm \leq m(X) \cdot \lambda_1^{-n} \cdot \text{var}(f)$$

and $\hat{P}^n(e_* f_n) \in \mathscr{H}^\infty(\mathscr{A})$. Since also

$$||| \hat{P}^n(e_* f_n) |||_{\mathscr{A}} \leq \text{const} \cdot ||f||_\infty \cdot ||| \hat{P}^n(e_* 1) |||_{\mathscr{A}} \leq ||f||_\infty \cdot \text{const} \cdot \lambda_1^n$$

by Lemma 6.5, we can apply Proposition 7.1(b) with $\hat{H} = \mathscr{H}^\infty(\mathscr{A})$, $H = L_m^1$, $\gamma_1 = \lambda_1 = r(P)$, $\hat{r}_{\text{ess}} = 1$, $\gamma_2 = \lambda_1^{-1}$, and $\beta = 1$, which results in the following conditions:

$$|\lambda| > \lambda_1^{\frac{1}{1+\alpha}}, \quad |\lambda| > \lambda_1^{\frac{\alpha}{1+\alpha}}.$$

The optimal choice for α is 1 in which case we obtain $|\lambda| > \sqrt{\lambda_1}$.

We resume: If $f \in L_m^1$ is of bounded variation and if $(P - \lambda \text{Id})^k f = 0$ for some $k \geq 0$ and $|\lambda| > \sqrt{\lambda_1}$, then $f = \pi_* \hat{f}$ for some $\hat{f} \in \hat{E}_\lambda(\mathscr{H}^\infty(\mathscr{A}))$. Also λ_1 is a simple eigenvalue for $P: L_m^1 \to L_m^1$ in view of Lemma 7.3, and if $M \nsubseteq k \cdot \mathbb{N}$ for all $k \geq 2$, then the irreducible part of $(\mathscr{D}, \to)$ has period 2, and $r(P)$ and $-r(P)$ are the only eigenvalues of maximal modulus for P. Unfortunately this result tells nothing about those λ with $1 = \hat{r}_{\text{ess}} < |\lambda| \leq \sqrt{\lambda_1}$. Better results cannot be expected, however, unless one can show directly that

$f \in \pi_*(\mathscr{H}^\infty(\mathscr{A}))$, as it has been done for certain interval maps in Lemma 11 of [Hofbauer/Keller, 1984].

7.7. **Example III** (piecewise monotonic transformations).

Suppose that $\mathscr{Z}$ is finite. In just the same way as in the foregoing example, but using the Lebesgue measure, which is a priori given, instead of the measure m constructed there, one can check (A6) when $H = L_m^1$, $\hat{H} = \mathscr{H}^\infty(\mathscr{A})$, $\gamma_1 = \lambda_1 = r(P) = 1$, $\hat{r}_{ess} \le g_\infty$, $\gamma_2 = g_\infty + \varepsilon$ (any $\varepsilon > 0$), $\beta = 1$, and $\alpha = 1$. The resulting condition on λ is $|\lambda| > \sqrt{g_\infty}$. Again there is a gap to $G_\mathscr{F} = \{|z| < g_\infty^{-1}\}$ (cf. 6.14).

Proposition 7.1(a) and Lemma 7.3 show that for $\lambda = 1$ there is a 1-1-correspondence between eigenfunctions of $\hat{P}: \mathscr{H}^\infty(\mathscr{A}) \to \mathscr{H}^\infty(\mathscr{A})$ and of $P: L_m^1 \to L_m^1$.

8. Invariant measures

A. Preliminaries.

Suppose that $(X, T, \mathscr{Z}, g)$ is a weighted, piecewise compact, $\mathscr{A}$-expanding system with a canonical Markov extension $(\hat{X}, \hat{T}, \hat{Z}, \hat{g})$ whose Markov diagram is uniformly forward finite. Let $P = P_g$, $\hat{P} = \hat{P}_{\hat{g}}$, and assume that there is a Borel probability measure m on X with $\int Pf \, dm = \int f \, dm$ for all $f \in L_m^1$. Let $\hat{m} = m \circ \pi_*$ be the "lift" of m to $\hat{X}$ (see Remark 7.4). If $\hat{r}_{as} < r(\hat{P})$ for some projection net $\mathscr{F}$, then Theorem 6.1 applies to $\hat{P}$. In particular $\lambda_1 = 1$, and for each $0 \le \hat{h} \in L_{\hat{m}}^1$ such that $\hat{P}\hat{h} = \hat{h}$ and $\int \hat{h} \, d\hat{m} = 1$ the probability measure $d\hat{\mu} = \hat{h} \, d\hat{m}$ is $\hat{T}$-invariant. It is easily seen that $d\mu := h \, dm$ is T-invariant, where $h = \pi_*(\hat{h})$, and that

$$(8.1) \qquad \int f \, d\mu = \int f \circ \pi \, d\hat{\mu} \quad \text{for all } f \in L_\mu^1.$$

8.1. Remark. T acts isometrically on L_μ^2, and its adjoint T^* satisfies $T^* f = P(f h)/h$. $T^* T = \text{Id}_{L_\mu^2}$, whereas TT^* is the orthogonal projection onto $T(L_\mu^2)$ (see Lemma 9 in [Hofbauer/Keller, 1982]). For $f \in L_\mu^2$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ we have $Tf = \lambda f$ if and only if $T^* \bar{f} = \lambda \bar{f}$ if and only if $P(\bar{f} h) = \lambda \bar{f} h$ (ibid., Theorem 2).

8.2. Lemma. (T, μ) *is weakly mixing if and only if*

$$(8.2) \qquad \int f \, d\mu = 0 \Rightarrow \pi_* \hat{E}_\lambda(f \circ \pi \cdot \hat{h}) \equiv 0 \qquad (f \in L_\mu^1, \lambda \in \Gamma)$$

Proof. Assume (8.2), let $f_1 \in L_\mu^\infty$, $f_2 \in L_\mu^1$, and let $\int f_2 \, d\mu = 0$. Then $\pi_*(f_2 \circ \pi \cdot \hat{h}) = f_2 h$, and

$$\int f_1 \circ T^n \cdot f_2 \, d\mu = \int f_1 \cdot \pi_* \hat{P}^n(f_2 \circ \pi \cdot \hat{h}) \, dm \to 0 \quad \text{as } n \to \infty$$

by Theorem 6.1(c), i.e. (T, μ) is weakly mixing.

Conversely, suppose that (T, μ) is weakly mixing. Fix $k \in \mathbb{N}$ such that $\lambda_i^k = 1$ for all $\lambda_i \in \Gamma$. Let $f_1 \in L_\mu^\infty$, $f_2 \in L_\mu^1$, and suppose that $\int f_2 \, d\mu = 0$. Then $f_2 \cdot h = \pi_*(f_2 \circ \pi \cdot \hat{h})$ and

$$\int f_1 \circ T^{kn} \cdot f_2 \, d\mu = \int f_1 \cdot P^{kn}(f_2 h) \, dm$$

$$= \int f_1 \cdot \pi_* \hat{P}^{kn}(f_2 \circ \pi \cdot \hat{h}) \, dm \to \sum_{\lambda \in \Gamma} \int f_1 \cdot \pi_* \hat{E}_\lambda (f_2 \circ \pi \cdot \hat{h}) \, dm$$

as $n \to \infty$. Since (T, μ) is weakly mixing, this implies $\sum_{\lambda \in \Gamma} \pi_* \hat{E}_\lambda(f_2 \circ \pi \cdot \hat{h}) \equiv 0$. Now use (4.13), Lemma 4.6, and (6.2) for $\hat{P}$ to show that $\pi_* \hat{E}_\mu e_* \pi_* \hat{E}_\lambda = \delta_{\mu, \lambda} \pi_* \hat{E}_\mu$ for $\lambda, \mu \in \Gamma$. Then (8.2) follows at once. $\square$

B. Equilibrium states.

We recall some concepts from ergodic theory specialized to piecewise invertible systems. General references are e.g. [Parry, 1969], [Walters, 1981], or [Petersen, 1983].

Suppose $(X, T, \mathscr{Z})$ is a generating, piecewise invertible system and $\mathscr{B}$ is the Borel-σ-algebra of X. By $\mathscr{M}(T)$ denote the set of T-invariant Borel probability measures on X. For $\nu \in \mathscr{M}(T)$ let

$$(8.3) \qquad g_\nu = \sum_{Z \in \mathscr{Z}} \chi_Z \cdot \mathbf{E}_\nu[\chi_Z | T^{-1}\mathscr{B}]$$

and

$$(8.4) \qquad h(\nu, T) = -\int \log g_\nu \, d\nu,$$

the *entropy* of (T, ν). (Note that $\nu\{g_\nu = 0\} = 0$.) From the general theory of entropy it follows that $h(T, \nu)$ does not depend on the particular choice of $\mathscr{Z}$ as long as $\mathscr{Z}$ generates.

Given a weight function $g \geq 0$ and $\nu \in \mathscr{M}(T)$, let

$$(8.5) \qquad F(\nu, T, g) = h(\nu, T) + \int \log g \, d\nu.$$

If $\nu\{g = 0\} > 0$, then $F(\nu, T, g) = -\infty$.

$\nu_0 \in \mathscr{M}(T)$ is an *equilibrium state* for $\log g$, if

$$(8.6) \qquad F(\nu_0, T, g) = \sup\{F(\nu, T, g): \nu \in \mathscr{M}(T)\}.$$

If (X, d) is a compact metric space and if T and $\log g$ are continuous, then the common value of both sides of (8.6) coincides with the pressure

$$\text{press}(T, \log g) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log p_n(T, \log g, \varepsilon),$$

where

$$p_n(T, \log g, \varepsilon) = \sup\left\{ \sum_{x \in E} g_n(x) : E \text{ is } (n, \varepsilon)\text{-separated} \right\}.$$

(Recall that $E \subseteq X$ is (n, ε)-separated, if whenever $x, y \in E$ and $x \neq y$, there exists some i with $0 \leq i \leq n - 1$ and $d(T^i x, T^i y) > \varepsilon$.) This variational principle is due to [Walters, 1976].

8.3. Theorem. *Let* $(X, T, \mathscr{Z}, g)$ *be as in part* **A** *of this section with the measures* m *and* μ *described there.*

(a) *μ is an equilibrium state for* $\log g$ *and* $F(\mu, T, g) = 0$ $(= \log \lambda_1)$.

(b) *Suppose* $\operatorname{supp}(m) = X$. *Then each equilibrium state* ν *for* $\log g$ *can be represented as* $d\nu = h_\nu\, dm$ *with* $h_\nu \in \pi_*(\operatorname{range}(\hat{E}_1))$, *and the number of ergodic equilibrium states for* $\log g$ *equals* $\operatorname{rank}(\pi_* \circ \hat{E}_1)$. *In particular, if* $\lambda_1 = 1$ *is a simple eigenvalue of* $\hat{P}_{\hat{g}}$, *then there is a unique equilibrium state for* $\log g$.

Proof. (a) (a slight generalisation of [Ledrappier, 1974]). For $\nu \in \mathscr{M}(T)$ let $P_\nu = P_{g_\nu}$, and observe that since we can choose $\mathbf{E}_\nu[\chi_Z | T^{-1}\mathscr{B}] = TT_\nu^*\chi_Z$,

$$(8.7) \qquad \int f_1 \cdot P_\nu f_2\, d\nu = \int f_1 \circ T \cdot f_2\, d\nu$$

if both sides of this equation are well defined. In particular, since ν is T-invariant, $P_\nu 1 = 1$ ν-a.e. Let α_n be a measurable function such that $\log \alpha_n \in L_\nu^1$. Then

(8.8)

$$F(\nu, T, g) = h(\nu, T) + \int \log g\, d\nu = \int \log \frac{g}{g_\nu}\, d\nu$$

$$= \frac{1}{n}\left(-\int \log \alpha_n\, d\nu + \int \log\left[\alpha_n \circ T^n \prod_{k=0}^{n-1} \frac{g \circ T^k}{g_\nu \circ T^k}\right] d\nu\right) \quad \text{as } \nu \circ T^{-1} = \nu$$

$$\leq \frac{1}{n}\left(-\int \log \alpha_n\, d\nu + \int \alpha_n \circ T^n \prod_{k=0}^{n-1} \frac{g \circ T^k}{g_\nu \circ T^k}\, d\nu - 1\right)$$

$$= \frac{1}{n}\left(-\int \log \alpha_n\, d\nu + \int P_\nu^n\left(\alpha_n \circ T^n \prod_{k=0}^{n-1} \frac{g \circ T^k}{g_\nu \circ T^k}\right) d\nu - 1\right) \quad \text{by (8.7)}$$

$$\leq \frac{1}{n}\left(-\int \log \alpha_n\, d\nu + \int P^n(\alpha_n \circ T^n)\, d\nu - 1\right)$$

$$= \frac{1}{n}\left(-\int \log \alpha_n\, d\nu + \int \alpha_n \cdot \pi_* \hat{P}^n(e_* 1)\, d\nu - 1\right)$$

Specialising to $\alpha_n = \exp(-\sqrt{n})$ we obtain

$$F(\nu, T, g) \leq \frac{1}{\sqrt{n}} + \frac{\exp(-\sqrt{n})}{n} \, ||| \hat{P}^n |||_{\mathscr{A}} \to 0 \quad \text{as } n \to \infty$$

because $||| \hat{P}^n ||| = O(n^{\sigma-1} \lambda_1^n) = O(n^{\sigma-1})$, where σ is the order of the pole λ_1 of the resolvent of $\hat{P}$.

On the other hand, using the convention $0 \cdot \log 0 = 0$, we have for $h = \pi_*(\hat{h})$: $h|\log h| \leq \max\{e^{-1}, ||| \hat{h} |||_{\mathscr{A}} \cdot \log ||| \hat{h} |||_{\mathscr{A}}\} < \infty$. Hence $h \cdot \log h \in L_m^1$, i.e. $\log h \in L_\mu^1$, and as $\mu \circ T^{-1} = \mu$,

$$\log \frac{h \circ T}{h} \in L_\mu^1 \qquad \text{and} \qquad \int \log \frac{h \circ T}{h}\, d\mu = 0.$$

Now

$$-F(\mu, T, g) = \int \log \frac{h \circ T \cdot g_\mu}{h \cdot g} d\mu$$

$$\leq \int \frac{h \circ T \cdot g_\mu}{h \cdot g} d\mu - 1 \leq \int \frac{h \circ T \cdot g_\mu}{g} dm - 1$$

$$= \int P \left(\frac{h \circ T \cdot g_\mu}{g} \right) dm - 1 = \int P_\mu(1) \cdot h \, dm - 1$$

$$= 0 \quad \text{as } P_\mu(1) = 1 \ \mu\text{-a.e.},$$

i.e. μ maximizes $F(\nu, T, g)$. This proves part (a).

We turn to (b): Since $\operatorname{supp}(m) = X$, we have $\operatorname{supp}(\hat{m}) = \hat{X}$, and hence Γ is a finite group and all $\lambda \in \Gamma$ are semisimple (Theorem 6.1). In particular there is $k > 0$ such that $\lambda^k = 1$ and $\sum_{i=0}^{k-1} \lambda^i = 0$ for all $1 \neq \lambda \in \Gamma$. Using the notation $\hat{\Psi} = \hat{P} - \sum_{\lambda \in \Gamma} \lambda \hat{E}_\lambda$ we can fix $\beta > 1$ and $0 < \gamma < 1$ such that $\beta r(\hat{\Psi})^k < \gamma < 1$. Let $h = \pi_* \hat{E}_1(e_* 1)$, $\alpha_n = \min\{h^{-1}, \beta^{[n/k]}\}$, and let ν be any equilibrium state for $\log g$. Then $F(\nu, T, g) = 0$, and by (8.8) we have for large j

$$0 = k \cdot F(\nu, T, g)$$

$$\leq \frac{1}{jk} \left(k \int \log \frac{1}{\alpha_{jk}} d\nu + k \int \alpha_{jk} \pi_* \hat{E}_1(e_* 1) d\nu - k + \sum_{i=jk}^{jk+k-1} \int \alpha_{jk} \pi_* \hat{\Psi}^i(e_* 1) d\nu \right)$$

$$\leq \frac{1}{j} \left(\int_{\{h > \beta^{-j}\}} \log h \, d\nu - j \log \beta \cdot \nu\{h \leq \beta^{-j}\} + \int \alpha_{jk} h \, d\nu - 1 + \gamma^j \right)$$

$$\leq \frac{1}{j} \int_{\{h > \beta^{-j}\}} \log h \, d\nu - \log \beta \cdot \nu\{h \leq \beta^{-j}\} + \frac{\gamma^j}{j}.$$

Hence

$$(8.9) \qquad \log \beta \cdot \nu\{h \leq \beta^{-j}\} \leq \frac{1}{j} \int_{\{h > \beta^{-j}\}} \log h \, d\nu + \frac{\gamma^j}{j}.$$

As $\log h$ is bounded from above, $-\infty \leq \int \log h \, d\nu < \infty$ is well defined. This shows $\nu\{h = 0\} = 0$ (let $j \to \infty$ in (8.9)).

We prove $\log h \in L_\nu^1$: Suppose for a contradiction that $\int \log h \, d\nu = -\infty$. Then $\int_{\{h > \beta^{-j}\}} \log h \, d\nu \leq 0$ for large j and (8.9) yields $\nu\{h \leq \beta^{-j}\} \leq \gamma^j$ for large j, which in turn implies $\int \log h \, d\nu > -\infty$. This allows the following

estimate:

$$1 = \int \frac{h}{h} d\nu = \int \frac{Ph}{h} d\nu = \int P\left(\frac{h}{h \circ T}\right) d\nu$$

$$= \int P_\nu \left(\frac{h \cdot g}{h \circ T \cdot g_\nu}\right) d\nu = \int \frac{h \cdot g}{h \circ T \cdot g_\nu} d\nu$$

$$\geq 1 + \int \log\left(\frac{h \cdot g}{h \circ T \cdot g_\nu}\right) d\nu = 1 + \int \log\left(\frac{g}{g_\nu}\right) d\nu$$

$$= 1 + F(\nu, T, g) = 1$$

with equality if and only if $(h \cdot g)/(h \circ T \cdot g_\nu) = 1$ ν-a.e. Hence $g_\nu = (g \cdot h)/(h \circ T)$ ν-a.e. Therefore we obtain for $f \in L_\nu^1$ and $n \in \mathbb{N}$

$$(8.10) \qquad \int f \, d\nu = \int P_\nu^n f \, d\nu = \int \frac{P^n(f h)}{h} d\nu$$

$$= \int \frac{\pi_* \hat{P}^n(f \circ \pi \cdot \hat{h})}{h} d\nu.$$

In view of Theorem 6.1(d) there are $\hat{T} - \hat{T}^{-1}$-invariant sets $\hat{X}_i$ and $\hat{P}$-invariant functions $\hat{h}_i \geq 0$ such that $\int_{\hat{X}_i} \hat{h}_j d\hat{m} = \delta_{i,j}$ and

$$\hat{E}_1 \hat{f} = \sum_{i=1}^d \int_{\hat{X}_i} \hat{f} \, d\hat{m} \cdot \hat{h}_i \quad \text{for } \hat{f} \in L_{\hat{m}}^1.$$

In particular $\hat{h} = \sum_{j=1}^d \alpha_j \hat{h}_j$ for some $\alpha_j \in \mathbb{R}$.

Suppose now f is bounded and constant on each cylinder $Z \in \mathscr{Z}_N$ for some $N > 0$. Then $\hat{P}^N(f \circ \pi \cdot \hat{h}) \in \mathscr{H}^\infty(\mathscr{A})$ and $\frac{1}{n} \sum_{k=N}^{n+N-1} \hat{P}^k(f \circ \pi \cdot \hat{h}) \to \hat{E}_1 \hat{P}^N(f \circ \pi \cdot \hat{h})$ in $\mathscr{H}^\infty(\mathscr{A})$. Hence, for such f, (8.10) implies

$$\int f \, d\nu = \int \frac{\pi_* \hat{E}_1 \hat{P}^N(f \circ \pi \cdot \hat{h})}{h} d\nu$$

$$= \sum_{i,j=1}^d \alpha_j \cdot \int_{\hat{X}_i} \hat{P}^N(f \circ \pi \cdot \hat{h}_j) d\hat{m} \cdot \int \frac{\pi_* \hat{h}_i}{h} d\nu$$

$$= \sum_{i=1}^d \alpha_i \int f \circ \pi \cdot \hat{h}_i \, d\hat{m} \cdot \int \frac{\pi_* \hat{h}_i}{h} d\nu$$

$$= \int f \cdot \left(\sum_{i=1}^d \alpha_i \int \frac{\pi_* \hat{h}_i}{h} d\nu \cdot \pi_* \hat{h}_i\right) dm$$

$$= \int f h_\nu \, dm$$

for some $h_\nu \in \pi_*(\text{range}(\hat{E}_1))$. Since $(X, T, \mathscr{Z})$ generates the Borel-σ-algebra of X, it follows that $d\nu = h_\nu dm$. $\square$

8.4. Remark. If $m\{x \in D : V_k(x) \not\subseteq D\} \to 0$ as $k \to \infty$ for all $D \in \mathscr{D}$, then Lemma 7.3 applies, and there are exactly $\text{rank}(\hat{E}_1)$ ergodic equilibrium states for $\log g$ in Theorem 8.3(b).

C. Exponential weak Bernoulli mixing.

Let $(X, T, \mathscr{Z}, g)$, m, and μ be as described in part A of this section. For finite or countable partitions $\mathscr{R}$ and $\mathscr{S}$ of X let

$$D_\mu(\mathscr{R}, \mathscr{S}) = \sum_{R \in \mathscr{R}, S \in \mathscr{S}} |\mu(R \cap S) - \mu(R)\mu(S)|.$$

$\mathscr{Z}$ is called a *weak Bernoulli partition* for (T, μ), if

$$\lim_{n \to \infty} b(n) = 0 \quad \text{where } b(n) = \sup_{k, l \in \mathbf{N}} D_\mu(\mathscr{Z}_k, T^{-(k+n)}\mathscr{Z}_l).$$

It is well known that

$$b(n) = 2 \cdot \sup_{k \in \mathbf{N}} \int \sup \left\{ |\mu(A|\mathscr{Z}_k) - \mu(A)| : A \in T^{-(n+k)}\mathscr{B}_{\mathscr{Z}} \right\} d\mu$$

where $\mathscr{B}_{\mathscr{Z}}$ is the σ-algebra generated by $\bigvee_{i=0}^{\infty} T^{-i}\mathscr{Z}$. $\frac{1}{2}b(n)$ is known as the mixing coefficient of *absolute regularity* of the $\mathscr{Z}$-valued stochastic process ξ_n, $\xi_n(x) = Z$ if $T^n(x) \in Z$. A discussion of this notion of mixing and a lot of references are provided by [Bradley, 1983].

8.5. Theorem. *Suppose that* (T, μ) *is weakly mixing. Then* $\mathscr{Z}$ *is a weak Bernoulli partition for* (T, μ) *and*

$$b(n) = O(r^n)$$

for each $r > \max\{|\lambda|/\lambda_1 : \lambda \in \sigma(\hat{P}), |\lambda| < \lambda_1\}$.

8.6. Remark. For finite state Markov chains an even stronger mixing property was proved by [Bowen, 1975]. For piecewise monotonic interval transformations (with g of bounded variation) a result like Theorem 8.5 can be found in [Hofbauer/Keller, 1982] and [Rychlik, 1983]. Here we adapt Rychlik's very elegant proof to our setting.

Proof of the theorem. Let $A = T^{-(n+k)}\check{A}$, $\check{A} \in \mathscr{B}_{\mathscr{Z}}$. On $B \in \mathscr{Z}_k$ we have

$$\mu(A|\mathscr{Z}_k) = \frac{1}{\mu(B)} \int_A \chi_B h\, dm = \frac{1}{\mu(B)} \int_{\check{A}} P^{n+k}(\chi_B h)\, dm.$$

Since $\mu(A) = \mu(\check{A}) = \int_{\check{A}} h\, dm$,

$$|\mu(A|\mathscr{Z}_k) - \mu(A)| \leq \int_{\check{A}} |P^{n+k}(\chi_B h)/\mu(B) - h|\, dm \quad \text{on } B$$

and hence

$$\frac{1}{2}b(n) \leq \sup_{k \in \mathbf{N}} \sum_{B \in \mathscr{Z}_k} \int |P^{n+k}(\chi_B h) - \mu(B) \cdot h|\, dm.$$

Let $\hat{B} = \pi^{-1}(B)$. Then $\chi_{\hat{B}} = \chi_B \circ \pi$, $\pi_*(\chi_{\hat{B}} \hat{h}) = \chi_B h$, and $P^{n+k}(\chi_B h) = \pi_* \hat{P}^{n+k}(\chi_{\hat{B}} \hat{h})$. Hence

$$\frac{1}{2}b(n) \leq \sup_{k \in \mathbf{N}} \sum_{B \in \mathscr{Z}_k} \int |\pi_* \hat{P}^{n+k}((\chi_{\hat{B}} - \mu(B))\hat{h})|\, dm$$

$$= \sup_{k \in \mathbf{N}} \sum_{B \in \mathscr{Z}_k} \int |\pi_* \hat{\Psi}^n \hat{P}^k((\chi_{\hat{B}} - \mu(B))\hat{h})|\, dm$$

by Lemma 8.2, where $\check{\Psi} = \hat{P} - \sum_{\lambda \in \Gamma} \lambda \hat{E}_\lambda$ as the $L^1_{\hat{m}}$-operator (cf. Theorem 6.1(c). Now

$$\frac{1}{2} b(n) \leq \sup_{k \in \mathbb{N}} \sum_{B \in \mathcal{Z}_k} \left\| \left\| \check{\Psi}^n \hat{P}^k ((\chi_{\hat{B}} - \mu(B)) \hat{h}) \right\| \right\|_{\mathscr{A}}$$

$$\leq \text{const} \cdot r^n \cdot \sup_{k \in \mathbb{N}} \sum_{B \in \mathcal{Z}_k} (\| \hat{P}^k (\chi_{\hat{B}} \cdot \hat{h}) \|_{\mathscr{A}} + \mu(B) \cdot \| \hat{h} \|_{\mathscr{A}})$$

$$\leq \text{const} \cdot r^n \left(\| \hat{h} \|_{\mathscr{A}} + \sup_{k \in \mathbb{N}} \sum_{\substack{\omega \text{ path in } \mathcal{Z} \\ \text{of length } k}} \| (\hat{h} \cdot \hat{g}_n) \circ T_\omega^{-1} \|_{\mathscr{A}} \right)$$

$$\leq \text{const} \cdot r^n \left(\| \hat{h} \|_{\mathscr{A}} + \sup_{k \in \mathbb{N}} \| \hat{P}^k (\hat{h}^*) \|_{\mathscr{A}} \right)$$

by Lemma 6.5 where

$$\hat{h}^* = \sum_{\hat{Z} \in \hat{\mathcal{Z}}} \chi_{\hat{Z}} \cdot \| \hat{h}_{|\hat{Z}} \|_{\mathscr{A}}.$$

Finally observe that $\sup_{k \in \mathbb{N}} \| \hat{P}^k \|_{\mathscr{A}} < \infty$ and $\hat{h}^* \in \mathscr{H}^\infty(\mathscr{A})$. $\square$

8.7. Example II (up-and-down-counter).

In (7.6) we saw that the counter $(\Omega(W_M), S)$ has a Borel measure m with $\text{supp}(m) = \Omega(W_M)$ and $\int P f \, dm = \lambda_1 \int f \, dm$ for all $f \in L^1_m$. λ_1 is a simple eigenvalue of $\hat{P}$. Replacing $g = 1$ by $g = \lambda_1^{-1}$, we can apply Theorem 8.3 and see that the counter has a unique measure μ of maximal entropy $\log \lambda_1$ for each subset M of $\mathbb{N}$. If $M \not\subseteq k \cdot \mathbb{N}$ for all $k \geq 2$, then 1 and -1 are the only peripheral eigenvalues of $\hat{P}_{\lambda_1^{-1}}$. Hence $(\Omega(W_M), S^2)$ has exactly two ergodic measures of maximal entropy, both weakly mixing, and hence exponentially weak Bernoulli mixing under S^2.

8.8. Example III (piecewise monotonic transformations).

Suppose that $\mathcal{Z}$ is finite. In Example 6.14 we showed, how Theorem 6.1 applies. In particular, if $\tilde{g} = 1/|\tilde{T}'|$, then all equilibrium states on X for $\log g$ are absolutely continuous with respect to the Lebesgue measure on X. The number of ergodic equilibrium states for $\log g$ equals $\text{rank}(\hat{E}_1)$, and in view of Example 7.7 this number is identical with the multiplicity of the eigenvalue 1 for the operator P acting on the space of functions of bounded variation on X or on $L^1_m(X)$ (cf. [Hofbauer/Keller, 1982]).

As the spaces $[0, 1] = \tilde{X}$ and X differ only by countably many points, " X " may be replaced by " $[0, 1]$ " in the above remarks.

9. Applications

A. Limiting behaviour of processes $S_n = \sum_{k=0}^{n-1} f \circ T^k$.

Let $(X, T, \mathscr{Z}, g)$ be as in §8.A, and assume that (T, μ) is weakly mixing. Fix $f \in L_\mu^2$ with $\int f \, d\mu = 0$ and suppose

$$(9.1) \qquad \sum_{k=1}^{\infty} \|(T^*)^k f\|_{L_\mu^2} < \infty .$$

$(Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ is a strictly stationary stochastic process with respect to the probability measure μ on X. Let $S_n = \sum_{k=0}^{n-1} Y_k$ and

$$(9.2) \qquad \sigma^2 = \mathbf{var}(Y_0) + 2 \cdot \sum_{k=1}^{\infty} \mathbf{cov}(Y_0, Y_k) < \infty.$$

An easy classical calculation [Kac, 1946] yields

$$(9.3) \qquad \lim_{n \to \infty} \mathbf{var}(S_n/\sqrt{n}) = \sigma^2 .$$

The exceptional case $\sigma^2 = 0$ is characterized by the following lemma (see Lemma 6 of [Rousseau-Egele, 1983]; we give a shorter proof).

9.1. Lemma. *Assume* (9.1). *Then* $\sigma^2 = 0$ *if and only if* $f = \phi - T\phi$ *for some* $\phi \in L_\mu^2$.

Proof. By (9.1), $f_\infty = \sum_{k=0}^{\infty} (T^*)^k f$ is in L_μ^2. A straightforward calculation yields

$$\sigma^2 = \int (f_\infty + T^* f_\infty)(f_\infty - T^* f_\infty) \, d\mu$$
$$= \|f_\infty\|_{L_\mu^2}^2 - \|T^* f_\infty\|_{L_\mu^2}^2$$
$$= \|f_\infty\|_{L_\mu^2}^2 - \|T T^* f_\infty\|_{L_\mu^2}^2 .$$

Since $T T^*$ is the orthogonal projection onto $T L_\mu^2$, $\sigma^2 = 0$ is equivalent to $f_\infty = T T^* f_\infty$, which is true if $f = \phi - T\phi$ and which, on the other hand, implies $f = f_\infty - T^* f_\infty = T T^* f_\infty - T^* f_\infty$. $\square$

Let $f_n = \mathbf{E}_\mu[f | \mathscr{Z}_n]$. A convenient tool to verify (9.1) is

9.2. Lemma. *Suppose* $f \in L_\mu^2$ *is bounded,* $\int f \, d\mu = 0$, *and*

$$(9.4) \qquad \|f_n - f\|_{L_\mu^2} = O(a^n)$$

for some $0 < a < 1$. *Then there is* $0 < b < 1$ *such that* $\|(T^*)^n f\|_{L_\mu^2} = O(b^n)$.

Proof. Fix $k \in \mathbb{N}$ such that $\lambda^k = 1$ for all $\lambda \in \Gamma$, and suppose that $n = 2lk$ for some $l \in \mathbb{N}$. Let $\hat{\Psi} = \hat{P} - \sum_{\lambda \in \Gamma} \lambda \hat{E}_\lambda$ (as L^1_m-operator). Then

$$\|(T^*)^n f_{lk}\|^2_{L^2_\mu} \leq \|f_{lk}\|_\infty \cdot \int |P^n(f_{lk} \cdot h)| \, dm$$

$$\leq \|f\|_\infty \cdot \int |\pi_* \hat{P}^n(f_{lk} \circ \pi \cdot \hat{h})| \, dm$$

$$\leq \|f\|_\infty \cdot \left(\sum_{\lambda \in \Gamma} \int |\pi_* \hat{E}_\lambda(f_{lk} \circ \pi \cdot \hat{h})| \, dm + \int |\pi_* \hat{\Psi}^n(f_{lk} \circ \pi \cdot \hat{h})| \, dm \right)$$

$$= \|f\|_\infty \cdot \int |\pi_* \hat{\Psi}^{lk} \hat{P}^{lk}(f_{lk} \circ \pi \cdot \hat{h})| \, dm$$

by Lemma 8.2 and as $\hat{\Psi}\hat{P} = \hat{\Psi}^2$,

$$\leq \|f\|_\infty \cdot \||\hat{\Psi}^{lk} \hat{P}^{lk}(f_{lk} \circ \pi \cdot \hat{h})\||_{\mathscr{A}}$$

$$\leq \|f\|_\infty \cdot \||\hat{\Psi}^{lk}\||_{\mathscr{A}} \cdot \|f\|_\infty \cdot \||\hat{P}^{lk}\hat{h}^*\||_{\mathscr{A}}$$

by Lemma 6.5, where $\hat{h}^* = \sum_{Z \in \hat{\mathscr{Z}}} \chi_Z \|\hat{h}_{|Z}\|_{\mathscr{A}}$,

$$= O(\||\hat{\Psi}^{n/2}\||_{\mathscr{A}}) \quad \text{as } \sup_n \||\hat{P}^n\||_{\mathscr{A}} < \infty.$$

Since the spectral radius of $\hat{\Psi}$ is less than 1, the lemma follows from $\|f_{lk} - f\|_{L^2_\mu} = O(a^{n/2})$ and the fact that $\|T^*\|_{L^2_\mu} = 1$. $\square$

There are (at least) four approaches to probabilistic limit theorems for the process S_n provided $\sigma^2 > 0$.

9.3. L^2-TECHNIQUE.

Assuming (9.4) and using Lemma 9.2 one can reduce the asymptotic normality of $S_n/\sqrt{n\sigma^2}$ to that of a backward martingale (see [Gordin, 1969]). This approach to the central limit theorem was used by [Keller, 1980] for piecewise monotonic interval maps.

9.4. $(f \circ T^n)_{n \geq 0}$ AS A FUNCTIONAL OF AN ABSOLUTELY REGULAR PROCESS.

Let $\xi_n(x) = Z \in \mathscr{Z}$ if $T^n x \in Z$. This defines a stationary $\mathscr{Z}$-valued process, which, by Theorem 8.5, is absolutely regular with exponential decrease of mixing coefficients. Assuming (9.4) and observing that $(X, T, \mathscr{Z})$ generates, the process $(Y_n)_{n \geq 0}$ can be represented as a sufficiently regular functional of the process $(\xi_n)_{n \geq 0}$, and one obtains a central limit theorem [28] with convergence rate $n^{-\nu}$ for some $0 < \nu < \frac{1}{2}$ and an almost sure invariance principle [29] for the process S_n that is good enough to imply loglog-laws and weak invariance principles. Details can be found in [Hofbauer/Keller, 1982] where this approach has been used for piecewise monotonic interval maps.

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[28] Cf. [Ibragimov/Linnik, 1971].
[29] Cf. [Philipp/Stout, 1975].

The other techniques rely on direct estimates of the Fourier- or Laplace-transforms of $S_n/\sqrt{n}$ or S_n. They apply only to "analytic" f, analytic in the sense that for each $Z \in \mathscr{Z}$ there is $\tilde{f}_Z \in \mathscr{H}^\infty(U_Z)$ such that $\tilde{f}_Z \circ s = f_{|Z}$ (cf. §5).

9.5. FOURIER-TRANSFORM TECHNIQUE.

Let $P_t = P_{g \cdot \exp(itf)}$ for $t \in \mathbf{R}$. Then Theorem 6.1 applies to P_t,

$$\int \exp(itS_n)\,d\mu = \int P^n(h \cdot \exp(itS_n))\,dm = \int P_t^n(h)\,dm,$$

and a perturbation expansion of the spectral representation of $\hat{P}_t$ at $t = 0$ leads to a convergence rate $n^{-1/2}$ in the central limit theorem, while this expansion for small $|t|$ together with the spectral theory of $\hat{P}_t$ for arbitrary t leads to a local limit theorem for S_n. [Rousseau-Egele, 1983] has carried out this program for piecewise monotonic interval maps.

9.6. LAPLACE-TRANSFORM TECHNIQUE.

We still assume f to be analytic in the above sense, but we skip the assumption $\int f\,d\mu = 0$. Let now $P_t = P_{\exp(tf)}$ for $t \in \mathbf{R}$. Then, similarly as above, $\int \exp(tS_n)\,dm = \int P_t^n(1)\,dm$, and Theorem 6.1(b) (or perturbation theory applied to $\hat{P}_0$) yields quasicompacity of $\hat{P}_t$ for small $|t|$, say $\hat{P}_t = \lambda_1(t)\hat{E}_1(t) + \hat{\Psi}(t)$ for $t_- < t < t_+$ and $\hat{E}_1(t)(\hat{f}) = \hat{h} \cdot \int \hat{f}\,d\hat{\mu}$. (Remember that (T,μ) is weakly mixing.)

Let $\psi_n(t) = \frac{1}{n}\log\int \exp(tS_n)\,dm$. Then

(a) $\psi_n(t) = \frac{1}{n}\log\int \pi_*\hat{P}_t^n(e_*1)\,dm \to \log\lambda_1(t) =: \psi(t) < \infty$ $(n \to \infty)$,

(b) $\psi(t)$ is real analytic on (t_-, t_+), since $\hat{P}_t$ is an analytic family of operators,

(c) using Chapter VII, §2.3 and formula (2.14) in [Kato, 1976] it is not hard to show that $\psi'(0) = \int f\,d\mu$ and $\psi''(0) = \sigma^2$.

If $\sigma^2 > 0$ one can choose t_-, t_+ so small that $\psi''(t) > 0$ for $t_- < t < t_+$. Let $\alpha_- = \psi'(t_-)$, $\alpha_+ = \psi'(t_+)$. Properties (a)–(c) are sufficient for a large deviations result, namely:

For all $\alpha \in (\psi'(0), \alpha_+)$

$$\lim_{n\to\infty} \frac{1}{n}\log m\left(\frac{S_n}{n} > \alpha\right) = -\mathrm{I}(\alpha) \in (-\infty, 0),$$

and for all $\alpha \in (\alpha_-, \psi'(0))$

$$\lim_{n\to\infty} \frac{1}{n}\log m\left(\frac{S_n}{n} < \alpha\right) = -\mathrm{I}(\alpha) \in (-\infty, 0),$$

where $\mathrm{I}(\alpha) = \alpha \cdot t_\alpha - \psi(t_\alpha)$ and t_α is the unique solution of $\psi'(t) = \alpha$. (Note that if $\int f\,d\mu = 0$, then $\mathrm{I}(\alpha) = \alpha^2/2\sigma^2 + O(\alpha^3)$ in the limit $\alpha \to 0$.)

Here we used [Plachky/Steinebach, 1975]. A good discussion of the Laplace-transform technique can be found in [Cox/Griffeaths, 1984].

B. Limiting behaviour of statistics based on $(f \circ T^n)_{n \geq 0}$.

The technique of representing $(f \circ T^n)_{n \geq 0}$ as a functional of an exponentially mixing absolutely regular process (cf. 9.4) can be used to derive limit theorems for statistics based on the sample $(f(x), \ldots, f \circ T^{n-1}(x))$, more complicated than the simple "mean-value statistic" S_n. This has been worked out for U-statistics by [Denker/Keller, 1985]. We do not go into details.

C. Extinction probabilities and transient behaviour.

Suppose $(X, T, \mathscr{Z}, g)$ is a weighted system and m a probability measure on X such that $\int Pf \, dm = \int f \, dm$ for all $f \in L^1_m$. Let A be a finite union of cylinders from $\mathscr{Z}$ and let

$$\tau(x) = \tau_A(x) = \min\{n \geq 0 : T^n x \in A\}.$$

(For other "traps" one must start with a refined partition $\mathscr{Z}$.) We investigate $p_n = m\{\tau \geq n\}$. Since $\tau \geq n$ is equivalent to $\frac{1}{n} \sum_{k=0}^{n-1} \chi_A \circ T^k \leq 0$, the large deviation result from (9.6) (if applicable to $f = \chi_A$) would yield $\lim_{n \to \infty} \frac{1}{n} \log p_n = -I(0)$. But as $f = \chi_A \geq 0$, $\lambda_1(t)$ is increasing in t and hence $\psi'(t) \geq 0$ such that $\alpha_- \geq 0$, i.e. $\alpha = 0$ is not in the domain of validity of the large deviation estimate. There is an alternative approach to this problem, however, which shares the large deviation flavour:

Let $g_A = g \cdot (1 - \chi_A)$. (Do not confuse this notation with the former g_n.) Suppose that Theorem 6.1 applies to $(\hat{X}, \hat{T}, \hat{\mathscr{Z}}, \hat{g}_A)$ (it need not apply to $(\hat{X}, \hat{T}, \hat{\mathscr{Z}}, \hat{g})$!). Then

$$p_n = \int \prod_{k=0}^{n-1} (1 - \chi_A) \circ T^k \, dm = \int P_g^n \left(\prod_{k=0}^{n-1} (1 - \chi_A) \circ T^k \right) dm$$

$$= \int P_{g_A}^n(1) \, dm = \int \pi_* \hat{P}_{\hat{g}_A}^n(e, 1) \, dm,$$

i.e. $\overline{\lim}_{n \to \infty} \frac{1}{n} p_n \leq \log r(\hat{P}_{\hat{g}_A})$.

Similarly, if $\lambda_{1,A} = r(\hat{P}_{\hat{g}_A})$ and $\hat{P}_{\hat{g}_A} \hat{h}_A = \lambda_{1,A} \hat{h}_A \geq 0$, then

$$p_n \geq ||| \hat{h}_A |||_{\mathscr{A}}^{-1} \cdot \int \prod_{k=0}^{n-1} (1 - \chi_A) \circ T^k \cdot \pi_*(\hat{h}_A) \, dm$$

$$= ||| \hat{h}_A |||_{\mathscr{A}}^{-1} \cdot \int \pi_* \hat{P}_{\hat{g}_A}^n(\hat{h}_A) \, dm$$

$$= ||| \hat{h}_A |||_{\mathscr{A}}^{-1} \cdot \lambda_{1,A}^n \cdot \int \pi_*(\hat{h}_A) \, dm,$$

and hence $\lim_{n \to \infty} \frac{1}{n} \log p_n = \log \lambda_{1,A}$ provided $\int \pi_*(\hat{h}_A) \, dm > 0$, e.g. if $\mathrm{supp}(m) = X$.

Theorem 8.3 provides an equilibrium state $\mu_A \in \mathcal{M}(T)$ with $F(\mu_A, T, g_A) = \log \lambda_{1,A}$. In particular, $\mu_A\{g_A \circ T^k = 0\} = 0$ for all $k \geq 0$, i.e. $\mathrm{supp}(\mu_A) =: X_A \subseteq \bigcap_{k=0}^{\infty} T^{-k}(X \setminus A)$, $g_{|X_A} > 0$, and

$$\begin{aligned}
\log \lambda_{1,A} &= F(\mu_A, T_{|X_A}, g_{|X_A}) \\
&= \sup\{F(\nu, T_{|X_A}, g_{|X_A}): \nu \in \mathcal{M}(T_{|X_A})\} \\
&= \mathrm{press}(T_{|X_A}, \log g_{|X_A})
\end{aligned}$$

provided X_A is a compact metric space (cf. §8.B).

We summarize these results in

9.7. Proposition. *If $X \setminus A$ is compact metric and if $\mathrm{supp}(m) = X$, then*

$$\lim_{n \to \infty} \frac{1}{n} \log m\{\tau_A \geq n\} = \mathrm{press}(T_{|X_A}, \log g_{|X_A}).$$

For constant g this was proved in [Keller, 1984].

If the trap A is very small, it may take quite a long time until a trajectory hits it with probability $\frac{1}{2}$, say, and it is interesting to characterize the behaviour of the orbits before extinction. We shall show that it is in some sense indistinguishable from the behaviour of typical orbits of the system (T, μ_A) where μ_A is an equilibrium state for $\log g_A$.

So suppose again that $(X, T, \mathcal{Z}, g)$ is a weighted system and m is a probability measure on X with $\mathrm{supp}(m) = X$ and $\int Pf\, dm = \int f\, dm$ for all $f \in L^1_m$. m is lifted to a measure $\hat{m} = m \circ \pi_*$ on $\hat{X}$. Let the trap A, the first exit time τ_A, and the weight function g_A be as above.

9.8. Proposition. *Suppose that Lemma 6.9 applies to the system $(\hat{X}, \hat{T}, \hat{\mathcal{Z}}, \hat{g}_A)$, that $\lambda_{1,A}$ is the only peripheral eigenvalue of $\hat{P}_{\hat{g}_A}$ and that $\lambda_{1,A}$ is semisimple, i.e. $\hat{P}_{\hat{g}_A} = \lambda_{1,A}\hat{E}_1 + \hat{\Psi}$ with $r(\hat{\Psi}) < \lambda_{1,A}$. Let $\hat{m}_A = (\hat{m} \circ \hat{E}_1(e_* 1))^{-1} \cdot \hat{m} \circ \hat{E}_1$, $m_A = \hat{m}_A \circ e_*$, and for $n \in \mathbb{N}$ let $m_n = m(\cdot \,|\, \tau_A \geq n)$.*

For $\Phi: X \to \mathbb{R}$ and $n \in \mathbb{N}$ let $\mathrm{Var}_n(\Phi) = \sup\{|\Phi(x) - \Phi(y)|: x, y \in Z \in \mathcal{Z}_n\}$. Then there are $S > 0$ and $0 < q < 1$ such that

$$\left| \int \Phi\, dm_n - \int \Phi\, dm_A \right| \leq S \cdot \|\Phi\|_\infty \cdot q^{n-k} + \mathrm{Var}_k(\Phi)$$

for all $0 < k < n$ and $\Phi: X \to \mathbb{R}$.

Proof. For $0 < k < n$ let

$$m_{k,n} = \sum_{Z \in \mathcal{Z}_k} m_A(Z) \cdot m_n(\cdot \,|\, Z),$$

$$\nu_{k,n} = \sum_{Z \in \mathcal{Z}_k} m_A(Z) \cdot m_n(\cdot \,|\, Z) \times m_A(\cdot \,|\, Z).$$

$m_{k,n}$ is a probability measure on X, $\nu_{k,n}$ on $X \times X$, and

$$\left| \int_X \Phi\, dm_n - \int_X \Phi\, dm_A \right|$$

$$\leq \left| \int_X \Phi\, dm_n - \int_X \Phi\, dm_{k,n} \right| + \left| \int_{X \times X} \Phi \circ \mathrm{pr}_1\, d\nu_{k,n} - \int_{X \times X} \Phi \circ \mathrm{pr}_2\, d\nu_{k,n} \right|$$

$$\text{where } \mathrm{pr}_1, \mathrm{pr}_2 \colon X \times X \to X \text{ are canonical projections,}$$

$$\leq \sum_{Z \in \mathcal{Z}_k} |m_n(Z) - m_A(Z)| \left| \int_Z \Phi\, dm_n(\cdot\,|Z) \right| + \mathrm{Var}_k(\Phi)$$

$$\leq 2\|\Phi\|_\infty \cdot \sup_B |m_n(B) - m_A(B)| + \mathrm{Var}_k(\Phi)$$

where the supremum extends over all B in the σ-algebra $\sigma(\mathcal{Z}_k)$ generated by $\mathcal{Z}_k$.

Next, if $p_n = m\{\tau \geq n\} > 0$, then

$$m_n(B) = p_n^{-1} \int_{\{\tau \geq n\}} \chi_B\, dm$$

$$= p_n^{-1} \int P^n \left(\chi_B \cdot \prod_{i=0}^{n-1} (1 - \chi_A) \circ T^i \right) dm$$

$$= p_n^{-1} \int \pi_* \hat{P}_{\hat{g}_A}^{n-k} \hat{f}\, dm \quad \text{where } \hat{f} = \hat{P}_{\hat{g}_A}^k (e_* \chi_B) \in \mathcal{H}^\infty(\mathcal{A}),$$

$$= \frac{\lambda_{1,A}^{n-k} \int \hat{E}_1 \hat{f}\, d\hat{m} + \int \hat{\Psi}^{n-k} \hat{f}\, d\hat{m}}{\lambda_{1,A}^n \int \hat{E}_1 (e_* 1)\, d\hat{m} + \int \hat{\Psi}^n (e_* 1)\, d\hat{m}}$$

$$= \frac{\lambda_{1,A}^n \int \hat{E}_1 (e_* \chi_B)\, d\hat{m} + \int \hat{\Psi}^{n-k} \hat{f}\, d\hat{m}}{\lambda_{1,A}^n \int \hat{E}_1 (e_* 1)\, d\hat{m} + \int \hat{\Psi}^n (e_* 1)\, d\hat{m}}.$$

Here we made use of the fact that $\hat{m} \circ \hat{E}_1$ is a σ-finite measure on $\hat{X}$ invariant under $\lambda_{1,A}^{-1} \hat{P}_{\hat{g}_A}$ (see Lemma 6.9).

Fix $0 < \tilde{q} < q < 1$ such that $\lambda_1^{-1} r(\hat{\Psi}) < \tilde{q}$. As

$$\| \hat{\Psi}^{n-k} \hat{f} \|_{\mathcal{A}} \leq \mathrm{const} \cdot (\lambda_{1,A} \tilde{q})^{n-k} \cdot \| \hat{f} \|_{\mathcal{A}} \leq \mathrm{const} \cdot (\lambda_{1,A} \tilde{q})^{n-k} (\lambda_{1,A} q/\tilde{q})^k$$

$$\leq \mathrm{const} \cdot \lambda_{1,A}^n q^{n-k} \quad \text{(use Lemma 6.5)}$$

and

$$\| \hat{\Psi}^n (e_* 1) \|_{\mathcal{A}} \leq \mathrm{const} \cdot (\lambda_{1,A} q)^n,$$

we finally get

$$m_n(B) = \int e_* \chi_B\, d\hat{m}_A \cdot (1 + O(q^n)) + O(q^{n-k})$$

$$= m_A(B) + O(q^{n-k})$$

uniformly in B. $\square$

9.9. Corollary. *Let* F_n *($n \in \mathbb{N}$) be a sequence of functions from* X^n *to* $\mathbb{R}$, *and assume that* F_n *is Lipschitz-continuous in each variable separately with a Lipschitz-constant* L_n, *in the sense that (for some* $\delta > 0$ *)* $|F_n(\ldots, x, \ldots) - F_n(\ldots, y, \ldots)| \leq L_n \cdot \Delta_\delta(x, y)$, *where* $\Delta_\delta(x, y) = \exp(-\delta \cdot \max\{n \in \mathbb{N} : \exists Z \in \mathscr{Z}_n \text{ such that } x, y \in Z\})$.

If $\sup_{n \in \mathbb{N}} \|F_n\|_\infty < \infty$ *and if* $\lim_{n \to \infty} L_n = 0$, *then*

$$\lim_{n \to \infty} \left| \int F_n(x, \ldots, T^{n-1}x) \, dm_n - \int F_n(x, \ldots, T^{n-1}x) \, dm_A \right| = 0.$$

Proof. Let $\Phi_n(x) = F_n(x, \ldots, T^{n-1}x)$. Then

$$\mathrm{Var}_k(\Phi_n) \leq L_n \cdot \left(\frac{1}{1 - e^{-\delta}} + n - k \right),$$

and choosing $k = k_n$ such that $n - k_n \to \infty$ and $L_n \cdot (n - k_n) \to 0$ as $n \to \infty$, the corollary follows from Proposition 9.8. $\square$

9.10. Remark. If $\Xi_n \colon X \to C[0, 1]$ is defined as the path obtained by linear interpolation of the points $(\frac{i}{n-1}, n^{-1/2} \sum_{j=0}^i f \circ T^j(x))$, $(i = 0, \ldots, n-1)$ for some Lipschitz-continuous function $f \colon X \to \mathbb{R}$ (in the above-defined sense), the corollary shows that the distributions of the Ξ_n under m_n and under m_A are asymptotically equivalent. The distribution of Ξ_n under m_A tends, in turn, to the distribution of a Brownian motion over $[0, 1]$ as $n \to \infty$ provided σ^2 as defined in (9.2) is positive, cf. 9.4.

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