PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

RUFUS BOWEN
CAROLINE SERIES

Markov maps associated with fuchsian groups

Publications mathématiques de l'I.H.É.S., tome 50 (1979), p. 153-170
<http://www.numdam.org/item?id=PMIHES_1979__50__153_0>

MARKOV MAPS ASSOCIATED WITH FUCHSIAN GROUPS

by Rufus BOWEN [1] *and* Caroline SERIES

Introduction.

There is a well known relation between the action $x \mapsto (ax+b)/(cx+d)$ of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{R}$ and continued fractions, namely if $x, y \in (0, 1)$ and $x = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \dots}}$, $y = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \dots}}$, then $x = gy$ for $g \in \mathrm{SL}(2, \mathbf{Z})$ if and only if there exist k, l, such that $(-1)^{k+l} = 1$ and $n_{k+r} = m_{l+r}$, for $r \geq 0$, cf. [14]. If we define $h : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ by $h(x) = -1/x$ for $x \in (-1, 1)$; $h(x) = x - 1$ for $x \geq 1$; $h(x) = x + 1$ for $x \leq -1$, then it is not hard to check using the above that $x = gy$, $g \in \mathrm{SL}(2, \mathbf{Z})$, if and only if there exist $n, m \geq 0$ such that $h^n(x) = h^m(y)$. Ergodic properties of continued fractions are usually studied using the first return map $h_0 : (0, 1) \to (0, 1)$ induced on $(0, 1)$ by h, $h_0(x) = (1/x) - [1/x]$, [1]. The important properties of h_0 are that it is expanding, Markov (see below), and satisfies Rényi's condition $\sup_{x \in (0,1)} |f''(x)|/|f'(x)|^2 < \infty$.

In this paper we shall show that if $\mathrm{SL}(2, \mathbf{Z})$ is replaced by any finitely generated discrete subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$ which acts on $\mathbf{R}$ with dense orbits, then one can associate to Γ a map $f = f_\Gamma : \mathbf{R} \cup \{\infty\} \to \mathbf{R} \cup \{\infty\}$ with properties analogous to those of h. For convenience we apply a conformal change of variable and replace the upper half plane by the unit disc D and $\mathbf{R}$ by the unit circle S^1, so that $f_\Gamma : S^1 \to S^1$. The map f_Γ is orbit equivalent to Γ on S^1; more precisely, except for a finite number of pairs of points $x, y \in S^1$, $x = gy$ with $g \in \Gamma$ if and only if there exist $n, m \geq 0$ such that $f^n(x) = f^m(y)$. The map f has the *Markov property* with respect to a finite or countable partition $\mathscr{P} = \{I_i\}_{i=1}^{\infty}$ of S^1 into intervals I_i, namely:

(Mi) f is strictly monotonic on each $I_i \in \mathscr{P}$ and extends to a C^2 function on $\bar{I}_i$. (In fact, f is equal to some fixed element of Γ on I_i.)

(Mii) If $f(I_k) \cap I_j \neq \emptyset$ then $f(I_k) \supset I_j$.

---

(1) Partially supported by NSF MCS74-19388. A01.

The map f also satisfies a transitivity condition:

(Miii) for all i, j, $\bigcup_{r=0}^{\infty} f^r(\mathrm{I}_i) \supset \mathrm{I}_j$,

and a finiteness condition:

(Miv) if int $\mathrm{I}_i=(a_i, b_i)$, then $\{\operatorname{Lim}_{h\to 0^+} f(a_i+h), \operatorname{Lim}_{h\to 0^-} f(b_i-h)\}_{i=1}^{\infty}$ is finite.

The groups Γ we are considering fall into two classes; Γ always has a fundamental region R in D consisting of a polygon bounded by a finite number of circular arcs orthogonal to S^1. If, as we are assuming, Γ has dense orbits on S^1, then $\overline{\mathrm{R}}\cap \mathrm{S}^1$ consists of a finite set of points called cusps which correspond to the parabolic elements of Γ (cf. § 1). The partition $\mathscr{P}$ defined above is finite if and only if Γ has no cusps.

If there are no cusps, f_Γ satisfies two additional properties:

(Ai) there exists $\mathrm{N}>0$ such that $\inf_{x\in(0,1)} |(f^{\mathrm{N}})'(x)|>\lambda>1$

and

(Aii) $\sup_{x\in \mathrm{S}^1}|f''(x)|/|f'(x)|^2<\infty$ (Rényi's condition).

If Γ has cusps, $\mathscr{P}$ is countable and f_Γ has periodic points x, $f_\Gamma^p(x)=x$, $(f_\Gamma^p)'(x)=1$. Therefore (Ai) fails. Also (Aii) is no longer trivial. In this situation we show that there is a subset $\mathrm{K}\subset \mathrm{S}^1$, consisting of a finite union of sets in $\mathscr{P}$, minus the countable set of points which eventually map onto one of the cusps, such that the induced map $f_\mathrm{K}:\mathrm{K}\to\mathrm{K}$, $f_\mathrm{K}(x)=f^{m(x)}(x)$, $m(x)=\inf\{m>0 : f^m(x)\in\mathrm{K}\}$, satisfies all the conditions (Mi)-(Miv), (Ai), (Aii) above.

We can now deduce results about the ergodic properties of f, and hence of Γ. We shall prove a modified version of a result due to Rényi (cf. [1]):

Theorem. — Suppose $f:\mathrm{S}^1\to\mathrm{S}^1$ *satisfies* (Mi)-(Miv), (Ai), (Aii) *above. Then* f *admits a unique finite invariant measure equivalent to Lebesgue measure.*

As a corollary, we obtain the well known result that Γ is ergodic with respect to Lebesgue measure. Since Γ and f are orbit equivalent on S^1 it follows from a result of Bowen [4] that the Γ action is hyperfinite, that is the Γ orbits can (up to sets of measure zero) be generated by the action of a single invertible map T.

For the Γ we are considering, the quotient space D/Γ is a Riemann surface of finite area, with possibly a finite number of ramification points $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n$ with ramification numbers $\nu_1\leqslant\nu_2\leqslant\ldots\leqslant\nu_n$. At P_i the total angle is $2\pi/\nu_i$, corresponding to an elliptic $(\nu_i<\infty)$ or parabolic $(\nu_i=\infty)$ element of Γ. The system $\{g; n; \nu_1, \ldots, \nu_n\}$ is called the *signature* of S_Γ, where g is the genus. The signature is restricted only by the topological constraint

$$2g-2+\sum_{i=1}^{n}(1-(1/\nu_i))>0.$$

If S_Γ and $S_{\Gamma'}$ are Riemann surfaces with the same signature, then there is a quasi-conformal map $S_\Gamma \to S_{\Gamma'}$ which induces an isomorphism $j:\Gamma\to\Gamma'$ and a homeomorphism $h: S^1 \to S^1$ such that $h(gx)=j(g)h(x)$, $x\in S^1$, $g\in\Gamma$, cf. [3, 3 *a*]; h is called the *boundary map* of j. By a result of Mostow [12] and Kuusalo [10], h is absolutely continuous if and only if it is a linear fractional transformation. This result, at least in the case when Γ has no cusps, follows from the theory of Gibbs states applied to f_Γ, $f_{\Gamma'}$, cf. [6].

§ 1 contains preliminaries on Fuchsian groups (*i.e.* discrete subgroups of $SL(2, \mathbf{R})$) and Markov maps. For more details on Markov maps, see [1] and [5]. In § 2 we construct the maps f_Γ subject to a certain geometric constraint (*) on the fundamental domain of Γ. In § 3 we show that, for a given signature $\{g; n; \nu_1, \ldots, \nu_n\}$ there is a Riemann surface with this signature whose fundamental domain in D can be taken to have property (*). In § 4, by using the boundary map h introduced above, we construct $f_{\Gamma'}$ for any $S_{\Gamma'}$ with the same signature as S_Γ.

The idea of using continued fractions to study $\Gamma=SL(2, \mathbf{Z})$ occurs in [2] and [8]. The case in which Γ has as fundamental domain a regular $4g$-sided polygon, corresponding to a surface of signature $\{g; 0; 0\}$ is treated in [13, 9]. The ideas of this paper appear in [6] for the special case $n=0$. We are indebted to Dennis Sullivan for some useful remarks.

The preparation of this paper has been overshadowed by Rufus' death in July. We had intended to write jointly: most of the main ideas were worked out together and I have done my best to complete them. In sorrow, I dedicate this work to his memory.

Caroline Series,
Berkeley, September 1978.

1. Preliminaries on Fuchsian Groups and Markov maps

A linear fractional transformation $\mathbf{C}\to\mathbf{C}$ is a map of the form $z\mapsto g(z)=\dfrac{az+b}{cz+d}$, with $ad-bc=1$. We have $g'(z)=(cz+d)^{-2}$. The circle $C=\{z: |cz+d|=1\}$ is called the *isometric circle* of g since g expands lengths within C and contracts outside. A linear fractional transformation is either *parabolic* (conjugate in the group of linear fractional transformations to $z\mapsto z+1$), *elliptic* (conjugate to $z\mapsto\lambda z$, $|\lambda|=1$), or *loxodromic* (conjugate to $z\mapsto\lambda z$, $|\lambda|\neq 1$). A loxodromic transformation is *hyperbolic* if λ is real and >0.

A discrete group Γ of linear fractional transformations is *Fuchsian* if its limit set (the set of accumulation points of orbits) is contained in the unit circle $S^1=\{z: |z|=1\}$, if the only loxodromic elements are hyperbolic, and if Γ maps the unit disc $D=\{z: |z|\leq 1\}$ to itself. Γ is of the *first kind* if its limit set is all of S^1, otherwise it is of the *second kind.*

A parabolic element in Γ has a unique fixed point on S^1, a hyperbolic element has two fixed points on S^1, and an elliptic element has one fixed point inside S^1 and one outside. The isometric circles of elements of Γ are circular arcs orthogonal to S^1.

We think of D as endowed with the Poincaré metric $ds=\frac{2|dz|}{1-|z|^2}$. The geodesics for this metric are circular arcs orthogonal to S^1. Most of the geometry we use is based on the fact that the Poincaré area of an n-sided geodesic polygon is $\pi(n-2)-\sum_{i=1}^{n}\alpha_i$, where α_i are the interior angles. In particular, two geodesics can intersect at most once, and if two geodesics make interior angles summing to more than π with a third, then the geodesics do not meet on the side of the interior angles. A polygon P is *geodesically convex* if the geodesic arc joining any two points in P lies in P.

Elements of Γ act as isometries of D with the Poincaré metric. A *fundamental region* for Γ is a set $R\subseteq D$ whose boundary has measure zero, such that no two interior points of R are conjugate under Γ and every point in D is conjugate to a point in $\overline{R}$. If Γ is finitely generated and of the first kind then it always has a fundamental region bounded by a finite number of geodesic arcs with vertices in or possibly on S^1. The images of R under Γ exactly fill up D. One way to construct such a region is to take the region outside all the isometric circles of Γ ([7], § 20). One can always assume that none of these circles are diameters of S^1. Each side s of R is identified with another side s', by a corresponding element $g(s)\in\Gamma$.

The set $\{g(s) : s \text{ a side of } R\}$ forms a set of generators for Γ ([7], § 23).

Let v_1 be a vertex of R and s_1 an adjacent side; then $v_2=g(s_1)(v_1)$ is another vertex and $s_2=g(s_1)(s_1)$ an adjacent side. Let s_2' be the other side of R adjacent to s_2. Let $v_3=g(s_2')(v_2)$, $s_3=g(s_2')(s_2')$, and define v_4, s_4, ... similarly. Eventually we will have $(v_{n+1}, s_{n+1})=(v_1, s_1)$ ([7], § 26); $v_1, v_2, \ldots, v_n$ is called the *vertex cycle* at v_1 and $g_1=g(s_1)$, $g_2=g(s_2)$, $\ldots$, $g_n=g(s_n)$ is the cycle of generators at v_1. Now $g_n g_{n-1}\ldots g_1$ fixes v_1. If $v_1\in \text{Int}\, D$, then $g_n\ldots g_1$ is elliptic and necessarily $(g_n\ldots g_1)^\nu=1$ for some integer ν. Such a point is called an *elliptic point* of order ν. If $v_1\in S^1$ then $g_n\ldots g_1$ is necessarily parabolic ([7], § 27). The relations $(g_n\ldots g_1)^\nu=1$, for all elliptic vertices v, form a complete set of relations for Γ [11]. Elliptic points of order ν in D correspond to ramification points with ramification number ν on the Riemann surface D/Γ. By convention, $\nu=\infty$ for vertices $v\in S^1$, and these vertices correspond to parabolic cusps on D/Γ.

Suppose conversely $\mathscr{P}$ is a geodesic polygon with a finite number of sides identified in pairs. Conditions can be given for $\mathscr{P}$ to be the fundamental region of a Fuchsian group, essentially that the angles at each vertex should after identification sum to an integral fraction of 2π. This result is due to Poincaré; for a precise statement, see [11].

When describing arcs on S^1, we always label in an anticlockwise direction, so that $\widehat{PQ}$ means the points lying between P and Q moving anticlockwise from P to Q. We write (PQ), [PQ], etc., to distinguish open and closed arcs on S^1.

We conclude this section with a proof of the theorem on Markov maps stated in the Introduction. In what follows, λ will denote Lebesgue measure on S^1.

Lemma (**1.1**). — *Let* $f: S^1 \to S^1$ *satisfy conditions* (Mi)-(Miv), (Ai), (Aii) *of the Introduction. Let* $W = \{\lim_{h\to 0^-} f(a_i - h), \lim_{h\to 0^+} f(b_i + h)\}_{i=1}^{\infty}$, *and let* $A_1, \ldots, A_p$ *be the intervals defined by the partition points* W. *Let* $E \subset S^1$ *be* f *invariant. Then* $\lambda(A_s \cap E) > 0$, $1 \leqslant s \leqslant p$.

Proof. — Fix s, $1 \leqslant s \leqslant p$. Since $\bigcup_{r=0}^{\infty} f^r(A_s) = S^1$, there exists r such that $\lambda(f^r(A_s \cap E)) > 0$.

Now A_r is partitioned into at most countably many intervals, on each of which f^r is C^2 with derivative bounded away from zero. Choose one such interval B with

$$\lambda(f^r B \cap E) > 0.$$

Then $\lambda(B \cap f^{-r}E) > 0$ since $f^r: B \to f^r B$ is C^2 with non-vanishing derivative. Thus, *a fortiori*, $\lambda(E \cap A_s) > 0$.

Theorem (**1.2**). — *Let* $f: S^1 \to S^1$ *satisfy conditions* (Mi)-(Miv), (Ai) *and* (Aii) *of the Introduction. Then there is a unique finite invariant measure* μ *for* f *equivalent to* λ.

Proof. — We use the notation of Lemma (1.1) and the Introduction. Let $\mathscr{P}^N = \bigvee_{n=0}^{N-1} f^{-N}\mathscr{P}$. An element $I^N \in \mathscr{P}^N$ is divided into (possibly empty) intervals I_r^N, $1 \leqslant r \leqslant p$, where $I_r^N = f^{-N}(A_r) \cap I^N$.

Suppose $\lambda(E \cap A_r) > 0$ and $\lambda(I_r^N) > 0$. Since $f^N: I_r^N \to A_r$ is a diffeomorphism:

$$\frac{\lambda(f^{-N}E \cap I_r^N)}{\lambda(E \cap A_r)} \Big/ \frac{\lambda(I_r^N)}{\lambda(A_r)} = \frac{(f^N)'(x_1)}{(f^N)'(x_2)} \quad \text{for some} \quad x_1, x_2 \in I^N.$$

Making use of the conditions (Ai), (Aii) on the derivatives of f, we obtain by a standard argument (cf. [1])

$$M^{-1} \leqslant \frac{(f^N)'(x_1)}{(f^N)'(x_2)} \leqslant M$$

for some $M > 0$ independent of E, N and r. Thus

(**1.2.1**) $$M^{-1}\lambda(E \mid A_r) \leqslant \lambda(f^{-N}E \mid I_r^N) \leqslant M\lambda(E \mid A_r).$$

Now suppose E is f invariant and $0 < \lambda(E) < 1$. By Lemma (1.1), $\lambda(E \cap A_r) > 0$, $1 \leqslant r \leqslant p$. Choose $a < \frac{1}{2M} \min_{1 \leqslant r \leqslant p} \lambda(E \mid A_r)$. Because of the expanding condition (Ai), $\mathscr{P}$ is a generator (cf. [1]) and so there is $I^N \in \mathscr{P}^N$ such that $\lambda(E \mid I^N) < a$. Since $I^N = \bigcup_{r=1}^{p} I_r^N$, we have $\lambda(E \mid I_r^N) < a$ for some r with $\lambda(I_r^N) > 0$. By (1.2.1)

$$2a < M^{-1} \min_{1 \leqslant r \leqslant p} \lambda(E \mid A_r) \leqslant \lambda(f^{-N}E \mid I_r^N) = \lambda(E \mid I_r^N) < a,$$

which is impossible. Hence $\lambda(E) = 0$ or 1, so λ is ergodic.

We wish to find an invariant measure for f. It is sufficient to show that there is $D>0$, so that $\lambda(f^{-N}E)/\lambda(E)\in[D^{-1}, D]$ independent of N, E. For we can then define $\mu(E)=\operatorname{LIM}_{N\to\infty}\lambda(f^{-N}E)$, where LIM is a Banach limit, and it follows as in [1] that μ is invariant and equivalent to λ. Ergodicity and uniqueness of μ follow from ergodicity of λ and invariance of μ.

Using the same type of estimates as above we find

$$M^{-1}\frac{\lambda(E\cap f^N I^N)}{\lambda(f^N I^N)}\leqslant\frac{\lambda(f^{-N}E\cap I^N)}{\lambda(I^N)}\leqslant M\frac{\lambda(E\cap f^N I^N)}{\lambda(f^N I^N)}. \tag{1.2.2}$$

Suppose $E\subset A_r$, so that $E\cap f^N I^N=\emptyset$ or $E\subset f^N I^N$. Since $\min_{1\leqslant r\leqslant p}\lambda(A_r)>0$,

$$M'^{-1}\lambda(E)\lambda(I^N)\leqslant\lambda(f^{-N}E\cap I^N)\leqslant M'\lambda(E)\lambda(I^N)$$

with $M'>0$, whenever $f^N I^N\cap A_r$. Thus

$$M''^{-1}\sum_{f^N I^N\supset A_r}\lambda(I^N)\leqslant\lambda(f^{-N}E)/\lambda(E)\leqslant\sum_{f^N I^N\supset A_r}\lambda(I^N)\quad\text{with}\quad M''>0. \tag{1.2.3}$$

Write $I^N=I(i_1\ldots i_N)$ if $f^r(I^N)\subset I_{i_r}$, $r=1,\ldots,N$. Then

$$\sum_{f^N I^N\supset A_r}\lambda(I^N)=\sum_{\{j:fI_j\supset A_r\}}\ \sum_{i_1,\ldots,i_{N-1}}\lambda(I(i_1\ldots i_{N-1}j))$$

and

$$\sum_{i_1,\ldots,i_{N-1}}\lambda(I(i_1\ldots i_{N-1}j))=\sum_{i_1,\ldots,i_{N-1}}\frac{\lambda(I(i_1\ldots i_{N-1}j))}{\lambda(I(i_1\ldots i_{N-1}))}\lambda(I(i_1\ldots i_{N-1}))$$

$$=\sum_{i_1,\ldots,i_{N-1}}\frac{(f^{N-1})'(x_1)}{(f^{N-1})'(x_2)}\frac{\lambda(I_j)}{\lambda(fI_{i_{N-1}})}\quad\text{with}\quad x_1, x_2\in I(i_1\ldots i_{N-1})$$

and hence, applying the usual type of estimates to $(f^{N-1})'$,

$$D^{-1}\lambda(I_j)\leqslant\sum_{i_1,\ldots,i_{N-1}}\lambda(I(i_1\ldots i_{N-1}j))\leqslant D\lambda(I_j),\quad D>0. \tag{1.2.4}$$

Since $0<b<\sum_{\{j:f(I_j)\supset A_r\}}\lambda(I_j)<1$ for all r, on substituting in (1.2.3) we obtain

$$D'^{-1}\leqslant\lambda(f^{-N}E)/\lambda(E)\leqslant D',\quad D'>0,$$

as required.

2. Construction of the maps f.

Let Γ be a finitely generated Fuchsian group of the first kind, acting in the unit disc D. Let R be a fundamental domain for Γ in D, with a finite set of sides $S=\{s_i\}_{i=1}^n$. Let $A(s)$ be the side of R identified with s, by an element $g(s)\in\Gamma$, and let $C(s)$ be the circle containing s orthogonal to S^1. Let N be the net in D consisting of all images of sides of R under elements of Γ. We will say R satisfies *property* (*) if:

(i) $C(s)$ is the isometric circle of $g(s)$.

(ii) $C(s)$ lies completely in N.

Theorem **(2.1)**. — *Let* Γ *be a finitely generated Fuchsian group of the first kind, with a fundamental region* R *satisfying* (*). *Then there is a Markov map* $f_\Gamma : S^1 \to S^1$ *which is orbit equivalent to* Γ *on* S^1. *Moreover:*

(a) *If* S_Γ *has no parabolic cusps, the Markov partition is finite and* f_Γ *satisfies properties* (Miii), (Miv), (Ai), (Aii) *of the introduction.*

(b) *If* S_Γ *has parabolic cusps, the Markov partition is countable. There is a subset* $K \subseteq S^1$, *consisting of a finite union of sets in the partition, minus the countable set of points which eventually map onto one of the cusps, such that the first return map induced by* f_Γ *on* K *has properties* (Miii), (Miv), (Ai) *and* (Aii).

For convenience we shall exclude for the moment the cases in which R is a triangle or has elliptic vertices of order 2; but see Remark following Lemma (2.5) below.

Lemma **(2.2)**. — *Suppose* R *is not a triangle. Then, if* s, s' *are non-consecutive sides of* R, $C(s)$ *and* $C(s')$ *do not intersect.*

Proof (see Figure 1).

Suppose that $C(s)$, $C(s')$ intersect in a point P. Let the sides of R between s and s', on the side of R closest to P, be labelled consecutively $s = s_0, s_1, \ldots, s_p = s'$. Let the vertices of R on s, s' closest to P be A, B respectively. Let γ be the geodesic

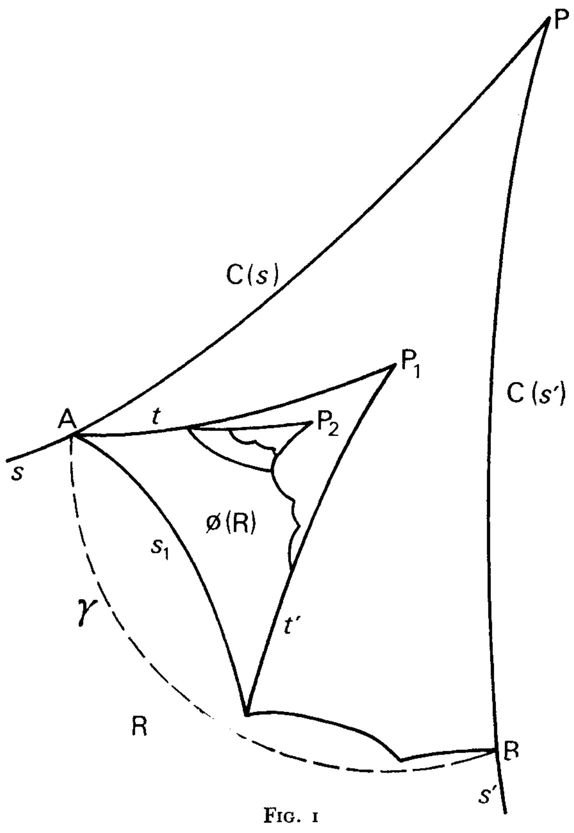

FIG. 1

containing the arc joining A, B. Since R is geodesically convex, $s_1, \ldots, s_{p-1}$ lie within the geodesic triangle APB. The circle $C(s_2)$ intersects at least one of $C(s)$, $C(s')$, since the endpoints of $C(s_2)$ lie outside APB (possibly with one endpoint at P) and $C(s_2)$ can intersect γ at most once. Proceeding inductively we see that without loss of generality we may assume that s and s' are separated by exactly one side s_1.

Let $\varphi(R)$ be the copy of R adjoining on the side s_1 of R. Let t, t' be the sides of $\varphi(R)$ adjacent to s_1. By (*), AP and BP are in N and so t, t' must either coincide with AP, BP or lie properly within APB. Moreover they must meet at a point P_1 within or on the boundary of APB, for otherwise one of t, t' would cut a side of APB twice, which is impossible. $\varphi(R)$ is not all of APB, since R is not a triangle.

Now repeat the argument within the triangle ABP_1 to obtain a copy $\psi(R)$ of R, adjacent to $\varphi(R)$, lying properly within ABP_1, and with non-adjacent sides u, u' meeting within ABP_1. Continuing in this way we obtain an infinite set of disjoint copies of R lying within the region ABP, which is impossible since ABP has finite (non-Euclidean) area.

Definition of f_Γ. — Let $s_1, \ldots, s_n$ be the sides of R, labelled in anti-clockwise order around the circle, and let $g_i = g(s_i)$ be the corresponding elements of Γ. Label the

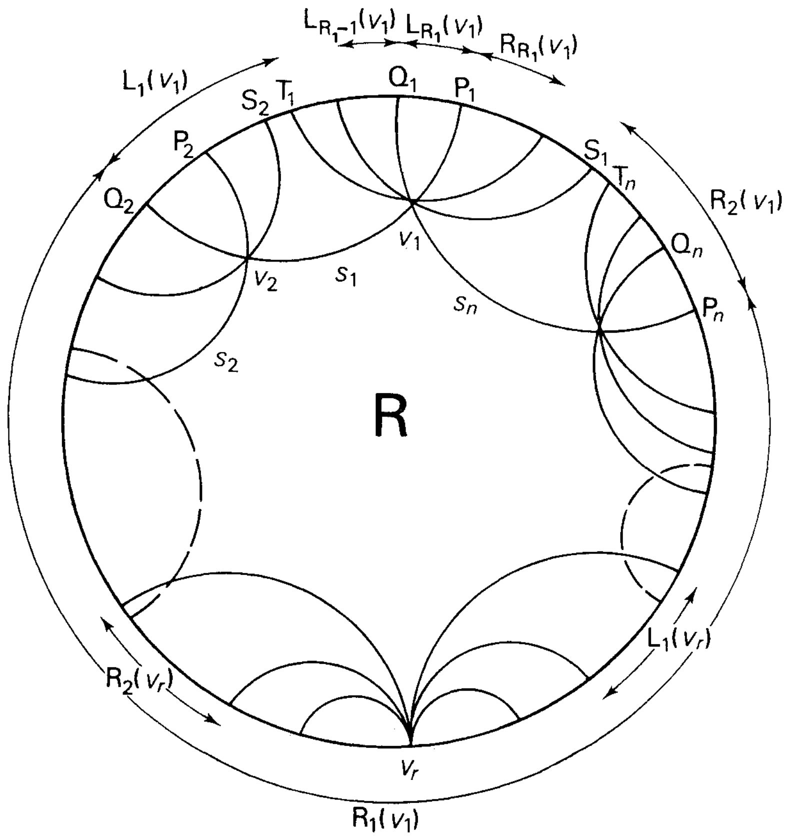

FIG. 2

end points of $C(s_i)$ on S^1, P_i, Q_{i+1} (with $Q_{n+1}=Q_1$), with P_i occurring before Q_{i+1} in the anti-clockwise order. By Lemma (2.2), these points must occur in the order P_1, Q_1, P_2, Q_2, ..., P_n, Q_n (see Figure 2). Define $f_\Gamma(x)=g_i(x)$ on the arc $[P_iP_{i+1})$.

Lemma (**2.3**). — *There is a finite or countable set* $W\subset S^1$ *with* $f(W)\subseteq W$ *which partitions* S^1 *into intervals; W is finite if and only if R has no parabolic vertices.*

Proof. — Let v_i be the vertex of R which is the intersection of s_{i-1}, s_i. Let $N(v_i)$ be the arcs in N which pass through v_i; by property (*) these are complete geodesics. Let $W(v_i)$ be the set of points where the arcs in $N(v_i)$ meet S^1, and let $2k_i=|W(v_i)|$, $1\leq k_i\leq\infty$, with $k_i=\infty$ if and only if v_i is a parabolic cusp. Label the arc $[P_iQ_i)$ as $L_{k_i}(v_i)$. Label the half-open arcs of S^1 cut off by successive points of $W(v_i)$ and proceeding in anti-clockwise order from Q_i, as $L_{k_i-1}(v_i)$, $L_{k_i-2}(v_i)$, ..., $L_1(v_i)$, and let T_i be the point of $W(v_i)$ immediately preceding Q_{i+1}, so that $L_1(v_i)=[T_iQ_{i+1})$. Label the arcs proceeding clockwise from P_i, as $R_{k_i}(v_i)$, $R_{k_i-1}(v_i)$, ..., $R_1(v_i)$. Let S_i be the point of $W(v_i)$ immediately preceding P_{i-1} in the clockwise order, so that $R_2(v_i)=[P_{i-1}S_i)$ and $R_1(v_i)=[Q_{i+1}P_{i-1})$. If v_i is parabolic, start the numbering with $[Q_{i+1}P_{i-1})=R_1(v_i)$ and $[T_iQ_{i+1})=L_1(v_i)$, see Figure 2.

Notice that T_i immediately precedes but does not coincide with S_{i+1} in the anti-clockwise order of the points of $W=\bigcup_{i=1}^{n}W(v_i)$ on S^1. This is because $\widehat{v_iT_i}$ and $\widehat{v_{i+1}S_{i+1}}$ are arcs through non-consecutive sides of R, and so do not intersect by Lemma (2.2).

Now pick $A\in W$ and suppose $A\in[P_iP_{i+1})$, so that $f_\Gamma(A)=g_iA$. Then

$$A\in W(v_i)\cup W(v_{i+1}).$$

If $A\in W(v_i)$, then $g_i(\widehat{v_iA})$ is an arc of N emanating from $g_i(v_i)$ and since $g_i(v_i)$ is a vertex of R, $g_i(A)\in W(g_i(v_i))\subseteq W$. Similarly, if $A\in W(v_{i+1})$, $g_i(\widehat{v_{i+1}A})$ is an arc of N emanating from $g(v_{i+1})$ which is also a vertex of R, so that $g_i(A)\subseteq W$.

We have shown that $f(W)\subseteq W$, which completes the proof.

Lemma (**2.4**). — *The map* f_Γ *and the group* Γ *are orbit equivalent on* S^1, *namely, except for the pairs* $(Q_i, g_{i-1}Q_i)$, *for* $i=1, \ldots, n$:

$$x=gy,\quad x,y\in S^1,\quad \textit{for some}\quad g\in\Gamma \Leftrightarrow \textit{there exist } n, m\geqslant 0 \quad \textit{such that}\quad f^n(x)=f^m(y).$$

Proof. — By definition of f_Γ, it is clear that $f^n(x)=f^m(y)\Rightarrow x=gy$, *with* $g\in\Gamma$. Since $\Gamma_0=\{g_i\}_{i=1}^n$ generates Γ, it is enough to show that $x=gy, g\in\Gamma_0\Rightarrow f^n(x)=f^m(y)$, for some $n, m\geqslant 0$. Since either $|g'(y)|\geq 1$ or $|g^{-1\prime}(x)|\geq 1$, either y lies within the isometric circle of g or x lies within the isometric circle of g^{-1}. If $g=g_i\in\Gamma_0$, then $g^{-1}=g_j\in\Gamma_0$ and so either $x\in[P_jQ_{j+1}]$ or $y\in[P_iQ_{i+1}]$. If $x\in[P_jP_{j+1})$ then

$$f_\Gamma(x)=g_j(x)=y$$

and if $y\in[P_iP_{i+1})$, $f_\Gamma(x)=g_i(x)=y$.

It remains to consider the case $x \in [P_{j+1}Q_{j+1}]$ or $y \in [P_{i+1}Q_{j+1}]$; in other words it is enough to show that if $x \in [P_iQ_i]$, then there are $p, q \geq 0$ such that $f^p(x) = f^q(g_{i-1}x)$. We will call a pair $(x, g_{i-1}(x))$, where $x \in [P_iQ_i]$, a *badly matched pair at* v_i.

Using the fact that all the mappings in Γ are conformal, and that $L_r(v_i) \subseteq [P_iP_{i+1})$, for $2 \leq r \leq k_i$, and $R_s(v_i) \subseteq [P_{i-1}P_i)$, for $2 \leq s \leq k_i - 1$, we see that

$$(2.4.1) \qquad f_{\Gamma | L_r(v_i)} = g_i, \quad f_\Gamma(L_r(v_i)) = L_{r-1}(g_i(v_i)), \quad \text{for} \quad 2 \leq r \leq k_i,$$

and
$$f_{\Gamma | R_s(v_i)} = g_{i-1}, \quad f_\Gamma(R_s(v_i)) = R_{s-1}(g_{i-1}(v_i)), \quad \text{for} \quad 2 \leq s \leq k_i.$$

Let $(x, g_{i-1}(x))$ be a badly matched pair at v_i. Write $f = f_\Gamma$, $w = w_1 = v_i$, $g = g_i$, $h = g_{i-1}$, $k = k_i$. Let the cycle of vertices starting with w and the side s_{i-1} be w_1, $w_2, \ldots, w_p$ and let the corresponding elements of Γ_0 be $h = h_1, h_2, \ldots, h_p$. Let $a = h_p h_{p-1} \ldots h_1$. Then $a^\nu = 1$, where ν is a positive integer, and by (*) $p\nu$ is even and $2k = p\nu$. Moreover $h_p = g^{-1}$ and the cycle starting from w with the side s_i is $w, w_p, \ldots, w_2$ with corresponding generators $g = h_p^{-1}, h_{p-1}^{-1}, \ldots, h_1^{-1} = h^{-1}$.

Suppose ν is even. By repeated applications of (2.4.1) we obtain

$$f(x) = h_p^{-1}(x), \quad f^2(x) = h_{p-1}^{-1}(x) h_p^{-1}(x), \ldots,$$
$$f^{k-1}(x) = h_2^{-1} \ldots h_p^{-1}(h_1^{-1} \ldots h_p^{-1})^{(\nu/2)-1}(x) = h_1 a^{-\nu/2}(x),$$

and
$$f(h_1(x)) = h_2(h_1(x)), \quad f^2(h_1(x)) = h_3 h_2 h_1(x), \ldots,$$
$$f^{k-1}(h_1(x)) = h_p \ldots h_2(h_1 h_p \ldots h_2)^{(\nu/2)-1}(h_1(x)) = a^{\nu/2}(x).$$

Moreover either

a) $\qquad f^{k-1}(x) \in [T_c P_{c+1})$, where $w_2 = v_c$, $c \in \{1, \ldots, n\}$, or

b) $\qquad f^{k-1}(x) \in [P_{c+1} Q_{c+1}]$.

In case *a)*, $f^k(x) = h_1^{-1} f^{k-1}(x) = a^{-\nu/2}(x)$ and so, since $a^\nu = 1$, $f^k(x) = f^{k-1}(h_1 x)$. In case *b)*, $y = f^{k-1}(x) = h_1 a^{-\nu/2}(x) \in [P_{c+1} Q_{c+1}]$ and $g_c(y) = h_1^{-1}(y) = a^{-\nu/2}(x) = f^{k-1}(h_1 x)$ and hence $(f^{k-1}(x), f^{k-1}(h_1 x))$ are badly matched at v_{c+1}.

If ν is odd, we use a similar argument. We now have

$$f^{k-1}(x) = h_{\frac{p}{2}+2}^{-1} \ldots h_p^{-1}(h_1^{-1} \ldots h_p^{-1})^{\frac{\nu-1}{2}}(x)$$
$$= h_{\frac{p}{2}+2}^{-1} \ldots h_p^{-1} a^{-\left(\frac{\nu-1}{2}\right)}(x)$$

and
$$f^{k-1}(h_1(x)) = h_{\frac{p}{2}} \ldots h_2(h_1 h_p \ldots h_2)^{\frac{\nu-1}{2}}(h_1(x))$$
$$= h_{\frac{p}{2}} \ldots h_1 a^{\frac{\nu-1}{2}}(x).$$

Therefore either

a) $f^{k-1}(x) \in [T_b P_{b+1})$, where $w_{\frac{p}{2}} = v_b$, $b \in \{1, \ldots, n\}$, or

b) $f^{k-1}(x) \in [P_{b+1} Q_{b+1}]$.

In case *a)*,

$$f^k(x) = h^{-1}_{\frac{p}{2}+1} f^{k-1}(x) = h^{-1}_{\frac{p}{2}+1} \ldots h^{-1}_{p} a^{-\left(\frac{\nu-1}{2}\right)}(x)$$
$$= h_{\frac{p}{2}} \ldots h_1 a^{\frac{\nu-1}{2}}(x)$$
$$= f^{k-1}(h_1(x))$$

and in case *b)*,

$$y = f^{k-1}(x) \in [P_{b+1} Q_{b+1}] \quad \text{and} \quad g_b = h^{-1}_{\frac{p}{2}+1},$$
$$g_b y = h^{-1}_{\frac{p}{2}+1} \ldots h_1^{-1} a^{-\left(\frac{\nu-1}{2}\right)}(x) = f^{k-1}(h_1(x))$$

so that $(f^{k-1}(x), f^{k-1}(h_1 x))$ are badly matched at v_{b+1}.

Now write $F_i = f^{k_i - 1}$, and observe $F_{i|[P_i Q_i]} = \gamma_i$ for some $\gamma_i \in \Gamma$. Let (x, x') be a badly matched pair at a vertex v_{i_1} of R. Then by the above, either

a) $F_{i_1}(x) \in [T_{i_2 - 1}, P_{i_2})$ for some $i_2 \in \{1, \ldots, n\}$, or

b) $F_{i_1}(x) \in [P_{i_2}, Q_{i_2}]$.

Moreover in case *a)*, there exist $p, q \geq 0$ such that $f^p(x) = f^q(x')$, and in *b)*, there exist $p, q \geq 0$ such that $(f^p(x), f^q(x'))$ are badly matched at v_{i_2}.

It is therefore enough to see that there exists $d > 0$ such that

$$F_{i_d} F_{i_{d-1}} \ldots F_{i_1}(x) \in [T_{i_d - 1} P_{i_d}).$$

Now provided $F_{i_r} F_{i_{r-1}} \ldots F_{i_1}(x) \in [P_{i_{r+1}} Q_{i_{r+1}}]$, $1 \leq r \leq s$, then

$$F_{i_s} \ldots F_{i_1 | [x Q_{i_1}]} = \gamma_{i_s} \ldots \gamma_{i_1},$$

and there exists $\mu > 1$ such that $|\gamma'_{i_r}(y)| \geq \mu > 1$ for $1 \leq r \leq s$, $y \in F_{i_{r-1}} \ldots F_{i_1}([x Q_{i_1}])$. Hence if $x \neq Q_{i_1}$, there exists d such that $F_{i_d} \ldots F_{i_1}([x Q_{i_1}])$ is longer than $[P_{i_{d+1}} Q_{i_{d+1}}]$, *i.e.* such that $F_{i_d} \ldots F_{i_1}(x) \in [T_{i_d - 1} P_{i_d})$. This completes the proof.

Notice that if v_i is a parabolic vertex, P_i and Q_i coincide and so there are no badly matched pairs at v_i.

Lemma **(2.5)**. — *Suppose* Γ *has no cusps. Then* f_Γ *satisfies conditions* (Miii) *and* (Miv) *of the Introduction.*

Proof. — Denote the intervals $[T_iP_{i+1})$ and $[Q_iS_{i+1})$ of Figure 2 by A_i, B_i respectively. Observe that by (2.4.1), if I_r is any interval in the partition defined by W, then there exists $p>0$ such that $f^p(I_r)$ contains an interval of the form A_i or B_j.

Using the fact that $f_{|A_i}=g_i=f_{|B_i}$, one sees that $f(A_i)$ covers all but two of the intervals $\{B_j\}_{j=1}^n$, $B_{r(i)}$ and $B_{r'(i)}$, say. Similarly $f(B_j)$ covers all of $\{A_i\}_{i=1}^n$ except $A_{s(j)}$ and $A_{s'(j)}$. Moreover if $i\neq i'$, then $\{r(i), r(i)\}\neq\{r(i'), r'(i')\}$, and similarly for $s(j)$, $s'(j)$. Thus for each i, $f^2(A_i)$ covers all except possibly one of the A_j's and similarly for $f^2(B_j)$. Suppose $f^2(A_i)$ does not cover A_j. Pick $k\neq i, j$. Then $A_k\subset f^2(A_i)$ and $f^2(A_k)\cup f^2(A_i)$ covers $\bigcup_{r=1}^{n} A_r$. Similarly $f^2(B_k)\cup f^2(A_i)$ covers $\bigcup_{r=1}^{n} B_r$.

Now $f(A_i)$ also covers all but the part of S^1 which lies inside the circular arcs $C(s_p)$, $C(s_{p+1})$, $C(s_{p+2})$ corresponding to three consecutive sides s_p, s_{p+1}, s_{p+2} of R. Since R has at least four sides one sees that $f(\bigcup_{r=1}^{n} B_r)=S^1$.

We have shown that f satisfies (Miii), and (Miv) is clear.

Remark. — If R has elliptic vertices of order two, we may proceed with the above construction omitting these vertices. If s is the side containing the elliptic vertex and g the elliptic element then we associate g to the entire side s.

If R is a triangle, the order of the points $W(v_i)$, $W(v_{i+1})$ around S^1 is altered if one of the angles is $\pi/2$. However a similar method to the above applies.

Neither of these two cases is involved in our construction of "canonical" fundamental regions below; they do however occur in the classical case $\Gamma=SL(2, \mathbf{Z})$ acting on the upper half-plane.

Since all the circles $C(s_i)$, $1\leq i\leq n$, are isometric circles for elements of Γ, it is clear that $|f'(x)|$ is bounded away from 1 on all intervals of the Markov partition formed by W except those of the form $[P_iQ_i)$. If v_i is not parabolic, we have $f([P_iQ_i))=L_{k_i-1}(v_i)$, and hence

Lemma (**2.6**). — *If* S_Γ *has no parabolic cusps, then* f_Γ *satisfies properties* (Ai), (Aii), *with* $N=2$.

Proof. — This follows by the above and the fact that W is finite.

This completes the proof of Theorem (2.1) *(a)*.

Suppose now R has parabolic vertices $v_{i_1}, \ldots, v_{i_r}$; v_{i_j} is a periodic point of order r_j for f_Γ, and $(f_\Gamma^{r_j})'(v_{i_j})=1$. The conditions of [5] apply and one may deduce (2.1) (b). It is however not hard to verify this result directly, as follows:

Let $K=S^1-\bigcup_{j=1}^{r}((\bigcup_{s=2}^{\infty}L_s(v_{i_j}))\cup(\bigcup_{t=3}^{\infty}R_t(v_{i_j})))-\bigcup_{j=1}^{r}\bigcup_{n=1}^{\infty}f_\Gamma^{-n}(v_{i_j})$. It is clear that $\inf_{x\in S^1}|(F^2)'(x)|\geq\lambda>1$ where $F=f_K$ is the map induced by f_Γ on K. Thus f_K satisfies (Ai). To prove (Aii) we use the following:

Lemma **(2.7).** — *The most general parabolic transformation* $T : D \to D$ *with a fixed point* $z_0 \in S^1$ *is of the form* $z \mapsto \frac{az+\bar{c}}{cz+\bar{a}}$, *where* $a\bar{a} - c\bar{c} = 1$ *and* $a + \bar{a} = \pm 2$, $z_0 = \frac{a-\bar{a}}{2c}$. *There is a linear fractional transformation* $P : \mathbf{C} \to \mathbf{C}$ *so that* $P(z_0) = 0$, $P(S^1) = \mathbf{R}$ *and* $S = PTP^{-1} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$.

Proof. — The first part follows from [7], Chapter 1, Theorem (2.3) and (1.5). We may clearly assume without loss of generality that $z_0 = i$. Then, if $a = x + iy$, we have $x = \pm 1$ and $y = c$. Without loss of generality assume $x = 1$. Let $P = \begin{pmatrix} 1 & -i \\ 1 & 1-i \end{pmatrix}$. Then $P(i) = 0$, $P(S^1) = \mathbf{R}$ and $PTP^{-1} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ as required.

Lemma **(2.8).** — *There is a point* $X \in S^1$, *so that if*

$$I = [iX] \subseteq S^1 \quad \textit{and} \quad n(x) = \sup\{n : T^n(x) \in I\}, \quad x \in I,$$

then

$$\sup_{x \in I} (|T^{n(x)''}(x)| / |T^{n(x)'}(x)|^2) < \infty.$$

Proof. — By Lemma (2.7), it is enough to show that if $J = \left[-\frac{1}{2y}, 0\right]$ and $m(x) = \sup\{m : S^m(x) \in J\}$ $x \in J$, then $\sup_{x \in J}(|S^{m(x)''}(x)| / |S^{m(x)'}(x)|^2) < \infty$.

Now $S^m = \begin{pmatrix} 1 & 0 \\ my & 1 \end{pmatrix}$ and $S^{-m}(-(2y)^{-1}) = -((m+2)y)^{-1})$. Therefore $m(x) = m$ precisely when $x \in [-((m+1)y)^{-1}, -((m+2)y)^{-1})$

$$S^{m'}(x) = (myx+1)^{-2}$$

and

$$S^{m''}(x)/S^{m'}(x) = (\log S^{m'}(x))' = -2mx(myx+1)^{-1}.$$

Thus

$$|S^{m''}(x)| / |S^{m'}(x)|^2 = |-2mx(myx+1)|.$$

If $x \in [-((m+1)y)^{-1}, -((m+2)y)^{-1})$, then we get

$$|S^{m''}(x)| / |S^{m'}(x)|^2 \leq 4m|y|(m+2)^{-1}.$$

The result follows.

To complete the proof of Theorem (2.1) (b) it remains to verify the conditions (Mi)-(Miv) for f_K. Label the intervals formed by the partition points W, $\{I_j\}_{j=1}^{\infty}$. By (2.4.1) the only intervals in K mapped outside K by f are those of the form $J_i = [T_i S_{i+1})$, $i = 1, \ldots, n$ (see Figure 2). The J_i are divided into a countable number of subintervals $\{J_{i,r}\}_{r=1}^{\infty}$, $J_{i,r} = f^{-1}(I_r) \cap J_i$. Let $\mathscr{P} = \{T_j\}_{j=1}^{\infty}$ be the partition of K formed by the $J_{i,r}$ and the I_j in K but not in $[T_i S_{i+1})$ for any i. On each T_j, f_K is equal to some fixed element of Γ and it is clear that $\mathscr{P}$ satisfies conditions (Mi) and (Mii) for f_K. If $f(J_{i,r}) \cap K = \emptyset$ then $f(J_{i,r}) = L_k(v)$ or $f(J_{i,r}) = R_k(v)$ for some $k > 0$ and some parabolic vertex v. Therefore, by (2.4.1) again and the definition of K, $f_K(J_{i,r}) = L_1(w)$ or $f_K(J_{i,r}) = R_2(w)$, where w is again a parabolic vertex. Therefore $f_K^r(T_j) \supset A_s$ or $f_K^r(T_j) \supset B_t$ for some r and some s, t. By exactly the same argument as in Lemma (2.5) we see that (Miii) holds for f_K. (Miv) is also clear from the above discussion.

3. Construction of Fundamental Domains.

In this section we construct, for any given signature $\{g; n; \nu_1, \ldots, \nu_n\}$ with $2g-2+\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)>0$, a fundamental domain for a surface of this signature which satisfies property (*). We begin with some lemmas:

Lemma (3.1). — *Let* C, C′ *be non-intersecting circular arcs in* D *orthogonal to* S^1 *and equidistant from the centre* o. *Let* P, Q *and* P′, Q′ *be points on* C, C′ *symmetrically placed with respect to* o. *Then there is a unique linear fractional transformation* $g: D \to D$ *carrying* C, P, Q *to* C′, P′, Q′ *respectively, and* C *is the isometric circle of* g.

Proof. — It is clear that the unique transformation carrying P to P′ and Q to Q′ fixing S^1 carries C to C′. P, Q and P′, Q′ divide C and C′ each into three arcs and the corresponding lengths are equal. If C were not the isometric circle of g, then, in order for these lengths to be preserved, the isometric circle would cut C on each of these arcs, which is impossible.

It is well known how to construct a regular $4g$-sided polygon with interior angles $\frac{\pi}{2g\nu}$, $\nu \geq 0$. Namely, if $4g$ symmetrically placed arcs are drawn orthogonal to S^1, and if their distance from o is allowed to increase from zero until the circles are touching, the angle between them decreases from $\frac{\pi(2g-1)}{2g}$ to zero, so that at some point it is $\frac{\pi}{2g\nu}$. This polygon satisfies the conditions of Poincaré's theorem [11], and is a fundamental region for a surface of signature $\{g; 0; 0\}$; moreover it satisfies property (*).

Notice that the same construction gives a regular $4g$-sided polygon of angle β, for any β, $0 \leq \beta < \pi(2g-1)/2g$.

Lemma (3.2). — *Let* C, C′ *be any two non-intersecting circular arcs orthogonal to* S^1 *and let angles* $x_0, y_0 \in [0, \pi/2]$ *be given. Then there is an oriented geodesic arc* K *joining* C *to* C′ *making angles* x_0, y_0 *with* C, C′ *respectively on the right hand side, and whose points of intersection with* C, C′ *lie to the left of the mid-points of* C, C′.

Proof. — Let T, T′ be the endpoints of the radii of S^1 through the mid-points of C, C′ and let M be the geodesic arc joining T and T′. Let L be an arc in a general position cutting C, C′ to the left of M, and let x, y be the angles cut off by M on C, C′ on the right hand side. Let ρ, ρ' be the (Euclidean) distances along C, C′ from the left end points to the points of intersection with L, and let ℓ, ℓ' be the distances to the points of intersection with M. Clearly M makes angles greater than $\pi/2$ with C, C′ on the right hand side.

Fix ρ and vary ρ'. When $\rho'=0$, $y=0$ and when $\rho'=\ell'$, $y \geq \pi/2$. Therefore there exists ρ' such that $y=y_0$. Let the corresponding value of x be $F(\rho)$.

Now vary ρ. When $\rho = 0$, $F(\rho) = 0$ and when $\rho = \ell$, $F(\rho) \geq \pi/2$. Therefore there exists ρ such that $F(\rho) = x_0$.

Lemma **(3.3).** — *Let* C *be a geodesic arc in* S^1, *and let* γ *be a geodesic arc cutting* C *at a point* P *and making an angle* β, $0 < \beta \leq \pi/2$, *with* C *on the side nearest the origin. Let* T *be the point of intersection of* C *with* S^1 *inside* γ, *and let* ρ *be the distance from* T *to* P *along* C. *Let* E *be the endpoint of* γ *outside* C. *Then as* $\rho \to 0$, E *approaches* T.

Proof. — For convenience we will apply a conformal map so that we are working on the upper half plane. We label the points as before. Let Y be the centre of the semicircular arc γ, lying on the real axis $\mathbf{R}$. Let S be the foot of the perpendicular from P to $\mathbf{R}$. It is clearly enough to see $YP \to 0$ as $\rho \to 0$. But $\rho > PS$, and $\sphericalangle PYS > 0$, and $YP = PS(\sin \sphericalangle PYS)^{-1}$.

Notice that if $\beta = 0$, we can draw arbitrarily small arcs γ touching C on S^1.

We are now ready to construct the required fundamental domains. We have already noted above the construction in case $n = 0$, $g > 1$.

If $n = 1$, $g > 0$, we construct similarly a regular $4g$-sided polygon with interior angle $\dfrac{\pi}{2g\nu_1}$. Label the sides in consecutive anti-clockwise order $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$. Let A_i, B_i be the transformations identifying a_i with a_i^{-1} and b_i with b_i^{-1} respectively. Identifying the sides gives a surface of signature $(g; 1; \nu_1)$. Moreover, by Lemma (3.1) the sides are isometric circles of the corresponding transformations; and the polygon has property (*) by symmetry.

Now suppose $n > 1$, $g > 0$. Since $\dfrac{\pi}{2g} < \dfrac{\pi(2g+1)}{2(g+1)}$, we can construct a regular $4(g+1)$-sided polygon of angle $\dfrac{\pi}{2g}$. Let $C_1, C_2, \ldots, C_8$ be geodesics through eight consecutive sides, oriented in an anticlockwise direction.

Remove arcs C_3 and C_7. C_2, C_4 and C_6, C_8 do not intersect by Lemma (2.2). We will join C_2 to C_4 by a chain of arcs making successive interior angles $\dfrac{\pi}{\nu_1}, \dfrac{\pi}{\nu_2}, \ldots, \dfrac{\pi}{\nu_n}$ (see Figure 3). Let S, T be the endpoints of C_2, C_4 respectively lying inside C_3. Divide the arc ST on S^1 into $n-1$ equal parts, at points $S, P_1, \ldots, P_{n-2}, T$. By Lemma (3.3) find a point Q_1 on C_2 close to S so that the arc γ_1 through Q_1 making an interior angle $\dfrac{\pi}{\nu_1}$ with C_2 has an end point within SP_1. Find Q_2 on γ_1 so that the arc γ_2 through Q_2 at an angle $\dfrac{\pi}{\nu_2}$ to γ_2 has an endpoint within SP_2. Repeat this to obtain arcs $\gamma_1, \ldots, \gamma_{n-2}$ making successive interior angles $\dfrac{\pi}{\nu_1}, \ldots, \dfrac{\pi}{\nu_{n-2}}$. Finally apply Lemma (3.2) to construct an arc γ_{n-1} making interior angles $\dfrac{\pi}{\nu_{n-1}}$ with γ_{n-2} and $\dfrac{\pi}{\nu_n}$ with C_4.

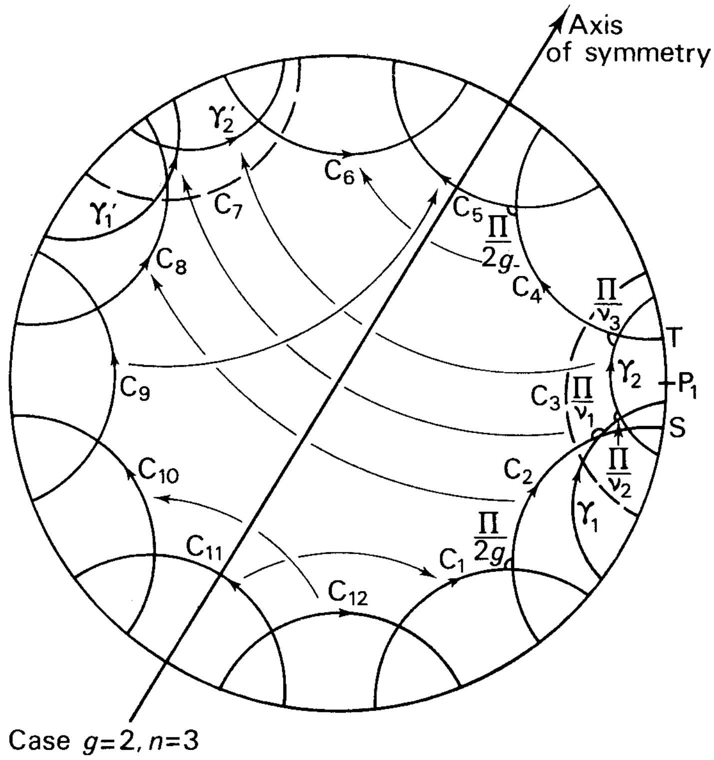

FIG. 3

Join C_8 and C_6 by a symmetrical sequence of arcs $\gamma'_1, \ldots, \gamma'_{n-1}$. Identify C_2 and C_3^{-1}, γ_1 and $\gamma_1^{-1}, \ldots, \gamma_{n-1}$ and γ'^{-1}_{n-1}, C_4 and C_6^{-1}, C_9 and C_5^{-1}. Also identify C_1 and C_{11}^{-1}, C_{10} and $C_{12}^{-1}, \ldots, C_{4g+1}$ and C_{4g+3}^{-1}, and C_{4g+2} and C_{4g+4}^{-1}.

The polygon thus formed satisfies the conditions of Poincaré's theorem [11].

The resulting surface has signature $\{g; n; \nu_1, \ldots, \nu_n\}$. Moreover, by Lemma (3.1) all sides of the fundamental polygon formed by $C_1, C_2, \gamma_1, \ldots, \gamma_{n-1}, C_4, C_5, C_6, \gamma'_{n-1}, \ldots, \gamma'_1, C_8, \ldots, C_{4g+4}$ are isometric circles of the corresponding transformations, and by symmetry the polygon satisfies (*).

If $n>3$, $g=0$, draw non-intersecting geodesics C_1, D_1 and their reflections C_2, D_2 in a diameter T of S^1, so that C_1, C_2 intersect at an angle $\frac{2\pi}{\nu_1}$ and D_1, D_2 at $\frac{2\pi}{\nu_2}$. Proceed as above to join C_1 to D_1 by arcs $\gamma_1, \ldots, \gamma_{n-3}$ making successive interior angles $\frac{\pi}{\nu_3}, \ldots, \frac{\pi}{\nu_n}$, and let $\gamma'_1, \ldots, \gamma'_{n-3}$ be the reflections of $\gamma_1, \ldots, \gamma_{n-3}$ in T. Identify the sides C_1, C_2; $\gamma_1, \gamma'_1; \ldots; \gamma_{n-3}, \gamma'_{n-3}$; D_1, D_2. The resulting polygon has all the required properties.

Finally, if $n=3$, $g=0$, we have $\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}<1$. Draw geodesics C_1; D_1 and their reflections C_2, D_2 in a diameter T, so that C_1 and C_2 intersect at an angle $\frac{2\pi}{\nu_1}$ at a point R on T, and D_1, D_2 at an angle $\frac{2\pi}{\nu_2}$ at a point S on T. When R, S are close to S^1 then C_1 and D_1, C_2 and D_2 do not intersect. As R, S move along T towards the centre o, C_1 and D_1 intersect at an angle which increases from o to $\pi\left(1-\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)$, as follows from a simple computation with C_1, C_2, D_1, D_2 in their limiting positions as diameters through o. Since $\pi\left(1-\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)>\frac{\pi}{\nu_3}$, there is an intermediate point where the angle of intersection is $\frac{\pi}{\nu_3}$, and the resulting polygon is the desired figure.

4. Boundary Maps

Now let S be an arbitrary Riemann surface with signature $\{g; n; \nu_1, \ldots, \nu_n\}$, $2g-2+\sum_{i=1}^{n}\left(1-\frac{1}{\nu_i}\right)>0$. Let S′ be a "canonical" surface of the same signature constructed as in § 3, with corresponding group Γ'. As in [3 *a*], p. 582 or [3], p. 268, there is a quasi-conformal map $g: S' \to S$. Pulling back the Beltrami differential of this map to D gives a symmetric Beltrami coefficient μ for Γ'. Let ω^μ be the associated μ-conformal automorphism of **C**. Then $\Gamma=\omega^\mu\Gamma'(\omega^\mu)^{-1}$ is a Fuchsian group defining the surface S and $j:\Gamma'\to\Gamma$, $j(g)=\omega^\mu g(\omega^\mu)^{-1}$, is an isomorphism. ω^μ restricts to a homeomorphism $h: S^1\to S^1$ such that $h(gx)=j(g)h(x)$. h is the *boundary map* described in the introduction.

We can now define the map f_Γ associated to Γ. Namely, $f_\Gamma=hf_{\Gamma'}h^{-1}$. It is clear that f_Γ satisfies (Mi)-(Miv). (Ai) for f_Γ follows exactly as in ([6], Lemma 3). (Aii) is immediate for compact S_Γ; in the non-compact case it follows from Lemmas (2.7) and (2.8).

This completes the construction described in the Introduction. The result of Mostow and Kuusalo mentioned in the Introduction follows from the existence of f_Γ, exactly as in [6].

REFERENCES

[1] R. Adler, Continued fractions and Bernoulli Trials, *Ergodic Theory Lecture Notes*, J. Moser, E. Phillips, S. Varadhan, Courant Institute, N.Y.U., 1975.
[2] E. Artin, Ein mechanisches System mit quasiergodischen Bahnen, *Collected Papers*, Addison Wesley, 1965, p. 499-501.
[3] L. Bers, Uniformization, moduli and Kleinian groups, *Bull. London Math. Soc.*, **4** (1972), 257-300.
[3 *a*] —, On Hilbert's 22nd problem, *Proc. Symp. Pure Math.*, **28** (1976), 559-609.
[4] R. Bowen, Anosov foliations are hyperfinite, *Ann. Math.*, **106** (1977), 549-565.
[5] —, Invariant measures for Markov maps of the interval, *Comm. Math. Phys.* (1979).
[6] —, Hausdorff dimension of quasi-circles, *Publ. Math. I.H.E.S.*, **50** (1979).

[7] L. R. FORD, *Automorphic Functions*, New York, McGraw-Hill, 1929 (1st edition).
[8] G. A. HEDLUND, A metrically transitive group defined by the modular group, *Amer. J. Math.*, **57** (1935), 668-678.
[9] —, On the metrical transitivity of the geodesics on closed surfaces of constant negative curvature, *Ann. Math.*, **35** (1934), 787-808.
[10] T. KUUSALO, Boundary mappings of geometric isomorphisms of Fuchsian groups, *Ann. Acad. Sci. Fennicae*, Ser A. Math., **545**, 1973, 1-7.
[11] B. MASKIT, On Poincaré's theorem for fundamental polygons, *Adv. Math.*, **7** (1971), 219-230.
[12] G. D. MOSTOW, *Strong rigidity of locally symmetric spaces*, Princeton Univ. Press, 1973, p. 178.
[13] J. NIELSEN, Untersuchungen zur Topologie der geschlossenen zweiseitige Flächen, *Acta Math.*, **50** (1927), 189-358.
[14] HARDY and WRIGHT, *An Introduction to the Theory of Numbers*, Oxford University Press.

Department of Mathematics,
University of Warwick
Coventry, CV4 7AL, England

Manuscrit reçu le 1er novembre 1978.