

# Markovian models of a transactional system supported by checkpointing and recovery strategies, Part 2: A model with a specified number of completed transactions between checkpoints

***Citation for published version (APA):***

Nicola, V. F. (1982). *Markovian models of a transactional system supported by checkpointing and recovery strategies, Part 2: A model with a specified number of completed transactions between checkpoints*. (EUT report. E, Fac. of Electrical Engineering; Vol. 82-E-129). Technische Hogeschool Eindhoven.

***Document status and date:***

Published: 01/01/1982

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

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Eindhoven  
University of Technology  
the Netherlands

## Department of Electrical Engineering

Markovian models of a transactional system  
supported by checkpointing and recovery  
strategies.

Part 2: A model with a specified number of  
completed transactions between checkpoints.

By  
V.F. Nicola

EUT Report 82-E-129  
ISBN 90-6144-129-3  
ISSN 0167-9708  
August 1982

Errata to: EUT Report 82-E-129  
 Markovian models of a transactional system  
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page      line or equation      (...) to be replaced with (...)

8	eq. (2.9)	$\frac{\gamma+\mu}{\mu_0}$	→	$\frac{\gamma+\mu_0}{\mu_0}$
13	1. 9	independently	→	independently
19	eq. (3.13)	$Q_j$	→	$Q_{j+1}$
20	eq. (3.19)	$p(r,j,j,i)$	→	$p(r,j,i)$
		$p(r,j,i)$	→	$p(r,j,j,i)$
31	1. 16	interprestred	→	interpreted

21      eq. (3.23)       $p(a, j+1, i) \rightarrow \left(\frac{\lambda+\mu}{\lambda}\right) p(a, j+1, i)$

36      L.2       $\left[\left(\frac{\gamma+\mu}{\mu}\right)^n - 1\right] \rightarrow \left(\frac{\mu}{\gamma+\mu}\right) \left[\left(\frac{\gamma+\mu}{\mu}\right)^n - 1\right]$

38      L.38      (4.10)      →      (4.9)

Eindhoven University of Technology Research Reports

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Electrical Engineering

Eindhoven

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CIP-gegevens

Nicola, V.F.

Markovian models of a transactional system supported by  
checkpointing and recovery strategies / by V.F. Nicola. -  
Eindhoven: University of technology.

Part 2: A model with a specified number of completed  
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(Eindhoven university of technology research reports; 82-E-129)

Met lit. opg., reg.

ISBN 90-6144-129-3

ISSN 0167-9708

SISO 656 UDC 519.71

Trefw.: regeltechniek / meettechniek.

Errata to: EUT Report 82-E-129  
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### Acknowledgement

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Abstract:

A new Markovian model for checkpointing and rollback recovery in a transactional system is considered, in which checkpoints are performed after the processing of a specified number of transactions. Failures may occur during any of the different modes of operation of the system (i.e. "available for processing transactions", "checkpointing" or "recovery after a failure"). The limiting state probabilities can be recursively expressed in terms of a finite set of boundary state probabilities. The set of boundary state probabilities can be determined by solving a set of linear equations. For two special cases; namely, heavily- and lightly-loaded situations, appropriate approximations will yield explicit forms for the system availability and the mean response time of a transaction.

Nicola, V.F.

MARKOVIAN MODELS OF A TRANSACTIONAL SYSTEM SUPPORTED BY CHECKPOINTING AND RECOVERY STRATEGIES. Part 2: A model with a specified number of completed transactions between checkpoints.

Department of Electrical Engineering, Eindhoven University of Technology, 1982.

EUT Report 82-E-129

Address of the author:

Group Measurement and Control,  
Department of Electrical Engineering,  
Eindhoven University of Technology,  
P.O. Box 513,  
5600 MB EINDHOVEN,  
The Netherlands



1. Introduction

A common strategy to keep the integrity of information and to enhance the reliability of operation in information processing and storage systems (database systems), is to save copies of the relevant information (i.e. the information needed to restore the system to its status at the time when the copy is made) in a secondary storage device (disk or tape) at successive instants of time. This saving process is called a checkpoint operation. During a checkpoint the system is unavailable for useful processing of transactions (a transaction may be defined as one or more tasks to be performed by the computer system). The processed transactions since the last checkpoint are recorded in a file called an audit trail.

Failures which invalidate the integrity of the information stored in the system, occur at random, due to hardware, software, program, operator, .... etc.

When a failure is detected (we assume that failures are detected as soon as they occur) and a corrective action is performed, a recovery operation is initiated. In a recovery operation a rollback procedure is performed which makes use of the saved information (the information saved during the last checkpoint operation) to restore the system to its status at the last checkpoint. The rollback procedure is followed by the reprocessing of all transactions which were processed since the last checkpoint.

The recovery operation is completed when reprocessing reaches the point at which the failure occurred (or was detected). During a recovery operation the system is unavailable for useful processing of transactions.

In this report we consider the case in which checkpoints are performed after the completion of a predetermined number of transactions. It is obvious that the more completed transactions between checkpoints, the greater will be the amount of time spent by the system in reprocessing during recoveries after random failures, and the fewer the completed transactions between checkpoints, the greater the amount of time spent

by the system in checkpointing.

Thus, it is reasonable to expect the existence of an optimum number of completed transactions between successive checkpoints.

Several authors [1,3,4,5,6,7,11,12] have presented models in which checkpoints are performed at subsequent time steps (independent of the number of completed transactions during these time intervals). They assumed certain time distribution for the interval between successive checkpoints and considered the problem of determining the optimum interval which maximizes the system availability (i.e. the fraction of time in which the system is available for useful processing).

Mikou and Tucci [10] considered a model in which checkpoints are performed after the completion of a fixed number of transactions. They proposed an M/M/1 queue subject to breakdowns as a model, and assumed that the departure process is a Poisson (i.e. exponential time distribution between successive completions of transactions). They also assumed a small failure rate and determined the optimum number of completions between checkpoints which maximizes the system availability.

In almost all previous work, a small failure rate was an essential assumption in order to keep the models simple and the analysis tractable.

In this report we present and analyse two Markovian models, for checkpointing and rollback recovery strategies, supporting a transactional system in saturated and non-saturated conditions.

Checkpoints are performed after the completion of a number of transactions. Failures may occur randomly at any state of the system operation (i.e. available, checkpointing and recovery).

A saturated condition arises when the system is operating in a batch environment in which transactions are processed one after another. The system may become unavailable for useful processing (due to checkpoints or recoveries) but it is never idle (as long as the batch is not completed). For such a system it is of interest to determine the optimum number of completed transactions between successive checkpoints

which maximizes the system availability (this minimizes the batch execution time).

A non-saturated condition arises when the system is operating in an on-line environment in which transactions arrive randomly at the system. They are processed according to a "First come - first served" discipline when the system is available for useful processing. The system may be unavailable for useful processing (due to checkpoints or recoveries) but transactions keep arriving randomly at the system. The system is idle when there are no transactions waiting for processing (or being processed) while the system is available. For such a system, it is of interest to determine the optimum number of completed transactions between successive checkpoints which maximizes the system availability or which minimizes the mean response time of a transaction.

In chapter 2, a model of the saturated system is considered; this model is analytically tractable. An expression for the system availability is obtained for a deterministic or a random number of completed transactions between successive checkpoints. The optimum number which maximizes the system availability is determined.

In chapter 3, a model of the non-saturated system is considered. Section 3.1 is devoted to the numerical computation of the limiting state-probabilities (and the performance variables). In section 3.2., a state-space analysis approach is used to derive expressions for the performance variables in terms of a set of state probabilities (boundary states). Explicit expressions for the performance variables are difficult to obtain in the general case.

In special cases, simplifying approximations will enable us to obtain explicit expressions for the performance variables. Two of these cases, namely heavily-loaded and lightly-loaded systems will be considered in chapter 4.

## 2. Model of the saturated system

In this chapter we introduce a mathematical model of the saturated system and consider its analysis. This model also corresponds to a system operating in a batch environment, where transactions are processed one after another. The system is never idle during a batch execution.

Each transaction requires an exponential service time with a mean  $\mu^{-1}$ . Checkpoints are performed after the completion of a fixed number ( $n$ ) of transactions (a random number ( $n$ ) will be considered later in this chapter). Checkpoint durations are exponential with a mean  $\beta^{-1}$ . Failures occur (and are instantaneously detected) according to a Poisson process at a rate  $\gamma$ .

When a failure is detected during the processing of the  $j$ -th transaction after the most recent checkpoint, a recovery action is initiated. It starts with a rollback operation which restores the system to its status at the most recent checkpoint. The rollback duration is exponential with a mean  $\mu_0^{-1}$ . This is followed by the reprocessing of  $j$  transactions corresponding (but not identical) to the transactions processed since the last checkpoint. Each transaction requires an exponential reprocessing time with a mean  $\mu^{-1}$  (here we assume identical processing and reprocessing time distributions of transactions).

A recovery operation is completed when reprocessing reaches the point at which the failure was detected.

Failures may occur during a checkpoint or a recovery operation (we assume identical failure processes during different system operations) in which case a corresponding (but not identical) operation is re-started.

Transaction processing is blocked during checkpointing and recovery operations.

A state transition diagram which represents the behaviour of this model is shown in figures (2.1) and (2.2), in which we make use of the following notations.

The state "c" corresponds to the checkpointing mode of operation.

The state "a,j" corresponds to the available mode of operation, during

the processing of the  $j$ -th transaction after the most recent checkpoint.

The state "a" corresponds to the set of states " $a,j$ ",  $j = 1,2,\dots,n$ .  $n$  is the number of completed transactions between successive checkpoints.

The state " $r,j,k$ " corresponds to the recovery mode of operation, in which  $j$  transactions have to be reprocessed, during the reprocessing of the  $k$ -th transaction ( $k = 0$  corresponds to a rollback operation).

The state " $r,j$ " corresponds to the set of states " $r,j,k$ ",  $k = 0,1,2,\dots,j$ .

The state " $r$ " corresponds to the set of states " $r,j$ ",  $j = 1,2,\dots,n$ .

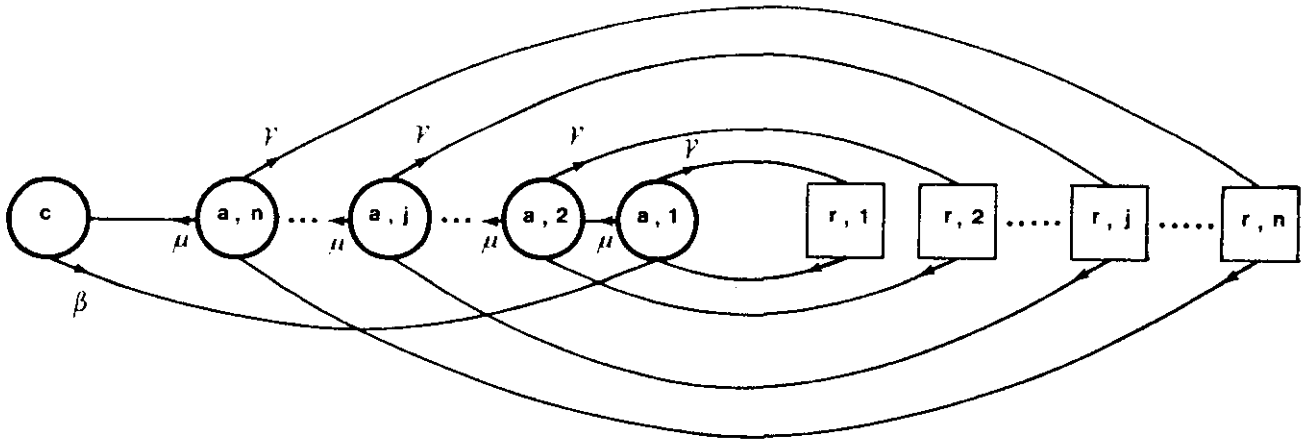


Fig. (2.1) State transition diagram representing the model of the saturated system with checkpointing and recovery operations.

The circle ( $\bigcirc$ ) stands for a state with exponential distribution of residence time. The square ( $\square$ ) stands for a state with general distribution of residence time.

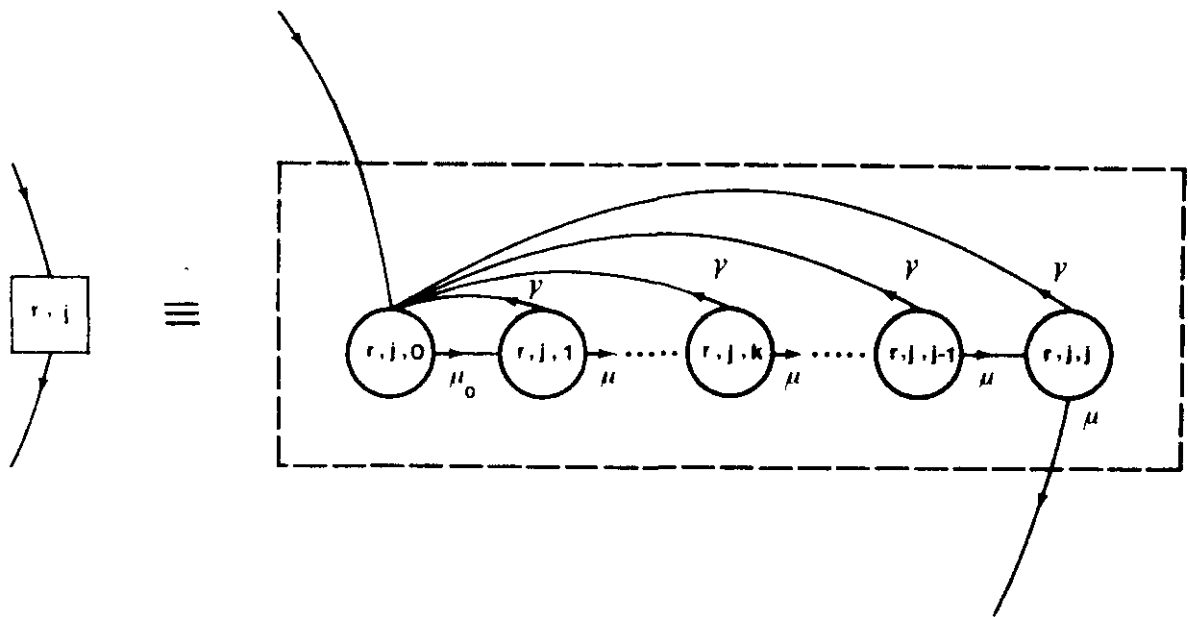


Fig. (2.2) State transition diagram representing the model of the rollback recovery operation followed by a failure detected during the processing of the  $j$ -th transaction after the last checkpoint.

## 2.1 Determination of the limiting state probabilities

In this section we determine analytically the limiting state probabilities in the model of the saturated system. This will yield an expression for the system availability (i.e. the fraction of time the system is available for processing transactions).

Consider the following state probabilities

$p(c)$  corresponding to the state "c"

$p(a, j)$  corresponding to the state "a, j",  $1 < j < n$

$p(r, j, k)$  corresponding to the state "r, j, k",  $0 < k < j, 1 < j < n$

$p(r, j)$  corresponding to the state "r, j",  $1 < j < n$

$p(r)$  corresponding to the state "r"

$p(a)$  corresponding to the state "a"

It follows, from earlier definitions of the states that

$$(2.1) \quad p(a) = \sum_{j=1}^n p(a, j)$$

$$(2.2) \quad p(r, j) = \sum_{k=0}^j p(r, j, k)$$

$$(2.3) \quad p(r) = \sum_{j=1}^n p(r, j)$$

The state probabilities  $p(r, j, k)$ ,  $0 < k < j$ , can be expressed in terms of the state probability  $p(a, j)$  as follows.

Transition balance at the state "r, j" yields

$$(2.4) \quad p(r, j, j) = \frac{\gamma}{\mu} p(a, j)$$

Transition balance at the states "r, j, k+1",  $k = j-1, j-2, \dots, 1$  yield the following recursive equations

$$(2.5) \quad p(r, j, k) = \left(\frac{\gamma + \mu}{\mu}\right) p(r, j, k+1), \quad k = j-1, j-2, \dots, 1$$

and

$$(2.6) \quad p(r, j, 0) = \left(\frac{\gamma + \mu}{\mu_0}\right) p(r, j, 1)$$

It follows from equations (2.4), (2.5) and (2.6) that

$$(2.7) \quad p(r, j, k) = \frac{\gamma}{\mu} \left(\frac{\gamma + \mu}{\mu}\right)^{j-k} p(a, j), \quad 1 \leq k \leq j$$

and

$$(2.8) \quad p(r, j, 0) = \frac{\gamma}{\mu_0} \left(\frac{\gamma + \mu}{\mu}\right)^j p(a, j)$$

Using equations (2.7) and (2.8) in equation (2.2) we get  $p(r, j)$  expressed in terms of  $p(a, j)$

$$(2.9) \quad p(r, j) = \left[\left(\frac{\gamma + \mu}{\mu_0}\right) \left(\frac{\gamma + \mu}{\mu}\right)^j - 1\right] p(a, j)$$

Equations (2.7), (2.8) and (2.9) hold for all  $j$ ,  $1 \leq j \leq n$ .

The state probabilities  $p(a, j)$ ,  $2 \leq j \leq n$ , can be expressed in terms of the state probability  $p(a, 1)$  as follows.

Transition balance at the states "a, j",  $j = 2, 3, \dots, n$ , yields the following recursive equations:

$$(2.10) \quad p(a, j) = p(a, j-1), \quad j = 2, 3, \dots, n,$$

It follows that

$$(2.11) \quad p(a, j) = p(a, 1), \quad 2 \leq j \leq n$$

and from equation (2.1) we get

$$(2.12) \quad p(a) = n p(a, 1)$$

The state probability  $p(c)$  can be expressed in terms of  $p(a, 1)$  as follows. Transition balance at the state "a, 1" yields



$$(2.13) \quad p(c) = \frac{\mu}{\beta} p(a,1)$$

Substituting from equations (2.9) and (2.11) in equation (2.3) yields

$$(2.14) \quad p(r) = \left[ \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] - n \right] p(a,1)$$

It follows from equations (2.12) and (2.14) that

$$(2.15) \quad p(a) + p(r) = \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] p(a,1)$$

Equations (2.13) and (2.15) together with the normalizing condition

$$p(a) + p(r) + p(c) = 1$$

yield an explicit expression for  $p(a,1)$  given by

$$(2.16) \quad p(a,1) = \left[ \frac{\mu}{\beta} + \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] \right]^{-1}$$

Now all state probabilities can be determined by substituting from equation (2.16) into their appropriate expressions.

## 2.2 The system availability and performance optimization

In this section we obtain expressions for the system availability for a fixed and a random number of completed transactions between successive checkpoints. The system availability (A) can be defined as follows:

$$A = \frac{E[a]}{E[a] + E[c] + E[r]}$$

where  $E[a]$  is the expected time spent by the system in the available state between successive checkpoints

$E[c]$  is the expected time spent by the system in the checkpointing state, and

$E[r]$  is the expected time spent by the system in the recovery state between successive checkpoints.

For a fixed number (n) of completed transactions between checkpoints,

we have

$$E[a] = \frac{n}{\mu}$$

$$E[c] = \frac{1}{\beta}$$

$$E[r, j] = \frac{\gamma}{\mu} \cdot \frac{1}{\gamma} \left[ \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\mu} \right)^j - 1 \right]$$

$$\begin{aligned} E[r] &= \sum_{j=1}^n E[r, j] \\ &= \frac{1}{\mu} \sum_{j=1}^n \left[ \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\mu} \right)^j - 1 \right] \\ &= \frac{1}{\mu} \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] - \frac{n}{\mu} \end{aligned}$$

It follows that the system availability  $A(n)$ , for a fixed  $n$ , is given by

$$(2.17) \quad A(n) = n \left[ \frac{\mu}{\beta} + \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] \right]^{-1}$$

which is equal to  $p(a)$  as determined from equations (2.12) and (2.16).

Differentiating equation (2.17) with respect to  $n$  and equating to zero yields

$$(2.18) \quad \left( \frac{\gamma + \mu}{\mu} \right)^n \left[ 1 - n \ln \left( \frac{\gamma + \mu}{\mu} \right) \right] = \left[ 1 - \frac{\mu}{\beta} \left( \frac{\mu_0}{\gamma + \mu_0} \right) \left( \frac{\gamma}{\gamma + \mu} \right) \right]$$

The optimal number  $\hat{n}$  which maximizes the system availability is the closest integer to the real solution  $\tilde{n}$  of equation (2.18) in  $n$ .

For small values of  $\gamma$ , an approximation of  $\hat{n}$  is the closest integer to the real value  $\tilde{n}$  given by

$$(2.19) \quad \bar{n} = \mu \left[ \frac{2}{\gamma\beta} \left( 1 - \frac{\gamma}{\mu_0} \right) \right]^{\frac{1}{2}}$$

Let  $n$  be a random number with the generating function defined by

$$G_n(z) \triangleq \sum_{k=0}^{\infty} p_k z^k$$

where  $p_k = P[n = k]$  is the probability that  $n$  takes the integer value  $k$  (thus  $\sum_{k=0}^{\infty} p_k = 1$ ).

The expected time spent by the system in different states is given by

$$E[a] = \frac{1}{\mu} \sum_{k=0}^{\infty} k p_k$$

$$E[c] = \frac{1}{\beta}$$

$$E[r] = \frac{1}{\mu} \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \sum_{k=0}^{\infty} \left( \frac{\gamma + \mu}{\mu} \right)^k p_k - 1 \right] - \frac{1}{\mu} \sum_{k=0}^{\infty} k p_k$$

An expression for the system availability (A) follows

$$(2.20) \quad A = \bar{n} \left[ \frac{\mu}{\beta} + \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ G_n \left( \frac{\gamma + \mu}{\mu} \right) - 1 \right] \right]^{-1}$$

with  $\bar{n} (= \sum_{k=0}^{\infty} k p_k)$  is the mean of the random integer  $n$ .

If  $n$  is a Poisson random integer with mean  $\bar{n}$ , then

$$\begin{aligned} G_n(z) &= \sum_{k=0}^{\infty} \frac{(\bar{n})^k}{k!} e^{-\bar{n}} z^k \\ &= e^{\bar{n}(z-1)} \end{aligned}$$

The system availability  $A(\bar{n})$ , for a Poisson random number of completed transactions between successive checkpoints is given by

$$(2.21) \quad A(\bar{n}) = \bar{n} \left[ \frac{\mu}{\beta} + \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ e^{\bar{n} \frac{\gamma}{\mu}} - 1 \right] \right]^{-1}$$

The optimum  $\hat{\bar{n}}$  which maximizes the system availability is the solution of the following equation

$$(2.22) \quad e^{\bar{n} \frac{\gamma}{\mu}} \left( 1 - \bar{n} \frac{\gamma}{\mu} \right) = \left[ 1 - \frac{\mu}{\beta} \left( \frac{\mu_0}{\gamma + \mu_0} \right) \left( \frac{\gamma}{\gamma + \mu} \right) \right]$$

For small values of  $\gamma$ , an approximation of  $\hat{\bar{n}}$  is given by equation (2.19).

### 3. Model of the non-saturated system

The mathematical model of the non-saturated system is similar to the mathematical model described in chapter 2, except for some essential differences which are mentioned here. This model corresponds to a system operating in an on-line environment, where transactions arrive randomly at the system and are processed according to a "First Come - First Served" discipline. The system is idle when it is available and there are no transactions to be processed. Transactions arrive according to a Poisson process at a rate  $\lambda$ , independently of the mode of the the system operation (i.e. available, checkpointing or recovery). They are processed at a rate  $\mu$  when the system is available.

Processed transactions since the most recent checkpoint are recorded in a file called an audit trail. They are reprocessed during a recovery operation when a failure is detected.

Checkpoints are performed after the completion of a fixed number ( $n$ ) of transactions.

Failures occur (and are instantaneously detected) according to a Poisson process at a rate  $\gamma$ , independently of the mode of the system operation. When a failure is detected during normal (available) operation, it is followed by a recovery operation (rollback and reprocessing of the recorded transactions in the audit trail). When a failure is detected during a checkpoint or a recovery operation, a corresponding (but not identical) operation is restarted.

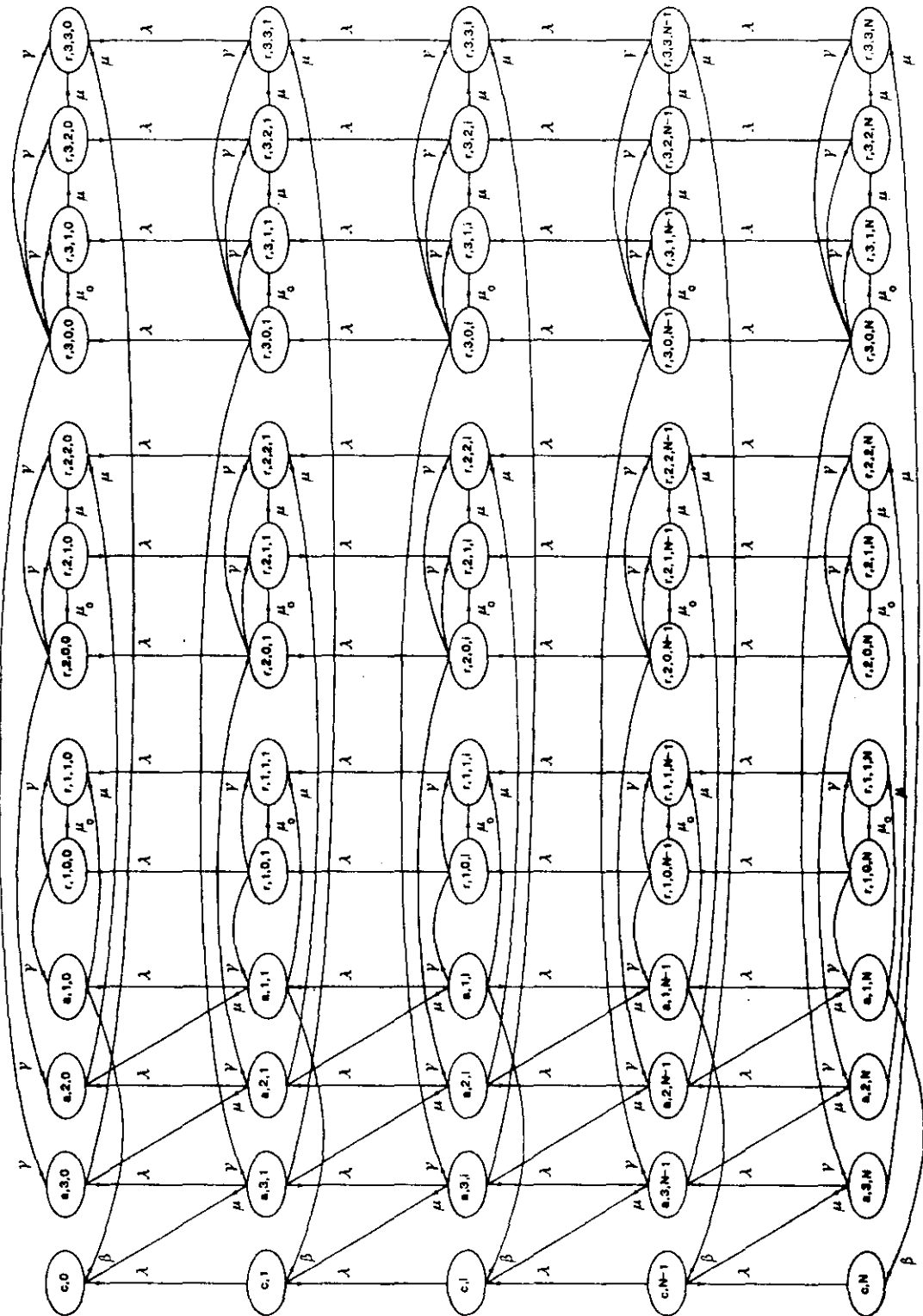
No transactions are processed during checkpointing or recovery operations.

A state transition diagram representing the model of the non-saturated system is shown in figure (3.1), in which the following notations are used.

The index "m" (m = a for available, c for checkpointing or r for recovery) indicates the mode of the system operation.

- The index "i"  $(0 < i < N)$  indicates the number of transactions in the system (queued and in processing).  $N$  is the size of the waiting room.
- The index "j"  $(1 < j < n)$  indicates the number of processed transactions since the most recent checkpoint (including the transaction in processing).  $n$  is the number of completed transactions between successive checkpoints.
- The index "k"  $(0 < k < j)$  indicates the number of reprocessed transactions in a recovery operation in which  $j$  transactions have to be reprocessed ( $k = 0$  corresponds to the rollback operation).
- The state "c,i" corresponds to the checkpointing mode of operation with  $i$  transactions in the system.  $p(c,i)$  is the associated probability.
- The state "a,j,i" corresponds to the available mode of operation during the processing of the  $j$ -th transaction after the most recent checkpoint, and with  $i$  transactions in the system.  $p(a,j,i)$  is the associated probability.
- The state "r,j,k,i" corresponds to the recovery mode of operation, in which  $j$  transactions have to be reprocessed, during the reprocessing of the  $k$ -th transaction and with  $i$  transactions in the system ( $k = 0$  corresponds to the rollback operation).  $p(r,j,k,i)$  is the associated probability.
- The state "r,j,i" corresponds to the set of states "r,j,k,i",  $k = 0, 1, 2, \dots, j$ .  $p(r,j,i)$  is the associated probability.

Fig. (3.1) State transitions diagram representing the model of the non-saturated system with checkpointing and recovery operations (number of completions between checkpoints  $(n) = 3$ , a waiting room of size  $N$ ).



### 3.1 Recursive computation of the limiting state probabilities

In this section we describe a numerical algorithm for the computation of the limiting state probabilities for the model of the non-saturated system with a limited waiting room equal to  $N$ .

Consider the Markov chain representing the model in fig. (3.1). This Markov chain contains  $D$  ( $D = \left(\frac{n^2 + 5n + 2}{2}\right) (N+1)$ ) states. These states can be determined by making use of  $(D-1)$  independent transition balance equations at  $(D-1)$  different states, together with the normalizing condition (all state probabilities sum to one). This forms a system of linear equations in the  $D$  unknown state probabilities. It is obvious that  $D$  can be large for small values of  $n$  and  $N$ . Significant reduction of the size of the system of linear equations can be achieved by making use of the model structure. The system in the  $D$  unknown state probabilities can be solved partially in a recursive manner. This results in a reduced system of linear equations in the  $n$  unknown boundary state probabilities ( $p(a, j, 0)$ ,  $j = 1, 2, \dots, n$ ).

In the remainder of this section we show how to express all state probabilities recursively in terms of the boundary state probabilities. For this we use  $(D-n)$  transition balance equations at  $(D-n)$  different states. The remaining  $(n-1)$  independent transition balance equations, together with the normalizing condition, form the reduced system in the  $n$  unknown boundary state probabilities. This system of  $n$  linear equations can be solved simulatenously to determine the values of the unknown boundary state probabilities. They can be used in the expressions of the other state probabilities (or the performance variables) to determine their actual values.

First we express the probabilities  $p(r, j, k, 0)$ ,  $k = 0, 1, 2, \dots, j$ , and  $p(r, j, 0)$  in terms of the probability  $p(a, j, 0)$ .

Transition balance at the states " $r, j, k+1, 0$ ",  $k = j-1, j-2, \dots, 1$ , yield the following recursive relations



$$(3.1) \quad p(r, j, k, 0) = \left( \frac{\lambda + \gamma + \mu}{\mu} \right) p(r, j, k+1, 0), \quad k = j-1, j-2, \dots, 1,$$

and

$$(3.2) \quad p(r, j, 0, 0) = \left( \frac{\lambda + \gamma + \mu}{\mu_0} \right) p(r, j, 1, 0)$$

From (3.1) and (3.2) we can express  $p(r, j, k, 0)$ ,  $0 < k < j-1$ , and  $p(r, j, 0)$  in terms of  $p(r, j, j, 0)$

$$(3.3) \quad p(r, j, k, 0) = \left( \frac{\lambda + \gamma + \mu}{\mu} \right)^{j-k} p(r, j, j, 0), \quad 1 < k < j-1$$

$$(3.4) \quad \frac{\mu_0}{\mu} p(r, j, 0, 0) = \left( \frac{\lambda + \gamma + \mu}{\mu} \right)^j p(r, j, j, 0)$$

and

$$(3.5) \quad p(r, j, 0) \triangleq \sum_{k=0}^j p(r, j, k, 0) \\ = \frac{\mu(1-Q_j)}{(\lambda + \gamma)Q_j} p(r, j, j, 0)$$

with

$$Q_k \triangleq \left( \frac{\mu_0}{\lambda + \gamma + \mu_0} \right) \left( \frac{\mu}{\lambda + \gamma + \mu} \right)^k$$

Note that  $Q_k$  is the probability of no failure or arrival during the rollback operation and the reprocessing of the first  $k$  transactions in a recovery operation.

Transition balance at the state " $r, j, 0$ " yields

$$(3.6) \quad p(r, j, j, 0) = \frac{\gamma}{\mu} p(a, j, 0) - \frac{\lambda}{\mu} p(r, j, 0)$$

Substitution from (3.5) in (3.6) yields an expression for  $p(r, j, j, 0)$  in terms of  $p(a, j, 0)$

$$(3.7) \quad p(r, j, j, 0) = \left(\frac{\gamma}{\mu}\right) \left(\frac{(\lambda+\gamma)Q_j}{\lambda+\gamma Q_j}\right) p(a, j, 0)$$

Finally we can express  $p(r, j, k, 0)$ ,  $k = 0, 1, 2, \dots, j$ , and  $p(r, j, 0)$  in terms of  $p(a, j, 0)$ , as follows

$$(3.8) \quad p(r, j, k, 0) = \left(\frac{\gamma}{\mu}\right) \left(\frac{(\lambda+\gamma)Q_k}{\lambda+\gamma Q_j}\right) p(a, j, 0), \quad 1 < k < j$$

$$(3.9) \quad p(r, j, 0, 0) = \left(\frac{\gamma}{\mu_0}\right) \left(\frac{(\lambda+\gamma)Q_0}{\lambda+\gamma Q_j}\right) p(a, j, 0),$$

and

$$(3.10) \quad p(r, j, 0) = \frac{\gamma(1-Q_j)}{(\lambda+\gamma Q_j)} p(a, j, 0)$$

The sum  $[p(r, j, 0) + p(a, j, 0)]$  will be used later in balance equations; it can be expressed in terms of  $p(a, j, 0)$ . From (3.10) we get

$$(3.11) \quad p(r, j, 0) + p(a, j, 0) = \left(\frac{\lambda+\gamma}{\lambda+\gamma Q_j}\right) p(a, j, 0)$$

Equations (3.8), (3.9), (3.10) and (3.11) hold for all  $j$ ,  $1 < j < n$ .

The probability  $p(c, 0)$  can be expressed in terms of the probability  $p(a, 1, 0)$  as follows.

Transition balance at the set of states "a, 1, 0" and "r, 1, 0", making use of (3.11), yields

$$(3.12) \quad p(c, 0) = \frac{\lambda}{\beta} \left(\frac{\lambda+\gamma}{\lambda+\gamma Q_1}\right) p(a, 1, 0)$$

Now we have expressed all state probabilities with the index "i" is equal to zero ( $i=0$ ) in terms of the boundary state probabilities  $p(a, j, 0)$ ,  $j = 1, 2, \dots, n$ .

The next step is to express the state probabilities  $p(a, j, 1)$ ,  $j = 1, 2, \dots, n$ , in terms of the boundary state probabilities. This can be accomplished as follows.

Transition balance at the set of states "a,j+1,0" and "r,j+1,0",  
 $1 < j < n-1$ , making use of (3.11), yields an expression for  $p(a,j,1)$ ,  
 $1 < j < n-1$ ,

$$(3.13) \quad p(a,j,1) = \left(\frac{\lambda}{\mu}\right) \left(\frac{\lambda+\gamma}{\lambda+\gamma Q_j}\right) p(a,j+1,0), \quad 1 < j < n-1$$

Transition balance at the state "c,0", using (3.12) yields an  
 expression for  $p(a,n,1)$  in terms of  $p(a,1,0)$

$$(3.14) \quad p(a,n,1) = \frac{\lambda}{\beta} \left(\frac{\lambda+\beta}{\mu}\right) \left(\frac{\lambda+\gamma}{\lambda+\gamma Q_1}\right) p(a,1,0)$$

The state probabilities  $p(r,j,k,i)$ ,  $0 < k < j$ ,  $1 < j < n$ , for  $i =$   
 $1, 2, \dots, N-1$ , can be expressed in terms of previously determined state  
 probabilities; namely,  $p(a,j,i)$  and  $p(r,j,k,i-1)$ ,  $k = 0, 1, \dots, j$ , as  
 follows.

Transition balance at the state "r,j,k+1,i",  $k = j-1, j-2, \dots, 1$ , yield  
 the following recursive relations

$$(3.15) \quad p(r,j,k,i) = \left(\frac{\lambda+\gamma+\mu}{\mu}\right) p(r,j,k+1,i) - \frac{\lambda}{\mu} p(r,j,k+1,i-1),$$

$$k = j-1, j-2, \dots, 1,$$

and

$$(3.16) \quad p(r,j,0,i) = \left(\frac{\lambda+\gamma+\mu}{\mu_0}\right) p(r,j,1,i) - \frac{\lambda}{\mu_0} p(r,j,1,i-1)$$

From (3.15) and (3.16) we can express  $p(r,j,k,i)$ ,  $0 < k < j-1$  and  
 $p(r,j,i)$  in terms of  $p(r,j,j,i)$  and previously determined probabilities

$$(3.17) \quad p(r,j,k,i) = \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{j-k} p(r,j,j,i)$$

$$- \frac{\lambda}{\mu} \sum_{\ell=k+1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-k-1} p(r,j,\ell,i-1)$$

,  $1 < k < j-1$

$$(3.18) \quad \frac{\mu_0}{\mu} p(r,j,0,i) = \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^j p(r,j,j,i)$$

$$- \frac{\lambda}{\mu} \sum_{\ell=1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-1} p(r,j,\ell,i-1)$$

and

$$\begin{aligned}
 (3.19) \quad p(r, j, j, i) &\stackrel{\Delta}{=} \sum_{k=0}^j p(r, j, k, i) \\
 &= \frac{\mu(1-Q_j)}{(\lambda+\gamma)Q_j} p(r, j, i) - \frac{\lambda}{\mu} \sum_{k=1}^{j-1} \sum_{\ell=k+1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-k-1} p(r, j, \ell, i-1) \\
 &\quad - \frac{\lambda}{\mu_0} \sum_{\ell=1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-1} p(r, j, \ell, i-1)
 \end{aligned}$$

Transition balance at the state "r, j, i" yields

$$(3.20) \quad p(r, j, j, i) = \frac{\lambda}{\mu} (p(r, j, i-1) - p(r, j, i)) + \frac{\gamma}{\mu} p(a, j, i)$$

Substitution from (3.19) in (3.20) yields an expression for  $p(r, j, j, i)$  in terms of previously determined probabilities

$$\begin{aligned}
 (3.21) \quad p(r, j, j, i) &= \left(\frac{\gamma}{\mu}\right) \frac{(\lambda+\gamma)Q_j}{(\lambda+\gamma)Q_j} [p(a, j, i) + \frac{\lambda}{\gamma} \left[ \frac{\lambda}{\mu} \sum_{k=1}^{j-1} \sum_{\ell=k+1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-k-1} p(r, j, \ell, i-1) \right. \\
 &\quad \left. + \frac{\lambda}{\mu_0} \sum_{\ell=1}^j \left(\frac{\lambda+\gamma+\mu}{\mu}\right)^{\ell-1} p(r, j, \ell, i-1) + p(r, j, i-1) \right]]
 \end{aligned}$$

Substitution from (3.21) in (3.17), (3.18) and (3.19) yields the desired expressions for  $p(r, j, k, i)$ ,  $0 < k < j$ , and  $p(r, j, i)$  in terms of previously determined probabilities; namely,  $p(a, j, i)$  and  $p(r, j, k, i-1)$ ,  $k = 0, 1, \dots, j$ .

The probability  $p(c, i)$ , for  $i = 1, 2, \dots, N-1$ , can be determined from the transition balance equation at the set of states "a, 1, i" and "r, 1, i", this yields

$$\begin{aligned}
 (3.22) \quad p(c, i) &= \frac{\lambda}{\beta} [p(a, 1, i) + p(r, 1, i) - p(a, 1, i-1) - p(r, 1, i-1)] \\
 &\quad + \frac{\mu}{\beta} p(a, 1, i), \quad \text{for } i=1, 2, \dots, N-1.
 \end{aligned}$$

In (3.22)  $p(c, i)$  is expressed in terms of previously determined probabilities.

The probability  $p(a, j, i+1)$ ,  $1 < j < n-1$ , for  $i = 1, 2, \dots, N-1$ , can be determined from the transition balance equation at the set of states "a, j+1, i" and "r, j+1, i", this yields

$$(3.23) \quad p(a, j, i+1) = \frac{\lambda}{\mu} [p(a, j+1, i) + p(r, j+1, i) - p(a, j+1, i-1) - p(r, j+1, i-1)], \quad 1 < j < n-1, \text{ for } i = 1, 2, \dots, N-1.$$

The probability  $p(a, n, i+1)$ , for  $i = 1, 2, \dots, N-1$ , can be determined from the transition balance equation at the state "c, i", this yields

$$(3.24) \quad p(a, n, i+1) = \left(\frac{\lambda+\beta}{\mu}\right) p(c, i) - \frac{\lambda}{\mu} p(c, i-1)$$

Thus from (3.23) and (3.24) we can express the probabilities  $p(a, j, i+1)$ ,  $1 < j < n$ , for  $i = 1, 2, \dots, N-1$ , in terms of previously determined probabilities.

The last probabilities to be determined are  $p(r, j, k, N)$ ,  $0 < k < j$ ,  $1 < j < n$  and  $p(c, N)$ . Equations (3.17), (3.18) and (3.19) do not hold for  $i = N$ .

Transition balance at the states "r, j, k+1, N",  $k = j-1, j-2, \dots, 1$  yield the following recursive relations

$$(3.25) \quad p(r, j, k, N) = \left(\frac{\gamma+\mu}{\mu}\right) p(r, j, k+1, N) - \frac{\lambda}{\mu} p(r, j, k+1, N-1),$$

$$k = j-1, j-2, \dots, 1$$

and

$$(3.26) \quad p(r, j, 0, N) = \left(\frac{\gamma+\mu}{\mu_0}\right) p(r, j, 1, N) - \frac{\lambda}{\mu_0} p(r, j, 1, N-1)$$

It follows from (3.25) and (3.26) that

$$(3.27) \quad p(r, j, k, N) = \left(\frac{\gamma+\mu}{\mu}\right)^{j-k} p(r, j, j, N) - \frac{\lambda}{\mu} \sum_{\ell=k+1}^j \left(\frac{\gamma+\mu}{\mu}\right)^{\ell-k-1}$$

$$p(r, j, \ell, N-1), \quad 1 < k < j-1.$$

$$(3.28) \quad \frac{\mu_0}{\mu} p(r, j, 0, N) = \left(\frac{\gamma + \mu}{\mu}\right)^j p(r, j, j, N) - \frac{\lambda}{\mu} \sum_{\ell=1}^j \left(\frac{\gamma + \mu}{\mu}\right)^{\ell-1} p(r, j, \ell, N-1)$$

and

$$(3.29) \quad p(r, j, N) = \frac{\mu(1-P_j)}{\gamma P_j} p(r, j, j, N) - \frac{\lambda}{\mu} \sum_{k=1}^{j-1} \sum_{\ell=k+1}^j \left(\frac{\gamma + \mu}{\mu}\right)^{\ell-k-1} p(r, j, \ell, N-1) - \frac{\lambda}{\mu_0} \sum_{\ell=1}^j \left(\frac{\gamma + \mu}{\mu}\right)^{\ell-1} p(r, j, \ell, N-1)$$

with  $P_j \triangleq \left(\frac{\mu_0}{\gamma + \mu_0}\right) \left(\frac{\mu}{\gamma + \mu}\right)^j$

Transition balance at the state "r, j, N" yields an expression for p(r, j, j, N) in terms of previously determined probabilities

$$(3.30) \quad p(r, j, j, N) = \frac{\gamma}{\mu} p(a, j, N) + \frac{\lambda}{\mu} p(r, j, N-1)$$

Substitution from (3.30) in (3.27), (3.28) and (3.29) yields expressions for p(r, j, k, N), 0 < k < j and p(r, j, N) in terms of previously determined probabilities; namely, p(a, j, N) and p(r, j, k, N-1), k = 0, 1, ..., j.

Transition balance at the state "c, N" yields an expression for p(c, N) in terms of p(c, N-1)

$$(3.31) \quad p(c, N) = \frac{\lambda}{\beta} p(c, N-1)$$

So far we have expressed, recursively, all the state probabilities of the Markov chain in fig. (3.1), in terms of the boundary state probabilities p(a, j, 0), 1 < j < n. There are n transition balance equations at the states "a, j, N", 1 < j < n, which were not used in the

recursive procedure. They contain  $(n-1)$  independent equations which can also be obtained from the transition balance equations at the sets of states "a,j,N" and "r,j,N", for  $j = 2,3,\dots,n$ ,

$$(3.32) \quad \mu p(a,j,N) = \lambda [p(a,j,N-1) + p(r,j,N-1)] , j = 2,3,\dots,n.$$

Equations (3.32) together with the normalizing equation (all state probabilities add up to one) form a system of  $n$  linear independent equations in the unknown boundary state probabilities,  $p(a,j,0)$ ,  $1 < j < n$ . This system of linear equations can be solved simultaneously to determine the values of the boundary state probabilities. These values can be substituted in the expressions of other state probabilities (or performance variables) to get their actual values.

A simple numerical algorithm for the recursive determination of the state probabilities is proposed in the following.

Any state probability ( $p$ ) in the Markov chain in fig. (3.1) can be written as a linear sum of the  $n$  boundary state probabilities  $p(a,j,0)$ ,  $1 < j < n$ , as follows

$$(3.33) \quad p = \sum_{j=1}^n g_j p(a,j,0)$$

where  $g_j$  is the coefficient of  $p(a,j,0)$  in the linear sum. It is then possible to determine the values of all the coefficients  $g_j$  for all the state probabilities in the Markov chain by letting  $p(a,j,0)$  be equal to one and all other boundary state probabilities set equal to zero. The recursive procedure, described above is then used to evaluate the coefficients  $g_j$ . By evaluating all the coefficients  $g_j$ , for  $j = 1,2,\dots,n$ , we have expressions for all state probabilities as a linear sum of the boundary state probabilities.

### 3.2 Analytical derivation of performance variables

It is of much interest to derive relations for some performance quantities such as the system availability and the average number of transactions in the system. In this section we use a state-space analysis approach to derive these relations. The resulting expressions for the

performance quantities are not explicit forms; they are functions of the system parameters as well as the boundary state probabilities  $p(a,j,0)$ ,  $1 < j < n$ .

In the following analysis we will consider the Markov chain of fig. (3.1) with an infinite state space (representing a system with unlimited waiting room,  $N = \infty$ ).

Define the following sets of states and the associated probabilities.

The set of states "c" corresponds to all the states "c,i", for  $i = 0, 1, \dots, \infty$ .

$A(c)$  is the associated probability.

The set of states "a,j" corresponds to all the states "a,j,i" for  $i = 0, 1, \dots, \infty$ .

$A(a,j)$  is the associated probability.  $1 < j < n$ .

The set of states "a" corresponds to all the sets of states "a,j", for  $j = 1, 2, \dots, n$ .

$A(a)$  is the associated probability.

The set of states "r,j,k" corresponds to all the states "r,j,k,i", for  $i = 0, 1, \dots, \infty$ .

$A(r,j,k)$  is the associated probability.  $0 < k < j$  and  $1 < j < n$ .

The set of states "r,j" corresponds to all the sets of states "r,j,k", for  $k = 0, 1, \dots, j$ .

$A(r,j)$  is the associated probability.  $1 < j < n$ .

The set of states "r" corresponds to all the sets of states "r,j", for  $j = 1, 2, \dots, n$ .

$A(r)$  is the associated probability.

The set of states "i" corresponds to all the sets of states "a,j,i", "r,j,i", for  $j = 1, 2, \dots, n$ , and the states "c,i".  $0 < i < \infty$ .

$p(i)$  is the associated probability.

Define the following quantities

$$B(c) \triangleq \sum_{i=1}^{\infty} p(c,i)$$



$$B(a, j) \triangleq \sum_{i=1}^{\infty} p(a, j, i), \quad 1 \leq j \leq n ,$$

$$B(r, j, k) \triangleq \sum_{i=1}^{\infty} p(r, j, k, i), \quad 0 \leq k \leq j \text{ and } 1 \leq j \leq n .$$

It follows that

$$(3.34) \quad A(c) = p(c, 0) + B(c)$$

$$(3.35) \quad A(a, j) = p(a, j, 0) + B(a, j), \quad 1 \leq j \leq n ,$$

$$(3.36) \quad A(a) = \sum_{j=1}^n A(a, j) ,$$

$$(3.37) \quad A(r, j, k) = p(r, j, k, 0) + B(r, j, k), \quad 0 \leq k \leq j \text{ and } 1 \leq j \leq n ,$$

$$(3.38) \quad A(r, j) = \sum_{k=0}^j A(r, j, k) , \quad 1 \leq j \leq n ,$$

$$(3.39) \quad A(r) = \sum_{j=1}^n A(r, j) .$$

Now, we proceed to relate the defined probabilities.

The probabilities  $A(r, j, k)$ ,  $0 \leq k \leq j$ , and thus  $A(r, j)$ , can be expressed in terms of the probability  $A(a, j)$  as follows.

Transition balance at the set of states "r, j" yields

$$(3.40) \quad A(r, j, j) = \frac{\gamma}{\mu} A(a, j)$$

Transition balance at the sets of states "r, j, k+1"  $k = j-1, j-2, \dots, 1$ , yields the following recursive relations.

$$(3.41) \quad A(r, j, k) = \left( \frac{\gamma + \mu}{\mu} \right) A(r, j, k+1) , \quad k = j-1, j-2, \dots, 1 ,$$

$$(3.42) \quad A(r, j, 0) = \left( \frac{\gamma + \mu}{\mu_0} \right) A(r, j, 1)$$

It follows from the equations (3.40), (3.41) and (3.42) that

$$(3.43) \quad A(r, j, k) = \frac{\gamma}{\mu} \left( \frac{\gamma + \mu}{\mu} \right)^{j-k} A(a, j) \quad , \quad 1 \leq k \leq j$$

$$(3.44) \quad A(r, j, 0) = \frac{\gamma}{\mu_0} \left( \frac{\gamma + \mu}{\mu} \right)^j A(a, j) \quad ,$$

and from (3.38) we have

$$(3.45) \quad A(r, j) = \left( \frac{1}{P_j} - 1 \right) A(a, j)$$

with

$$P_k \triangleq \left( \frac{\mu_0}{\gamma + \mu_0} \right) \left( \frac{\mu}{\gamma + \mu} \right)^k$$

Note that  $P_k$  is the probability of no failure during the rollback operation and the reprocessing of the first  $k$  transactions in a recovery operation.

Equations (3.42), (3.44) and (3.45) hold for all  $j$ ,  $1 \leq j \leq n$ .

The probabilities  $B(a, j)$ ,  $2 \leq j \leq n$ , can be expressed in terms of the probability  $B(a, 1)$  by taking transition balance at the sets of states "a, j",  $j = 2, 3, \dots, n$ . This yields the following recursive relation

$$(3.46) \quad B(a, j) = B(a, j-1) \quad , \quad j = 2, 3, \dots, n$$

It follows that

$$(3.47) \quad B(a, j) = B(a, 1) \quad , \quad 2 \leq j \leq n \quad ,$$

The probabilities  $B(a, j)$ ,  $1 \leq j \leq n$ , can be determined by summing the transition balance equations between the sets of states "i" and "i+1", for  $i = 0, 1, \dots, \infty$ , (this equation holds only for a system with an infinite waiting room).

$$(3.48) \quad \lambda = \mu \sum_{j=1}^n B(a, j)$$

Using (3.47), we obtain

$$(3.49) \quad B(a, j) = \frac{\lambda}{n\mu} \quad , \quad 1 \leq j \leq n.$$

The probabilities  $A(a, j)$ ,  $1 \leq j \leq n$ , can be written as follows

$$(3.50) \quad A(a, j) = \frac{\lambda}{n\mu} + p(a, j, 0) \quad , \quad 1 \leq j \leq n$$

And the system availability  $A(a)$  is expressed in terms of the boundary state probabilities  $p(a, j, 0)$ ,  $1 \leq j \leq n$ .

$$(3.51) \quad A(a) = \frac{\lambda}{\mu} + \sum_{j=1}^n p(a, j, 0)$$

Transition balance at the set of states "c", and using (3.49), yields the following for the probability  $A(c)$

$$(3.52) \quad A(c) = \frac{\lambda}{n\beta}$$

The probability  $A(r)$  follows from (3.39), (3.45) and (3.50).

$$(3.53) \quad A(r) = \frac{\lambda}{n\gamma} \left(\frac{\gamma+\mu_0}{\mu_0}\right) \left(\frac{\gamma+\mu}{\mu}\right) \left[\left(\frac{\gamma+\mu}{\mu}\right)^n - 1\right] - \frac{\lambda}{\mu} + \sum_{j=1}^n \left(\frac{1}{P_j} - 1\right) p(a, j, 0)$$

The normalizing equation  $A(c) + A(r) + A(a) = 1$  yields the following relation

$$(3.54) \quad \sum_{j=1}^n \frac{p(a, j, 0)}{P_j} = 1 - \left(\frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left(\frac{\gamma+\mu_0}{\mu_0}\right) \left(\frac{\gamma+\mu}{\mu}\right) \left[\left(\frac{\gamma+\mu}{\mu}\right)^n - 1\right]\right)$$

The condition for ergodicity follows from the fact that for a stable system  $p(a, j, 0) > 0$ , for  $j = 1, 2, \dots, n$ , (a sufficient condition). This yields a necessary and sufficient condition (using (3.54)) given by

$$(3.55) \quad \frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left(\frac{\gamma+\mu_0}{\mu_0}\right) \left(\frac{\gamma+\mu}{\mu}\right) \left[\left(\frac{\gamma+\mu}{\mu}\right)^n - 1\right] < 1$$

Now we proceed to derive an expression for the average number of transactions in the system. First we introduce the following definitions

$$N(c) \triangleq \sum_{i=1}^{\infty} i p(c,i) ,$$

$$N(a,j) \triangleq \sum_{i=1}^{\infty} i p(a,j,i) , \quad 1 < j < n ,$$

$$N(a) \triangleq \sum_{j=1}^n N(a,j) ,$$

$$N(r,j,k) \triangleq \sum_{i=1}^{\infty} i p(r,j,k,i) , \quad 0 < k < j \text{ and } 1 < j < n ,$$

$$N(r,j) \triangleq \sum_{k=0}^j N(r,j,k) , \quad 1 < j < n ,$$

$$N(r) \triangleq \sum_{j=1}^n N(r,j) ,$$

From the definition of the "i" set of states, p(i) is the probability that there are i transactions in the system. It follows that

$$p(i) \triangleq p(c,i) + \sum_{j=1}^n [p(a,j,i) + \sum_{k=0}^j p(r,j,k,i)] ,$$

and the average number of transactions in the system  $\bar{N}$  is given by

$$\bar{N} \triangleq \sum_{i=1}^{\infty} i p(i)$$

Using the above definitions we have for  $\bar{N}$ , the following relation

$$(3.56) \quad \bar{N} = N(c) + N(a) + N(r)$$

In the following, we relate the quantities defined above in order to obtain an expression for  $\bar{N}$ .

N(c) can be expressed in terms of N(a,1) as follows.

Transition balance at the sets of states "a,1,i" and "r,1,i",

$1 < i < \infty$ , yields the following equation

$$(3.57) \quad \beta p(c,i) = \mu p(a,1,i) + \lambda [p(a,1,i) + p(r,1,i) - p(a,1,i-1) - p(r,1,i-1)]$$

Multiplying (3.57) by  $i$  and summing for  $i = 1, 2, \dots, \infty$  yields

$$(3.58) \quad N(c) = \frac{\mu}{\beta} N(a,1) - \frac{\lambda}{\beta} \left( \frac{A(a,1)}{P_1} \right)$$

in which we make use of (3.45).

$N(a,j)$ ,  $2 \leq j \leq n$ , can be expressed in terms of  $N(a,1)$  as follows.

Transition balance at the sets of states " $a,j,i$ " and " $r,j,i$ ",  $1 \leq i \leq \infty$ , for  $j = 2, 3, \dots, n$ , yields the following equations.

$$(3.59) \quad \mu p(a,j,i) = \mu p(a,j-1,i+1) - \lambda [p(a,j,i) + p(r,j,i) - p(a,j,i-1) - p(r,j,i-1)], \quad j = 2, 3, \dots, n,$$

Multiplying (3.59) by  $i$  and summing for  $i = 1, 2, \dots, \infty$ , yields the following recursive relation

$$(3.60) \quad N(a,j) = N(a,j-1) + \frac{\lambda}{\mu} \left[ A(a,j) + A(r,j) - \frac{1}{n} \right], \quad j = 2, 3, \dots, n.$$

It follows (using (3.45)) that

$$(3.61) \quad N(a,j) = N(a,1) + \frac{\lambda}{\mu} \sum_{k=2}^j \left[ \frac{A(a,k)}{P_k} - \frac{1}{n} \right], \quad 2 \leq j \leq n.$$

Thus we have for  $N(a)$ , the following

$$(3.62) \quad \begin{aligned} N(a) &= n N(a,1) + \frac{\lambda}{\mu} \sum_{j=2}^n \sum_{k=2}^j \left[ \frac{A(a,k)}{P_k} - \frac{1}{n} \right] \\ &= n N(a,1) + \frac{\lambda}{\mu} \sum_{k=2}^n (n-k+1) \left[ \frac{A(a,k)}{P_k} - \frac{1}{n} \right] \end{aligned}$$

The quantities  $N(r,j,k)$ ,  $0 \leq k \leq j-1$ , and  $N(r,j)$  can be expressed in terms of  $N(r,j,j)$  as follows.

Transition balance at the states "r,j,k+1,i", for k = j-1, j-2, ..., 0, 1 < j < n and 1 < i < ∞, yields the following recursive equations

$$(3.63) \quad \mu p(r, j, k, i) = (\lambda + \gamma + \mu) p(r, j, k+1, i) - \lambda p(r, j, k+1, i-1),$$

for k = j-1, j-2, ..., 1, 1 < j < n and 1 < i < ∞,

and

$$(3.64) \quad \mu_0 p(r, j, 0, i) = (\lambda + \gamma + \mu) p(r, j, 1, i) - \lambda p(r, j, 1, i-1),$$

1 < j < n and 1 < i < ∞.

Multiplying (3.63) and (3.64) by i and summing, for i = 1, 2, ..., ∞, yields the following recursive relations

$$(3.65) \quad N(r, j, k) = \left(\frac{\gamma + \mu}{\mu}\right) N(r, j, k+1) - \frac{\lambda}{\mu} A(r, j, k+1),$$

for k = j-1, j-2, ..., 1 and 1 < j < n,

and

$$(3.66) \quad N(r, j, 0) = \left(\frac{\gamma + \mu}{\mu_0}\right) N(r, j, 1) - \frac{\lambda}{\mu_0} A(r, j, 1), \quad 1 < j < n.$$

It follows from (3.65) and (3.66) (using (3.43) and (3.44)) that

$$(3.67) \quad N(r, j, k) = \left(\frac{\gamma + \mu}{\mu}\right)^{j-k} \left[ N(r, j, j) - (j-k) \left(\frac{\gamma}{\gamma + \mu}\right) \frac{\lambda}{\mu} A(a, j) \right]$$

for k = 1, 2, ..., j and 1 < j < n.

$$(3.68) \quad \frac{\mu_0}{\mu} N(r, j, 0) = \left(\frac{\gamma + \mu}{\mu}\right)^j \left[ N(r, j, j) - j \left(\frac{\gamma}{\gamma + \mu}\right) \frac{\lambda}{\mu} A(a, j) \right],$$

1 < j < n

With some manipulations (3.67) and (3.68) yield

$$(3.69) \quad N(r, j) = \frac{\mu}{\gamma} \left( \frac{1}{P_j} - 1 \right) N(r, j, j) - \frac{\lambda}{\gamma} \frac{A(a, j)}{P_j}$$

\*  $\left[ \frac{\mu}{\gamma} + j \frac{\gamma}{\gamma + \mu} - \frac{\mu_0}{\gamma + \mu_0} \right], \quad 1 < j < n,$

with -----

$$P_j \triangleq \left(\frac{\mu_0}{\gamma + \mu_0}\right) \left(\frac{\mu}{\gamma + \mu}\right)^j$$

$N(r, j, j)$  can be expressed in terms of  $N(a, j)$ , by using the transition balance equations at the sets of states " $r, j, i$ ",  $1 < j < n$  and  $1 < i < \infty$ ,

$$(3.70) \quad \mu p(r, j, j, i) = \gamma p(a, j, i) - \lambda [p(r, j, i) - p(r, j, i-1)] ,$$

$$1 < j < n \quad \text{and} \quad 1 < i < \infty .$$

Multiplying (3.70) by  $i$  and summing, for  $i = 1, 2, \dots, \infty$ , yields

$$(3.71) \quad N(r, j, j) = \frac{\gamma}{\mu} N(a, j) + \frac{\lambda}{\mu} A(r, j) \quad , \quad 1 < j < n .$$

Substitution from (3.71) into (3.69) yields for  $N(r, j)$  the following

$$(3.72) \quad N(r, j) = \left(\frac{1}{P_j} - 1\right) N(a, j) + \frac{\lambda}{\gamma} \frac{A(a, j)}{P_j} \left[ \left(\frac{1-P_j}{P_j}\right) - \left(j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0}\right) \right],$$

$$1 < j < n$$

$N(r, j)$  can also be written in the form

$$(3.73) \quad N(r, j) = \left(\frac{1}{P_j} - 1\right) N(a, j) + \lambda \sum_{k=0}^j A(r, j, k) \cdot \bar{t}(r, j, k)$$

with

$$\bar{t}(r, j, 0) = \frac{1}{\gamma P_j} (1 - P_j)$$

$$\bar{t}(r, j, k) = \frac{1}{\gamma P_j} \left(1 - \left(\frac{\mu}{\gamma+\mu}\right)^{j-k+1}\right)$$

$\bar{t}(r, j, k)$  can be interpreted as the expected time spent in the set of states " $r, j$ " with " $r, j, k$ " as an initial state.

From (3.72) we obtain for  $N(r)$ , the following

$$(3.74) \quad N(r) = \sum_{j=1}^n \left(\frac{1}{P_j} - 1\right) N(a, j) + \frac{\lambda}{\gamma} \sum_{j=1}^n \frac{A(a, j)}{P_j}$$

$$* \left[ \left(\frac{1-P_j}{P_j}\right) - \left(j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0}\right) \right]$$

It follows directly from (3.74) that

$$(3.75) \quad N(r) + N(a) = \sum_{j=1}^n \frac{N(a,j)}{P_j} + \frac{\lambda}{\gamma} \sum_{j=1}^n \frac{A(a,j)}{P_j} \\ * \left[ \left( \frac{1-P_j}{P_j} \right) - \left( j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0} \right) \right]$$

Substitution from (3.61) in (3.75) yields

$$(3.76) \quad N(r) + N(a) = N(a,1) \left( \sum_{j=1}^n \frac{1}{P_j} \right) + \frac{\lambda}{\mu} \sum_{j=2}^n \frac{1}{P_j} \sum_{k=2}^j \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \\ + \frac{\lambda}{\gamma} \sum_{j=1}^n \frac{A(a,j)}{P_j} \left[ \left( \frac{1-P_j}{P_j} \right) - \left( j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0} \right) \right]$$

An expression for  $\bar{N}$  (the average number of transactions in the system) in terms of  $N(a,1)$  follows from (3.76) and (3.58)

$$(3.77) \quad \bar{N} = \left( \frac{\mu}{\beta} + \sum_{j=1}^n \frac{1}{P_j} \right) N(a,1) - \frac{\lambda}{\beta} \frac{A(a,1)}{P_1} \\ + \frac{\lambda}{\mu} \sum_{j=2}^n \frac{1}{P_j} \sum_{k=2}^j \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \\ + \frac{\lambda}{\gamma} \sum_{j=1}^n \frac{A(a,j)}{P_j} \left[ \left( \frac{1-P_j}{P_j} \right) - \left( j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0} \right) \right]$$

Now we get an expression for  $N(a,1)$  in terms of  $\bar{N}$ .

Transition balance between the sets of states "i" and "i-1",  $1 < i < \infty$ , yields the following equation

$$(3.78) \quad \lambda p(i-1) = \mu \sum_{j=1}^n p(a,j,i) \quad , \quad 1 < i < \infty.$$

Multiplying (3.78) by i and summing, for  $i = 1, 2, \dots, \infty$ , yields

$$(3.79) \quad N(a) = \frac{\lambda}{\mu} (\bar{N} + 1)$$



From (3.79) and (3.62) we have for  $N(a,1)$  the following

$$(3.80) \quad N(a,1) = \frac{\lambda}{n\mu} \left[ (\bar{N}+1) - \sum_{k=2}^n (n-k+1) \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \right]$$

Substituting from (3.80) into (3.77) yields the following expression for  $\bar{N}$

$$(3.81) \quad \bar{N} = \left[ 1 - \left( \frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left( \frac{\gamma+\mu_0}{\mu_0} \right) \left( \frac{\gamma+\mu}{\mu} \right) \left[ \left( \frac{\gamma+\mu}{\mu} \right)^n - 1 \right] \right) \right]^{-1}$$

$$* \left\{ \left( \frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left( \frac{\gamma+\mu_0}{\mu_0} \right) \left( \frac{\gamma+\mu}{\mu} \right) \left[ \left( \frac{\gamma+\mu}{\mu} \right)^n - 1 \right] \right) \right.$$

$$* \left( 1 - \sum_{k=2}^n (n-k+1) \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \right)$$

$$+ \frac{\lambda}{\mu} \sum_{j=2}^n \frac{1}{P_j} \sum_{k=2}^j \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) - \frac{\lambda}{\beta} \left( \frac{A(a,1)}{P_1} \right)$$

$$\left. + \frac{\lambda}{\gamma} \sum_{j=1}^n \frac{A(a,j)}{P_j} \left( \left( \frac{1-P}{P_j} \right)^j - \left( j \frac{\gamma}{\gamma+\mu} + \frac{\gamma}{\gamma+\mu_0} \right) \right) \right\} .$$

with  $A(a,j)$ ,  $0 < j < n$ , as given in (3.50).

Equation (3.81) expresses  $\bar{N}$  in terms of the boundary state probabilities  $p(a,j,0)$ ,  $1 < j < n$ .

Note that the denominator of the expression for  $\bar{N}$  should be greater than zero for a stable system. This yields the same condition for ergodicity as that obtained in (3.55).

Although an explicit form for  $\bar{N}$  is quite difficult in the general case, it is possible in some special cases (or limiting situations) to obtain an explicit form for  $\bar{N}$ . In the next chapter two such cases will be discussed.

#### 4. Special cases

A model of the non-saturated system, introduced in chapter 1, was analysed in chapter 3. In section 3.2, expressions for performance variables such as the system availability and the average number of transaction in the system were obtained in terms of the boundary state probabilities for a system with an infinite waiting room. In this chapter two special cases of this system are considered, namely, heavily- and lightly-loaded situations. In those cases, simplifying assumptions can be made which are approximately valid. These approximations enable us to obtain explicit expressions for the performance variables.

##### 4.1 Heavily-loaded system:

Consider the model of section 3.2. In heavy-load conditions the boundary state probabilities  $p(a, j, 0)$ ,  $1 < j < n$ , approach zero. Referring to equation (3.50), we make the following approximate assumption

$$(4.1) \quad A(a, j) \cong A(a, 1) \quad , \quad 2 < j < n .$$

From equation (3.45), it follows that

$$(4.2) \quad A(a) + A(r) \cong A(a, 1) \sum_{j=1}^n \frac{1}{P_j} \\ \cong \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] A(a, 1)$$

From (4.2) and (3.52) and the normalizing equation  $A(c) + A(r) + A(a) = 1$ , we get for  $A(a, 1)$ , the following

$$(4.3) \quad A(a, 1) \cong \left( 1 - \frac{\lambda}{n\beta} \right) \left( \frac{\gamma}{\gamma + \mu} \right) \left( \frac{\mu_0}{\gamma + \mu_0} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right]^{-1}$$

An expression for the system availability is readily obtained

$$(4.4) \quad A(a) \equiv (n - \frac{\lambda}{\beta}) (\frac{\gamma}{\gamma+\mu}) (\frac{\mu_0}{\gamma+\mu_0}) [(\frac{\gamma+\mu}{\mu})^n - 1]^{-1}$$

The system is stable if  $A(a) > \frac{\lambda}{\mu}$ . This yields the condition

$$\frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} (\frac{\gamma+\mu_0}{\mu_0}) (\frac{\gamma+\mu}{\mu}) [(\frac{\gamma+\mu}{\mu})^n - 1] < 1$$

which is identical to the condition in (3.55).

In order to get an explicit expression for  $\bar{N}$ , we make use of the assumption (4.1) in equation (3.81). In the following we evaluate some terms in (3.81).

$$\begin{aligned} \text{The term } (t_1) : & \left[ 1 - \sum_{k=2}^n (n-k+1) \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \right] \\ & = \left[ 1 - \sum_{j=2}^n \sum_{k=2}^j \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \right] \\ & \cong \frac{n+1}{2} - \left( \frac{\gamma+\mu_0}{\mu_0} \right) \left( \frac{\gamma+\mu}{\mu} \right)^2 \left( \frac{\mu}{\gamma} \right) \left[ \left( \frac{\gamma+\mu}{\gamma} \right) \left[ \left( \frac{\gamma+\mu}{\mu} \right)^{n-1} - 1 \right] - (n-1) \right] A(a,1) \end{aligned}$$

$$\begin{aligned} \text{The term } (t_2) : & \left[ \sum_{j=2}^n \frac{1}{P_j} \sum_{k=2}^j \left( \frac{A(a,k)}{P_k} - \frac{1}{n} \right) \right] \\ & \cong \left[ \left( \frac{\gamma+\mu_0^2}{\mu_0} \right) \left( \frac{\gamma+\mu}{\mu} \right)^4 \left( \frac{\mu}{\gamma} \right)^2 \left[ \left( \frac{\gamma+\mu}{\gamma+2\mu} \right) \left[ \left( \frac{\gamma+\mu}{\mu} \right)^{2(n-1)} - 1 \right] - \left( \frac{\gamma+\mu}{\mu} \right)^{n-1} \right] \right] A(a,1) \\ & \quad + \frac{1}{n} \left( \frac{\gamma+\mu_0}{\mu_0} \right) \left( \frac{\gamma+\mu}{\gamma} \right)^2 \left[ \left( \frac{\gamma+\mu}{\mu} \right)^{n-1} \left( 1 - \frac{(n-1)\gamma}{\mu} \right) - 1 \right] \end{aligned}$$

$$\begin{aligned} \text{The term } (t_3) : & \left[ \sum_{j=1}^n \frac{A(a,j)}{P_j^2} \right] \\ & \cong \left( \frac{\gamma+\mu_0^2}{\mu_0} \right) \left( \frac{\gamma+\mu}{\mu} \right)^2 \left( \frac{\mu}{\gamma} \right) \left( \frac{\mu}{\gamma+2\mu} \right) \left[ \left( \frac{\gamma+\mu}{\mu} \right)^{2n} - 1 \right] A(a,1) \end{aligned}$$

$$\begin{aligned} \text{The term } (t_4) &: \left[ - \left( \frac{\gamma}{\gamma + \mu} \right) \sum_{j=1}^n \frac{jA(a, j)}{P_j} \right] \\ &\equiv \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right) - n \left( \frac{\gamma}{\mu} \right) \left( \frac{\gamma + \mu}{\mu} \right)^{n-1} \right] A(a, 1) \end{aligned}$$

$$\begin{aligned} \text{The term } (t_5) &: \left[ - \left( \frac{2\gamma + \mu_0}{\gamma + \mu_0} \right) \sum_{j=1}^n \frac{A(a, j)}{P_j} \right] \\ &\equiv - \left( \frac{2\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] A(a, 1) \end{aligned}$$

Substituting from the terms evaluated above into (3.81) yields an explicit expression for  $\bar{N}_h$ , the heavy load approximation of the average number of transactions in the system.

$$\begin{aligned} (4.5) \quad \bar{N}_h &\equiv \left[ 1 - \left( \frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\mu} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] \right) \right]^{-1} \\ &\quad * \left\{ \left( \frac{\lambda}{n\beta} + \frac{\lambda}{n\gamma} \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\mu} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] \right) t_1 \right. \\ &\quad + \frac{\lambda}{\mu} \left( t_2 - \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\beta} \right) A(a, 1) \right) \\ &\quad \left. + \frac{\lambda}{\gamma} (t_3 + t_4 + t_5) \right\} \end{aligned}$$

with  $A(a, 1)$  given in (4.3).

#### 4.2 Lightly-loaded system

Consider the model of section 3.2. In very light load conditions the probability that there is more than one transaction in the system is negligible. It is then reasonable to make the following approximate assumption

$$(4.6) \quad p(i) \stackrel{\Delta}{=} p(c, i) + \sum_{j=1}^n [p(a, j, i) + p(r, j, i)] \cong 0, \text{ for } i > 1,$$

with  $p(i)$ ,  $p(c, i)$ ,  $p(a, j, i)$  and  $p(r, j, i)$  as defined in chapter 3.

The Markov chain representing the system operation can be reduced to the one shown in fig. (4.1).

The following are the probabilities corresponding to the different states in fig. (4.1).

$$E(c) = [A(c) - B(c)] = p(c,0)$$

$$E(a,j) = [A(a,j) - B(a,j)] = p(a,j,0) \quad , \quad 1 < j < n \quad ,$$

$$E(r,j) = [A(r,j) - B(r,j)] = \sum_{k=0}^j p(r,j,k,0) \quad , \quad 1 < j < n \quad ,$$

with

$$B(r,j) \triangleq \sum_{k=0}^j B(r,j,k) \quad , \quad 1 < j < n \quad ,$$

and  $A(c)$ ,  $B(c)$ ,  $A(a,j)$ ,  $B(a,j)$ ,  $A(r,j)$  and  $B(r,j,k)$  as defined in section 3.2.

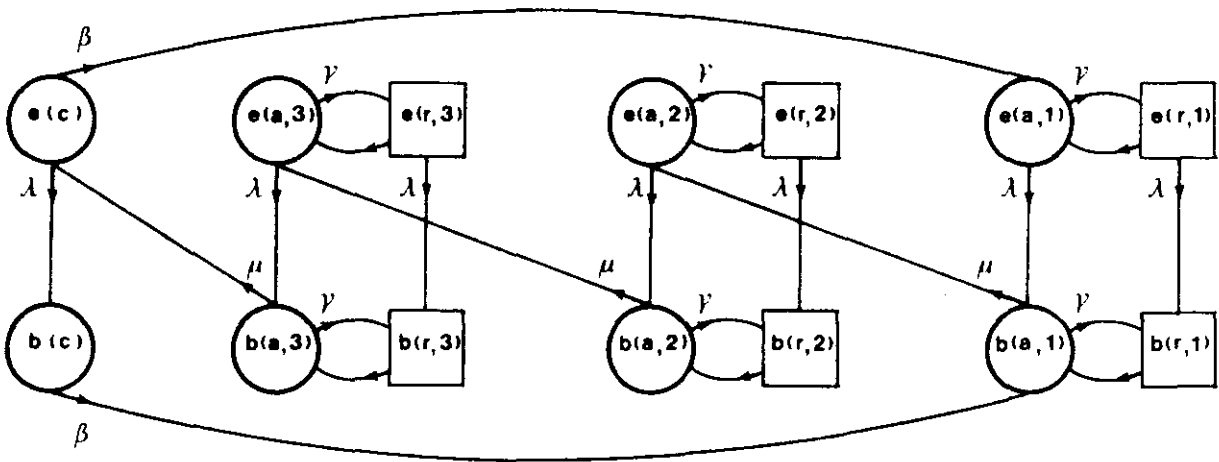


Fig. (4.1) An equivalent state transition diagram for a very lightly-loaded system (with  $n = 3$  and  $\square$  as introduced in figures (2.1) and (2.2)).

Define, also, the following sets of states

The set  $e^j$  corresponding to the sets  $e^{a,j}$  and  $e^{r,j}$ , with the associated probability  $E(j)$ ,  $1 < j < n$ .

The set  $b^j$  corresponding to the sets  $b^{a,j}$  and  $b^{r,j}$ ,  $1 < j < n$ .

Transition balance at the sets of states  $e^j$  and  $b^{j-1}$ , for  $j = 3, 4, \dots, n$ , yield the following recursive relation

$$(4.7) \quad E(j) \cong E(j-1) \quad , \quad j = 3, 4, \dots, n$$

$$\stackrel{\Delta}{=} E \quad , \quad 2 < j < n$$

Transition balance at the sets of states  $e^2$  and  $b^1$  and  $b^c$  yields

$$(4.8) \quad E(1) + E(c) \cong E(2)$$

$$\cong E$$

Transition balance at the sets of states  $e^c$  and  $b^n$  yields

$$(4.9) \quad E(c) \cong \frac{\lambda}{\lambda + \beta} E$$

It follows from the transition balance at the state  $b^c$  that

$$(4.10) \quad B(c) \cong \frac{\lambda}{\beta} \left( \frac{\lambda}{\lambda + \beta} \right) E$$

Thus, we have, for  $A(c)$ , the following

$$(4.11) \quad A(c) \cong \frac{\lambda}{\beta} E$$

Transition balance at the sets of states  $b^c$ ,  $b^1$  and the sets of states  $b^j$ , for  $j = 2, 3, \dots, n$  yield the following relation

$$(4.12) \quad B(a, j) \cong \frac{\lambda}{\mu} E \quad , \quad 1 < j < n$$

From equations (3.11), (3.12) and (4.10) we have

$$(4.13) \quad E(a, j) = \left( \frac{\lambda + \gamma Q_j}{\lambda + \gamma} \right) E, \quad 2 \leq j \leq n$$

and

$$(4.14) \quad E(a, 1) = \left( \frac{\beta}{\lambda + \beta} \right) \left( \frac{\lambda + \gamma Q_1}{\lambda + \gamma} \right) E$$

with

$$Q_j \triangleq \left( \frac{\mu_0}{\lambda + \gamma + \mu_0} \right) \left( \frac{\mu}{\lambda + \gamma + \mu} \right)^j$$

We rewrite equation (3.45) in section 3.2 as follows

$$(4.15) \quad A(a, j) + A(r, j) = \frac{A(a, j)}{P_j}, \quad 1 \leq j \leq n$$

with

$$P_j \triangleq \left( \frac{\mu_0}{\gamma + \mu_0} \right) \left( \frac{\mu}{\gamma + \mu} \right)^j$$

From equations (4.12), (4.13) and (4.14) we get

$$(4.16) \quad A(a, j) = \left( \frac{\lambda}{\mu} + \frac{\lambda + \gamma Q_j}{\lambda + \gamma} \right) E, \quad 2 \leq j \leq n,$$

and

$$(4.17) \quad A(a, 1) = \left( \frac{\lambda}{\mu} + \left( \frac{\beta}{\lambda + \beta} \right) \left( \frac{\lambda + \gamma Q_1}{\lambda + \gamma} \right) \right) E$$

In order to get an expression for E we substitute from (4.11), (4.15), (4.16) and (4.17) in the normalizing equation

$$A(c) + \sum_{j=1}^n \frac{A(a, j)}{P_j} = 1$$

After some manipulations we obtain for E, the following expression

$$(4.18) \quad E = \left\{ \frac{\lambda}{\beta} + \left( \frac{\lambda}{\mu} \right) \left( \frac{\lambda + \gamma + \mu}{\lambda + \gamma} \right) \left( \frac{\gamma + \mu_0}{\mu_0} \right) \left( \frac{\gamma + \mu}{\gamma} \right) \left[ \left( \frac{\gamma + \mu}{\mu} \right)^n - 1 \right] \right. \\ \left. + \left( \frac{\gamma}{\lambda + \gamma} \right) \left( \frac{\gamma + \mu_0}{\lambda + \gamma + \mu_0} \right) \left( \frac{\gamma + \mu}{\lambda} \right) \left[ 1 - \left( \frac{\gamma + \mu}{\lambda + \gamma + \mu} \right)^n \right] - \left( \frac{\lambda}{\lambda + \beta} \right) \left( \frac{\lambda + \gamma Q_1}{\lambda + \gamma} \right) \frac{1}{P_1} \right\}^{-1}$$

The light-load approximation of the average number of transactions in the system,  $\bar{N}_\ell$ , is finally obtained

$$(4.19) \quad \bar{N}_\ell = \sum_{i=1}^{\infty} i p(i) \cong p(1) = 1 - p(0) \\ \cong (1 - nE)$$

with E given from (4.18).



## 5. Conclusions

A new Markovian model of a transactional computer system supported with checkpointing and rollback recovery strategies is presented. In this model checkpoints are performed after the completion of a number of transactions. Failures occur randomly at any mode of the system operation (i.e. available, checkpointing and recovery). Although we have assumed identical failure rates at different modes of operation, the same model can be analysed for different failure rates at different modes of operation.

Transactions arrive randomly at the system during different modes of the system operation. They are processed according to a FCFS discipline when the system is available.

Two models were analysed. The first model is for a saturated system. This model is analytically tractable. Explicit forms for the system availability are obtained for fixed and random numbers of completed transactions between checkpoints. The optimum number which maximizes the system availability is determined.

The second model is for a non-saturated system. For this model, explicit analytical forms for the performance variables in the general case are not possible; they are expressed in terms of the boundary state probabilities. A numerical algorithm is proposed to compute the limiting state probabilities and, thus, the performance variables. The algorithm is partly recursive and requires the solution of a system of linear equations in the unknown boundary state probabilities.

It is important to notice that the same numerical procedure can be used for the computation of the state probabilities in the case of state-dependent model parameters.

Considerable simplifications can be made in some special cases, due to approximate assumptions. These assumptions enable us to obtain explicit forms for the performance variables. Two such cases; namely, heavily and lightly loaded systems are treated. It is worthwhile analysing the model for some other interesting special cases, e.g. when the failure rate is small during the available mode of operation or when the completion process of transactions is approximated by a Poisson process.

The present model offers a more realistic and more accurate analysis for the system operation than previously published models. It gives the possibility of investigating the validity of other models with more restrictive (simplifying) assumptions.

So far, in most of the existing models, a Poisson failure process is assumed. It is of much interest to introduce the time and load dependent behaviour of the failure process and consider techniques to determine an optimum checkpointing strategy. This will be a considerable step towards more realistic modelling of existing systems.

#### Acknowledgement

I would like to thank Prof. ir. F.J. Kylstra for many valuable comments and corrections. I would also like to express my gratitude to Dr. ir. J. v.d. Wal for very helpful discussions.

I am grateful to Mrs. B. Cornelissen for her patience and skill in the preparation of the typescript.

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