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# MARKOV PROCESSES ASSOCIATED WITH CERTAIN INTEGRO-DIFFERENTIAL OPERATORS

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**Introduction.** Let us consider a linear operator $L$:

$$(1) \qquad Lv(s,x)=\frac{1}{2}\sum_{ij} a_{ij}(s,x)D_iD_jv(s,x)+\sum_i b_i(s,x)D_iv(s,x)$$
$$+\int[v(s,x+u)-v(s,x)-I_{(|u|\leqq 1)}(u,\nabla v(s,x))]S(s,x,du),$$

where $D_i=\partial/\partial x_i$, $\nabla=(D_i)$, $a$ is a non-negative definite $d\times d$-matrix, $b$ is a $d$-vector and $S$ is a *Lévy kernel*, that is, a kernel satisfying $S(s,x,\{0\})=0$ and

$$\int_{R^d}|u|^2\wedge 1\ S(s,x,du)<\infty$$

In case $S\equiv 0$, the Markov process on $R^d$ having $L$ as its generator can be constructed by solving Ito's stochastic equation

$$(2) \qquad dX_t=a(t,X_t)^{1/2}dB_t+b(t,X_t)dt,$$

where $B_t$ is the $d$-dimensional Brownian motion. It is well known that if $a$ and $b$ are continuous in $(s,x)$, then there exists at least a solution of (2). Roughly speaking, the problem whether the equation (2) has a unique solution in the sense of probability law corresponds to the analytical problem whether the equation

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=f$$

has solutions $v$ for each $\lambda$ ($\geqq$ some $\lambda_0$) and for sufficiently wide class of functions $f$. We know that if $a$ is continuous in $(s,x)$ and is positive definite, then the above equation has a solution $v$ (in the sense of distribution) for each $\lambda>0$ and test function $f$. Using this fact, Strook-Varadhan [8] proved that the equation (2) has a unique solution in the sense of probability law. Their method is basically analytic but it also needs some probabilistic arguments. They introduced and made use of certain martingale equations equivalent to the equation (2).

In this paper, we shall consider the case where $S$ is not always 0. In §2, three equivalent martingale formulations of the stochastic equation associated with the operator $L$ are defined, and we call each of them '*the* $(a,b,S)$-*stochastic equation*'. The main aim of this paper is to prove the uniqueness and existence of the solution of the $(a, b, S)$-stochastic equation. Our results are as follows. To explain them, let us introduce some conditions:

I. for each bounded domain $D\subset R_+\times R^d$, there exist positive constants $a_1$ and $a_2$ such that $a_1|\theta|^2\leqq(\theta, a(s, x)\theta)\leqq a_2|\theta|^2$ for each $\theta\in R^d$ and $(s, x)\in D$,

II. $\lim_{x'\to x}\sup_{s\leqq T}\sum_{ij}|a_{ij}(s, x')-a_{ij}(s, x)|=0$ for each $T$ and $x\in R^d$; $b(s, x)$ is locally bounded; and, for each bounded domain $D\subset R_+\times R^d$, there exists a measure $\bar{S}(du)$ such that

$$\int|u|^2\wedge 1\ \bar{S}(du)<\infty$$

and $\bar{S}(du)-S(s, x, du)$ is a non-negative measure for $(s, x)\in D$,

III. for each bounded interval $[0, T]$, there exists a constant $K$ such that

$$S(s, x, \{|u|>1\})\leqq K \text{ and}$$

$$|(x, b(s, x))|+\text{trace}\ a(s, x)+\int_{|u|\leqq 1}|u|^2 S(s, x, du)\leqq K(1+|x|^2\log^+|x|)$$

for all $(s, x)\in[0, T]\times R^d$. (We shall give a more general condition in §5.) If conditions I and II are satisfied, then the $(a, b, S)$-stochastic equation has at most a solution; if condition I, II and III are satisfied, then there exists uniquely a solution of the equation; and if condition II and III are satisfied and $a, b, S$ are continuous in $x$, then the equation has at least a solution.

The proof of the uniqueness is progressed in the same way as in Strook-Varadhan [8]. But, in our case, there arise some difficulties, for the condition II is very weak and $a(s, x)$ is not always continuous in $s$. Our proof of the existence is based on Hille-Yosida's semi-group theory. The merit of the way is in the fact that one can weaken continuity condition for $a$, $b$ and $S$ (especially with respect to the time variable $s$).

Finally we should mention that Tsuchiya [10] disposed of a similar problem in a different context. In his case, $a=a(x)$, $b=b(x)$ and $S=K(x, u)|u|^{-d-\alpha}du$ where $K$ is a positive and bounded function and $1<\alpha<2$, and he considered two cases: i) $a(x)$ is positive definite, ii) $a(x)$ is identically 0. In Case i), his results are included in ours. But Case ii) is quite different; Tsuchiya solved the problem by making use of a purturbation method based on the $\alpha$-stable process.

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## 1. Notation and preliminalies

We shall consider the space $W_t^s$ of trajectories consisting of right continuous functions admitting limits from the left, defined on $[s, t]\subset R_+$, with values in $R^d$. The value of a funciton $w$ at time $t$ is denoted by $x_t(w)$. Let $\boldsymbol{W}_t^s$ be the $\sigma$-field generated by $(x_\tau;\ s\leqq\tau\leqq t)$. The function space $W_t^s$ becomes a complete separable metric space by the Billingsley metric and the $\sigma$-field $\boldsymbol{W}_t^s$ conincides with the $\sigma$-field of Borel sets of the metric space. Let $W^s$ be the space of functions $w$ on $[s, \infty)$ satisfying $w\ (\tau\wedge t)\in W_t^s$ for each $t\geqq s$. The $\sigma$-field $\boldsymbol{W}_t^s$ is identified with the $\sigma$-field in $W^s$ generated by $(x_\tau;\ s\leqq\tau\leqq t)$.

**Lemma 1.1.** (Skorokhod [7]) *Let $\boldsymbol{Q}$ be a family of probability measures on $(W_t^s,\ \boldsymbol{W}_t^s)$ If the family $\boldsymbol{Q}$ satisfies the condition:*

$$\lim_{l\uparrow\infty}\sup_{Q\in\boldsymbol{Q}} Q\ (\sup_\tau |x_\tau|>l)=0 \text{ and}$$

$$\lim_{\delta\downarrow 0}\sup_{Q\in\boldsymbol{Q}}\sup_{|\tau-\tau'|\leqq\delta} Q\ (|x_\tau-x_{\tau'}|>\varepsilon)=0 \text{ for any } \varepsilon>0,$$

*then it is possible to pick up a sequence $(Q^n)$ from $\boldsymbol{Q}$ and is possible to construct a sequence $(X_\tau^n)$ of processes and a process $X_\tau$ which are defined on a certain probability space $(\Omega', \boldsymbol{F}', P')$ such that a) for each $n$, the processes $(x, Q^n)$ and $(X_\tau^n, P')$ are equivalent, b) the process $X_\tau$ is stochastically continuous and the sequence $(X_\tau^n)$ of random variables converges to the random variable $X_\tau$ in probability for any $\tau$.*

We call $T(w)$ an *s-stopping time* if $s\leqq T$ and the set $\{T\leqq t\}$ is $\boldsymbol{W}_t^s$-measurable for all $t\geqq s$. The $\sigma$-field $\boldsymbol{W}_T^s$, defined as the collection of sets $\{A\in\boldsymbol{W}^s;\ A\cap(T\leqq t)\in\boldsymbol{W}_t^s \text{ for all } t\geqq s\}$, coincides with the $\sigma$-field generated by $(x_{t\wedge T};t\geqq s)$. The $\sigma$-field $\boldsymbol{W}_T^s$ admits *'regular conditional probabilities'*. (*see* Parthasarathy [6])

**Lemma 1.2.** (Strook-Varadhan [8]) *Let $T$ be a finite s-stopping time. Let $Q'$ be a probability measure on $(W^s,\ \boldsymbol{W}^s)$, and for each $w\in W^s$, let $Q''_w$ be a probability measure on $(W^s,\ \boldsymbol{W}^{T(w)})$ where $\boldsymbol{W}^{T(w)}$ is the $\sigma$-field generated by $(x_t\ (w');\ t\geqq T(w))$. Suppose that*

1) $Q''_w(w';x_{T(w)}(w)=x_{T(w)}(w'))=1$ *for each* $w\in W$,

2) *for each* $t\geqq s$ *and* $A\in\boldsymbol{W}^t$, $w\rightsquigarrow Q''_w(A)\,I_{(T(w)\leqq t)}$ *is* $\boldsymbol{W}_t^s$*-measurable.*

*Then there exists a unique probability measure $Q$ on $(W^s,\ \boldsymbol{W}^s)$ such that $Q=Q'$ on $\boldsymbol{W}_T^s$, and for almost all $w$ (w.r.t. $Q'$) the regular conditional probability of $Q$ given the $\sigma$-field $\boldsymbol{W}_T^s$ coincides with $Q''_w$ on $\boldsymbol{W}^{T(w)}$.*

Let $(\Omega,\boldsymbol{F},P)$ be a complete probability space with a non-decreasing and right continuous family $(\boldsymbol{F}_t)_{t\geqq 0}$ of sub-$\sigma$-fields of $\boldsymbol{F}$. Without loss of generality, we can suppose that the $\sigma$-field $\boldsymbol{F}_0$ contains all the negligible sets of $\boldsymbol{F}$. Moreover, we assume that the family $(\boldsymbol{F}_t)$ has no time of discontinuity. In the rest of this section, we suppose that each real valued process is adapted to the family $(F_t)$ and its paths are right continuous, have limits from the left and equals 0 at time 0. We shall pick up some notation and remarks from Meyer [5]- I.

**Notation and remarks.** We say that a process $X_t$ is *'natural'* if $\Delta X_T \equiv X_T - X_{T-} = 0$ for any totally inaccesible stopping time $T$. Each martingale is *quasi-left-continuous*, for the family $(\mathbf{F}_t)$ has no time of discontinuity. Therefore each martingale has no common jump with any natural process. We say that a process $X_t$ is *'locally integrable'* if there exists a sequence $(T_n)$ of stopping times such the $T_n \uparrow \infty$ as $n \to \infty$ and each stopped process $X_{t\wedge T_n}$ is uniformly integrable.

1) We denote by $\boldsymbol{A}_{loc}$ the space of all processes whose total variations on any bounded intervals are finite. Each process $X_t \in \boldsymbol{A}_{loc}$ is decomposed to the sum of a continuous process and a purely discontinuous process as follows:

$$X_t = [X_t - \sum_{\tau \leqq t} \Delta X_\tau] + \sum_{\tau \leqq t} \Delta X_\tau \qquad \text{where } \Delta X_\tau = X_\tau - X_{\tau-}$$

We denote the subspace of $\boldsymbol{A}_{loc}$ consisting of all continuous (resp. purely discontinuous) processes by $\boldsymbol{A}^c_{loc}$ (resp. $\boldsymbol{A}^d_{loc}$). The decomposition $\boldsymbol{A}_{loc} = \boldsymbol{A}^c_{loc} + \boldsymbol{A}^d_{loc}$ is always direct, and the decomposition $\boldsymbol{A}^n_{loc} = \boldsymbol{A}^c_{loc} + \boldsymbol{A}^{nd}_{loc}$ is also direct, where the spaces with superfix *'n'* express the subspaces consisting of natural processes. In the sequel, we shall consider the subspace $A^q_{loc} = \{A_t \in \boldsymbol{A}_{loc};\ A_t$ is quasi-left continuous$\}$ instead of the space $\boldsymbol{A}_{loc}$. If $A_t \in \boldsymbol{A}^q_{loc}$ is locally integrable, then we can uniquely choose $B_t \in \boldsymbol{A}^c_{loc}$ such that $A_t - B_t$ is a locally integrable martingale. We denote the process $B_t$ (resp. $A_t - B_t$) by $\langle A_t \rangle$ (resp. ${}^cA_t$).

2) We denote the space of all locally integrable martingales by $\boldsymbol{M}_{loc}$. Let $X_t$ be in $\boldsymbol{M}_{loc}$ and let a sequence $(T_n)$ of stopping times carry all the jumps of $X_t$, that is to say, $P[T_n = T_m < \infty] = 0$ for any $n \neq m$ and $\{(w, t);\ X_t(w) \neq X_{t-}(w)\} \subset \bigcup_{n=1}^{\infty}(\text{graph } T_n)$. We call the process $\sum_{n=1}^{\infty} {}^c(\Delta X_{T_n} I_{(T_n \leqq t)})$, which is formally denoted by ${}^c(\sum_{\tau \leqq t} \Delta X_\tau)$, the *'purely discontinuous local maringale'* part of $X_t$. Let $\boldsymbol{M}^c_{loc}$ (resp. $\boldsymbol{M}^d_{loc}$) be the subspace consisting of all continouus (resp. purely discontinuous) local martingales, then we have $\boldsymbol{M}_{loc} = \boldsymbol{M}^c_{loc} + \boldsymbol{M}^d_{loc}$ (direct sum). If $M_t$ and $N_t$ in$\boldsymbol{M}_{loc}$ are locally square integrable, then there exists a unique $A_t \in \boldsymbol{A}^c_{loc}$ such that $M_t N_t - A_t \in \boldsymbol{M}_{loc}$. The process $A_t$ is denoted by $\langle M_t, N_t \rangle$.

Every process $X_t = M_t + A_t$ with $M_t \in \boldsymbol{M}_{loc}$ and $A_t \in \boldsymbol{A}_{loc}$ is called a *'weak local semi-martingale'*.

Let $C^n(D)$ (resp. $C^{n,b}(D)$) be the space of functions on a domain $D$ whose $r$-th $(0 \leqq r \leqq n)$ derivatives are continuous (resp. bounded and continuous). The formula of change of variables on semi-martingale (*see* Kunita-Watanabe [4] and Meyer [5]) gives a base of the discussions in this paper.

**Theorem 1.3.** (Kunita-Watanabe's formula) *Let $X_t=(X_t^i)$ be a d-dimensional weak local semi-martingale such that*

$$X_t^i-X_0^i=M_t^i+N_t^i+A_t^i+B_t^i \text{ with } M_t^i\in \boldsymbol{M}_{loc}^c,\ N_t^i\in \boldsymbol{M}_{loc}^d,\ A_t^i\in \boldsymbol{A}_{loc}^c \text{ and } B_t^i\in \boldsymbol{A}_{loc}^{qd},$$

*and that $\Delta N_t^i \Delta B_t^j=0$ for all i, j and t. If $F(x)\in C^2(R^d)\cap C^{1,b}(R^d)$, then the process $F(X_t)-F(X_0)$ has the unique decomposition:*

*$F(X_t)-F(X_0) = M_t'+N_t'+A_t'$ with $M_t'\in \boldsymbol{M}_{loc}^c$, $N_t'\in \boldsymbol{M}_{loc}^d$ and $A_t'\in \boldsymbol{A}_{loc}^c$, where*

$$M_t'=\sum_{i=1}^d \int_0^t D_iF(X_\tau)\, dM_\tau^i,$$

$$N_t'={}^c\Big(\sum_{\substack{\tau\leq t\\ \Delta N_\tau\neq 0}}[F(X_\tau)-F(X_{\tau-})]\Big)+{}^c\Big(\sum_{\substack{\tau\leq t\\ \Delta B_\tau\neq 0}}[F(X_\tau)-F(X_{\tau-})]\Big),$$

$$A_t'=\int_0^t \Big(\frac{1}{2}\sum_{i,j=1}^d D_iD_jF(X_\tau)\, d\langle M_\tau^i, M_\tau^j\rangle+\sum_{i=1}^d D_iF(X_\tau)\, dA_\tau^i\Big)$$
$$+\Big\langle \sum_{\substack{\tau\leq t\\ \Delta N_\tau\neq 0}}[F(X_\tau)-F(X_{\tau-})-\sum_{i=1}^d D_iF(X_{\tau-})\Delta N_\tau^i]\Big\rangle+\Big\langle \sum_{\substack{\tau\leq t\\ \Delta B_\tau\neq 0}}[F(X_\tau)-F(X_{\tau-})]\Big\rangle.$$

The following lemma is useful.

**Lemma 1.4.** *If a process $X_t$ is uniformly integrable and $E[X_T]=0$ for any bounded stopping time T, then the process is a martingale.*

Proof. Let $s<t$ and $A\in \boldsymbol{F}_s$. Considering the stopping time $T=sI_A+tI_{A^c}$, we have $E[X_T]=E[X_sI_A]+E[X_tI_{A^c}] = 0$. On the other hand $E[X_t]=E[X_tI_A]+E[X_tI_{A^c}]=0$. Thus, $E[X_tI_A] = E[X_sI_A]$.

**Notation.** The spaces $\boldsymbol{A}_{loc}$, $\boldsymbol{M}_{loc}$ *etc.* are also defined for $d$-dimensional real valued processes (resp. complex valued processes) in such a way that each coordinate (resp. real and imaginary parts of each coordinate) belongs to the spaces $\boldsymbol{A}_{loc}$, $\boldsymbol{M}_{loc}$ *etc.* of the previous sense. We often denote the spaces $\boldsymbol{A}_{loc}$, $\boldsymbol{M}_{loc}$ *etc.* and the expectation $E$ with respact to $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P)$ by $\boldsymbol{A}_{loc}(\boldsymbol{F}_t, P)$, $\boldsymbol{M}_{loc}(\boldsymbol{F}_t, P)$ *etc.* and by $E_P$ respectively, lest we should get confused.

Let $S$ and $S'$ be kernels. It is simply denoted by $S\leq S'$ the fact that $S'-S$ is non-negative. And $|S-S'|$ stands for the total variation of the signed measure $S'-S$.

## 2. Martingale problems of stochastic equations

Let $(\Omega, \boldsymbol{F}, P)$ and $(\boldsymbol{F}_t)_{t\geq 0}$ be the object as we stated in §1. Let us choose a $d$-dimensional Brownian motion $B_t$, a *Poisson random measure* $p\ (dt, dz)$ on $R_+ \times R^d$ with parameter $dt \times dz/|z|^{d+1}$ and the *Poisson martingale measure* $q\ (dt, dz)$. These are defined as follows: for each Borel set $A$ in $R^d - \{0\}$, the process $p\ ([0, t] \times A)$ is a Poisson process adapted to $(F_t)$ which is independent of $B_t$ and satisfies that

$$E[p([0, t] \times A)] = \int_0^t \int_A \frac{d\tau dz}{|z|^{d+1}}.$$

And $q\ (dt, dz) = p\ (dt, dz) - dt\ dz/|z|^{d+1}$.

For each $s \in R_+$, $x$, $z \in R^d$, let $\sigma\ (s, x)$ denote a $d \times d$-matrix, $b\ (s, x)$ and $c\ (s, x, z)$ d-vectors. We suppose that $\sigma$, $b$ and $c$ are Borel measurable. Let $T$ be an $s$-stopping time and let $(X_t;\ t \in [s, T])$ be an $R^d$-valued right continuous process admitting limits from the left and adapted to $(F_t)$, where $[s, T] = \{t < \infty;\ s \leqq t \leqq T\}$. From now on, we shall assume these conditions without any assignment.

DEFINITION. We call a process $X_t$ a solution of *the original* $(\sigma, b, c)$-*stochastic equation* starting from $(s, x)$, constructed over $(\Omega, F, F_t, P, B_t, p, q)$ if

S0) $X_t = x + \int_s^t \sigma(\tau, X_\tau) dB_\tau + \int_s^t b(\tau, X_\tau) d\tau + \int_s^t \int c^1(\tau, X_\tau, z) q(d\tau, dz)$

$+ \int_s^t \int c_1(\tau, X_\tau, z) p(d\tau, dz)$ for all $t \in [s, T]$, where $c^1 = cI_{(|c| \leqq 1)}$ and $c_1 = cI_{(|c| > 1)}$.

We assume that

$$\int_s^t \left\{ \text{trace } (\sigma\sigma^*)(\tau, X_\tau) + |b(\tau, X_\tau)| + \int |c(\tau, X_\tau, z)|^2 \wedge 1 \frac{dz}{|z|^{d+1}} \right\} d\tau < \infty$$

for all $t \in [s, T]$, where the domain of integration by $dz$ is the set $R^d - \{0\}$.

Let $a(s, x)$ be a measurable and non-negative definite $d \times d$-matrix and $S\ (s, x, du)$ be a *Lévy kernel*. Let us introduce an integro-differential operator $L$ defined by, for each $v \in C^{2,b}\ (R_+ \times R^d)$,

$$Lv(s, x) = \frac{1}{2} \sum_{ij} a_{ij} D_i D_j v(s, x) + \sum_i b_i D_i v(s, x)$$
$$+ \int [v(s, x+u) - v(s, x) - I_{(|u| \leqq 1)}(u, \nabla v(s, x))]\ S(s, x, du).$$

Set $\Phi_\theta(s, x) = e^{-i(\theta, x)} L e^{i(\theta, \cdot)}(s, x)$. Then we have

$$\Phi_\theta(s, x) = -\frac{1}{2}(\theta, a(s, x)\theta) + i(\theta, b(s, x)) + \int [e^{i(\theta, u)} - 1 - I_{(|u| \leqq 1)} i(\theta, u)] S(s, x, du).$$

Now suppose that $a = \sigma\sigma^*$ ($\sigma^*$ stands for the transposed matrix of $\sigma$),

$$S(s, x, D) = \int_{c(s,x,z)\in D} \frac{dz}{|z|^{d+1}} \text{ for each open set } D\subset R^d - \{0\},$$

and that $(X_t, P; t\in[s, T])$ is a solution of the original $(\sigma, b, c)$—stochastic equation starting from $(s, x)$. Then we have

$$v(t, X_t) - v(s, x) - \int_s^t \left(\frac{\partial}{\partial\tau} + L\right) v(\tau, X_\tau) d\tau \in \boldsymbol{M}_{loc}(\boldsymbol{F}_t, P)$$

for all $v\in C^{2,b}(R_+\times R^d)$. This can be verified by the Kunita-Watanabe formula (Th. 1. 3.).

**Notation and remarks.** Let $J(dt, du)$ be the random measure defined by

$$J(dt, du) = \sum_{s\in dt} I_{(\Delta X_s \in du - \{0\})}.$$

Then it holds that

$$E\left[\int_s^{T'} \int h(t, X_{t-}, u) J(dt, du)\right] = E\left[\int_s^{T'} \int h(t, X_t, u) S(t, X, du) dt\right].$$

for any non-negative and measurable function $h(s, x, u)$ and $s$-stopping time $T'$ $(T'\leqq T)$; where if any one of the members is integrable, so is the other. Set, for $p=1, 2$,

$$\boldsymbol{H}^p = \left\{h(s, x, u); \int_s^t \int |h(\tau, X_\tau, u)|^p S(\tau, X_\tau, du) d\tau < \infty \ a.\,e. \text{ for all } t\in[s, T]\right\}.$$

If $h$ is a function of $\boldsymbol{H}^1$, then the purely discontinuous process

$$\int_s^t \int h(\tau, X_\tau, u) J(d\tau, du)$$

is locally integrable. By Lemma 1.4, we have

$$\left\langle \int_s^t \int h(\tau, X_\tau, u) J(d\tau, du) \right\rangle = \int_s^t \int h(\tau, X_\tau, u) S(\tau, X_\tau, du) d\tau.$$

Set ${}^cJ(dt, du) = J(dt, du) - S(t, X_t, du)dt$. Then stochastic integrals by ${}^cJ$ are local martingales, therefore we shall say that ${}^cJ$ is a *'martingale measure'*. If $h$ is a function of $\boldsymbol{H}^2$, then the locally square integrable martingale:

$$\int_s^t \int h(\tau, X_\tau, u) {}^cJ(d\tau, du) = {}^c\Big(\sum_{\substack{s<\tau\leq t \\ \Delta X_\tau \neq 0}} h(\tau, X_{\tau-}, \Delta X_\tau)\Big)$$

is defined. And if $h_1$ and $h_2$ are elements of $\boldsymbol{H}^2$, then we have

$$\left\langle \int_s^t \int h_1 {}^cJ, \int_s^t \int h_2 {}^cJ \right\rangle = \int_s^t \int h_1 h_2 S \, d\tau.$$

In order to give the martingale formulation of stochastic equation, we need the following theorem. ($a$, $b$, $S$ are given and $L$, $\Phi_\theta$ are defined as before.)

**Throrem 2.1.** *Let $(X_t, P; t\in[s, T])$ be a process such that* $\lim_{l\to\infty} P[\sup_{\tau\leq t}|X_\tau|>l]=0$ *and*

$$\int_s^t \{\text{trace } a(\tau, X_\tau)+|b(\tau_\tau, X_\tau)|+\int|u|^2\wedge 1\, S(\tau, X_\tau, du)\}d\tau<\infty$$

*for all $t\in[s, T]$. Then the following three conditions are equivalent.*

S1) *For any $v(s, x)\in C^{2,b}(R_+\times R^d)$,*

$$v(t, X_t)-v(s, x)-\int_s^t\Big(\frac{\partial}{\partial\tau}+L\Big)v(\tau, X_\tau)d\tau\in \boldsymbol{M}_{loc}(\boldsymbol{F}_t, P).$$

S2) *For all $\theta\in R^d$,*

$$\exp[i(\theta, X_t-x)-\int_s^t \Phi_\theta(\tau, X_\tau)d\tau]-1\in \boldsymbol{M}_{loc}(\boldsymbol{F}_t, P).$$

S3) *$X_t$ is a weak local semi-martingale which has the following property:*

a) *for any positive measurable function $h$ and for any $s$-stopping time $T'$ ($T'\leq T$),*

$$E\Big[\int_s^{T'}\int h(\tau, X_\tau, u)J(d\tau, du)\Big]=E\Big[\int_s^{T'} h(\tau, X_\tau, u)S(\tau, X_\tau, du)d\tau\Big]$$

*if a measure $J$ of jumps of the process $X_t$ is defined as before. That is to say, $J(dt, du)-S(t, X_t, du)\,dt$ is a martingale measure (which we shall denote by ${}^cJ(dt, du)$).*

b) *there exists a d-dimensional process $M_t=(M_t^i)$ such that*

$$M_t^i\in \boldsymbol{M}_{loc}^c(\boldsymbol{F}_t, P),\ \langle M_t^i, M_t^j\rangle=\int_s^t a_{ij}(\tau X_\tau)\,d\tau,$$

*and the process $X_t$ can be decomposed as follows:*

$$X_t=x+M_t+\int_s^t b(\tau, X_\tau)d\tau+\int_s^t\int_{|u|\leq 1}u\,{}^cJ(d\tau, du)+\int_s^t\int_{|u|>1}uJ(d\tau, du).$$

Proof. 1° Condition S1) implies the next condtion:

S2′) for all $\theta\in R^d$,

$$\exp[i(\theta, X_t-x)]-1-\int_s^t\exp[i(\theta, X_\tau-x)]\Phi_\theta(\tau, X_\tau)d\tau\in \boldsymbol{M}_{loc}(\boldsymbol{F}_t,P).$$

By the Kunita-Watanabe formula, it is easily proved that Condition S2) and S2′) are equivalent.

2° Next, we shall show that Condition S2′) implies Condition S3). Set

$$T_n = \inf\{t\in[s, T];|X_t|>n\}.$$

Each stopped process $X_{t\wedge T_n}$ can be decomposed as follows:

$$X_{t\wedge T_n} = Y_t^n + \Delta X_{T_n} I_{(t\geq T_n)}, \ \sup_t |Y_t^n| \leqq n$$

If $|\theta|<2\pi/n$, then $i(\theta, Y_t^n)=\log \exp[i(\theta, Y_t^n)]$. Since the process $\exp[i(\theta, Y_t^n)]$ is a weak local semi-martingale, the process $i(\theta, Y_t^n)$ is a weak local semi-martingale. From the fact that $T_n \uparrow T$, the process $X_t$ is a weal local semi-martingale. It is easy to show that the process $X_t$ has no natural jump, in other words, the process $X_t$ is quasi-left-continuous. Thus the process $X_t$ is decomposed as follows:

$$X_t = x+M_t+A_t+N_t+B_t,\ M_t^t\in M_{loc}^c,\ A_t^t\in A_{loc}^c,\ N_t^t\in M_{loc}^d,\ B_t^t\in A_{loc}^{qd}$$

We suppose that $|\Delta N_t|\leqq 1$ (resp. $|\Delta B_t|>1$) if $\Delta N_t\neq 0$ (resp. $\Delta B_t\neq 0$).
By the Kunita-Watanabe formula, we have

$$e^{i(\theta, X_t-x)}-1=[\text{a local martingale taking value 0 at time } s]$$

$$+\int_s^t e^{i(\theta, X_\tau-x)}\left[-\frac{1}{2}d\langle(\theta, M_\tau), (\theta, M_\tau)\rangle+i(\theta, dA_\tau)\right]$$

$$+\langle\sum_{\substack{s<\tau\leqq t\\ \Delta X_\tau\neq 0}} e^{i(\theta, X_{\tau-}-x)}[e^{i(\theta, \Delta X_\tau)}-1-i(\theta, \Delta N_\tau)]\rangle.$$

Condition S2′) and the uniqueness of Meyer's decomposition imply that, for each $\theta\in R^d$, the process

$$Z_t^\theta=\left[-\frac{1}{2}\langle(\theta, M_t), (\theta, M_t)\rangle+i(\theta, A_t)+\langle\sum_{\substack{s<\tau\leqq t\\ \Delta X_\tau\neq 0}}(e^{i(\theta, \Delta X_\tau)}-1-i(\theta, \Delta N_\tau))\rangle\right]$$

$$-\int_s^t \Phi_\theta(\tau, X_\tau)d\tau$$

is identically zero. An elementary computation shows that

$$\langle\sum_{\substack{s<\tau\leqq t\\ \Delta X_\tau\neq 0}}\sin^4(\beta, \Delta X_\tau)\cdot e^{i(\alpha, \Delta X_\tau)}\rangle-\int_s^t\int \sin^4(\beta, u)\cdot e^{i(\alpha, u)}S(\tau, X_\tau, du)\,d\tau$$

$$=(Z_t^{4\beta+\alpha}-4Z_t^{2\beta+\alpha}+6Z_t^\alpha-4Z_t^{-2\beta+\alpha}+Z_t^{-4\beta+\alpha})/2^4=0.$$

Thus, for each $s$-stopping time $T'$ $(T'\leqq T)$,

$$\mathrm{E}\left[\int_s^{T'}\int \sin^4(\beta, u)\cdot e^{i(\alpha, u)}J(dt, du)\right]=\mathrm{E}\left[\int_s^{T'}\int \sin^4(\beta, u)\cdot e^{i(\alpha, u)}S(t, X_t\, du)dt\right].$$

(If any one of the members is integrable, so is the other.) Set

$$S_n = \inf \{t \in [s, T']; \int_s^t \int |u|^2 \wedge 1 S(\tau, X_\tau, du) d\tau > n\}, \text{ and}$$

$H = \{$bounded measurable function $h(u)$ on $R^d$;

$$\mathrm{E}\left[\int_s^{S_n} \int \sin^4(\beta, u) h(u) J(dt, du)\right] = \mathrm{E}\left[\int_s^{S_n} \int \sin^4(\beta, u) h(u) S(t, X_t, du) dt\right].$$

We note that the class $C = \{\exp[i(\alpha, u)];\ \alpha \in R^d\}$ is included in $H$. It is easy to show that $H$ is closed under the formation of limits of uniform or bounded monotone sequences. Since $C$ is closed under the multiplication and contains 1, $H$ contains all the $\sigma(C)$-measurable bounded functions, where $\sigma(C)$ is the $\sigma$-field generated by the functions of $C$. Thus $H$ contains all the bounded measurable functions. As $\beta$ and $n$ are arbitrary, we have

$$\mathrm{E}\left[\int_s^{T'} \int h(u) J(dt, du)\right] = \mathrm{E}\left[\int_s^{T'} \int h(u) S(t, X_t\, du) dt\right]$$

for each non-negative measurable function $h$ on $R^d$. Property S3)-a) follows immediately from this fact. On the other hand it hold that

$$0 = \operatorname{Re} Z_t^\theta = -\frac{1}{2}\langle(\theta, M_t), (\theta, M_t)\rangle + \frac{1}{2}\int_s^t (\theta, a(\tau, X_\tau)\theta) d\tau$$
$$+ \langle \int_s^t \int (\cos(\theta, u) - 1) J(d\tau, du)\rangle - \int_s^t \int (\cos(\theta, u) - 1) S(\tau, X_\tau, du) d\tau,$$

that is to say,

$$\langle(\theta, M_t), (\theta, M_t)\rangle = \int_s^t (\theta, a(\tau, X_\tau)\theta) d\tau.$$

It is immediate to show that

$$A_t = \int_s^t b(\tau, X_\tau) d\tau,\ N_t = \int_s^t \int_{|u| \leq 1} u^c J(d\tau, du),\ B_t = \int_s^t \int_{|u| > 1} u J(d\tau, du).$$

3° By the Kunita-Watanabe formula, it is easily proved that Condition S3) implies Condition S1). Q.E.D.

DEFINITION. We say that a process $(X_t, P; t \in [s, T])$ is a solution of *the martingale problem of the* $(a, b, S)$*-stochastic equation* starting from $(s, x)$ if $X_t$ satisfies either one of the conditions of Theorem 2. 1.

A solution of the original stochastic equation is also a solution of the associated martingale problem. We can prove the converse of this fact under some restriction for $a$ and $S$.

**Theorem 2.2** *Let* $(\Omega, F_t, P, X_t)$ *be a martingale solution of the* $(a, b, S)$*-*

*stochastic equation starting from* $(s, x)$. *If* $a(s, x)$ *is strictly positive, and the kernel* $S(s, x, du)$ *has no point mass and its support is* $R^d-\{0\}$, *then there exist* $\sigma$, $c$, *and* $(B_t, p(dt, dz), q(dt, dz))$ *such that*

a) $\sigma$ *and* $c$ *are Borel measurable,* $\sigma\sigma^*=a$ *and*

$$\int_{c(s,x,z)\in D}\frac{dz}{|z|^{d+1}} = S(s, x, D) \text{ for each open set } D\subset R^d-\{0\}.$$

b) $B_t$ *(resp.* $p$, $q$*) is a Brownian motion (resp. a Poisson random measure, a Poisson martingale measure) on the probabillity space* $(\Omega, F_t, P)$. $B_t$, $p$ *and* $q$ *satisfy the same condition as we stated at the beginning of this section.*

c) $X_t$ *is a solution of the original* $(\sigma, b, c)$*-stochastic equation starting from* $(s, x)$ *constructed over* $(\Omega, F_t, P, B_t, p, q)$.

We omit the proof, since we never make use of the original stochastic equation.

## 3. Operator $L$ and transition probabilities

I. *(Some inequalities for solutions of a parabolic equation)*

Let $a(s)$ be a measurable $d\times d$ symmetric matrix such that

$$a_1|\theta|^2\leqq(\theta, a(s)\theta)\leqq a_2|\theta|^2, \quad 0<a_1\leqq a_2<\infty,$$

for all $\theta\in R^d$ and $s\in R_+$. We shall consider the parabolic equation:

$$\left(\lambda-\frac{\partial}{\partial s}-\frac{1}{2}\sum_{ij}a_{ij}(s)D_iD_j\right)g(s, x)=f(s, x), \quad \lambda>0.$$

Define an operator $G_\lambda$ acting on a suitable class of functions on $R_+\times R^d$ by

$$G_\lambda f(s, x)=\int_0^\infty\int_{R^d}\frac{\det U(s, t)}{(2\pi t)^{d/2}}e^{-|U(s,t)y|^2/2t-\lambda t}f(s+t, x-y)\,dt\,dy$$

where $U(s, t)=\left(\frac{1}{t}\int_s^{s+t}a(\tau)d\tau\right)^{-1/2}$.

We say that a function $f$ on $R_+\times R^d$ belongs to the class $C_K^\infty$ (or is a test function) if and only if $f$ is a $C^\infty(R_+\times R^d)$-function with compact support. We can verify that if $f\in C_K^\infty$, then $g(s, x)=G_\lambda f(s, x)$ is a solution of the parabolic equation. Set

$$J_\lambda(s, t, x)=\frac{\det U(s, t)}{(2\pi t)^{d/2}}e^{-|U(s,t)x|^2/2t-\lambda t}.$$

Then we have, for each $f\in C_K^\infty$,

$$G_\lambda f(s, x) = \int_0^\infty \int J_\lambda(s, t, y) f(s+t, x-y) dt\, dy ,$$

$$D_i G_\lambda f(s, x) = \int_0^\infty \int D_i J_\lambda(s, t, y) f(s+t, x-y) dt\, dy .$$

Moreover, for the set $B_\varepsilon = \{(t, y) \in R_+ \times R^d ; t > \varepsilon \text{ or } |U(s, t)y|^2 > \varepsilon\}$, the integral:

$$\iint_{B_\varepsilon} D_i D_j J_\lambda(s, t, y) f(s+t, x-y) dt\, dy$$

exists for each $\varepsilon > 0$ and it converges to a certain function as $\varepsilon \downarrow 0$. Let us denote the limit by $I_\lambda^{ij} f$. We can verify that

$$D_i D_j G_\lambda f(s, x) = I_\lambda^{ij} f(s, x) + c^{ij}(s) f(s, x)$$

where $c^{ij}(s)$ is a function on $R_+$ such that there exists a finite upper bound of $|c^{ij}|$ independent of $\lambda$ and $f$. (*see* Bers-John-Schechter [1] p–226)

REMARK. We can verify that if $a(s)$ is continuous, then the functions $G_\lambda f$, $D_i G_\lambda f$, $D_i D_j G_\lambda f$ and $\frac{\partial}{\partial s} G_\lambda f$ are continuous in $(s, x)$ for $f \in C_K^\infty$, and that these functions converge to 0 as $s + |x|^2 \to \infty$.

Let us introduce some norms for functions on $R_+ \times R^d$.

$$||f|| = \sup_{s,x} |f(s, x)|$$

$$H^\alpha(f) = \sup_{s,x,x'} \frac{|f(s, x) - f(s, x')|}{|x - x'|^\alpha} \qquad (0 < \alpha < 1)$$

$$|f|_{L^p} = \left(\int_0^\infty \int |f(s, x)|^p ds\, dx\right)^{1/p} \qquad (1 < p < \infty)$$

In the sequel, these $\alpha$ and $p$ are fixed, The following theorem can be proved by a similar way to the proof by Jones [2] and in Bers *et. al.* [1].

**Theorem 3.1** *If $f$ is a function of $C_K^\infty$ and $\lambda r^2 \geqq 1$, then*
1) $H^\alpha(I_\lambda^{ij} f) \leqq c.\ H^\alpha(f)$ and $||I_\lambda^{ij} f|| \leqq c.\ r^\alpha H^\alpha(f)$ ,
2) $|I_\lambda^{ij} f|_{L^p} \leqq c.\ |f|_{L^p}$ ,
*where the c.'s stand for certain constants independent of $\lambda$, $f$ and $r$.*

REMARK. Let $|f|_{C_r^\alpha}$ be the norm $||f|| + r^\alpha H^\alpha(f)$. Then there exist constants $c_\alpha$ and $c_p$ depending only on $d, \alpha, p, a_1$ and $a_2$ such that

$$\sum_{ij} |D_i D_j G_\lambda f|_{C_r^\alpha} \leqq c_\alpha |f|_{C_r^\alpha} \text{ and } \sum_{ij} |D_i D_j G_\lambda f|_{L^p} \leqq c_p |f|_{L^p}$$
for $f \in C_K^\infty$ and $\lambda r^2 \geqq 1$ .

In the following two lemmas, we suppose that $\lambda r^2 \geqq 1$ and $f \in C_K^\infty$, and the $c.$'s stand for certain constants independent of $\lambda$, $f$, and $r$. We will omit the proofs of these lemmas, because these are not so difficult.

**Lemma 3.2**

1) $\|G_\lambda f\| \leqq r^2 \|f\|$ *and* $\|D_i G_\lambda f\| \leqq c.\ r\|f\|$,
2) $H^\alpha(G_\lambda f) \leqq r^2 H^\alpha(f)$ *and* $H^\alpha(D_i G_\lambda f) \leqq c.\ rH^\alpha(f)$,
3) $|G_\lambda f|_{L^p} \leqq r^2 |f|_{L^p}$ *and* $|D_i G_\lambda f|_{L^p} \leqq c.\ r|f|_{L^p}$.

**Lemma 3.3**

1) *If* $2-d/p>0$, *then* $\|G_\lambda f\| \leqq c.\ r^{2-d/p}|f|_{L^p}$,
*and if* $1-d/p>0$, *then* $\|D_i G_\lambda f\| \leqq c.\ r^{2-d/p}|f|_{L^p}$.
2) *If* $1+\alpha-d/p>0$, *then* $H^{1-\alpha}(G_\lambda f) \leqq c.\ r^{1+\alpha-d/p}|f|_{L^p}$,
*and if* $\alpha-d/p>0$, *then* $H^{1-\alpha}(D_i G_\lambda f) \leqq c.\ r^{\alpha-d/p}|f|_{L^p}$.
3) $|D_i G_\lambda f(\cdot, \cdot+u) - D_i G_\lambda f(\cdot, \cdot)|_{L^p} \leqq c.\ |u|\,|f|_{L^p}$.

II. (*Á priori estimate for the operator L*)

Let $a(s, x)$, $b(s, x)$, $S(s, x, du)$ be as follows:

a) there exist positive constants $a_1$ and $a_2$ such that

$$a_1|\theta|^2 \leqq (\theta, a(s, x)\theta) \leqq a_2|\theta|^2$$

for all $\theta \in R^d$ and $(s, x) \in R_+ \times R^d$,

b) $\sup\limits_{x,s} |b(s, x)| < \infty$,

c) there exists a measure $\bar{S}(du)$ such that

$$\int |u|^2 \wedge 1\ \bar{S}(du) < \infty \text{ and } S(s, x, du) \leqq \bar{S}(du) \text{ for all } (s, x) \in R_+ \times R^d.$$

Let $L$ be an operator defined by

$$Lv(s, x) = \frac{1}{2}\sum_{ij} a_{ij}(s, x) D_i D_j v(s, x) + \sum_i b_i(s, x) D_i v(s, x) + \int [v(s, x+u) - v(s, x) - I_{(|u| \leqq 1)}(u, \nabla v(s, x))] S(s, x, du).$$

Let $x_0$ be an arbitrary but fixed point of $R^d$ and set

$$L_0 = \frac{1}{2}\sum_{ij} a_{ij}(s, x_0) D_i D_j \text{ and } T_\lambda = (L - L_0) G_\lambda,$$

where $G_\lambda$ is an operator defined simiarly to that in Subsection I using $a(s, x_0)$ in place of $a(s)$. And denote by $K_\lambda$ a formal expression $G_\lambda[I - T_\lambda]^{-1}$.

**Assumption.** Let $c_\alpha$ and $c_p$ be the constants defined in Subsection I.

($A_\alpha$) $c_\alpha \max\limits_{ij} \sup\limits_{s,x} |a_{ij}(s, x) - a_{ij}(s, x_0)| \leqq 1$.

($A_p$) $c_p \max\limits_{ij} \sup\limits_{s,x} |a_{ij}(s, x) - a_{ij}(s, x_0)| \leqq 1$.

**Lemma 3.4** *If Assumption* $(A_p)$ *is satisfied, then there exists a function* $\sigma_p(r)$ *on* $R_+$ *such that* $\lim_{r\downarrow 0}\sigma_p(r)=0$ *and*

$$\frac{1}{2}\max_{ij}\sup_{x,s}|a_{ij}(s,x)-a_{ij}(s,x_0)|\sum_{ij}|D_iD_jG_\lambda f|_{L^p}+\max_i\|b_i\|\sum_i|D_iG_\lambda f|_{L^p}$$
$$+\left|\int|G_\lambda f(s,x+u)-G_\lambda f(s,x)-I_{(|u|\leqq 1)}(u,\nabla G_\lambda f(s,x))|\bar{S}(du)\right|_{L^p}$$
$$\leqq\left(\frac{1}{2}+\sigma_p(r)\right)|f|_{L^p}.$$

*In particular,* $|T_\lambda f|_{L^p}\leqq\left(\frac{1}{2}+\sigma_p(r)\right)|f|_{L^p}$ *for* $f\in C_K^\infty$ *and* $\lambda r^2\geqq 1$.

Proof. In this proof, the $c.$'s stand for certain constants independent of $\lambda$, $f$ and $r$. For simplicity, we set $a'(s,x)=a(s,x)-a(s,x_0)$ and $g(s,x)=G_\lambda f(s,x)$.

By Theorem 3.1 and Lemma 3.2, we have

$$\frac{1}{2}\max_{ij}\|a'_{ij}\|\sum_{ij}|D_iD_jg|_{L^p}+\max_i\|b_i\|\sum_i|D_ig|_{L^p}$$
$$\leqq\frac{1}{2}\max_{ij}\|a'_{ij}\|c_p|f|_{L^p}+\max_i\|b_i\|c_1r|f|_{L^p}\leqq\left(\frac{1}{2}+c.r\right)|f|_{L^p}.$$

On the other hand,

$$\left|\int_{|u|>1}|g(s,x+u)-g(s,x)|\bar{S}(du)\right|_{L^p}^p$$
$$\leqq\int_0^\infty\int ds\,dx\left(\int_{|u|>1}\bar{S}(du)\right)^{p/q}\int_{|u|>1}|g(s,x+u)-g(s,x)|^p\bar{S}(du)\quad\left(\frac{1}{p}+\frac{1}{q}=1\right)$$
$$\leqq c_2\int_{|u|>1}(2|g|_{L^p})^p\bar{S}(du)\leqq c_3|g|_{L^p}^p\leqq c.(r^2|f|_{L^p})^p\qquad\text{(by Lemma 3.2)}.$$

Moreover,

$$\left|\int_{|u|\leqq r}|g(s,x+u)-g(s,x)-(u,\nabla g(s,x))|\bar{S}(du)\right|_{L^p}^p$$
$$\leqq\int_0^\infty\int ds\,dx\int_0^1 d\theta\left(\int_{|u|\leqq r}|u|\,|\nabla g(s,x+\theta u)-\nabla g(s,x)|\bar{S}(du)\right)^p$$
$$\leqq\int_0^\infty\int ds\,dx\int_0^1 d\theta\left(\int_{|u|\leqq r}|u|^2\bar{S}(du)\right)^{p/q}\int_{|u|\leqq r}|u|^2\left(\frac{|\nabla g(s,x+\theta u)-\nabla g(s,x)|}{|u|}\right)^p\bar{S}(du)$$
$$\leqq\left(\int_{|u|\leqq r}|u|^2\bar{S}(du)\right)^{p/q}\int_0^1 d\theta\int_{|u|\leqq r}|u|^2\left(\frac{|\nabla g(s,x+\theta u)-\nabla g(s,x)|_{L^p}}{|u|}\right)^p\bar{S}(du)$$
$$\leqq c.\left(\int_{|u|\leqq r}|u|^2\bar{S}(du)\right)^p|f|_{L^p}^p\qquad\text{(by Lemma 3.3–3))}.$$

Similarly we have

$$\left|\int_{r<|u|\leqq 1}|g(s,x+u)-g(s,x)-(u,\nabla g(s,x))|\bar{S}(du)\right|_{L^p}^p$$
$$\leqq\int_0^\infty\int dx\,ds\int_0^1 d\theta\left(\int_{r<|u|\leqq 1}|u|\,|\nabla g(s,x+\theta u)-\nabla g(s,x)|\bar{S}(du)\right)^p$$

$$\leqq \left(\int_{r<|u|\leqq 1} |u| \bar{S}(du)\right)^{p/q} \int_0^1 d\theta \left(\int_{r<|u|\leqq 1} 2|u| |\nabla g|_{L^p}^p \bar{S}(du)\right)$$

$$\leqq c. \left(\int_{r<|u|\leqq 1} |u| \bar{S}(du)\right)^p (r|f|_{L^p})^p \qquad \text{(by Lemma 3.2)}.$$

Combining these inequalities, we see that

$$\left|\int_{|u|\leqq 1} |g(s, x+u)-g(s, x)-(u, \nabla g(s, x))| \bar{S}(du)\right|_{L^p}$$

$$\leqq c. \int_{|u|\leqq 1} |u|^2 \left(\frac{r}{|u|}\wedge 1\right) \bar{S}(du) |f|_{L^p}.$$

The right-hand side of this inequality tends to 0 as $r \downarrow 0$. Thus we can choose $\sigma_p(r)$ of the form

$$\sigma_p(r) = c. \left(r + \int_{|u|\leqq 1} |u|^2 \left(\frac{r}{|u|}\wedge 1\right) \bar{S}(du)\right). \qquad \text{Q.E.D.}$$

The kernel $S$ is said to be continuous if

$$\lim_{s'\to s,\, x'\to x} \int |u|^2 \wedge 1 \, |S(s', x', du)-S(s, x, du)| = 0 \qquad \text{for all } (s, x)\in R_+\times R^d.$$

**Lemma 3.5.** *Suppose that Assumption* ($A_\alpha$) *is satisfied. If* $a(s, x)$, $b(s, x)$ *and* $S(s, x, du)$ *are continuous in* $(s, x)$, *and if there exists a constant* $h_\alpha$ *such that*

$$\max_{ij} |a_{ij}(s, x')-a_{ij}(s, x)| + \max_i |b_i(s, x')-b_i(s, x)|$$
$$+\int |u|^2\wedge 1 \, |S(s, x', du)-S(s, x, du)| \leqq h_\alpha |x'-x|^\alpha \qquad \textit{for any } s, x', x,$$

*then there exists a function* $\sigma_\alpha(r)$ *on* $R_+$ *such that* $\lim_{r\downarrow 0} \sigma_\alpha(r)=0$ *and*

$$|T_\lambda f|_{C_r^\alpha} \leqq \left(\frac{1}{2}+\sigma_\alpha(r)\right)|f|_{C_r^\alpha}$$

*for* $f\in C_K^\infty$ *and* $\lambda r^2 \geqq 1$.

We can prove this lemma by means of Theorem 3.1 and Lemma 3.2. Although the computation is rather long, it is a routine work; so we omit the proof. We only mention that we can choose $\sigma_\alpha(r)$ of the form

$$\sigma_\alpha(r) = c. \left(r + r^\alpha h_\alpha + \int_{|u|\leqq 1} |u|^2 \left(\frac{r}{|u|}\wedge 1\right) \bar{S}(du)\right).$$

Let us introduce two norms for functions on $R_+\times R^d$.

$$|f|_{L^{2,p}} = |f|_{L^p} + \sum_i |D_i f|_{L^p} + \sum_{ij} |D_i D_j f|_{L^p},$$
$$|f|_{C^{2+\alpha}} = \|f\| + \sum_i \|D_i f\| + \sum_{ij} \|D_i D_j f\| + \sum_{ij} H^\alpha(D_i D_j f)$$

Completing the space $C_K^\infty$ by the norms $|\cdot|_{L^p}$, $|\cdot|_{L^{2,p}}$, $|\cdot|_{C_1^\alpha}$ and $|\cdot|_{C^{2+\alpha}}$, we get Banach spaces $L^p$, $L^{2,p}$, $C_*^\alpha$ and $C_*^{2+\alpha}$, respectively.

REMARKS. 1) Let Assumption $(A_p)$ be satisfied and let $r_p$ be a positive constant such that $\sigma_p(r_p)<\frac{1}{2}$. Then $|T_\lambda|_{L^p}<1$ for $\lambda\geqq r_p^{-2}$. Therefore the operators

$$[I-T_\lambda]^{-1}: L^p\to L^p \quad \text{and} \quad K_\lambda: L^p\to L^{2,p}$$

are well defined,

2) Suppose that the conditions in Lemma 3.5 are satisfied, and let $r_\alpha$ be a positive constant such that $\sigma_\alpha(r_\alpha)<\frac{1}{2}$. Then $|T_\lambda|_{C_T^\alpha}<1$ for $\lambda\geqq r_\alpha^{-2}$. Therefore, considering the first remark in Subsection I, the operators

$$[I-T_\lambda]^{-1}: C_*^\alpha\to C_*^\alpha \text{ and } K_\lambda: C_*^\alpha\to C_*^{2+\alpha}$$

are well defined.

**Theorem 3.6.** 1) *Suppose that Assumption* $(A_p)$ *is satisfied. If* $f\in L^p$ *and* $\lambda\geqq r_p^{-2}$, *then* $v=K_\lambda f\in L^{2,p}$ *and this is a solution (in the distribution sense) of the equation*

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=f.$$

*Moreover if* $p>d$ *and* $\frac{d}{p}<\alpha<1$, *then there exists a constant* $c$ *(independent of* $f$*) such that*

$$||v||+\sum_i ||D_i v||+\sum_i H^{1-\alpha}(D_i v)\leqq c|f|_{L^p}.$$

2) *Suppose that conditions in Lemma* 3.5 *are satisfied. If* $f$ *is a* $C_*^\alpha$*-function such that the support of the funciton* $\sup_x |f(s,x)|$ *is a compact set in* $R_+$, *then* $v=K_\lambda f$ *is a* $C_*^{2+\alpha}$*-function satisfying the equation*

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=f$$

*for any* $\lambda>0$, *and* $||v||\leqq 1/\lambda||f||$.

Proof. 1) Set $f'=[I-T_\lambda]^{-1}f$. Then $f'\in L^p$ and $K_\lambda f=G_\lambda f'$. Therefore

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=\left[\left(\lambda-\frac{\partial}{\partial s}-L_0\right)-(L-L_0)\right]G_\lambda f'=[I-T_\lambda]f'=f.$$

The rest of the assertion follows from Lemma 3.3.

2) Set $\lambda'=r_\alpha^{-2}$. If $f$ is a function concerned with, then the $C_*^{2+\alpha}$-function $K_\lambda f$:

$$K_\lambda f(s,x)=e^{-(\lambda'-\lambda)s}K_{\lambda'}f_1(s,x) \text{ with } f_1(s,x)=e^{(\lambda'-\lambda)s}f(s,x)$$

is well defined for each $\lambda>0$. It is easy to show that $v=K_\lambda f$ satisfies the equation

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=f.$$

By the maximum principle of parabolic-type equation, we see easily the last assertion. Q.E.D.

III. (*Construction of transition probabilities*)

Let $a, b, S, a_1, a_2, \bar{S}, c_\alpha, c_p, r_p$ and other notations be the same as in Subsection II. In this subsection, we suppose that Assumption $(A_\alpha)$ and $(A_p)$ are satisfied. Let us denote the Radon-Nikodym derivative of $S(s, x, du)$ given $\bar{S}(du)$ by $K(s, x, u)$.

It is easy to construct a sequence $(a^n, b^n, S^n)$ satisfying a)—e).

a) $a_1|\theta|^2\leqq(\theta, a^n(s, x)\theta)\leqq a_2|\theta|^2$ for all $\theta$ and $(s, x)\in R_+\times R^d$, and $a^n$ satisfies Assumption $(A_\alpha)$ and $(A_p)$.

b) $||a^n_{ij}||\leqq||a_{ij}||$, $||b^n_i||\leqq||b_i||$ and there exists $K^n$ such that $0\leqq K^n(s, x, u)\leqq 1$ and $S^n(s, x, du)=K^n(s, x, u)\bar{S}(du)$.

c) There exists a bounded set $D^n\subset R_+\times R^d$ such that $a^n(s, x)-a^n(s, x_0)=b^n(s, x)=K^n(s, x, u)=0$ for $(s, x)\notin D^n$.

d) $a^n$, $b^n$ and $S^n$ are continous in $(s, x)$, and there exists a constant $h^n$ such that

$$\sum_{ij}|a^n_{ij}(s, x')-a^n_{ij}(s, x)|+\sum_i|b^n_i(s, x')-b^n_i(s, x)| \\ +\int|u|^2\wedge 1\,|K^n(s, x', u)-K^n(s, x, u)|\,\bar{S}(du)\leqq h^n_\alpha|x'-x|^\alpha$$

for all $s, x', x$.

e) $a^n_{ij}(s, x)\to a_{ij}(s, x)$, $b^n_i(s, x)\to b_i(s, x)$ *a.e.* with respect to $dsdx$, and $K^n(s, x, u)\to K(s, x, u)$ *a.e.* with respect to $ds\,dx\,\bar{S}(du)$.

We define operators $L^n$ and $K^n_\lambda$ similarly to $L$ and $K_\lambda$ respectively by using $(a^n, b^n, S^n)$ in place of $(a, b, S)$.

**Lemma 3.7.** *Let $f\in L^p\cap L^\infty$ and let $(f^m)$ be a sequence of $C^\alpha_*$-functions with compact supports such that $f^m\to f$ in $L^p$ and $||f^m||\leqq||f||$. Then $K^n_\lambda f^m$ is a $C^{2+\alpha}_*\cap L^{2,p}$-function. And if $\lambda\geqq r_p^{-2}$, then*

$$|K^n_\lambda f^m-K^n_\lambda f|_{L^{2,p}}\to 0 \text{ as } m\to\infty, \quad |K^n_\lambda f-K_\lambda f|_{L^{2,p}}\to 0 \text{ as } n\to\infty .$$

*Moreover if $2p>d$ and $\lambda\geqq r_p^{-2}$, then*

$$||K^n_\lambda f^m-K^n_\lambda f||\to 0 \text{ as } m\to\infty, \quad ||K^n_\lambda f-K_\lambda f||\to 0 \text{ as } n\to\infty .$$

Proof. By Theorem 3.6 we see that $K^n_\lambda f^m\in C^{2+\alpha}_*\cap L^{2,p}$. It is immediate to show that $\lim_{m\to\infty}|K^n_\lambda f^m-K^n_\lambda f|_{L^{2,p}}=0$ for $\lambda\geqq r_p^{-2}$. By Lemma 3.3, this implies that $\lim_{m\to\infty}||K_\lambda^n f^m - K_\lambda^n f||=0$ for $\lambda \geqq r_p^{-?}$ and $2p>d$. On the other hand, for each $L^{2,p}$-function $v$, $\lim_{n\to\infty}|(L^n-L)v|_{L^p}=0$. In fact, for example,

$$I^n = \iint ds\,dx\left[\int |v(s,x+u)-v(s,x)-I_{(|u|\leq 1)}(u,\nabla v(s,x))|\,|S^n(s,x,dv)-S(s,x,du)|\right]^p$$
$$=\iint ds\,dx\left[\int |u|^2\wedge 1 h(s,x,u)\,|K^n(s,x,u)-K(s,x,u)|\,\bar{S}(du)\right]^p$$
$$\leqq c.\int |u|^2\wedge 1\left[\iint h(s,x,u)^p\,|K^n(s,x,u)-K(s,x,u)|^p ds\,dx\right]\bar{S}(du)\,,$$

where the $c.$ stands for a constant independent of $n$ and

$$h(s,x,u) = |v(s,x+u)-v(s,x)-I_{(|u|\leq 1)}(u,\nabla v(s,x))|/(|u|^2\wedge 1)\,.$$

It is easy to show that $\sup_u |h(s,x,u)|_{L^p}<\infty$. As $|K^n-K|\leqq 2$ and $K^n\to K$ *a.e.*, we conclude that $\lim_{n\to\infty} I^n=0$. In particular, $\lim_{n\to\infty}|(L^n-L)K_\lambda|_{L^p}=0$ for $\lambda\geqq r_p^{-2}$. Since $K_\lambda^n f=K_\lambda[I-(L^n-L)K_\lambda]^{-1}f$, we have $\lim_{n\to\infty}|K_\lambda^n f-K_\lambda f|_{L^{2,p}}=0$ for $\lambda\geqq r_p^{-2}$. By Lemma 3.3, we have also $\lim_{n\to\infty}||K_\lambda^n f-K_\lambda f||=0$ for $\lambda\geqq r_p^{-2}$ and $2p>$d.

Q.E.D.

**Lemma 3.8.** *Let $v$ be $C_*^{2+\alpha}$-function with compact support and let*

$$f=\left(\lambda-\frac{\partial}{\partial s}-L\right)v \qquad (\lambda>0)\,.$$

*Then $f$ is an $L^p\cap L^\infty$-function such that the support of $\sup_x|f(s,x)|$ is compact, and $v=K_\lambda f$.*

Proof. Set

$$g^n=\left(\lambda-\frac{\partial}{\partial s}-L^n\right)v=f-(L^n-L)v\,.$$

Then $g^n$ is an $L^p\cap C_*^\alpha$-function such that the support of $\sup_x|g^n(s,x)|$ is compact. Thus

$$v=K_\lambda^n g^n=K_\lambda^n(f-(L^n-L)v)\,.$$

From $\lim_{n\to\infty}|(L^n-L)v|_{L^p}=0$, which have proved in the proof of Lemma 3.7, it follows that $\lim_{n\to\infty}|K_\lambda^n(L^n-L)v|_{L^{2,p}}=0$. In fact, in the case $\lambda\geqq r_p^{-2}$, this follows from Theorem 3.6; and in the case $\lambda<r_p^{-2}$, it follows from Theorem 3.6 and the next remark that if $f'$ is an $L^p$-function such that the support of $\sup_x|f'(s,x)|$ is compact, then we have

$$K_\lambda^n f'(s,x)=e^{-(\lambda'-\lambda)s}K_{\lambda'}^n f_1'(s,x) \quad \text{with} \quad f_1'(s,x)=e^{(\lambda'-\lambda)s}f'(s,x)\,,$$

where $\lambda'=r_p^{-2}$. On the other hand $\lim_{n\to\infty}|K_\lambda^n f-K_\lambda f|_{L^{2,p}}=0$ by Lemma 3.7 (and the above remark). Thus $v=K_\lambda f$. Q.E.D.

Let $C_0$ be the class of all continuous function $f(s, x)$ on $R_+\times R^d$ such that $\lim_{s+|x|^2\to\infty} f(s, x)=0$. In the rest of this section, we assume that $2p>d$.

By Lemma 3.7 and Theorem 3.6, we see that if $\lambda\geqq r_p^{-2}$ and $f\in L^p\cap L^\infty$, then $K_\lambda f$ is an $L^{2,p}\cap C_0$-function and

$$||K_\lambda f|| = \lim_{n,m}||K_\lambda^n f^m|| \leqq \lim_m \frac{1}{\lambda}||f^m|| \leqq \frac{1}{\lambda}||f||,$$

where $(f^m)$ is a sequence approximating $f$, having the same properties as in Lemma 3.7. Let $K_\lambda'$ be a unique extention of $K_\lambda$ onto $C_0$. Then $||K_\lambda'||\leqq 1/\lambda$. It is easy to show that

$$K_\lambda^n f^m - K_\mu^n f^m = -(\lambda-u)K_\lambda^n K_\mu^n f^m \quad (\lambda,\ \mu\geqq 0), \text{ and } K_\lambda^n f^m\geqq 0 \quad \text{if } f^m\geqq 0\,.$$

This implies that, for each $\lambda$, $\mu\geqq r_p^{-2}$,

$$K_\lambda' - K_\mu' = -(\lambda-\mu)K_\lambda' K_\mu' \text{ and } K_\lambda'\geqq 0\,.$$

Set $\boldsymbol{K}=K_\lambda(L^p\cap C_0)$ (which is independent of $\lambda$), and let $A=\frac{\partial}{\partial s}+L$. The family $\boldsymbol{K}$ is dense in $C_0$. In fact, let $v$ and $f$ be the functions considered in Lemma 3.8 and let $(f^m)$ be a sequence approximating $f$ which has been considered in Lemma 3.7. Then we have $\lim_{m\to\infty}||K_\lambda f^m - v|| = \lim_{m\to\infty}||K_\lambda f^m - K_\lambda f||=0$ for $\lambda\geqq r_p^{-2}$ (by Lemma 3.7 and Lemma 3.8). This implies that $\boldsymbol{K}$ is dense in $C_0$. It is easy to show that

$$(\lambda-A)K_\lambda = I \text{ on } L^p\cap C_0, \text{ and } K_\lambda(\lambda-A) = I \text{ on } \boldsymbol{K}\,.$$

Therefore

$$(\lambda-A)K_\lambda' = K_\lambda'(\lambda-A) = I \text{ on } \boldsymbol{K}\,.$$

By Hille-Yosida's semi-group theory, there exists a closed extension $A'$ of the operator $A$ and a positive contraction semi-group $(P_t)$ on $C_0$ whose infinitesimal generator is $A'$ and whose resolevents are $(K_\lambda')$. That is,

$$K_\lambda' f = (\lambda-A')^{-1}f = \int_0^\infty e^{-\lambda t}P_t f\,dt\,, \quad f\in C_0 \text{ and } \lambda\geqq r_p^{-2}\,.$$

Let $\rho(\xi)$ be a $C^\infty(R^1)$-function such that $0\leqq\rho(\xi)\leqq 1$, $\rho(\xi)=1$ for $|\xi|\leqq\frac{1}{2}$ and $\rho(\xi)=0$ for $|\xi|\geqq 1$.

**Lemma 3.9.** *Suppose that the domain of the operator $P_t$ is extended to $C^{0,b}(R_+\times R^d)$ by such a way that $P_t f=\lim_{m\to\infty} P_t f^m$ for each $f\in C^{0,b}(R_+\times R^d)$ where*

$$f^m(s, x) = f(s, x)\rho\left(\frac{s}{m^2}\right)\rho\left(\frac{|x|}{m}\right).$$

*Then* 1) $P_t 1 = 1$, *and* 2) *if* $f$ *is a bounded continuous function on* $R_+$ *with compact support, then* $P_t f(s, x) = f(s+t)$ ($P_t f$ *is independent of* $x$).

Proof. 1) Let $1^m = \rho(s/m^2)\rho(|x|/m)$. Then we have

$$\left(\lambda - \frac{\partial}{\partial s} - L\right)1^m = \lambda 1^m + g^m, \quad g^m \in L^p \cap L^\infty \text{ and } \lim_{m\to\infty} \|g^m\| = 0.$$

By Lemma 3.8, $1^m = \lambda K_\lambda 1^m + K_\lambda g^m$. Since $\|K_\lambda g^m\| \leqq \frac{1}{\lambda}\|g^m\|$, it holds that $\lim_{m\to\infty} K_\lambda 1^m = \frac{1}{\lambda}$. And so we have

$$\int_0^\infty e^{-\lambda t} P_t 1 dt = \int_0^\infty e^{-\lambda t} \lim_{m\to\infty} P_t 1^m dt = \lim_{m\to\infty} \int_0^\infty e^{-\lambda t} P_t 1^m dt = \lim_{m\to\infty} K_\lambda' 1^m = \frac{1}{\lambda},$$

for $\lambda \geqq r_p^{-2}$. Since $P_t 1 \leqq 1$, it must hold that $P_t 1 = 1$.

2) Let $h(s) = \int_s^\infty e^{-\lambda(t-s)} f(t) dt = \int_0^\infty e^{-\lambda t} f(s+t) dt$ and $h^m(s, x) = h(s)\rho(|x|/m)$. Then we have

$$\left(\lambda - \frac{\partial}{\partial s} - L\right)h^m = f(s)\rho\left(\frac{|x|}{m}\right) + g^m,\ g^m \in L^p \cap L^\infty \text{ and } \lim_{m\to\infty} \|g^m\| = 0.$$

By a similar way to the item 1), we have

$$\int_0^\infty e^{-\lambda t} P_t f dt = \lim_{m\to\infty} K_\lambda' f^m = h(s) = \int_0^\infty e^{-\lambda t} f(s+t) dt,$$

for all $\lambda \geqq r_p^{-2}$. This implies that $P_t f(s, x) = f(s+t)$. Q.E.D.

There exists a kernel $p_t'(s, x; d\tau, dy)$ such that

$$P_t f(s, x) = \int p_t'(s, x; d\tau, dy) f(\tau, y)$$

for each bounded continuous function $f$ on $R_+ \times R^d$. By Lemma 3.9, we see that

$$p_t'(s, x; R_+, dy) = p_t'(s, x; \{t+s\}, dy), \quad p_t'(s, x; R_+, R^d) = 1.$$

Set $p_t'(s, x; R_+, dy) = p(s, x; t+s, dy)$. Then $p(s, x; t, dy)$ is a kernel such that

*a*) $p(s, x; t, R^d) = 1$,

*b*) $p(s, x; s, dy) = \delta_x(dy)$ ($\delta$-measure at $x$),

*c*) $p(s, x; t, dy) = \int p(s, x; r, dz) p(r, z; t, dy)$ for $s < r < t$, and

*d*) $P_{t-s} f(s, x) = \int p(s, x; t, dy) f(y)$ for each $f \in C^{0,b}(R^d)$.

## 4. Uniqueness of martingale solution

In this section, we suppose that the following condition is satisfied.

**Condition (U)**

1) For each bounded domain $D$, there exist constants $0<a_1\leqq a_2$ such that

$$a_1|\theta|^2\leqq(\theta, a(s, x)\theta)\leqq a_2|\theta|^2$$

for all $\theta\in R^d$ and $(s, x)\in D$, Moreover $\lim_{x'\to x}\sup_{s\leqq T}\sum_{ij}|a_{ij}(s, x')-a_{ij}(s, x)|=0$ for any $T$.

2) $b(s, x)$ is locally bounded.

3) For each bounded domain $D$, there exists a measure $S(du)$ such that

$$\int|u|^2\wedge 1\,\bar{S}(du)<\infty,\text{ and } S(s, x, du)\leqq\bar{S}(du) \text{ for } (s, x)\in D\,.$$

Let $T$ be a positive constant, and $\rho(\xi)$ the function defined in §3—III. Let $y$ be an arbitrary but fixed point of $R^d$, and set

$$^Ra(s, x) = a(s\wedge T, y)+[a(s\wedge T, x)-a(s\wedge T, y)]\rho\left(\frac{|x-y|}{R}\right),$$

$$^Rb(s, x) = b(s\wedge T, x)\rho\left(\frac{|x-y|}{R}\right),$$

$$^RS(s, x, du) = S(s\wedge T, x, du)\rho\left(\frac{|x-y|}{R}\right).$$

In §3—I, we have learnt that if $a_1|\theta|^2\leqq(\theta,a(s)\theta)\leqq a_2|\theta|^2$ for all $\theta$ and $s$, then there exist constants $c_p(a_1, a_2)$ and $c_\alpha(a_1, a_2)$ such that

$$\sum_{ij}|D_iD_jG_\lambda|_{L^p}\leqq c_p(a_1, a_2),\quad \sum_{ij}|D_iD_jG_\lambda|_{C_r^\alpha}\leqq c_\alpha(a_1, a_2)\qquad (\lambda r^2\geqq 1),$$

where $G_\lambda$ is the Green operator associated with the parabolic operator

$$\lambda-\frac{\partial}{\partial s}-\frac{1}{2}\sum_{ij}a_{ij}(s)D_iD_j\,.$$

Let $\underline{a}$ and $\bar{a}$ be positive constants such that

$$\underline{a}|\theta|^2\leqq(\theta,{}^Ra(s, x)\theta)\leqq\bar{a}|\theta|^2,$$

for any $\theta\in R^d$, $R\leqq 1$, $(s, x)\in R_+\times R^d$. There exists a positive constant $R(y)\leqq 1$ such that

$$c_\alpha(\underline{a}, \bar{a})\vee c_p(\underline{a}, \bar{a})\max_{ij}\sup_{s,x',x}|{}^{R(y)}a_{ij}(s, x')-{}^{R(y)}a_{ij}(s, x)|\leqq 1\,.$$

It is possible to assume, without any loss of generality, that $R(y)$ is measurable and $1/R(y)$ is locally bounded.

As we have proved in §3—III, there exists a system of transition probabilities $p(s, x; t, dy)$ associated with the operator $L$ with coefficients $({}^{R(y)}a, {}^{R(y)}b, {}^{R(y)}S)$. Therefore it is possible to construct a strong Markov process $(x_t, Q_{s,x}; t\in[s, \infty))$ on the measurable space $(W^s, \boldsymbol{W}^s, \boldsymbol{W}_t^s)$ such that

$$E_{s,x}[f(x_t)\,|\,\boldsymbol{W}_r^s] = \int p(r, x_r; t, dy)f(y) \qquad (s\leqq r\leqq t)$$

for each bounded Borel function $f$ on $R^d$.

In the next lemma, $({}^{R(y)}a, {}^{R(y)}b, {}^{R(y)}S)$ are simply denoted by $(a, b, S)$. And $L$ and $K_\lambda$ are the operators associated with these coefficients.

**Lemma 4.1.** *The strong Markov process $(x_t, Q_{s,x})$ is a martingale solution of the $(a, b, S)$- stochastic equation starting from $(s, x)$.*

Proof. Suppose that $2p>d$, $\lambda\geqq r_p^{-2}$ and $f\in L^p\cap C_0$ (*see* §3.) Then we have

$$K_\lambda f(s, x) = \int_s^\infty e^{-\lambda(t-s)}E_{s,x}[f(t, x_t)]dt\,.$$

Set $\rho_N(x)=\rho(|x|/N)$ and

$$\boldsymbol{H}= \left\{f\in L^\infty; K_\lambda(f\rho_N)(s, x) = E_{s,x}\left[\int_s^\infty e^{-\lambda(t-s)}(f\rho_N)(t, x_t)dt\right]\right\}.$$

Then $\boldsymbol{H}$ contains 1 and all test functions. If $f_n\in\boldsymbol{H}$ and $f_n\to f$ in sup-norm, then $f_n\rho_N\to f\rho_N$ in $L^p$ and in sup-norm, and hence $f\in\boldsymbol{H}$. Similarly, if $f_n\in\boldsymbol{H}$, $f_n\geqq 0$ and $f_n\uparrow f(f\in L^\infty)$, then $f\in\boldsymbol{H}$. Therefore $\boldsymbol{H}$ must contain all the bounded measurable functions. From this fact, it is easily verified that if $\lambda\geqq r_p^{-2}$ and $f\in L^p\cap L^\infty$, then

$$K_\lambda f(s, x) = E_{s,x}\left[\int_s^\infty e^{-\lambda(t-s)}f(t, x_t)dt\right].$$

This fact and the Markov property of $(x_t, Q_{s,x})$ imply that the process

$$e^{-\lambda(t-s)}K_\lambda f(t, x_t)-K_\lambda f(s, x)+\int_s^t e^{-\lambda(\tau-s)}f(\tau, x_\tau)d\tau$$

is a square integrable martingale for each $\lambda\geqq r_p^{-2}$ and $f\in L^p\cap L^\infty$. Let $v$ be a $C^2$-function with compact support and let

$$\left(\lambda-\frac{\partial}{\partial s}-L\right)v=f.$$

Then $f\in L^p\cap L^\infty$ and $v=K_\lambda f$ (by Lemma 3.8). Thus the process

$$e^{-\lambda(t-s)}v(t, x_t)-v(s, x)+\int_s^t e^{-\lambda(\tau-s)}f(\tau, x_\tau)d\tau$$

is a square integrable martingale. It is easy to show that this property holds for any $C^{2,b}(R_+\times R^d)$-function $v$. Moreover the property $\lim_{l\to\infty} Q_{s,x}[\sup_t|x_t|>l]=0$

follows easily from the above facts. Consequently, the process $(x_t, Q_{s,x})$ is a martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$.

Q.E.D.

REMARK. In the above proof, we used the constant $r_p$ which was not well identified. Here we shall give a discussion upon this. In this remark, let us denote $({}^{R(y)}a, {}^{R(y)}b, {}^{R(y)}S)$ by $(a, b, S)$. And let $G_\lambda^z$ be the Green operator associated with the parabolic operator

$$\lambda-\frac{\partial}{\partial s}-\frac{1}{2}\sum_{ij} a_{ij}(s, z)D_iD_j = \lambda-\frac{\partial}{\partial s}-L^z\,.$$

There exists a measure $\bar{S}(du)$ (depending on $R(y)$) such that

$$\int |u|^2\wedge 1\ \bar{S}(du)<\infty, \quad S(s, x, du)\leqq \bar{S}(du) \text{ for each } (s, x)\in R_+\times R^d\,.$$

The definition of $R(y)$ shows the existence of positive constants $\gamma$ and $r_p$ such that $0<\gamma<1$ and

$$\frac{1}{2}\max_{ij}\|a_{ij}(s, x)-a_{ij}(s, z)\|\sum_{ij}|D_iD_jG_\lambda^z f|_{L^p}+\max_i\|b_i\|\sum_i|D_iG_\lambda^z f|_{L^p}$$
$$+\left|\int|G_\lambda^z f(s, x+u)-G_\lambda^z f(s, x)-I_{(|u|\leqq 1)}(u, \nabla G_\lambda^z f(s, x))|\bar{S}(du)\right|_{L^p}\leqq\gamma|f|_{L^p}$$

for each $\lambda\geqq r_p^{-2}, f\in L^p$ and $z\in R^d$.

In the following theorem, $(a, b, S)$ means the coefficient $({}^{R(y)}a, {}^{R(y)}b, {}^{R(y)}S)$, and $G_\lambda^z$, $\bar{S}(du)$, $\gamma$ and $r_p$ are the objects defined in the above remark.

**Theorem 4.2.** *The martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$ exists uniquely, and this is a strong Markov process.*

Proof. 1° Let $(x_t, Q_{s,x})$ be any martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$. Let us define an operator $V_\lambda$ acting on bounded measurable functions $f$ on $R_+\times R^d$ by the formula:

$$V_\lambda f(s, x) = E_{s,x}\left[\int_s^\infty e^{-\lambda(t-s)}f(t, x_t)dt\right].$$

We shall prove that if $p>d$, $\lambda\geqq r_p^{-2}$ and $f\in L^p\cap L^\infty$, then $K_\lambda f(s,x)=V_\lambda f(s,x)$, where $K_\lambda$ is the operator associated with the coefficient $(a, b, S)$. From this fact, the assertion of the theorem follows immediately.

2° There exists a Brownian motion $B_t$ such that

$$x_t = x+\int_s^t a(\tau, x_\tau)^{1/2}dB_\tau+\int_s^t b(\tau, x_\tau)d\tau$$
$$+\int_s^t\int_{|u|\leqq 1} u\ {}^cJ(d\tau, du)+\int_s^t\int_{|u|>1} u\ J(d\tau, du)$$

where $J$ is the measure of jumps of $x_t$ and ${}^cJ$ is the martingale measure associated with $J$ and $S$ (*see* §2). Let $\pi_n$ be a function satisfying $\pi_n(t)=s+\frac{\nu}{n}$ when $s+\frac{\nu}{n}<t\leqq s+\frac{\nu+1}{n}$, and define a new process $x_t^n$ by the formula:

$$x_t^n = x+\int_s^t a(\tau, x_{\pi_n(\tau)})^{1/2}dB_\tau+\int_s^t b(\tau, x_\tau)d\tau$$
$$+\int_s^t\int_{1/n<|u|\leqq 1} u\ {}^cJ(d\tau, du)+\int_s^t\int_{|u|>1} u\ J(d\tau, du)\,.$$

Let $V_\lambda^n$ be an operator defined by

$$V_\lambda^n f(s, x) = E_{s,x}\left[\int_s^\infty e^{-\lambda(t-s)}f(t, x_t^n)dt\right].$$

It is not so difficult to prove that, for each $\varepsilon>0$ and $T'<\infty$,

$$\lim_{\delta\downarrow 0}\sup_{\substack{0\leqq t,t'\leqq T'\\ |t-t'|<\delta}} Q_{s,x}[|x_t-x_{t'}|>\varepsilon] = 0\,.$$

This property and the martingale inequality imply that

$$\lim_{n\to\infty} Q_{s,x}[\sup_{0\leqq t\leqq T'}|x_t^n-x_t|>\varepsilon] = 0\,.$$

Thus we have $\lim\limits_n V_\lambda^n f(s, x)=V_\lambda f(s, x)$ for each bounded continuous function $f$.

3° We shall prove that there exists a constant $N_\lambda^n$ such that $||V_\lambda^n f||\leqq N_\lambda^n|f|_{L^p}$ for each $f\in C_*^\alpha\cap L^p$, $p>d$ and $\lambda\geqq r_p^{-2}$.

Set $v(z; s, x)=G_\lambda^z f(s, x)$ for $f\in C_*^\alpha\cap L^p$, then $v(z; \cdot, \cdot)\in C_*^{2+\alpha}\cap L^{2,p}$ for each $z\in R^d$. Let us denote $s+\frac{\nu}{n}$ by $t_\nu$ and $v(x_{t_\nu}; s, x)$ by $v_\nu(s, x)$. Then the process

$$e^{-\lambda(t-s)}v_\nu(t, x_t^n)-e^{-\lambda(t_\nu-s)}v_\nu(t_\nu, x_{t_\nu}^n)+\int_{t_\nu}^t e^{-\lambda(\tau-s)}f(\tau, x_\tau^n)d\tau$$
$$+\int_{t_\nu}^t e^{-\lambda(\tau-s)}\{\sum_i b_i(\tau, x_\tau)D_i v_\nu(\tau, x_\tau^n)+\int_{|u|>1/n}[v_\nu(\tau, x_\tau^n+u)-v_\nu(\tau, x_\tau^n)$$
$$-I_{(|u|\leqq 1)}(u, \nabla v_\nu(\tau, x_\tau^n))]S(\tau, x_\tau,du)\}\,d\tau$$

is a square integrable martingale on the time interval $(t_\nu, t_{\nu+1}]$. Therefore,

$$||V_\lambda^n f||\leqq \sum_{\nu=0}^\infty (e^{-\lambda(t_{\nu+1}-s)}-e^{-\lambda(t_\nu-s)})\sup_z||v(z;\cdot,\cdot)||$$
$$+\int_s^\infty e^{-\lambda(t-s)}\{\sum_i||b_i||\sup_z||D_iv(z;\cdot,\cdot)||+2\sup_z||v(z;\cdot,\cdot)||\int_{|u|>1}\bar{S}(du)$$
$$+2\sum_i\sup_z||D_iv(z;\cdot,\cdot)||n\int_{|u|\leqq 1}|u|^2\bar{S}(du)\}\,dt.$$

Since, by Lemma 3.3, there exists a constant $c$ independent of $f$ and $z$ such that

$$\sum_i ||D_i v(z;\cdot,\cdot)|| + ||v(z;\cdot,\cdot)|| \leqq c|f|_{L^p},$$

there exists a constant $N_\lambda^n$ such that $||V_\lambda^n f|| \leqq N_\lambda^n |f|_{L^p}$ for each $f \in C_*^\alpha \cap L^p$.

4° We shall prove that there exists a constant $N_\lambda$ such that $||V_\lambda f|| \leqq N_\lambda |f|_{L^p}$ for each $f \in C_*^\alpha \cap L^p$ provided that $\lambda \geqq r_p^{-2} (p>d)$.

Let $f \in C_*^\alpha \cap L^p$ and $v(s, x) = G_\lambda^z f(s, x)$ where $z$ is an arbitrary but fixed point of $R^d$.

Set

$$h(s, x) = \max_{ij} ||a_{ij}(s, x) - a_{ij}(s, z)|| \sum_{ij} |D_i D_j v(s, x)| + \max_i ||b_i|| \sum_i |D_i v(s, x)|$$
$$+ \int |v(s, x+u) - v(s, x) - I_{(|u| \leqq 1)}(u, \nabla v(s, x))| \bar{S}(du).$$

Since the process

$$e^{-\lambda(t-s)} v(t, x_t^n) - v(s, x) + \int_s^t e^{-\lambda(\tau-s)} f(\tau, x_\tau^n) d\tau$$
$$-\int_s^t e^{-\lambda(\tau-s)} \Big\{ \frac{1}{2} \sum_{ij} (a_{ij}(\tau, x_{\pi_n(\tau)}) - a_{ij}(\tau, z)) D_i D_j v(\tau, x_\tau^n) + \sum_i b_i(\tau, x_\tau) D_i v(\tau, x_\tau^n)$$
$$+ \int_{1/n<|u|} [v(\tau, x_\tau^n + u) - v(\tau, x_\tau^n) - I_{(|u| \leqq 1)}(u, \nabla v(\tau, x_\tau^n))] S(\tau, x_\tau, du) \Big\} d\tau$$

is a square integrable martingale, we have $||V_\lambda^n f|| \leqq ||v|| + ||V_\lambda^n h||$. Let $N_\lambda^n$ be the smallest constant such that $||V_\lambda^n f|| \leqq N_\lambda^n |f|_{L^p}$ for any $C_*^\alpha \cap L^p$-function $f$, and let $N_\lambda$ be a constant such that $||G_\lambda^z f|| \leqq (1-\gamma) N_\lambda |f|_{L^p}$ for any $L^p \cap L^\infty$-function $f$. Since $|h|_{L^p} \leqq \gamma |f|_{L^p}$, we have

$$||V_\lambda^n f|| \leqq (1-\gamma) N_\lambda |f|_{L^p} + N_\lambda^n |h|_{L^p} \leqq (1-\gamma) N_\lambda |f|_{L^p} + \gamma N_\lambda^n |f|_{L^p}$$

This implies that $N_\lambda^n \leqq (1-\gamma) N_\lambda + \gamma N_\lambda^n$, and so $N_\lambda^n \leqq N_\lambda$. Therefore, for any $f \in C_*^\alpha \cap L^p$,

$$||V_\lambda f|| = \lim_n ||V_\lambda^n f|| \leqq N_\lambda |f|_{L^p}$$

5° The inequality $||V_\lambda f|| \leqq N_\lambda |f|_{L^p}$ holds good for any $L^p \cap L^\infty$-function $f$. This may be proved by making use of general results in measure theory. Therefore we omit the proof.

6° Let $p>d$, $\lambda \geqq r_p^{-2}$, $f \in C_*^\alpha \cap L^p$ and set $v = G_\lambda^z$ where $z$ is an arbitrary fixed point. Then the process

$$e^{-\lambda(t-s)} v(t, x_t) - v(s, x) + \int_s^t e^{-\lambda(\tau-s)} (\lambda - \frac{\partial}{\partial \tau} - L) v(\tau, x_\tau) d\tau$$

is a square integrable martingale. Therefore

$$v(s, x)=E_{s,x}\left[\int_s^\infty e^{-\lambda(t-s)}(\lambda-\frac{\partial}{\partial t}-L)v(t, x_t)dt\right] = V_\lambda(I-T_\lambda^z)f(s, x),$$

where $T_\lambda^z=(L-L^z)G_\lambda^z$. On the other hand $G_\lambda^z=K_\lambda(I-T_\lambda^z)$. Hence $K_\lambda g= V_\lambda g$ for each function $g$ of the form: $g=(I-T_\lambda^z)f, f\in C_*^\alpha \cap L^p$. The set of such functions $g$ is dense in $L^p \cap L^\infty$ with $L^p$-norm, and so $K_\lambda f=V_\lambda f$ for each $f\in L^p\cap L^\infty$ for $p>d$ and $\lambda\geqq r_p^{-2}$. Q.E.D

REMARK. Let $\zeta$ be a bounded $s$-stopping time and $p>d$, then

$$K_\lambda f(\zeta, x_\zeta) = E_{s,x}\left[\int_\zeta^\infty e^{-\lambda(t-\zeta)}f(t, x_t)dt \mid \boldsymbol{W}_\zeta^s\right] \ a.e. \ (Q_{s,x})$$

for each $f\in L^p\cap L^\infty$ and $\lambda\geqq r_p^{-2}$. (This can be proved similarly to the above theorem.)

In the following lemma, we do not assume that $(a, b, S)$ satisfy Condition (U). (And $(a, b, S)$ does not mean the coefficient $({}^{R(y)}a, {}^{R(y)}b, {}^{R(y)}S)$.)

**Lemma 4.3** *We assume that $(a, b, S)$ satisfy the condition*

$$\max\|a_{ij}\|+\max\|b_i\|+\|\int|u|^2\wedge 1\ S(s, x, du)\|<\infty.$$

*Let $T$ be a bounded s-stopping time. Let $Q'$ be a probability measure on $(W^s, \boldsymbol{W}^s)$ such that $(x_t, Q'; t\in[s, T])$ is a martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$. Suppose that $Q_w''$ $(w\in W^s)$ is a probability measure on $(W^s, \boldsymbol{W}^{T(w)})$ a. e. $w$ $(Q')$ such that $(x_t, Q_w''; t\in[T(w), \infty))$ is a martingale solution of the $(a, b, S)$-stochastic equation starting from $(T(w), x_{T(w)})$, and $w \rightsquigarrow Q_w''(A)I_{(T(w)\leqq t)}$ is $\boldsymbol{W}_t^s$-measurable for each $A\in \boldsymbol{W}^t$ $(t\geqq s)$. Then there exists a unique probability measure $Q$ on $(W^s, \boldsymbol{W}^s)$ such that $Q=Q'$ on $\boldsymbol{W}_T^s$ and the regular conditional disfribution of $Q$ given $\boldsymbol{W}_T^s$ equals $Q_w''$ on $\boldsymbol{W}^{T(w)}$. If $Q$ is this probability, then the process $(x_t, Q; t\in[s, \infty))$ is a martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$.*

Proof. The first assertion is the conclusion of Lemma 1.2. Extend $Q_w''$ onto $\boldsymbol{W}^s$ so that $Q[A\mid \boldsymbol{W}_T^s]=Q_w''[A]$ for all $A\in \boldsymbol{W}^s$. Then, for each $A\in \boldsymbol{W}^s$ and $B\in \boldsymbol{W}_T^s$, we have

$$Q[A\cap B] = \int_B Q_w''[A]Q'(dw).$$

Let $\Phi_\theta(s, x)$ be Ito's differential associated with $(a, b, S)$ (*see* the paragraph under the first definition in §2), and set

$$M_t^r = \exp[i(\theta, x_t-x_r)-\int_r^t\Phi_\theta(\tau, x_\tau)d\tau], \ s\leqq r\leqq t.$$

From the assumption, $M^s_{T\wedge t}$ (resp. $M^{T\wedge t}_t$) is a square integrable martingale with respect to $(\boldsymbol{W}^s_t, Q')$ (resp. $(\boldsymbol{W}^T_{T\vee t}, Q''_w)$). Let $A\in \boldsymbol{W}^s_r$ be of the form: $A=A^1_\nu\cap A^2_\nu$, where

$$A^1_\nu=\{x_{s_1}\in F_1, \cdots, x_{s_\nu}\in F_\nu\},\ A^2_\nu=\{x_{s_{\nu+1}}\in F_{\nu+1}, \cdots, x_{s_n}\in F_n\},\ s\leqq s_1<\cdots<s_n\leqq r.$$

Then we have,

$$\begin{aligned}\int_{(r>T)} Q''_w[I_A M^s_t]Q' &= \sum_\nu \int_{(s_\nu\leqq T<s_{\nu+1})} Q''_w[I_A M^s_t]Q' \\ &= \sum_\nu \int_{(s_\nu\leqq T<s_{\nu+1})\cap A^1_\nu} M^s_T Q''_w[M^T_t I_{A^2_\nu}]Q' \\ &= \sum_\nu \int_{(s_\nu\leqq T<s_{\nu+1})\cap A^1_\nu} M^s_T Q''_w[M^T_r I_{A^2_\nu}]Q' \\ &= \sum_\nu \int_{(s_\nu\leqq T<s_{\nu+1})} Q''_w[I_A M^s_r]Q' = \int_{(T<r)} Q''_w[I_A M^s_r]Q'.\end{aligned}$$

on the other hand,

$$\begin{aligned}\int_{(r\leqq T)} Q''_w[I_A M^s_t]Q' &= \int_{(r\leqq T)\cap A} M^s_{T\wedge t}\, Q''_w[M^{T\wedge t}_t]Q' \\ &= \int_{(r\leqq T)\cap A} M^s_{T\wedge t}Q' = \int_{(r\leqq T)\cap A} M^s_t Q'.\end{aligned}$$

Combining these equalities, we have $E_Q[I_A M^s_t]=E_Q[I_A M^s_r]$. This equality holds good for each $A\in \boldsymbol{W}^s_r$, provided that $s\leqq r\leqq t$. This implies that $M^s_t$ is a square integrable martingale (for each $\theta\in R^d$) with respect to $(\boldsymbol{W}^s_t, Q)$. Thus $(x_t, Q; t\in[s, \infty))$ is a martingale solution of the $(a, b, S)$-stochastic equation. Q.E.D.

**Theorem 4.4.** *If* $(a, b, S)$ *satisfy Condition* (U), *then a martingale solution* $(x_t, Q_{s,x}; t\in[s, T])$ *of the* $(a, b, S)$*-stochastic equation starting from* $(s, x)$ *is uniquely determined for any* $(s, x)\in R_+\times R^d$. *It is all the same when* $T$ *is an s-stopping time with respect to to the family* $(\boldsymbol{W}^s_t)$.

Proof. For the simplicity let us suppose that $T$ is a constant.

1° Set $T_1=\inf\{t\in[s, T]; |x_t-x|>1/2R(x)\}$. Let $(x_t, Q'; t\in[s, T])$ be a martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$. Then $(x_t, Q'; t\in[s, T_1])$ is a martingale solution of the $({}^{R(x)}a, {}^{R(x)}b, {}^{R(x)}S)$-stochastic equation starting from $(s, x)$. By Lemma 4.1, there exists a martingale solution $(x_t, Q''_{s',x'}; t\in[s', \infty))$ of the $({}^{R(x)}a, {}^{R(x)}b, {}^{R(x)}S)$-stochastic equation starting from $(s', x')$. Since the probability measure $Q''_{s',x'}$ is Borel measurable in $(s'\ x')$, $Q''_w=Q''_{T_1(w), x_{T_1(w)}}$ satisfies the conditions of Lemma 4.3. And so, there exists a martingale solution $(x_t, Q; t\in[s, \infty))$ of the $({}^{R(x)}a, {}^{R(x)}b, {}^{R(x)}S)$-stochastic equation starting from $(s, x)$ such that $Q=Q'$ on $\boldsymbol{W}^s_{T_1}$ and $Q[\cdot\mid \boldsymbol{W}^s_{T_1}]=Q''_w[\cdot]$ on $\boldsymbol{W}^{T_1}$. By Theorem 4.2, a martingale solution $(x_t, Q)$ is uniquely determined. Thus the

restricted measure $Q'|W^s_{T_1}$ is uniquely deternined.
2° Let us define a non-decreasing sequence of $s$-stopping times by

$$T_{n+1} = \inf\{t\in(T_n, T]; |x_t - x_{T_n}| > 1/2\, R(x_{T_n})\}.$$

Then $\lim_{n\to\infty} Q'[T_n < T]=0$ holds, for the process $(x_t, Q', t\in[s, T])$ is right continous, stochastically bounded and $1/R(y)$ is locally bounded. By applying the same method used in 1° for the $({}^{R(x_{T_1})}a, {}^{R(x_{T_1})}b, {}^{R(x_{T_1})}S)$-stochastic equation, it can be shown that the restricted measure $Q'|W^s_{T_2}$ is uniquely determined. Inductively, we conclude that $Q'|W^s_{T_n}$ is uniquely determined for each $n$. Q.E.D.

## 5. Existence of martingale solutions

Let T be a positive constant. We shall introduce a new condition.

**Condition (B)**

1) There exists a constant $K$ such that $S(s, x, \{|u|>1\})\leqq K$ for all $(s, x)\in [0.T]\times R^d$
2) There exists a $C^1(R_+)$-function $k(\xi)$ satisfying the following conditions.
   a) $k(0)>0$, $k'(\xi)\geqq 0$ and $k(\xi)$ is a concave function.
   b) $\int_1^\infty \frac{d\xi}{\xi k(\xi)} = \infty$.
   c) $|(x, b(s, x))| + \text{trace } a(s, x) + \int_{|u|\leqq 1} |u|^2 S(s, x, du) \leqq k(|x|)(1+|x|^2)$ for all $(s, x)\in[0, T]\times R^d$.
3) For each bounded domain $D\subset R^d$, there exists a measure $\bar{S}(du)$ such that $\int |u|^2\wedge 1\, \bar{S}(du)<\infty$ and $S(s, x, du)\leqq \bar{S}(du)$ for $(s, x)\in[0, T]\times D$.

(The function $\log(e+\xi)$ is an example satisfying a) and b) of Condition (B)—2).)

Let $\rho(\xi)$ be the function defined in §3—III. Set

$${}^N a(s, x) = a(s\wedge T, 0) + (a(s\wedge T, x) - a(s\wedge T, 0))\rho(|x|/N),$$

$${}^N b(s, x) = b(s\wedge T, x)\rho(|x|/N) \text{ and } {}^N S(s, x, du) = S(s\wedge T, x, du)\rho(|x|/N).$$

**Lemma 5.1.** *Let $(a, b, S)$ satisfy Condition* (B), *and let $(x_t, {}^N Q; t\in[s, T])$ be a martingale solution of the $({}^N a, {}^N b, {}^N S)$-stochastic equation starting from $(s, x)$. Then*

1) $\lim_{l\to\infty} \sup_N {}^N Q[\sup_t |x_t| > l]=0$, *and*
2) $\lim_{\delta\to 0} \sup_N \sup_{|t-t'|\leqq\delta} {}^N Q[|x_t - x_{t'}|>\varepsilon]=0$ *for each* $\varepsilon>0$.

Proof. 1° Set $T_0=s$, $T_{n+1}=\inf\{t\in(T_n, T]; |\Delta x_t|>1\}$. Since

$$^NQ[T_n<T]={}^NQ[\int_s^T\int_{|u|>1}J(dt, du)>n]\leqq\frac{(T-s)K}{n}$$

where $J$ is the measure of jumps of $x_t$ (*see* §2), we have $\lim_{n\to\infty}\sup_N {}^NQ[T_n<T]=0$. If the property $\lim_{l\to\infty}\sup_N {}^NQ[\sup_t|x_{t\wedge T_n}|>l]=0$ implies the property $\lim_{l\to\infty}\sup_N {}^NQ[\sup_t|x_{t\wedge T_{n+1}}|>l]=0$, then assertion 1) of the lemma holds.

2° There exists a non-negative function $h(\xi)\in C^2(R_+)$ such that $h'(+0)=h''(+0)=0$ and

$$h(\xi)=\int_0^\xi\frac{\eta d\eta}{(1+\eta^2)k(\eta)}\quad\text{on }[1,\infty).$$

Let $^NL$ be an operator defined by

$$^NLv(s, x)=\frac{1}{2}\sum_{ij}{}^Na_{ij}(s, x)\mathrm{D}_i\mathrm{D}_jv(s, x)+\sum_i{}^Nb_i(s, x)D_iv(s, x)$$
$$+\int_{|u|\leqq1}\{v(s, x+u)-v(s, x)-(u, \nabla v(s, x))\}{}^NS(s, x, du).$$

An elementary computation shows that there exist constants $H$ and $H_1$ such that

i) $|{}^NLh(|x|)|\leqq H$ and

ii) $h'(|x|)^2\frac{(x, {}^Na(s, x)x)}{|x|^2}+\int_{|u|\leqq1}(h(|x+u|)-h(|x|))^2\,{}^NS(s, x, du)\leqq H_1$

for each $(s, x)\in[0, T]\times R^d$ and $N$.

3° Let us denote $(t\vee T_n)\wedge T_{n+1}$ by $t_n$. Let us introduce a new process $y_t$:

$$y_t=y_{T_n}-\int_{T_n}^{t_n}\int_{|u|>1}J(d\tau, du).$$

Then the process $y_t$ has the Meyer decomposition (with respect to the measure $^NQ$

$$y_t=x_{T_n}+\int_{T_n}^{t_n}dM_\tau^N+\int_{T_n}^{t_n}{}^Nb(\tau, x)d\tau+\int_{T_n}^{t_n}\int_{|u|\leqq1}u^cJ^N(d\tau, du),$$

where

$$M_t^{N,i}\in\boldsymbol{M}_{loc}^c(\boldsymbol{W}_t^s, {}^NQ),\ M_t^{N,i}M_t^{N,j}-\int_s^t{}^Na_{ij}(\tau, x_\tau)d\tau\in\boldsymbol{M}_{loc}^c(\boldsymbol{W}_t^s, {}^NQ)\text{ and}$$
$$^cJ^N(dt, du)=J(dt, du)-{}^NS(t, x_t, du)dt.$$

From inequality i) and the Kunita-Watanabe formula,we have

$$h(|y_t|)\leqq h(|x_{T_n}|)+\int_{T_n}^{t_n} h'(|x_\tau|)\frac{(x_\tau, dM_\tau^N)}{|x_\tau|}$$
$$+\int_{T_n}^{t_n}\int_{|u|\leqq 1} h(|x_t+u|)-h|x_\tau|))^c J^N(d\tau, du)+H(t_n-T_n).$$

Let us denote the right-hand side of the inequality by $z_t$, then the process $z_t$ is a positive submartingale with respect to the measure ${}^NQ$. By the martingale inequality,

$$ {}^NQ[\sup_t |y_t|>l\,|\,W_{T_n}^s] = {}^NQ[\sup_t h(|y_t|)\geqq h(l)\,|\,W_{T_n}^s]$$
$$\leqq \frac{4}{h(l)^2}(2h(|x_{T_n}|)^2+3H^2T^2+2H_1T).$$

Since $h(\infty)=\infty$, the right-hand side of the above inequality tends to 0 as $l\to\infty$.

On the other hand, $\sup_t |x_{t_n}|\leqq\sup_t |y_t|+|\Delta x_{T_{n+1}}|$, and

$$ {}^NQ[|\Delta x_{T_{n+1}}|>l\,|\,W_{T_n}^s] = {}^NE[\int_{T_n}^{T_{n+1}}\int_{|u|>l} {}^NS(t, y_t, du)dt\,|\,W_{T_n}^s]$$
$$\leqq T \sup_{0\leqq s\leqq T,|x|\leqq\sup_t|y_t|}\int_{|u|>l} {}^NS(s, x, du).$$

By Condition (B)—3), it hods that

$$\lim_{l\to\infty}\sup_N \sup_{(s,x)\in[0,T]\times D}\int_{|u|>l} {}^NS(s, x, du)=0 \text{ for each bounded domain } D\subset R^d.$$

Thus, if $\lim_{l\to\infty}\sup_N {}^NQ[|x_{T_n}|>l]=0$, then $\lim_{l\to\infty}\sup_N {}^NQ[\sup_t|y_t|>l]=0$; and if $\lim_{l\to\infty}\sup_N {}^NQ[\sup_t|y_t|>l]=0$, then $\lim_{l\to\infty}\sup_N {}^NQ[\sup_t|x_{t_n}|>l]=0$. Thus assertion 1) is verified.

4° Set $U_0=s$ and $U_{n+1}=\inf\{t\in(U_n, T];|x_t|>n\}$. Then $\lim_{n\to\infty}\sup_N {}^NQ[U_n<T]=0$.

The process $(x_t, {}^NQ)$ has the Meyer decomposition:

$$x_t-x=\int_s^t dM_\tau^N+\int_s^t {}^Nb(\tau, x_\tau)d\tau+\int_s^t\int_{|u|\leqq 1} u\,{}^cJ^N(d\tau, du)+\int_s^t\int_{|u|>1} u\,J(d\tau, du).$$

Therefore, for each $\varepsilon>0$ and $s\leqq t<t'\leqq T$,

$$ {}^NQ[|x_{t'}-x_t|>\varepsilon]\leqq {}^NQ[|\int_{t\wedge U_n}^{t'\wedge U_n}\{dM_\tau^N+{}^Nb(\tau, x_\tau)d\tau+\int_{|u|\leqq 1} u\,{}^cJ^N(d\tau, du)\}|>\varepsilon]$$
$$+{}^NQ[\int_t^{t'}\int_{|u|>1} J(d\tau, du)\neq 0]+{}^NQ[U_n<T].$$

The second term of the right-hand side of this inequality tends to 0 uniformly in $N$ as $|t-t'|\downarrow 0$ because of the following inequality

$$^{N}Q[\int_{t}^{t'}\int_{|u|>1}u\,J(d\tau, du)\neq 0]\leqq {}^{N}E[\int_{t}^{t'}\int_{|u|>1}J(d\tau, du)]\leqq K(t'-t).$$

By Condition (B)—2), the first term also tends to 0 uniformly in $N$ as $|t-t'|$ $\downarrow 0$. Consequently, the second assertion of this lemma holds good. Q.E.D.

**Theorem 5.2.** *If* $(a, b, S)$ *satisfies Condition* (U) *and* (B), *then there exists (uniquely) a martingale solution* $(x_t, Q_{s,x}\,;\, t\in[s, T])$ *of the* $(a, b, S)$*-stochastic equation starting form* $(s, x)$. *And the solution is a strong Markov process.*

Proof. By Theorem 4.2 and Lemma 4.3 (see also the proof of Theorem 4.4), it is possible to construct (uniquely) a martingale solution $(x_t, {}^{N}Q_{s,x}; t\in[s, T])$ of the $({}^{N}a, {}^{N}b, {}^{N}S)$-stochastic equation starting form $(s, x)$. Let $T_0=s$, $T_{N+1}=\operatorname{int}\{t\in(T_N, T]; |x_t|>N/2\}$. By the uniqueness of the martingale solution of the $({}^{N}a, {}^{N}b, {}^{N}S)$-stochatic equation, we have ${}^{N}Q_{s,x}={}^{N+1}Q_{s,x}$ on the $\sigma$-field $W^{s}_{T_N}$. Thus there exists a probability measure $Q_{s,x}$ on the $\sigma$-field $W^{s}_{\vee T_N}$ such that $Q_{s,x}={}^{N}Q_{s,x}$ on the $\sigma$-field $W^{s}_{T_N}$. The process $(x_t, Q_{s,x}; t\in[s, \vee T_N])$ is a martingale solution of the $(a, b, S)$-stochastic equation starting from $(s, x)$. Since, by Lemma 5.1,

$$\lim_{N\to\infty} Q_{s,x}[T_N<T]=\lim_{N\to\infty}\sup_{M\geqq N}{}^{M}Q_{s,x}[T_N<T]\leqq\lim_{N\to\infty}\sup_{M}\ {}^{M}Q_{s,x}[\sup_{t}|x_t|>N/2]=0,$$

the process $(x_t, Q_{s,x})$ is a martingale solution of the $(a, b, S)$-stochatsic equation on the time interval $[s, T]$. It is easy to show that this process is a strong Markov process. Q.E.D.

**Condition** (C)

1) $\lim_{x'\to x}\sup_{0\leqq s\leqq T}\sum_{ij}|a_{ij}(s, x')-a_{ij}(s, x)|=0$ for each $x\in R^d$.

2) b$(s, x)$ is locally bounded, and

$$\lim_{x'\to x}\{\sum_{i}|b_i(s, x')-b_i(s, x)|+\int|u|^2\wedge 1\,|S(s, x', du)-S(s, x, du)|\}=0$$

for all $(s, x)\in[0, T]\times R^d$

**Lemma 5.3,** *If Condition* (B) *and* (C) *are satisfied, then there exists a martingale solution* $(x_t, Q_{s,x}; t\in[s, T])$ *of the* $({}^{N}a, {}^{N}b, {}^{N}S)$*-stochastic equation starting from* $(s, x)$.

Proof. We shall omit the super-prefix $N$ of ${}^{N}a$, ${}^{N}b$, and ${}^{N}S$ in this proof.

1° If $a^{(m)}=a+1/m\cdot I$, then there exists a martingale solution $(x_t, Q^{(m)})$ of the $(a^{(m)}, b, S)$-stochastic equation starting from $(s, x)$, by Theorem 5.2. Similarly to the proof of Lemma 5.1, we can prove that

1) $\lim_{l\to\infty}\sup_m Q^{(m)}[\sup_t |x_t|>l]=0$, and

2) $\lim_{\delta\downarrow 0}\sup_m \sup_{|t'-t|\leqq\delta} Q^{(m)}[|x_{t'},-x_t|>\varepsilon]=0$ for any $\varepsilon>0$.

By Lemma 1.1, it is possible to extract a subsequence $Q^n=Q^{(m_n)}$ from the sequence $Q^{(m)}$ and it is possible to construct a sequence $(X_t, X_t^n: n=1, 2, \cdots)$ of processes on a certain probability space $(\Omega, \boldsymbol{F}_t, P)$ such that the processes $(x_t, Q^n)$ and $(X_t^n, P)$ are equivalent for each $n$, and the random sequence $X_t^n$ converges in probability to $X_t$ for each $t\in[s, T]$. Let $(x_t, Q_{s,x})$ be the process on the base space $(W^s, \boldsymbol{W}^s, \boldsymbol{W}_t^s)$ equivalent to the process $(X_t, P)$.

2° Let $\Phi_\theta(s, x)$ be Ito's differential associated with $(a, b, S)$ (defined in the paragraph under the first definition of §2). Then

$$E_P[\{\exp[i(\theta,X_t^n-X_r^n)-\int_r^t\Phi_\theta(\tau, X_\tau^n)d\tau-\frac{1}{m_n}|\theta|^2(t-r)]-1\}f_1(X_{s_1}^n)\cdots f_k(X_{s_k}^n)]=0.$$

for any $s\leqq s_1\leqq\cdots\leqq s_k\leqq r<t$, and for any $f_1, \cdots, f_k\in C^{0,b}(R^d)$. Therefore

$$\begin{aligned}&E_P[\{\exp[i(\theta, X_t-X_r)-\int_r^t\Phi_\theta(\tau, X_\tau)d\tau]-1\}f_1(X_{s_1})\cdots f_k(X_{s_k})]\\ &\leqq c\varlimsup_{n\to\infty} E_P\Big[\int_r^t|\Phi_\theta(\tau, X_\tau^n)-\Phi_\theta(\tau, X_\tau)|d\tau\Big]\end{aligned}$$

where $c$ is a constant depending only on $||f_j||$ $(j=1, \cdots, k)$, $t-r$ and $||\Phi_\theta||$.

By Condition (C), $\lim_{n\to\infty} E_P[|\Phi_\theta(\tau, X_\tau^n)-\Phi_\theta(\tau, X_\tau)|]=0$ for each $\tau\in[r, t]$.Thus,

$$\lim_{n\to\infty}\int_r^t E_P[|\Phi_\theta(\tau, X_\tau^n)-\Phi_\theta(\tau, X_\tau)|]d\tau=0.$$

Consequently, for any $s\leqq r<t$ and $\theta\in R^d$,

$$E_P[\exp(i(\theta, X_t-X_r)-\int_r^t\Phi_\theta(\tau, X_\tau)d\tau)|\boldsymbol{F}_r]=1.$$

This limplies that the process $(X_t, P)$ or $(x_t, Q_{s,x})$ is a martingale solution of the $(a, b, S)$-stochastic equation starting form $(s, x)$. Q.E.D.

**Theorem 5.4** *If Condition* (B) *and* (C) *are satisfied, then there exists a martingale solution* $(x_t, Q_{s,x}; t\in[s, T])$ *of the* $(a, b, S)$*-stochastic equation starting from* $(s, x)$.

Proof. By Lemma 5.3, we can construct a martingale solution $(x_t, {}^NQ_{s'x'}$; $t\in[s', T])$ of the $({}^Na, {}^Nb, {}^NS)$-stochastic equation starting form $(s'\, x')$ for each $(s'\, x')\in[s, T]\times R^d$. Set $T_0=s$ and $T_{N+1}=\inf\{t\in(T_N, T]; |x_t|>N/2\}$. By Lemma 4.3, there exists a probability measure ${}^2Q'_{s,x}$ on the $\sigma$-field $\boldsymbol{W}_{T_2}^s$ such that ${}^2Q'_{s,x}={}^1Q_{s,x}$ on the $\sigma$-field $\boldsymbol{W}_{T_1}^s$ and ${}^2Q'_{s,x}[\cdot|\boldsymbol{W}_{T_1}^s]={}^2Q_{T_1,x_{T_1}}[\cdot]$ on the

$\sigma$-field $W^{T_1}_{T_2}$. Inductively, we can construct a sequence ${}^N Q'_{s,x}$ of probability measures on the $\sigma$-fields $W^s_{T_N}$ such that ${}^{N+1}Q'_{s,x}={}^N Q'_{s,x}$ on the $\sigma$-field $W^s_{T_N}$ and ${}^{N+1}Q'_{s,x}[\cdot \mid W^s_{T_N}]={}^{N+1}Q_{T_N,x_{T_N}}[\cdot]$ on the $\sigma$-field $W^{T_N}_{T_{N+1}}$. Let $Q_{s,x}$ be the probability measure on the $\sigma$-field $W^s_{\vee T_N}$ such that $Q_{s,x}={}^N Q'_{s,x}$ on the $\sigma$-field $W^s_{T_N}$. The method used in the proof of Theorem 5.2. yields us the fact that the process $(x_t, Q)_{s,x}$ is a martingale solution of the $(a, b, S)$-stocahstic equation starting form $(s,x)$ Q.E.D.

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