Markov traces and knot invariants related to Iwahori-Hecke algebras of type B

Meinolf Geck and Sofia Lambropoulou

1 Introductory remarks on knots, braids and trace functions

In classical knot theory we study knots inside the 3-sphere modulo isotopy. Using the Alexander and Markov theorem, we can translate this into a purely algebraic setting in terms of Artin braid groups modulo an equivalence relation generated by 'Markov moves' (one of which is usual conjugation inside the braid group). V.F.R. Jones [5] used this fact in 1984 for constructing a new knot invariant through trace functions on the associated Iwahori-Hecke algebras of type A with suitable properties that reflect the above Markov moves.

Jones's work led to questions of developing knot theory corresponding to other types of Coxeter groups. It is proved in [6] that there exist braid structures related to arbitrary 3-manifolds, which in addition satisfy appropriate Markov-isotopy equivalence; also that, if the 3-manifold is a solid torus, then the sets of related braids form groups, which are in fact the Artin-Tits braid groups related to the B-type Coxeter groups. These results together with a linear trace that we found in 1991 are used in [7] for constructing a 4-variable analogue of the homfly-pt (2-variable Jones) polynomial for oriented knots inside a solid torus. We proved the existence of this trace (see [7]) by following and adapting to the B-type case Jones's proof of the existence of Ocneanu's trace in [5], Theorem 5.1.

The aim of this paper is to give a full classification of *all* linear traces on Iwahori-Hecke algebras of type B which support the *Markov property*, see Definition 4.1 and Theorem 4.3. This uses in an essential way the results in [3] about trace functions on arbitrary Iwahori-Hecke algebras associated with finite Coxeter groups. This method yields an alternative proof of the above special trace (cf. also [3], (4.2), where an alternative proof for Ocneanu's original trace is given.)

In Section 5 we discuss the knot theory of a solid torus and we explain why the related braid groups are in fact the Artin-Tits groups of B-type. We also give the Markov equivalence of these braids, so that the equivalence classes correspond bijectively to isotopy classes of knots in the solid torus (detailed account and proofs of these results can be found in [6] or [7]). The Markov equivalence is in terms of two isotopy moves which are reflected precisely in the

(B_n) 

$n \geq 1$

Figure 1:

definition of the Markov property for our traces. Then we normalize properly the constructed traces in order to obtain *all* homfly-pt analogues related to the B-type Iwahori-Hecke algebras, for oriented knots inside the solid torus. Finally, we give the skein rules and initial conditions that characterize these invariants diagramatically.

In [2] the first author uses the results of this paper to provide a full classification of Markov traces for Iwahori-Hecke algebras of type D (see 4.7 for precise statement of the main result). Moreover, in a further 'vertical' development (cf. [8]) the second author considered *all* Hecke-type quotients of the Artin-Tits braid group of B-type and constructed Markov traces and knot invariants on all levels, the basic level being the Iwahori-Hecke algebras of B-type and the results in [7].

We shall now explain in more detail our results. Let us consider the following Dynkin diagram.
The symbols $t, s_1, \ldots, s_{n-1}$ labelling the nodes form generators for the corresponding Artin-Tits braid group $\tilde{W}_n$ and the finite Coxeter group W_n of type B_n. The braid group $\tilde{W}_n$ has the defining relations

$$\begin{array}{rcll} s_1ts_1t & = & ts_1ts_1 & \\ ts_i & = & s_it & \text{if} \quad i>1 \\ s_is_j & = & s_js_i & \text{if} \quad |i-j|>1 \\ s_is_{i+1}s_i & = & s_{i+1}s_is_{i+1} & \text{if} \quad 1 \leq i \leq n-2 \end{array}$$

Relations of these types will be called *braid relations.* As proved in [6] the elements of $\tilde{W}_n$ can be represented geometrically by braids in S^3 on $n+1$ strands in which the first strand remains pointwise fixed. When we refer to this geometric interpretation of $\tilde{W}_n$ we shall denote it by $B_{1,n}$. Below we illustrate the generators s_i, t and the element $t_i' = s_i \cdots s_1 t s_1^{-1} \cdots s_i^{-1}$ in $B_{1,n}$, and also an example of an element in $B_{1,5}$.

If in addition to the braid relations we impose the quadratic relations that each generator has order 2, then we obtain the finite factor group W_n.

The corresponding Iwahori-Hecke algebra H_n is obtained as a quotient of the group algebra of $\tilde{W}_n$ by factoring out the quadratic relations

$$t^2 = (Q-1)t + Q \cdot 1 \quad \text{and} \quad g_i^2 = (q-1)g_i + q \cdot 1 \text{ for all } i,$$

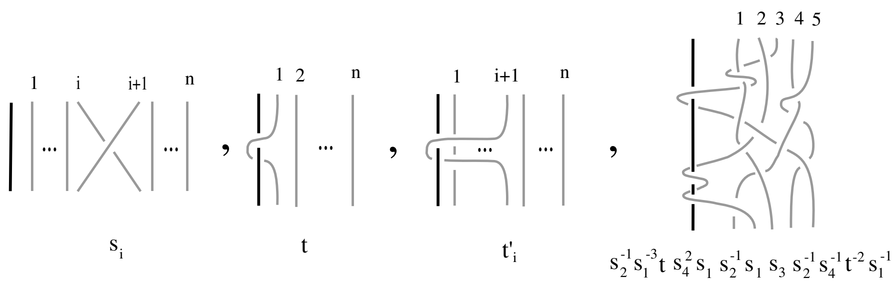

Figure 2:

where we denote the image of s_i in H_n simply by g_i, and where q, Q are fixed parameters from the ground ring. The algebra H_n is finite-dimensional, with a basis $\{g_w\}$ labelled by the elements of W_n. Now the idea is to construct invariants of knots in the solid torus using trace functions on $\tilde{W}_n$ which factor through H_n and which respect the braid equivalence on $\tilde{W}_n$. The latter is generated by the following two moves (cf. Theorem 5.2).

(i) Conjugation: if $\alpha, \beta \in \tilde{W}_n$ then $\alpha \sim \beta^{-1}\alpha\beta$.

(ii) Markov moves: if $\alpha \in \tilde{W}_n$ then $\alpha \sim \alpha {s_n}^{\pm 1} \in \tilde{W}_{n+1}$.

Thus, the problem is reduced to studying trace functions τ on $H := \bigcup_{n=1}^{\infty} H_n$ which satisfy the rule $\tau(hg_n) = z\tau(h)$, where z is a fixed parameter in the ground ring over which the algebra H is defined, and $h \in H_n$. (Note that such an h is a linear combination of basis elements g_w which do not involve the generator g_n.) This rule is what we call 'the Markov property' for trace functions on H.

A general scheme for constructing trace functions on H_n (in fact, for Iwahori-Hecke algebras of any given type) has been developed in [3]. Firstly, it is known that any trace is determined by its values on basis elements corresponding to a set of representatives of the conjugacy classes of W_n. However, it is not true in general that basis elements corresponding to conjugate group elements are also conjugate in the algebra, and so, to compute the trace of an arbitrary element is no more a trivial task. In [3], there is an explicit algorithm for computing the value of a given trace on an arbitrary basis element g_w from the values on basis elements corresponding to elements of *minimal length* in the various conjugacy classes.

In W_n, let $t_i := s_i \cdots s_1 t s_1 \cdots s_i$ and let a positive, respectively negative, block be an element of the form

$$s_{i+1} s_{i+2} \cdots s_{i+m} \quad \text{respectively} \quad t_i s_{i+1} \cdots s_{i+m}, \quad \text{where } m, i \geq 0.$$

Then an element of minimal length in a conjugacy class is a product of negative blocks ordered by increasing length, followed by various positive blocks also in increasing length (for example, $tt_1s_2t_3s_4s_6s_8s_9s_{10} \in W_{11}$). The main idea of this paper is that, in order to determine the Markov traces on H, we lift the elements t_i in W_n to the elements $t'_i := g_i \cdots g_1 t g_1^{-1} \cdots g_i^{-1}$ in H_n (instead of the obvious lifting to $g_i \cdots g_1 t g_1 \cdots g_i$). This will enable us to parametrize in Section 4 all possible Markov traces with parameter z by the initial conditions

$$\tau(t'_0 t'_1 t'_2 \cdots t'_{k-1}) = y_k \quad \text{for all } k \geq 1,$$

where $y_1, y_2 \ldots$ are arbitrary elements from the ground ring. In particular, if $h = d_1 \cdots d_n \in H$ with $d_i \in \{1, g_{i-1}, t'_{i-1}\}$, a lifting of a minimal length representative, we define a trace function τ on H by the rule $\tau(h) = z^{a(h)} y_{b(h)}$, where $a(h)$ is the number of d_i which are in the set $\{g_1, \ldots, g_{n-1}\}$ and $b(h)$ is the number of d_i which are of the form t'_i (cf. Theorem 4.3). From our definition it is clear that τ satisfies the Markov property for these elements, and the problem is then to show that this property holds on all elements of H. This will require some technical preliminaries which are provided in Sections 2 and 3. The final proof will then be given in Section 4.

2 Computations in the braid group of type B_n

It is the purpose of this section to reformulate some of the results in [3] on elements in signed block form in terms of the braid group. These will then carry through to the Iwahori-Hecke algebra level and will be used in the existence and uniqueness proof for the analogues of Ocneanu's trace for type B.

2.1. Let W_n and $\tilde{W}_n$ as in Section 1. For each element w in W_n (or in $\tilde{W}_n$) we define the length, $l(w)$, to be the smallest non-negative integer k such that w can be written as a product of k generators (or their inverses). Such an expression of w of minimal possible length will be called a reduced expression for w. (See [1], Chap. IV, §1.1.) The exchange condition for Coxeter groups implies that, if we are given two reduced expressions of an element in W_n as products in the generators $t, s_1, \ldots, s_{n-1}$ then the corresponding expressions in the braid group $\tilde{W}_n$ are also equal (see [1], Chap. IV, §1, Proposition 5).

By convention, we let $W_0 = \{1\}$. Then, for all $n \geq 1$, the group W_{n-1} is a parabolic subgroup of W_n obtained by removing the node with label s_{n-1} (cf. [1], Chap. IV, §1.8). Hence, we have natural embeddings $W_0 \subset W_1 \subset W_2 \subset \ldots$ and we let $W := \bigcup_n W_n$. We define

$$t_i := s_i s_{i-1} \cdots s_1 t s_1 \cdots s_{i-1} s_i \in W_n \quad \text{for all } 0 \leq i \leq n-1, \text{ where } t_0 := t.$$

Then the set of distinguished right coset representatives of W_{n-1} in W_n is given as follows.

$$\begin{array}{rcl} \mathcal{R}_n := \{ & 1, \quad t_{n-1}, & \\ & s_{n-1}s_{n-2}\cdots s_{n-k} & (1 \leq k \leq n-1), \\ & s_{n-1}s_{n-2}\cdots s_{n-k}t_{n-k-1} & (1 \leq k \leq n-1)\} \end{array}$$

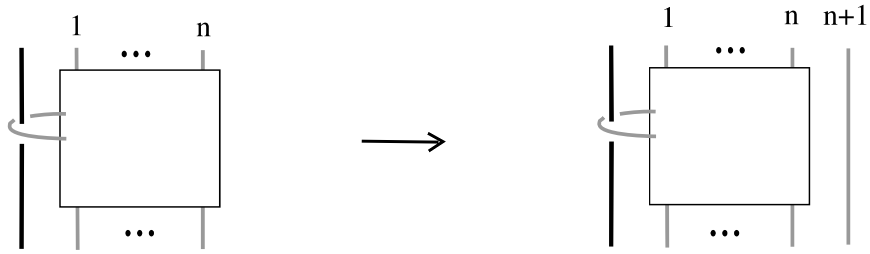

Figure 3:

(Note that $\mathcal{R}_1 = \{1, t\}$.) Then each element $w \in W_n$ can be written uniquely in the form $w = r_1 \cdots r_n$ with $r_i \in \mathcal{R}_i$. Such an expression of w is reduced, that is, we have $l(w) = l(r_1) + \ldots + l(r_n)$.

Finally, let $\mathcal{D}_n := \{1, s_{n-1}, t_{n-1}\} \subseteq \mathcal{R}_n$. Then $\mathcal{D}_n$ is the set of distinguished double coset representatives of W_{n-1} in W_n (see [1], Chap. IV, §1, Ex. 3).

Each $r \in \mathcal{R}_n$ can now be written uniquely in the form $r = dr'$ where $d \in \mathcal{D}_n$ and $r' = 1$ or $r' = s_{n-2} \cdots s_{n-k}$ or $r' = s_{n-2} \cdots s_{n-k} t_{n-k-1}$. In particular, we have $r' \in \mathcal{R}_{n-1}$.

2.2. We shall now lift these elements to the braid group $\tilde{W}_n$. First note that we also have natural embeddings $\tilde{W}_0 \subset \tilde{W}_1 \subset \tilde{W}_2 \subset \ldots$ and we let $\tilde{W} := \bigcup_n \tilde{W}_n$. Geometrically the embedding of $B_{1,n}$ into $B_{1,n+1}$ is described by the following picture.

We would like to define the analogue of t_i to be a conjugate of t where the conjugating element is of the form $s_1^{\pm 1} \cdots s_i^{\pm 1}$. In accordance with the geometric considerations in Section 1, we choose the exponents ± 1 so as to obtain the element t_i' already encountered above:

$$t_i' := s_i s_{i-1} \cdots s_1 t s_1^{-1} \cdots s_{i-1}^{-1} s_i^{-1} \in \tilde{W}_n \quad \text{for all } 0 \le i \le n-1.$$

(All inverses on the right hand side of t.) Then each t_i' maps to t_i under the canonical surjection $\tilde{W}_n \to W_n$. We let $\mathcal{D}_n' \subseteq \mathcal{R}_n' \subseteq \tilde{W}_n$ be the analogous sets as above, where each t_i is replaced by t_i'.

For any i, j the elements t_j and t_i commute with each other (cf. [3], (2.3)). For t_j' and t_i', this will only be true up to possibly changing some inverses in the definition of these elements. We will not completely formalize this, but only give the following additional definition. For any $j \in \{0, \ldots, i\}$ we denote

$$\begin{aligned} t_{i,j}' &:= s_i^{\pm 1} \cdots s_{j+1}^{\pm 1} s_j s_{j-1} \cdots s_1 t s_1^{-1} \cdots s_{j-1}^{-1} s_j^{-1} s_{j+1}^{\mp 1} \cdots s_i^{\mp 1} \\ &= s_i^{\pm 1} \cdots s_{j+1}^{\pm 1} t_j' s_{j+1}^{\mp 1} \cdots s_i^{\mp 1}. \end{aligned}$$

(That is, all inverses up to index j are in the right position, and for each bigger index, the inverse may be put either on the right or on the left hand side.)

Finally, we let $\mathcal{D}'_{n,j}$ be the set consisting of 1, s_{n-1} and all possible elements of the form $t'_{n-1,j}$. Similarly, we define $\mathcal{R}'_{n,j}$. As a convention, we will usually denote the elements in $\mathcal{D}'_{n,0}$ by the symbol d^*_n.

The next result will show that, in particular, a product of the form $t'_i t'_j$ with $i < j$ can be written as $t'_{j,i} t'_j$. Relations of this kind will be used frequently in the sequel.

Lemma 1. *The following relations hold in* $\tilde{W}_n$.

(a) $s_i t'_m = t'_m s_i$ *and* $s_i^{-1} t'_m = t'_m s_i^{-1}$ *for all* $i < m$ *and* $i > m+1$.

(b) $t'_i t'_m = s_m \cdots s_{i+2} s_{i+1}^{-1} s_i \cdots s_1 t s_1^{-1} \cdots s_i^{-1} s_{i+1} s_{i+2}^{-1} \cdots s_m^{-1} t'_i$ *for* $i < m$.
(The inverse at index $i+1$ *changes in* t'_m*)*

(c) $s_{m-1} \cdots s_{m-k} t'_{m-k-1} t'_m = t'_{m,m-1} s_{m-1} \cdots s_{m-k} t'_{m-k-1}$ *for* $0 \le k \le m-1$.
(The inverse at index m *changes in* t'_m*)*

Proof. The defining relations for $\tilde{W}_n$ imply that

$$\begin{aligned} t s_i^{-1} &= s_i^{-1} t \quad &&\text{if} \quad i > 1 \\ s_i s_j^{-1} &= s_j^{-1} s_i \quad &&\text{if} \quad |i-j| > 1 \\ s_i s_{i+1} s_i^{-1} &= s_{i+1}^{-1} s_i s_{i+1} \quad &&\text{if} \quad 1 \le i \le n-2 \\ s_1^{-1} t s_1 t &= t s_1 t s_1^{-1}. && \end{aligned}$$

The assertions of the Lemma now readily follow by straightforward computations. (Notice that these relations could be alternatively checked easily using the geometric interpretations given above.) □

2.4. Consider an element of the form $d_1 \cdots d_n \in \tilde{W}_n$ with $d_i \in \mathcal{D}'_i$ for all i. If we collect together non-trivial terms with consecutive indices we obtain a decomposition of this element as a product of signed blocks. More precisely, a positive respectively negative block of length $m+1 \ge 0$ in $\tilde{W}_n$ is an element of the form

$$s_{i+1} s_{i+2} \cdots s_{i+m} \qquad \text{respectively} \qquad t'_i s_{i+1} \cdots s_{i+m}$$

where $m, i \ge 0$. If we denote such an element by $b(i,m)$ then

$$d_1 \cdots d_n = b(i_1, m_1) b(i_2, m_2) b(i_3, m_3) \cdots, \qquad \text{where } i_2 > i_1 + m_1, i_3 > i_2 + m_2 \text{ etc.}$$

By [3], §2, each conjugacy class in the Coxeter group W_n contains an element in signed block form, and such an element is of miminal length in its class if and only if all negative blocks are in the beginning, ordered by increasing length. In order to reduce an arbitrary element in W_n to such a minimal form, it is necessary to interchange by conjugation two consecutive blocks in the signed block form of an element in W_n. Our aim here is to show that similar relations also hold in the braid group.

Lemma 2. (cf. [3], Proposition 2.4.)
Let $y = (s_{i+m+1} \cdots s_{i+1})(s_{i+m+2} \cdots s_{i+2}) \cdots (s_{i+m+k+1} \cdots s_{i+k+1})$ *for* $i, k, m \geq 0$. *Geometrically* y *is a half-twist of* $m+1$ *consecutive strands around the next* $k+1$ *consecutive strands in the classical braid group.*

(a) Let

$$\begin{aligned} w &= (s_{i+1}s_{i+2} \cdots s_{i+m})(s_{i+m+2}s_{i+m+3} \cdots s_{i+m+k+1}) \quad \textit{and} \\ v &= (s_{i+1}s_{i+2} \cdots s_{i+k})(s_{i+k+2}s_{i+k+3} \cdots s_{i+k+m+1}). \end{aligned}$$

Then $y^{-1}wy = v$ *in the braid group.*

(b) Let

$$\begin{aligned} w &= (s_{i+1}s_{i+2} \cdots s_{i+m})(t'_{i+m+1}s_{i+m+2}s_{i+m+3} \cdots s_{i+m+k+1}) \quad \textit{and} \\ v &= (t'_i s_{i+1}s_{i+2} \cdots s_{i+k})(s_{i+k+2}s_{i+k+3} \cdots s_{i+k+m+1}). \end{aligned}$$

Then $y^{-1}wy = v$ *in the braid group.*

(c) Let

$$w = (t'_i s_{i+1}s_{i+2} \cdots s_{i+m})(t'_{i+m+1}s_{i+m+2}s_{i+m+3} \cdots s_{i+m+k+1})$$

for some $m > k$. *Then* $y^{-1}wy = v$ *in the braid group where*

$$v = (t'_i s_{i+1}s_{i+2} \cdots s_{i+k})(t'_{i+k+1,i}s_{i+k+2}s_{i+k+3} \cdots s_{i+k+m+1})$$

for some $t'_{i+k+1,i}$ *(see the proof below).*

In (b), (c), the length of v *is strictly shorter than the length of* w.

Proof. Geometrically, (a), (b) and (c) follow immediately by looking at the corresponding braid pictures and comparing their closures; note that in each case we obtain links of two components. For an algebraic proof, we use the similar relations in [3], Proposition 2.4, on the level of W_n. We have to slightly modify those arguments in order to derive relations in $\tilde{W}_n$.

(a) In [3], Proposition 2.4(a), it is shown that the equation $wy = vy$ holds in W_n and that the expressions on both sides are reduced. Hence the left hand side can be transformed to the right hand side by a finite sequence of braid relations. It follows that the equation $wy = vy$ also holds in the braid group.

(b) We write $y = (s_{i+m+1} \cdots s_{i+1})y'$. Then

$$y^{-1}t'_{i+m+1}y = y^{-1}s_{i+m+1} \cdots s_{i+1}t'_i s_{i+1}^{-1} \cdots s_{i+m+1}^{-1}y = y'^{-1}t'_i y' = t'_i.$$

For the last equality, note that y' is a product of generators s_j with $j > i+1$ and that all of these commute with t'_i.

Now $(s_{i+1} \cdots s_{i+m})$ commutes with t'_{i+m+1}. So we can write $w = t'_{i+m+1}w_a$ and $v = t'_i v_a$ where w_a and v_a are as in part (a) of the Lemma. Hence we deduce, using (a), that

$$y^{-1}wy = (y^{-1}t'_{i+m+1}y)(y^{-1}w_a y) = t'_i v_a = v.$$

(c) We rearrange the given reduced expression for y as

$$y = (s_{i+m+1} \cdots s_{i+m+k+1})(s_{i+m} \cdots s_{i+m+k}) \cdots (s_{i+1} \cdots s_{i+k+1})$$

and write $y = y'(s_{i+1} \cdots s_{i+k+1})$. Now note that y' is a product of generators s_j with $j > i+1$ and that all of these commute with t'_i. Then we compute that

$$\begin{aligned} y^{-1}t'_i y &= (s_{i+k+1}^{-1} \cdots s_{i+1}^{-1}) {y'}^{-1} t'_i y' (s_{i+1} \cdots s_{i+k+1}) \\ &= s_{i+k+1}^{-1} \cdots s_{i+1}^{-1} t'_i s_{i+1} \cdots s_{i+k+1} \\ &=: t''_{i+k+1,i}. \end{aligned}$$

We can write $w = t'_i w_b$ where w_b is as in part (b). Thus, we deduce, using (b), that

$$y^{-1}wy = (y^{-1}t'_i y)(y^{-1}w_b y) = (t''_{i+k+1,i} t'_i s_{i+1} \cdots s_{i+k})(s_{i+k+2} \cdots s_{i+k+m+1}).$$

Using Lemma 2.3(b), we compute that $t''_{i+k+1,i} t'_i = t'_i t''_{i+k+1,i+1}$ where the inverses in $t''_{i+k+1,i+1}$ at all indices bigger than $i+1$ are on the left hand side of t. A final calculation then shows that

$$t''_{i+k+1,i+1}(s_{i+1} \cdots s_{i+k}) = (s_{i+1} \cdots s_{i+k})(s_{i+k+1} s_{i+k}^{-1} \cdots s_{i+1}^{-1} t'_i s_{i+1} \cdots s_{i+k} s_{i+k+1}^{-1}).$$

If we denote the second factor on the right hand by $t'_{i+k+1,i}$ then $y^{-1}wy = v$ as required. The proof is complete. □

Remarks 1. (a) One can also show that, if the conjugation in (b), (c) of the above Lemma is performed step by step (one generator of y at a time) then this sequence of conjugations can be arranged in such a way that the length of the elements does not increase at each step (cf. [3], Proposition 2.4(b,c)).

(b) Consider the elements

$$w = (s_1)(s_2^{-1} s_1 t s_1^{-1} s_2) \quad \text{and} \quad v = ts_2 \quad (\text{in } \tilde{W}_3).$$

If all the inverses in w were on the right hand side of t then this element would be conjugate to v in $\tilde{W}_3$, by part (b) of the above Lemma (with $m = 1$, $k = i = 0$). However, one can show that w and v are not conjugate in $\tilde{W}_n$, and not even in the associated Iwahori-Hecke algebra. This example indicates that the statements in the above Lemma are as strong as possible, and that we cannot distribute the inverses in some arbitrary order around t when we want to conjugate elements in block form.

Note, however, that the oriented links obtained by closing the braids corresponding to the above elements are isotopic (see (5.4) below). In particular, the knot invariants constructed in Definition 5.3 must have the same value on them.

3 Trace functions on the Iwahori-Hecke algebra of type B

In this section, we introduce the Iwahori-Hecke algebra H_n of type B_n as a quotient of the braid group algebra of $\tilde{W}_n$. We also show how the main results of [3] on determining trace functions on H_n can be adapted to our present situation where reduced expressions for representatives of minimal length in the conjugacy classes involve some inverses.

3.1. Let A be a commutative ring with 1 and $Q, q \in A$ two fixed invertible elements. Then H_n is an associative algebra over A. It can be described as a quotient of the group algebra of the braid group $\tilde{W}_n$ (over A) obtained by factoring by the ideal generated by all elements of the form $t^2 - (Q-1)t - Q \cdot 1$, $s_i^2 - (q-1)s_i - q \cdot 1$ for $i = 1, \ldots, n-1$. We denote the image of t under the canonical map $A\tilde{W}_n \to H_n$ again by t, and the image of s_i by g_i, for all i. Then the generators $t, g_1, \ldots, g_{n-1}$ of H_n satisfy braid relations completely analogous to the braid relations for the generators $t, s_1, \ldots, s_{n-1}$ of $\tilde{W}_n$. In addition, we have the following quadratic relations.

$$t^2 = (Q-1)t + Q \cdot 1 \quad \text{and} \quad g_i^2 = (q-1)g_i + q \cdot 1 \text{ for all } i.$$

Let $w \in W_n$ and assume that we are given a reduced expression for w as a product of generators $t, s_1, \ldots, s_{n-1}$. Then the corresponding element of H_n in terms of the generators $t, g_1, \ldots, g_{n-1}$ is independent of the chosen reduced expression for w. We may therefore denote this element in H_n unambiguously by g_w. It is known that the set of elements $\{g_w \mid w \in W_n\}$ forms an A-basis of H_n. (For all these facts see [1], Chap. IV, §2, Ex. 23.) We then also have the following relations.

$$g_w g_{w'} = g_{ww'} \quad \text{if } l(ww') = l(w) + l(w').$$

The embeddings $W_0 \subset W_1 \subset W_2 \subset \ldots$ of (2.1) induce corresponding embeddings of algebras $H_0 \subset H_1 \subset H_2 \subset \ldots$ and we shall denote

$$H := \bigcup_{n \geq 0} H_n.$$

3.2. The fact that q is invertible in A implies that the generators g_i are also invertible in H. In fact, we have that

$$g_i^{-1} = q^{-1} g_i + (q^{-1} - 1) \cdot 1 \in H_n.$$

Thus, the images of the elements $t_i', t_{i,j}' \in \tilde{W}_n$ under the map $A\tilde{W}_n \to H_n$ are well-defined elements in H_n, and we shall denote them by the same symbols. We also write

$$\mathcal{D}_n' = \{1, g_{n-1}, t_{n-1}'\}$$

and, similarly, for $\mathcal{R}'_n$, $\mathcal{D}'_{n,i}$ and $\mathcal{R}'_{n,i}$ (cf. (2.2)). With these conventions, all results about commutation and conjugation of the various special elements considered in the previous section carry over without change to H_n.

Let $w \in W_n$ and write $w = r_1 \cdots r_n$ with $r_i \in \mathcal{R}_i$ for all i. Since this expression is reduced we also have $g_w = g_{r_1} \cdots g_{r_n}$. For each r_i let $r'_i \in \mathcal{R}'_i$ be the corresponding element in H_n (where the s_j are replaced by g_j, and t_j by t'_j). Let $n_w \geq 0$ be the total number of inverses in the terms $r'_1, \ldots, r'_n$. Using the above inversion formula it then follows that

$$g_w = q^{n_w} r'_1 \cdots r'_n + A\text{-linear combination of elements } g_v \text{ with } l(v) < l(w).$$

One consequence of this is the fact that the elements $\{r'_1 \cdots r'_n \mid r'_i \in \mathcal{R}'_i\}$ form an A-basis of H_n, and if we order the elements of W_n by increasing length then the matrix performing the base change to the old basis $\{g_{r_1 \cdots r_n} \mid r_i \in \mathcal{R}_i\}$ is triangular with powers of q along the diagonal.

3.3. Let $\{C\}$ be the set of conjugacy classes of W_n and let w_C be an element of minimal length in C which admits a decomposition as a product of negative blocks (ordered by increasing length) followed by various positive blocks (see (2.4)). We can even fix a unique choice of w_C if we also require that the positive blocks have increasing length. We then define an element $g_C \in H_n$ by taking an expression $w_C = d_1 \cdots d_n$ (with $d_i \in \mathcal{D}_i$) and replacing each s_i by g_i and each t_i by t'_i. As in (3.2) we have (with $n_C := n_{w_C}$) that

$$g_{w_C} = q^{n_C} g_C + A\text{-linear combination of elements } g_w \text{ with } l(w) < l(w_C).$$

A trace function on H_n is an A-linear map $\varphi : H_n \to A$ such that $\varphi(hh') = \varphi(h'h)$ for all $h, h' \in H_n$. By [3], each trace function on H_n is uniquely determined by its values on the elements g_{w_C}, for all C. Conversely, given a set of elements $a_C \in A$, one for each conjugacy class C, there exists a unique trace function φ on H_n such that $\varphi(g_{w_C}) = a_C$ for all C. Using the above relations, we deduce that these results on the determination of trace functions remain valid when we replace each g_{w_C} by g_C, for all C.

The following result will show how to reduce the computation of the value of a trace function on any element to the values on elements in signed block form.

Proposition 1. *For each $h \in H_n$ there exists a finite (non-empty) subset $I(h) \subseteq A \times \mathcal{D}'_{1,0} \times \ldots \times \mathcal{D}'_{n,0}$ such that*

$$\varphi(h) = \sum_{(r, d_1^*, \ldots, d_n^*) \in I(h)} r\varphi(d_1^* \cdots d_n^*),$$

for all trace functions φ on H_n.

Proof. The result clearly holds if $n = 1$. Now let $1 < j \leq n$ and assume that we have already found a finite (non-empty) subset $I_j \subseteq H_j \times D'_{j+1,j-1} \times \ldots \times \mathcal{D}'_{n,j-1}$ such that

$$\varphi(h) = \sum_{(h_j, d'_{j+1}, \ldots, d'_n) \in I_j} \varphi(h_j d'_{j+1} \cdots d'_n),$$

for all trace functions φ on H_n. We will proceed by downward induction on j. For $j = n$ there is nothing to prove. We now show how to obtain an analogous statement with j replaced by $j-1$. This is done as follows.

Consider one element $(h_j, d'_{j+1}, \ldots, d'_n) \in I_j$. The element h_j is an A-linear combination of basis elements g_w with $w \in W_j$. By (3.2), this can be rewritten as an A-linear combination of products $r'_1 \cdots r'_j$ with $r'_i \in \mathcal{R}'_i$. Collecting terms with a fixed value of r'_j, we obtain a finite (non-empty) subset $R(h_j) \subseteq H_{j-1} \times \mathcal{R}'_j$ such that

$$h_j = \sum_{(h_{j-1}, r'_j) \in R(h_j)} h_{j-1} r'_j.$$

We can write $r'_j = d'_j r''_{j-1}$ with $d'_j \in \mathcal{D}'_j$ and $r''_{j-1} \in \mathcal{R}'_{j-1}$ (cf. (2.1)). Now the element r''_{j-1} either is a product of various generators $g_1, \ldots, g_{j-2}$ or is like the element considered in Lemma 2.3(c). In any case, it commutes with d'_{j+1} up to (possibly) changing some inverses at indices bigger than $j-2$, and similarly for $d'_{j+2}, \ldots, d'_n$. Thus, we conclude that

$$h_j d'_{j+1} \cdots d'_n = \sum_{(h_{j-1}, r'_j) \in R(h_j)} h_{j-1} d'_j d''_{j+1} \cdots d''_n r''_{j-1},$$

where $d''_{j+1} \in \mathcal{D}'_{j+1,j-2}, \ldots, d''_n \in \mathcal{D}'_{n,j-2}$. So we have that

$$\varphi(h) = \sum_{(h_j, d'_{j+1}, \ldots, d'_n) \in I_j} \; \sum_{(h_{j-1}, r'_j) \in R(h_j)} \varphi(r''_{j-1} h_{j-1} d'_j d''_{j+1} \cdots d''_n).$$

We can combine the index sets to a new set $I_{j-1} \subseteq H_{j-1} \times \mathcal{D}'_{j,j-2} \times \ldots \times \mathcal{D}'_{n,j-2}$ and arrive at a new expression as above with j replaced $j-1$. We repeat this process until we arrive at $j = 1$. Then $H_1 = \langle 1, t \rangle = \langle \mathcal{D}'_1 \rangle$, and we are done. $\square$

In fact, the above proof also yields the following extension.

Corollary 1. *Let $h \in H_n$ and $d_{n+1} \in \mathcal{D}'_{n+1}, \ldots, d_{n+m} \in \mathcal{D}'_{n+m}$ (for some $m \geq 0$). Then, for all trace functions φ on H_n, we have that*

$$\varphi(h d_{n+1} \cdots d_{n+m}) = \sum_{(r, d^*_1, \ldots, d^*_n) \in I(h)} r \varphi(d^*_1 \cdots d^*_n d^*_{n+1} \cdots d^*_{n+m}),$$

for some elements $d^*_{n+1} \in \mathcal{D}'_{n+1,0}, \ldots, d^*_{n+m} \in \mathcal{D}'_{n+m,0}$ (depending on the various elements in $I(h)$).

4 Markov traces for Iwahori-Hecke algebras of type B

Jones writes in [5], p.346, that there should be analogues of Ocneanu's trace for Iwahori-Hecke algebras other than those of type A. The trace given in [7] was the first such analogue for B-type Iwahori-Hecke algebras. The aim of this section is to classify *all* such 'Markov' traces on Iwahori-Hecke algebras of type B, based on the results in the previous sections.

Definition 1. *Let $z \in A$ and $\tau : H \to A$ be an A-linear map. Then τ is called a Markov trace (with parameter z) if the following conditions are satisfied.*

(1) τ is a trace function on H.

(2) $\tau(1) = 1$ (normalization).

(3) $\tau(hg_n) = z\tau(h)$ for all $n \geq 1$ and $h \in H_n$.

We note that all generators g_i (for $i = 1, 2, \ldots$) are conjugate in H. In particular, any trace function on H must have the same value on these elements. This explains why the parameter z is independent of n in rule (3) of this definition.

Let us consider the subalgebra H' of H generated by $g_1, g_2, \ldots$. Then H' is the algebra considered by Jones in [5], §5 (infinite union over all Iwahori-Hecke algebras of type A_n with parameter q). Moreover, the restriction of any Markov trace τ on H to H' yields Ocneanu's original trace as in [5], Theorem 5.1, which is uniquely determined by the parameter z.

The next result describes a set of elements in H which is sufficient to determine a Markov trace τ. We shall see that this set is in fact as small as possible.

Lemma 3. *Let $\tau : H \to A$ be a Markov trace (with parameter $z \in A$).*

(a) If $n \geq 1$, $m \geq 0$ and $h \in H_n$, then

$$\begin{aligned} \tau(hg_n t'_{n+1} \cdots t'_{n+m}) &= z\tau(ht'_n \cdots t'_{n+m-1}) \quad \textit{and} \\ \tau(ht'_{n+1} \cdots t'_{n+m}) &= \tau(ht'_n \cdots t'_{n+m-1}) \end{aligned}$$

(b) If $h = d_1 \cdots d_n \in H$ where $d_i \in \mathcal{D}'_i$ for all i then

$$\tau(h) = z^{a(h)} \tau(t'_0 t'_1 \cdots t'_{b(h)-1})$$

where $a(h)$ is the number of d_i which are in the set $\{g_1, \ldots, g_{n-1}\}$ and $b(h)$ is the number of d_i which are conjugate to t.

(c) τ is uniquely determined by its values on the elements in the set

$$\{t'_0 t'_1 \cdots t'_{k-1} \mid k = 1, 2, \ldots\}.$$

Proof. To prove the first relation in (a) we will proceed by induction on m. If $m = 0$ then we can apply directly rule (3) in Definition 4.1. Now let us assume that $m > 0$. We have to evaluate the expression

$$\tau(hg_n t'_{n+1} \cdots t'_{n+m}).$$

We write $t'_{n+1} = g_{n+1} t'_n g_{n+1}^{-1}$ and observe that g_{n+1}^{-1} commutes with $t'_{n+2}, \ldots, t'_{n+m}$ by Lemma 2.3(a). Since τ is a trace our expression is equal to

$$\tau(g_{n+1}^{-1} hg_n g_{n+1} t'_n t'_{n+2} \cdots t'_{n+m}).$$

Now h lies in H_n, that is, h only involves the generators $t, g_1, \ldots, g_{n-1}$. It follows that h commutes with g_{n+1}^{-1}. Using moreover the braid relation $g_{n+1}^{-1} g_n g_{n+1} = g_n g_{n+1} g_n^{-1}$, the above expression can be rewritten as

$$\tau(h g_n g_{n+1} g_n^{-1} t'_n t'_{n+2} \cdots t'_{n+m}).$$

If we write $t'_n = g_n t'_{n-1} g_n^{-1}$, the left hand term g_n will cancel and then g_{n+1} commutes with t'_{n-1}. Now our expression reads

$$\tau(h g_n t'_{n-1} g_{n+1} g_n^{-1} t'_{n+2} \cdots t'_{n+m}).$$

The element g_n^{-1} commutes with all terms to the right of it. Hence our expression is equal to

$$\tau(g_n^{-1} h g_n t'_{n-1} g_{n+1} t'_{n+2} \cdots t'_{n+m}).$$

We write $h' := g_n^{-1} h g_n t'_{n-1}$ and observe that this element lies in H_{n+1}. So we can apply the induction and obtain that

$$\tau(h g_n t'_{n+1} \cdots t'_{n+m}) = \tau(h' g_{n+1} t'_{n+2} \cdots t'_{n+m}) = z\tau(h' t'_{n+1} \cdots t'_{n+m-1}).$$

We insert the expression for h' again, note that g_n^{-1} commutes with $t'_{n+1} \cdots t'_{n+m-1}$, and conclude that

$$\tau(h' t'_{n+1} \cdots t'_{n+m}) = \tau(h g_n t'_{n-1} g_n^{-1} t'_{n+1} \cdots t'_{n+m-1}) = \tau(h t'_n t'_{n+1} \cdots t'_{n+m-1}).$$

Putting things together we see that this completes the proof of the first relation. The proof of the second is achieved by an analogous computation (with g_n replaced by 1). In order to prove (b) we consider an element

$$h = d_1 \cdots d_n \in H_n \quad \text{where } d_i \in \mathcal{D}'_i \text{ for all } i.$$

Using (a) and induction on n we deduce that

$$\tau(h) = z^{a(h)} \tau(t'_0 t'_1 \cdots t'_{b(h)-1})$$

where $a(h), b(h)$ are defined as above. Using (3.3) we conclude that these equations determine τ uniquely, proving (c). □

Theorem 7. *Let $z, y_1, y_2 \ldots \in A$. Then there exists a unique Markov trace τ on H with parameter z such that*

$$\tau(t'_0 t'_1 t'_2 \cdots t'_{k-1}) = y_k \quad \textit{for all } k \geq 1.$$

Proof. Uniqueness was already proved in Lemma 4.2. Using the facts summarized in (3.3) we can define a trace function τ on H satisfying $\tau(1) = 1$ and

$$\tau(g_C) = z^a y_b$$

where $g_C = d_1 \cdots d_n$ with $d_i \in \mathcal{D}'_i$ and the elements $a = a(g_C)$, $b = b(g_C)$ are defined as in Lemma 4.2. Thus, the existence of a trace function τ on H satisfying conditions (1), (2) in Definition 4.1 is already established. The problem is to show that (3) holds.

This will be done by an induction, as follows. For $N \geq 0$ let $H(\leq N)$ be the A-subspace of H generated by all elements g_w with $w \in W$ and $l(w) \leq N$. We shall prove the following claim, for all $N \geq 0$.

(*) *Let $h \in H_n$ and $d^*_{n+i} \in \mathcal{D}'_{n+i,0}$ (for some $n, m \geq 1$ and $i = 1, \ldots, m$) such that $hd^*_{n+1} \cdots d^*_{n+m} \in H(\leq N)$. Then*

$$\begin{aligned} \tau(hd^*_{n+1} \cdots d^*_{n+m}) &= \tau(hd_{n+1} \cdots d_{n+m}) \quad \text{and} \\ \tau(hg_n d^*_{n+2} d^*_{n+3} \cdots d^*_{n+m}) &= z\tau(hd^*_{n+2} \cdots d^*_{n+m}). \end{aligned}$$

(Here, we used the following convention. For each i, we denote by d_i the analogous element as d^*_i with the inverses (if any) on the right hand side of t.) If this is proved for all N then τ will satisfy condition (3) in Definition 4.1 as a special case. Clearly, (*) is true for $N = 0$. Now let $N > 0$. We proceed in a number of steps.

Step 1. At first we show that $\tau(hd^*_{n+1} \cdots d^*_{n+m}) = \tau(hd_{n+1} \cdots d_{n+m})$. By induction on m we may assume that $d^*_{n+2} = d_{n+2}, \ldots, d^*_{n+m} = d_{n+m}$. We can also assume that

$$d^*_{n+1} = T_{n+1,i} := g_n \cdots g_{i+1} g_i^{\mp 1} t'_{i-1,0} g_i^{\pm 1} g_{i+1}^{-1} \cdots g_n^{-1} \quad \text{for some } i \leq n.$$

(That is, the inverses are already fixed at indices bigger than i.) Then $T_{n+1,i-1}$ is the analogous element where the inverse at index i is also correct. Now assume that the sign in $g_i^{\mp 1}$ is -1. We will show that $\tau(hT_{n+1,i} d_{n+2} \cdots d_{n+m}) = \tau(hT_{n+1,i-1} d_{n+2} \cdots d_{n+m})$. This is done as follows. In $T_{n+1,i}$ and $T_{n+1,i-1}$ we replace g_i^{-1} by $(q^{-1}-1)g_i + q^{-1} \cdot 1$. Then $hT_{n+1,i} d_{n+2} \cdots d_{n+m} = (q^{-1}-1)S_1 + q^{-1}S_2$ and $hT_{n+1,i-1} d_{n+2} \cdots d_{n+m} = (q^{-1}-1)S_1 + q^{-1}S_3$ where

$$\begin{aligned} S_1 &= h(g_n \cdots g_{i+1} g_i t'_{i-1,0} g_i g_{i+1}^{-1} \cdots g_n^{-1}) d_{n+2} \cdots d_{n+m}, \\ S_2 &= h(g_n \cdots g_{i+1} t'_{i-1,0} g_i g_{i+1}^{-1} \cdots g_n^{-1}) d_{n+2} \cdots d_{n+m}, \\ S_3 &= h(g_n \cdots g_{i+1} g_i t'_{i-1,0} g_{i+1}^{-1} \cdots g_n^{-1}) d_{n+2} \cdots d_{n+m}. \end{aligned}$$

Note that $S_2, S_3 \in H(\leq N-1)$ so that we can use our inductive hypotheses in the evaluation of τ on these elements. Let us first consider S_2. At first, we note that $g_n \cdots g_{i+1}$ commutes with $t'_{i-1,0}$. So we obtain that

$$\begin{aligned} \tau(S_2) &= \tau(ht'_{i-1,0}(g_n \cdots g_{i+1} g_i g_{i+1}^{-1} \cdots g_n^{-1}) d_{n+2} \cdots d_{n+m}) \\ &= \tau(ht'_{i-1,0}(g_i^{-1} \cdots g_{n-1}^{-1} g_n g_{n-1} \cdots g_i) d_{n+2} \cdots d_{n+m}). \end{aligned}$$

Now $g_{n-1} \cdots g_i$ commutes with all terms to the right of it, by Lemma 2.3(a). Hence we conclude that

$$\tau(S_2) = \tau((g_{n-1} \cdots g_i) ht'_{i-1,0}(g_i^{-1} \cdots g_{n-1}^{-1}) g_n d_{n+2} \cdots d_{n+m}).$$

We write the argument of τ as $h' g_n d_{n+2} \cdots d_{n+m}$ with $h' \in H_n$. Since this element lies in $H(\leq N-1)$ we can use induction to deduce that

$$\tau(S_2) = \tau(h' g_n d_{n+2} \cdots d_{n+m}) = z\tau(h' d_{n+2} \cdots d_{n+m}).$$

Now we observe that h' is a conjugate of $ht'_{i-1,0}$ and the conjugating element $g_{n-1}\cdots g_i$ commutes with $d_{n+2}\cdots d_{n+m}$, again using Lemma 2.3(a). Thus, we finally compute that

$$\tau(S_2) = z\tau(ht'_{i-1,0}d_{n+2}\cdots d_{n+m}).$$

Now we follow the same procedure with S_3 and find that

$$\begin{aligned}\tau(S_3) &= \tau(h(g_n\cdots g_{i+1}g_ig_{i+1}^{-1}\cdots g_n^{-1})t'_{i-1,0}d_{n+2}\cdots d_{n+m})\\ &= \tau(h(g_i^{-1}\cdots g_{n-1}^{-1}g_ng_{n-1}\cdots g_i)t'_{i-1,0}d_{n+2}\cdots d_{n+m}).\end{aligned}$$

Now we would like to interchange $t'_{i-1,0}$ and $d_{n+2}\cdots d_{n+m}$ but this is not necessarily possible on the algebra level. However, it still works modulo the kernel of τ, as the following argument shows. We write $t'_{i-1,0} = x^{-1}tx$ where $x = g_1^{\pm 1}\cdots g_{i-1}^{\pm 1}$. By Lemma 2.3(a), the element x will commute with $d_{n+2}\cdots d_{n+m}$. The next factor, t, only commutes with $d_{n+2}\cdots d_{n+m}$ up to (possibly) changing some inverses. Thus, the term on the right hand side of the above equality is equal to

$$\tau(txh(g_i^{-1}\cdots g_{n-1}^{-1}g_ng_{n-1}\cdots g_i)x^{-1}d'_{n+2}\cdots d'_{n+m}),$$

for some $d'_{n+2}\in\mathcal{D}'_{n+2,0},\ldots,d'_{n+m}\in\mathcal{D}'_{n+m,0}$. Now we can apply our inductive hypothesis to conclude that this equals

$$\tau(txh(g_i^{-1}\cdots g_{n-1}^{-1}g_ng_{n-1}\cdots g_i)x^{-1}d_{n+2}\cdots d_{n+m}).$$

Now, again by Lemma 2.3(a), the element x^{-1} will commute with the terms to the right of it. So, finally, we obtain that

$$\tau(S_3) = \tau(t'_{i-1,0}h(g_i^{-1}\cdots g_{n-1}^{-1}g_ng_{n-1}\cdots g_i)d_{n+2}\cdots d_{n+m}).$$

Once more, we use Lemma 2.3(a) to conclude that $g_{n-1}\cdots g_i$ commutes with all terms to the right of it. So we arrive at the equality

$$\tau(S_3) = \tau((g_{n-1}\cdots g_i)t'_{i-1,0}h(g_i^{-1}\cdots g_{n-1}^{-1})g_nd_{n+2}\cdots d_{n+m}).$$

In this situation, we can argue similarly as in the evaluation of $\tau(S_2)$. This evaluation will result in an analogous expression as before, but with the terms h and $t'_{i-1,0}$ interchanged. Thus, we conclude that

$$\tau(S_3) = z\tau(t'_{i-1,0}hd_{n+2}\cdots d_{n+m}).$$

Arguing as above, we see that $t'_{i-1,0}$ and $d_{n+2}\cdots d_{n+m}$ can be interchanged modulo the kernel of τ. So, eventually, we find that $\tau(S_2)=\tau(S_3)$ and, hence, that $\tau(hT_{n+1,i}d_{n+2}\cdots d_{n+m}) = \tau(hT_{n+1,i-1}d_{n+2}\cdots d_{n+m})$, as required. The proof of the assertion of Step 1 is now completed by repeating the whole process with $T_{n+1,i-1}$ and so on, until all inverses are fixed.

Step 2. Now we consider the case where $h = d_1^*\ldots d_n^*$, for some $d_i^*\in\mathcal{D}'_{i,0}$. As before, let d_i be the element of $\mathcal{D}'_i$ corresponding to d_i^* where all inverses (if any) are in the right position. Using Step 1, we first see that $\tau(d_1^* \cdots d_{n+m}^*) = \tau(d_1 \cdots d_{n+m})$. Now, we will prove that

$$\tau(d_1^* \cdots d_{n+m}^*) = z^a y_b,$$

where $a = a(d_1 \cdots d_{n+m})$ and $b = b(d_1 \cdots d_{n+m})$. (Note that, in the definition of a, b in Lemma 4.2(b), it does not matter whether we take d_i^* or d_i.) As explained in (2.4), (3.3), the element $d_1 \cdots d_{n+m}$ can be regarded as a product of positive and negative blocks. If this element is equal to g_C for some conjugacy class C then we are done by the defining equation for τ. If this is not the case, then some positive block is followed by a negative block or some negative block is followed by a strictly shorter negative block. We can then use Lemma 2.5 to conjugate our element to $d_1' \cdots d_{n+m}' \in H_n(\leq N-1)$ with $d_i' \in \mathcal{D}_{i,0}'$ and with the same signed block structure as before, that is, the values of a and b haven't changed after this conjugation. Thus, we conclude that

$$\tau(d_1^* \cdots d_{n+m}^*) = \tau(d_1 \cdots d_{n+m}) = \tau(d_1' \cdots d_{n+m}') = z^a y_b,$$

where the last equality is by induction on N. Our claim is proved.

Step 3. Finally, let $h \in H_n$ and $d_{n+i}^* \in \mathcal{D}_{n+i,0}'$ for $2 \leq i \leq m$. Now, we show that

$$\tau(hg_n d_{n+2}^* d_{n+3}^* \cdots d_{n+m}^*) = z\tau(hd_{n+2}^* \cdots d_{n+m}^*).$$

Using Step 1, we have that $\tau(hg_n d_{n+2}^* \cdots d_{n+m}^*) = \tau(hg_n d_{n+2} \cdots d_{n+m})$ where $d_{n+i} \in \mathcal{D}_{n+i}'$ corresponds to d_{n+i}^* as above. Using Corollary 3.5, the latter trace is equal to

$$\sum_{(r,d_1^*,\ldots,d_n^*)\in I(h)} r\tau(d_1^* \cdots d_n^* g_n d_{n+2}^* \cdots d_{n+m}^*),$$

for some $d_{n+2}^* \in \mathcal{D}_{n+2,0}', \ldots, d_{n+m}^* \in \mathcal{D}_{n+m,0}'$ (depending on the various elements in $I(h)$). Let us consider one term in this sum, corresponding to $(r, d_1^*, \ldots, d_n^*) \in I(h)$. We shall write $d_1^* \cdots d_n^* g_n d_{n+2}^* \cdots d_{n+m}^*$ in the form $h_1 g_n h_2$. Let $a = a(h_1 g_n h_2)$, $b = b(h_1 g_n h_2)$ and $a' = a(h_1 h_2)$, $b' = b(h_1 h_2)$. Then, clearly, $a' = a - 1$ and $b = b'$. Using this and Step 2, the value of τ on our element is given by

$$rz^a y_b = rz^{a'} y_{b'} z.$$

Thus, for each term in the above sum, we obtain a factor z as the expense of cancelling the factor g_n in that term. We conclude that

$$\tau(hg_n d_{n+2}^* \cdots d_{n+m}^*) = z\tau(hd_{n+2}^* \cdots d_{n+m}^*).$$

The proof is complete. □

4.4. In the course of the above proof, we have shown the following remarkable property of a Markov trace τ.

Let $h \in H_n$ and $d_{n+1}^* \in \mathcal{D}_{n+1,0}', \ldots, d_{n+m}^* \in \mathcal{D}_{n+m,0}'$ for some $m \geq 1$. Then

$$\tau(hd_{n+1}^* \cdots d_{n+m}^*) = \tau(hd_{n+1} \cdots d_{n+m}),$$

where, as before, d_i is the element in $\mathcal{D}'_i$ corresponding to d_i^*.

For $h \in H_n$ let $I(h)$ be the corresponding set given by Proposition 3.4. For each $i = (r, d_1^*, \ldots, d_n^*) \in I(h)$ we write $r(i) = r$, $a(i) = a(d_1 \cdots d_n)$ and $b(i) = b(d_1 \cdots d_n)$. Then the above property in combination with Proposition 3.4 and Lemma 4.2 yields the following rule for computing $\tau(h)$.

$$\tau(h) = \sum_{i \in I(h)} r(i) z^{a(i)} y_{b(i)}.$$

Note that an algorithm for computing $I(h)$ is given by the inductive proof of Proposition 3.4.

Proposition 2. *Let $z, y \in A$ and $\tau : H \to A$ be a Markov trace with parameter z and such that $\tau(t'_0 t'_1 \cdots t'_{k-1}) = y^k$ for all $k \geq 1$. Then*

$$\tau(h t'_{n,0}) = y\tau(h) \quad \textit{for all } n \geq 0 \textit{ and } h \in H_n.$$

Proof. Following the steps in the proof of Theorem 4.3 we see that it is sufficient to consider the case where $h = d_1^* \cdots d_n^*$ with $d_i^* \in \mathcal{D}'_{i,0}$ for all i. The result in this case follows from Lemma 4.2. □

Remarks 2. (a) Given elements $z, y \in A$, the existence of a Markov trace $\tau = \tau_{z,y}$ as in the previous Proposition can also be proved along the lines of the approach followed by Jones [5], §5, in the proof of Ocneanu's Theorem for Iwahori-Hecke algebras of type A_n. Such a proof is sketched in [7] (see also [6], §3.3). It is based on the observation that, for all $n \geq 0$, the map

$$C_n : H_n \oplus H_n \oplus H_n \otimes_{H_{n-1}} H_n \longrightarrow H_{n+1}, \quad a + b + c \otimes d \mapsto a + bt'_n + cg_n d$$

defines an isomorphism of (H_n, H_n)-bimodules. Analogously to the proof of [5], (5.1), we can now define an A-linear map τ inductively on $H = \bigcup_n H_n$ by the rules: $\tau(1) = 1$, $\tau(bt'_n) = y\tau(b)$ and $\tau(cg_n d) = z\tau(cd)$ where $b, c, d \in H_n$ and $n \geq 0$. It follows easily that this map satisfies conditions (2) and (3) of Definition 4.1, and the problem then is to show that (1) holds, that is, to show that τ is a trace function. This verification is a lengthy and tedious calculation that we do not want to reproduce here. We do not see, however, how this method could be modified so as to give an alternative proof of our more general Theorem 4.3, too. Yet another construction of the trace in [7] was given by T. tom Dieck in [9] using Turaev's R–matrix approach (see [11]). It would be interesting to find such an R–matrix interpretation of our more general traces in Theorem 4.3, too.

(b) For the definition of Markov traces it doesn't matter whether we use the elements t_i or t'_i in the signed block form of elements. From Theorem 4.3 we see that this only plays a role in the formulation of the *initial conditions* determining the trace, and it would not be clear how to do this in terms of the elements t_i. This gives an explanation for using t'_i rather than t_i.

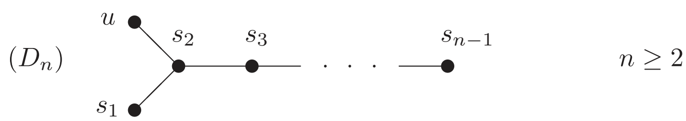

Figure 4:

4.7. We can use the results of this section to obtain a classification of Markov traces for Iwahori–Hecke algebras of type D, in the following way. First note that if we define $u := ts_1t$ then the elements $u, s_1, \ldots, s_{n-1}$ generate a subgroup $W'_n \subset W_n$ which is the finite Coxeter group of type D_n with relations given by the following diagram.
We shall use the convention that $W'_1 = \{1\}$.

The above embedding also works on the level of the Iwahori–Hecke algebras if we let $Q = 1$. Indeed, denoting $u := tg_1t \in H_n$ in this case, we compute that $u^2 = (q-1)u + q \cdot 1$. Thus, the elements $u, g_1, \ldots, g_{n-1}$ generate a subalgebra $H'_n \subset H_n$ which is the Iwahori–Hecke algebra of type D_n. As in the B-type case, we have embeddings $W'_1 \subset W'_2 \subset \ldots$ and $H'_1 \subset H'_2 \subset \ldots$, and we denote $H' := \bigcup_{n \geq 1} H'_n$. In analogy to Definition 4.1 we say that a trace function $\tau : H' \to A$ is a Markov trace with parameter $z \in A$ if $\tau(1) = 1$ and $\tau(hg_n) = z\tau(h)$ for all $n \geq 1$ and $h \in H'_n$. Now we can state (for the proof see [2], Section 6):

(a) *Every Markov trace on H' is the restriction of a Markov trace on H (with the same parameter $z \in A$).*

(b) *A Markov trace τ on H' (with parameter $z \in A$) is uniquely determined by its values on the elements in the set*

$$\{u'_1 \cdots u'_{2k-1} \mid k = 1, 2, \ldots\}$$

where $u'_i := g_i \cdots g_2 u g_1^{-1} g_2^{-1} \cdots g_i^{-1}$ for all $i \geq 1$.

The elements u'_i play an analogous role as the elements t'_i in the B-type case. Note that under the above embedding $H' \subset H$ we have $u'_i = tt'_i$ for all $i \geq 1$.

5 The knot invariants related to the Hecke algebras of B-type

5.1. Knots and links inside a solid torus T can be represented unambiguously by 'mixed' knots/links in S^3 which contain one oriented, unknotted, pointwise fixed component (the core of the complementary unknotted solid torus in S^3). An example of a mixed link is illustrated below.

So, two links L_1, L_2 in T are isotopic if and only if their corresponding mixed links in S^3 are, through an isotopy that keeps the specified unknotted component pointwise fixed. By applying to an oriented mixed link an appropriate

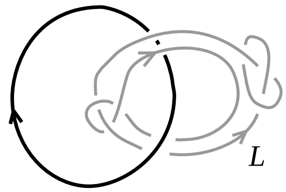

Figure 5:

braiding we can then turn it into a 'mixed' braid (a braid that keeps the specified component pointwise fixed in the first position), so that the closure of this braid is isotopic to our mixed link. An example of a mixed braid is illustrated in the Introduction. The set of all mixed braids on n strands (where the numbering excludes the first fixed one) form the group $B_{1,n}$, the geometric version of $\tilde{W}_n$. Moreover, similarly to braid equivalence in S^3 we also have Markov equivalence for mixed braids. (For details and proofs of the above the reader is referred to [6] or [7].) Namely we have the following.

Theorem 10. (cf. [7], Theorem 3.)
Let L_1, L_2 be two oriented links in T and β_1, β_2 be mixed braids in $\bigcup_{n=1}^{\infty} B_{1,n}$ corresponding to L_1, L_2. Then L_1 is isotopic to L_2 in T if and only if β_1 is equivalent to β_2 in $\bigcup_{n=1}^{\infty} B_{1,n}$ under equivalence generated by the braid relations together with the following two moves:

(i) Conjugation: If $\alpha, \beta \in B_{1,n}$ then $\alpha \sim \beta^{-1}\alpha\beta$.

(ii) Markov moves: If $\alpha \in B_{1,n}$ then $\alpha \sim \alpha {s_n}^{\pm 1} \in B_{1,n+1}$.

As already noted in Section 1, there is a strong resemblance between the Markov moves and the special property of a Markov trace. Let now π denote the canonical quotient map $A\tilde{W}_n \to H_n$ given in (3.1), and denote the generators of H_n by $t, g_1, \ldots, g_{n-1}$ as above. Let also $\tau : H = \bigcup_{n\geq 0} H_n \to A$ be the Markov trace (with parameter $z \in A$) with initial conditions $\tau(t'_0 t'_1 t'_2 \cdots t'_{k-1}) = y_k$ for all $k \geq 1$. Then a braid in $B_{1,n}$ can be mapped through $\tau \circ \pi$ to an expression in the variables $q, Q, z, y_1, y_2, \ldots$. For an element $\alpha \in B_{1,n}$ we shall denote by $\widehat{\alpha}$ its closure. Then, according to Theorem 5.2, in order to obtain an isotopy invariant X for oriented knots in T we only need to normalize τ so that

$$X(\widehat{\alpha}) = X(\widehat{\alpha s_n}) = X(\widehat{\alpha s_n^{-1}}).$$

This normalization has been done in [7], (5.1), where Jones's normalization of Ocneanu's trace (cf. [5]) was followed. For this purpose, we have to take some care in the choice of A and the parameter z. We let A be the field of rational functions over $\mathbb{Q}$ in indeterminates $\sqrt{\lambda}, \sqrt{q}, \sqrt{Q}, y_1, y_2, \ldots$, and we let

$$z := \frac{1-q}{q\lambda - 1}.$$

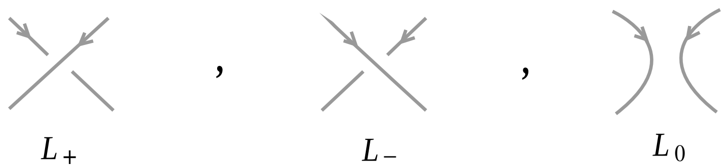

Figure 6:

(The reason for having square roots of q and Q will become clear in the recursive formulae in (5.4) below; a square root of λ is already required in the normalization of τ.)

Definition 2. (cf. [7], Definition 1.) For α, τ, π as above let

$$X_{\widehat{\alpha}} = X_{\widehat{\alpha}}(q, Q, \sqrt{\lambda}, y_1, y_2, \ldots) = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} (\sqrt{\lambda})^e \, \tau(\pi(\alpha)),$$

where e is the exponent sum of the s_i's that appear in α. (As noted in [7], the t_i''s can be ignored in the estimation of e as they do not affect it.) Then $X_{\hat{\alpha}}$ depends only on the isotopy class of $\widehat{\alpha}$, as a mixed link representing an oriented link in T.

If we look at $y_1, y_2, \ldots$ as parameters then this Definition supplies a family of invariants and Theorem 4.3 implies that these are the only possible analogues of the 2-variable Jones polynomial for oriented knots inside a solid torus, which are related to the Iwahori-Hecke algebras of type B. Note also that, if $\alpha \in B_{1,n}$ is a product of the generators s_i or their inverses (i.e, α does not involve the generator t) then $X_{\hat{\alpha}}$ is a rational function of $\sqrt{\lambda}$ and q only, and it is exactly the same as the invariant in [5], Definition 6.1. Geometrically, this means that if an oriented knot in T can be enclosed in a 3-ball then the above invariant applied to this knot will yield the 2-variable Jones polynomial (homfly-pt) for the knot, seen as a knot in S^3.

5.4. We shall now show how to interpret the above in terms of knot diagrams, and how to calculate alternatively the above solid torus knot invariants using *initial conditions* and applying *skein relations* on the mixed link diagrams.

Let L_+, L_-, L_0 be oriented mixed link diagrams that are identical, except in one crossing, where they are as depicted below:

Let also M_+, M_-, M_0 be oriented mixed link diagrams that are identical, except in the regions depicted below:

In [7], (5.2) it is shown that the knot invariant defined there (which is a special member of the family of invariants defined above), satisfies the two recursive linear formulae.

$$\frac{1}{\sqrt{q}\sqrt{\lambda}} X_{L_+} - \sqrt{q}\sqrt{\lambda}\, X_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})\, X_{L_0}$$

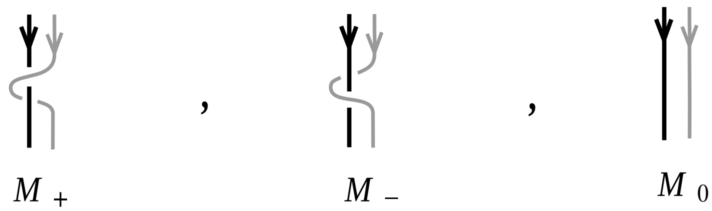

Figure 7:

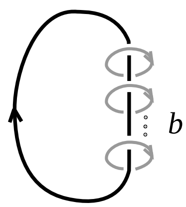

Figure 8:

$$\frac{1}{\sqrt{Q}}\, X_{M_+} - \sqrt{Q}\, X_{M_-} = (\sqrt{Q} - \frac{1}{\sqrt{Q}})\, X_{M_0}$$

These are the two skein relations that derive from the defining quadratic relations of H, and the first one of them is the well-known skein rule used for the evaluation of the homfly-pt polynomial. The same reasoning applies to any invariant of Definition 5.3 above.

Take now a braid α in $B_{1,n}$. Then $\pi(\alpha)$ is an A-linear combination of elements in the basis of H_n. (In terms of diagrams, we have used on $X_{\widehat{\alpha}}$ the skein relations above.) This fact and Proposition 3.4 imply now that, under conjugation and the skein relations, $X_{\widehat{\alpha}}$ can be further written as an A-linear combination of the values of X on diagrams of the form $\widehat{d_1^* \cdots d_n^*}$, where d_i^* is either $1, s_i$ or $t'_{i,0}$ (with inverses mixed up). If now b is the number of $t'_{i,0}$'s in $d_1^* \cdots d_n^*$, then $\widehat{d_1^* \cdots d_n^*}$ is a mixed link on b components, such that each component links once (in a positive sense) with the special fixed one, but they are otherwise unlinked with each other. Therefore $\widehat{d_1^* \cdots d_n^*}$ is isotopic to the mixed link $t'\widehat{t'_1 \cdots t'_{b-1}}$, pictured below.

Now $\tau(t't'_1 \cdots t'_{k-1}) = y_k$ is one of the initial conditions from Theorem 4.3, and using Definition 5.3 we can calculate $X_{t'\widehat{t'_1 \cdots t'_{k-1}}}$. Hence, we proved that the two skein rules together with the following (infinitely many) initial conditions

$$X_{\widehat{1}} = 1 \quad (1 \in B_{1,1}), \qquad X_{\widehat{\alpha_k}} = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{k-1} y_k$$

(with $\alpha_k = t't'_1 \cdots t'_{k-1} \in B_{1,k}$ for all $k \geq 1$) determine uniquely the invariant X.

Remarks 3. Let $\alpha \in B_{1,n}$. The above discussion shows *geometrically* that, firstly, $\tau \circ \pi(\alpha)$ can be calculated as a linear combination of terms $\tau(d_1^* \cdots d_n^*)$ with $d_i^* \in \mathcal{D}'_{n,0}$ and, secondly, that $\tau(d_1^* \cdots d_n^*) = z^a y_b$ where a is the number of s_i's and b is the number of conjugates of t in this element. This is the exact counterpart of the purely algebraic argument given before in (4.4).

Also, notice that the set of mixed links of the form $\widehat{t't'_1 \cdots t'_{k-1}}$ form the basis of the submodule of the 3rd skein module of the solid torus (as calculated by Turaev in [10] and by Hoste and Kidwell in [4]), that is related to the Iwahori-Hecke algebras of type B.

Finally, the defining equation in Definition 5.3 already shows that $X_{\widehat{\alpha}}$ is a polynomial in $Q^{\pm}, y_1, y_2, \ldots$. If we also perform the change of variables $x := \sqrt{q\lambda}$ and $r := \sqrt{q} - \frac{1}{\sqrt{q}}$ then the first skein rule can be rewritten as

$$\frac{1}{x} X_{L_+} - x X_{L_-} = r X_{L_0}.$$

As in [5], Proposition 6.2, this allows us to deduce that $X_{\widehat{\alpha}}$ also is a Laurent polynomial in the variables x and r.

Example 1. We shall now evaluate explicitly $X_{\widehat{\alpha}}$ for some special choices of α as a Laurent polynomial in $x, r, Q, y_1, y_2, \ldots$.

(a) If α is a product of generators s_i or their inverses (and does not involve t) then $X_{\widehat{\alpha}}$ equals the known 2-variable invariant for oriented knots inside S^3 (see the remarks following Definition 5.3).

(b) Consider $\alpha = s_1 s_2^{-1} s_1 t s_1^{-1} s_2$ and $\alpha' = t s_2$ in $B_{1,3}$ (see Remarks 2.6(b)). Using (4.4) we directly find that $\tau \circ \pi(\alpha) = \tau \circ \pi(\alpha') = z y_1$. The exponent sum e of the factors s_i equals 1 in both cases. Therefore, we also have

$$\begin{aligned} X_{\widehat{\alpha}} = X_{\widehat{\alpha}'} &= \left(\frac{1 - \lambda q}{\sqrt{\lambda}(1-q)} \right)^2 \sqrt{\lambda} z y_1 \\ &= \frac{\lambda q - 1}{\sqrt{\lambda}(1-q)} \sqrt{\lambda} y_1 && \text{(inserting } z\text{)} \\ &= \frac{1 - x^2}{xr} y_1 && \text{(inserting } x \text{ and } r\text{).} \end{aligned}$$

(c) For a general braid $\alpha \in B_{1,n}$, we first have to consider its image in H_n and express it as a linear combination in the standard basis of H_n:

$$h := \pi(\alpha) = \sum_{w \in W} a_w g_w \quad \text{with } a_w \in A.$$

Then we could compute the set $I(h)$ defined in Proposition 3.4 and finally use the recipe given in (4.4) to evaluate the trace $\tau(h)$. Now the computation of the set $I(h)$ involves performing a base change from the standard basis of H_n to the new basis consisting of the elements $r'_1 \cdots r'_n$, with $r'_i \in \mathcal{R}'_i$ (see (3.2) and the proof of Proposition 3.4). In practice, however, this will be quite cumbersome. A more economic way is by using the class polynomials of [3].

Recall from [*loc. cit.*] that for each $w \in W_n$ there exist elements $f_{w,C} \in \mathbb{Z}[q,Q]$ such that $\varphi(g_w) = \sum_C f_{w,C}\varphi(g_{w_C})$ (sum over all conjugacy classes C of W_n), for all trace functions φ on H_n. In [*loc. cit.*], Section 1, there is also given a recursive formula for computing $f_{w,C}$. Assume then that we know (for our given braid α) the coefficients $a_w \in A$ and the class polynomials $f_{w,C}$ for all w such that $a_w \neq 0$. Then we have

$$\tau \circ \pi(\alpha) = \sum_C \left(\sum_{w \in W_n} a_w f_{w,C} \right) \tau(g_{w_C}).$$

Thus, we are reduced to calculating, once and for all, the values of τ on basis elements corresponding to representatives w_C of minimal length in the conjugacy classes C of W_n. These classes are parametrized by pairs of partitions (π_1, π_2) such that the total sum of the parts of π_1 and π_2 equals n and where each part of π_1 (respectively π_2) corresponds to a negative (respectively positive) block. The formulae become more complicated as the number of negative blocks involved in w_C gets larger. Below we give the trace values for $n \leq 4$ and all w_C which contain at most three negative blocks. (The only class C for which we don't give the value is the one with representative $tt_1t_2t_3$; for each $n \geq 2$, we consider only those w_C which are not already contained in $B_{1,n-1}$.)

$B_{1,1}$: $\tau(1) = 1, \quad \tau(t) = y_1.$

$B_{1,2}$: $\tau(g_1) = z, \quad \tau(tg_1) = zy_1,$
$\tau(tt_1) = ((q-1)(Q-1)y_1 + (q-1)Q)z + qy_2.$

$B_{1,3}$: $\tau(tg_2) = zy_1, \quad \tau(g_1g_2) = z^2, \quad \tau(tg_1g_2) = z^2y_1,$
$\tau(tt_1g_2) = z\tau(tt_1)$ (see $B_{1,2}$),
$\tau(tt_1t_2) = ((q-1)(Q^2-Q+1)y_1 + Q(q-1)(Q-1))(q^3-1)z^2$
$\quad + (Qy_1 + (Q-1)y_2)(q^3-1)qz + q^3y_3.$

$B_{1,4}$: $\tau(g_1g_3) = z^2, \quad \tau(g_1g_2g_3) = z^3,$
$\tau(tg_1g_3) = \tau(tg_2g_3) = z^2y_1, \quad \tau(tg_1g_2g_3) = z^3y_1,$
$\tau(tt_1g_3) = z\tau(tt_1), \quad \tau(tt_1g_2g_3) = z^2\tau(tt_1)$ (see $B_{1,2}$),
$\tau(tg_1t_2g_3) = ((Q-1)y_1 + Q)(q-1)(q^2+1)z^3 + q(q^2-q+1)y_2z^2,$
$\tau(tt_1t_2g_3) = z\tau(tt_1t_2)$ (see $B_{1,3}$),
$\tau(tt_1t_2t_3)$ (four negative blocks).

Let us now consider the example of the braid $\alpha \in B_{1,5}$ given in the Introduction. By first using the defining properties for τ, we can eliminate the generators g_4 and g_3 and are reduced to computing

$$\tau(\alpha) = z^2\tau(g_2^{-1}g_1^{-3}tg_1g_2^{-1}g_1g_2^{-1}t^{-2}g_1^{-1}).$$

In principle, this can be done by following the above scheme (but the result will be quite cumbersome and we do not want to print it here). For a similar computation see [7], Example pp. 237. Based on the programs in [3] it is straightforward to implement the above algorithmic description in a computer program.

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M.G.: UFR de Mathématiques, Université Paris 7, 2 Place Jussieu, 75251 Paris, France, E-mail geck@mathp7.jussieu.fr
S.L.: Mathematisches Institut, Göttingen Universität, 3–5 Bunsenstrasse, 37073 Göttingen, Germany, E-mail sofia@cfgauss.uni-math.gwdg.de