Research Article GU J Sci 35(1): 293-312 (2022) DOI: 10.35378/gujs.741755

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Marshall Olkin Alpha Power Extended Weibull Distribution: Different Methods of Estimation based on Type I and Type II Censoring

Ehab M. ALMETWALLY *

Faculty of Business Administration, Delta University of Science and Technology, Mansoura, 35511, Egypt

Highlights

- A new extended Weibull distribution is offered.
- Essential properties are studied.
- MOAPEW based on Type I and Type II censored samples are examined.
- Parameter estimation using classical methods and Bayesian were obtained.
- Simulation studies Application and the application of real data are used.

Article Info

Received: 23 May 2020
Accepted: 24 Feb 2021

Keywords

Marshall–Olkin Alpha Power
Maximum product spacing
Bayesian estimation
Type I censored
Type II censored

Abstract

In this paper, we insert and study a novel five-parameter extended Weibull distribution denominated as the Marshall–Olkin alpha power extended Weibull (MOAPEW) distribution. This distribution's statistical properties are discussed. Maximum likelihood estimations (MLE), maximum product spacing (MPS), and Bayesian estimation for the MOAPEW distribution parameters are obtained using Type I and Type II censored samples. A numerical analysis using Monte-Carlo simulation and real data sets are realized to compare various estimation methods. The supremacy of this novel model upon some famous distributions is explicated using different real datasets as it appears the MOAPEW model achieves a good fit for these applications.

1. INTRODUCTION

Statistical distributions are commonly used in real-life phenomena to explain them. For this reason, it is of great importance to the theory of distributions to generate novel distributions. Several researchers produced and studied novel distributions from classical ones. Many groups of generalized distributions were grown and applied for various events explanation in real-life in the last few years. These generalized distributions are favored because they have more parameters and thus more versatility to model life data.

A number of distributions were made using pdf or cdf based on a model with $g(t)$ like a probability density function (pdf), and $G(t)$, which is cumulative distribution function (cdf), and also the survival functions as a basic distribution to add a new model. The Marshall-Olkin distribution family has been introduced in [1], which depends on adding an extra parameter to the distribution family and is known as extended distributions. $G(t)$ and $g(t)$ are the cdf and pdf of a given random variable. Then the Marshall-Olkin (MO) family cdf and pdf can be written as, respectively, by:

$$F(t) = \frac{G(t)}{\theta + (1-\theta)G(t)}, \qquad \theta > 0, \tag{1}$$

and

* e-mail: ehabxp_2009@hotmail.com

$$f(t)=\frac{\theta g(t)}{[\theta+(1-\theta)G(t)]^2}. \tag{2}$$

The MO family used to propose a new distribution with a different behavioral spectrum. See [2] introduced Marshall-Olkin extended Weibull distribution (MOW), [3] introduced Marshall-Olkin extended Lomax distribution, [4] discussed Marshall-Olkin logistic processes, [5] obtained a new Marshall-Olkin extended generalized linear, exponential distribution family, and [6] introduced Marshall-Olkin generalized Pareto distribution.

In addition, the alpha power (AP) transformation by addition of a new parameter to obtain a distribution family was discussed by [7]. If $G(t)$ is a cdf of any distribution, $W(t)$ is a cdf that is obtained by Equation (3):

$$W(t)=\begin{cases}\dfrac{\alpha^{G(t)}-1}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ G(t) & if\ \alpha=1\end{cases}, \tag{3}$$

and the corresponding pdf takes the form

$$w(t)=\begin{cases}\dfrac{\ln(\alpha)\,g(t)\alpha^{G(t)}}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ g(t) & if\ \alpha=1\end{cases}. \tag{4}$$

For example, [8] introduced alpha power Weibull (APW) distribution, [9] introduced new AP transformed family of distributions, [10, 11] introduced AP transformed Lindley and inverse Lindley respectively, [12] introduced AP transformed power Lindley distribution and [13] introduced AP inverse Weibull distribution, have done a lot of works on distributions based on AP transformation.

The Marshall Olkin Alpha Power (MOAP) family has recently been introduced in [14]. This is a new method for inserting an extra parameter for more flexibility into a family of distributions. By taking the cdf of AP family as the baseline cdf in cdf of MO family, then the cdf of MOAP family received by the Equation (5):

$$F_{MOAP}(t)=\begin{cases}\dfrac{\alpha^{G(t)}-1}{(\alpha-1)\left(\theta+\frac{(1-\theta)}{(\alpha-1)}(\alpha^{G(t)}-1)\right)} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ G(t) & if\ \alpha=1\end{cases}, \tag{5}$$

its pdf can be expressed as

$$f_{MOAP}(t)=\begin{cases}\dfrac{\theta\ln(\alpha)}{\alpha-1}\dfrac{g(t)\alpha^{G(t)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}(\alpha^{G(t)}-1)\right)^2} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ g(t) & if\ \alpha=1\end{cases}. \tag{6}$$

In [15] introduced a new generalization of the Pareto distribution based on the MOAP family. In [16] introduced MOAP inverse exponential distribution. [17] introduced a new Weibull distribution based on MOAP family. In [18], a new logistics differential model is based on the MOAP family. [19] introduced MOAP Inverse Weibull distribution.

In [20] discussed extended Weibull (EW) distribution. The cdf and pdf of EW distribution with parameters β,λ, and δ are as follows:

$$G(t;\beta,\lambda,\delta)=1-e^{\lambda\delta\left(1-e^{\left(\frac{t}{\lambda}\right)^{\beta}}\right)};\quad t\geq 0,\beta,\lambda,\delta>0, \tag{7}$$

$$g(t;\beta,\lambda,\delta)=\delta\beta\left(\frac{t}{\lambda}\right)^{\beta-1}e^{\left(\frac{t}{\lambda}\right)^{\beta}+\lambda\delta\left(1-e^{\left(\frac{t}{\lambda}\right)^{\beta}}\right)}. \tag{8}$$

There are numerous situations in life testing and reliability experiments where observations are missed or set aside from experimentation prior to failure. For all experimental observations, the tester is unable to obtain full data about times of failure. Censored information can decrease time and costs for any experiment. The test of censoring has several forms. Type I censored and Type II censored are the most relevant and used schemes. For example, see [21].

First of all, this paper aims to insert and introduce, depending on the MOAP family, a new lifetime distribution known as MOAP extended Weibull (MOAPEW) distribution. Considerable statistical characteristics of MOAPEW are shown. Second, the estimation of parameters for the distribution of the MOAPEW is explored using non-Bayesian (MLE and MPS) and Bayesian estimations. Third, considering parameter estimates of the MOAPEW distribution with censored samples of Type I and Type II scheme. The simulation of Monte-Carlo is conducted to assess the accuracy of estimators. Two real data analyses are developed to verify the model and scheme's veracity.

The structure of this paper is as follows, and a novel distribution is introduced and outlined in section 2. Reliability analyses are taken from section 3, although certain statistical properties of MOAPEW are discussed in section 4. Section 5 presents parameter estimation of MOAPEW with complete, Type I and Type II censored samples. The simulation analysis of Monte-Carlo can be found in section 6. In section 7, the implementations of two real data sets are studied. Lastly, it addresses the conclusion of this analysis.

2. DISTRIBUTION DESCRIPTION

The MOAPEW distribution and its sub-models have been introduced in this section.

2.1. MOAPEW Distribution

MOAPEW distribution was developed using the MOAP family and EW distribution. Let a random variable t is distributed MOAPEW with parameter $(\alpha,\beta,\theta,\lambda,\delta)$. Referring to Equations (6), (7) and (8), the pdf of MOAPEW can be written as

$$f(t,\Psi)=\frac{\theta\ln(\alpha)}{\alpha-1}\frac{\delta\beta\left(\frac{t}{\lambda}\right)^{\beta-1}e^{\left(\frac{t}{\lambda}\right)^{\beta}}\mathfrak{E}(t;\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}-1\right)\right)^{2}}\;;\alpha\neq 1 \tag{9}$$

where $\Psi=(\alpha,\beta,\theta,\lambda,\delta)$, $\Psi>\mathbf{0}$, and $\mathfrak{E}(t;\beta,\lambda,\delta)=e^{\lambda\delta\left(1-e^{\left(\frac{t}{\lambda}\right)^{\beta}}\right)}$. By using Equations (5) and (7), the cdf of MOAPEW is as following

$$\mathbb{F}(t,\Psi)=\frac{\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}-1}{(\alpha-1)\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}-1\right)\right)}\;;\;\alpha\neq 1. \tag{10}$$

Figure 1 displays different shape for pdf of the MOAPEW with the following values for certain parameters

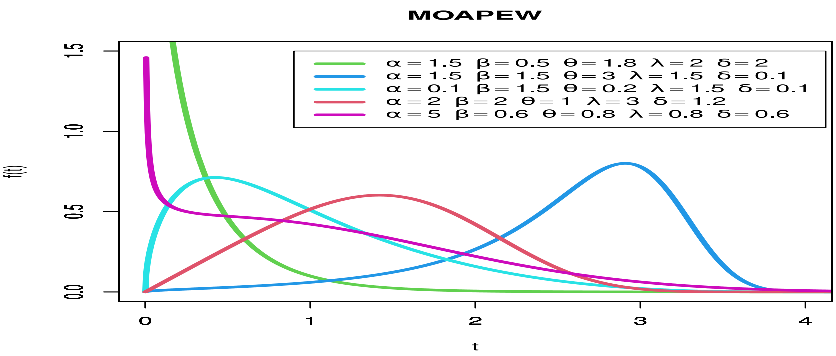

Figure 1. *Different shapes for the pdf function*

2.2. Sub-Models From MOAPEW

The MOAPEW distribution can be used to create a variety of sub-models, as appear in Table 1

Table 1. *New sub-models from MOAPEW distribution*

Models	α	β	θ	λ	δ
MOAP extended exponential (MOAPEE)	α	1	θ	λ	δ
MOAP Rayleigh (MOAPR)	α	β	θ	2	δ
AP extended Weibull (APEW)	α	β	1	λ	δ
MO extended Weibull (MOEW)	1	β	θ	λ	δ

3. ANALYSIS OF RELIABILITY

The survival function of MOAPEW can be written by the following Equation:

$$S(t,\Psi)=\frac{\theta\left[\alpha-\alpha^{\left(1-\mathfrak{E}(t;\beta,\lambda,\delta)\right)}\right]}{(\alpha-1)\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(t;\beta,\lambda,\delta)\right)}-1\right)\right]}. \tag{11}$$

The hazard function of a MOAPEW distribution can be written by the following Equation:

$$h_{MOAPW}(t,\Psi)=\frac{\ln(\alpha)}{\alpha-\alpha^{\left(1-\mathfrak{E}(t;\beta,\lambda,\delta)\right)}}\frac{\delta\beta\left(\frac{t}{\lambda}\right)^{\beta-1}e^{\left(\frac{t}{\lambda}\right)^{\beta}}\alpha^{\left(1-\mathfrak{E}(t;\beta,\lambda,\delta)\right)}}{\left[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-\mathfrak{E}(t;\beta,\lambda,\delta)\right)}-1\right)\right]}. \tag{12}$$

The hazard functions of the MOAPEW distribution with various true values of the parameter are shown in Figure 2.

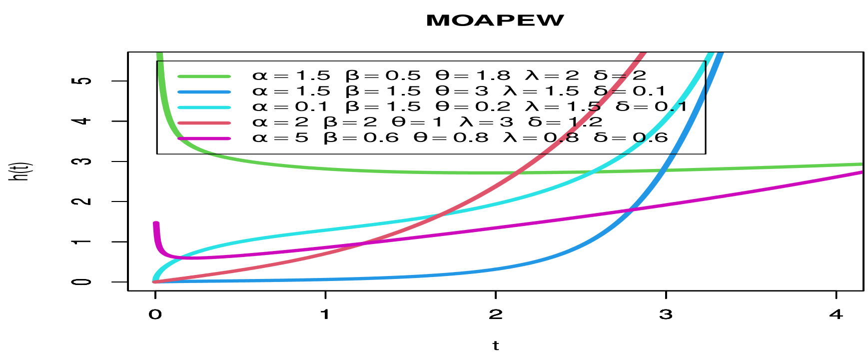

Figure 2. *Different shapes for MOAPEW*

We infer from the study of Figures 1 and 2 that the MOAPEW distribution can work as a reliable distribution for modeling and fitting many kinds of data, such as skewed and different data that isn't consistent with other traditional distributions. The reversed hazard function of the MOAPEW distribution is shown as below:

$$rh_{MOAPW}(x,\Psi)=\frac{\theta\ln(\alpha)}{\alpha-1}\frac{\delta\beta\left(\frac{t}{\lambda}\right)^{\beta-1}e^{\left(\frac{t}{\lambda}\right)^{\beta}}\alpha^{(1-\mathfrak{E}(t;\beta,\lambda,\delta))}}{\left(\alpha^{(1-\mathfrak{E}(t;\beta,\lambda,\delta))}-1\right)\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{(1-\mathfrak{E}(t;\beta,\lambda,\delta))}-1\right)\right]}. \qquad (13)$$

The reversed hazard functions of MOAPEW distribution with some values of the parameters are displayed in Figure 3.

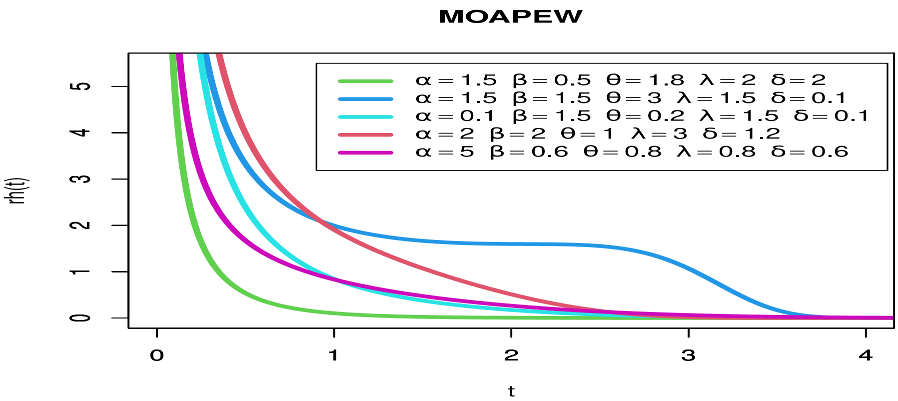

Figure 3 *Different shapes for reversed hazard function*

For the MOAPEW delivery stress-strength reliability scale, let X and Y are random variables (RV) with independent intensity and stress observed from the distribution of MOAPEW. The stress-strength reliability R is then evaluated as:
$R=P(Y<X)=\int_{x=0}^{\infty}\left\{\int_{y=0}^{x}f(y;\Psi_2)\,dy\right\}f(x;\Psi_1)\,dx$. Then, we have

$$R=\frac{\theta_1\delta_1\beta_1\ln(\alpha_1)}{(\alpha_1-1)(\alpha_2-1)}\int_0^{\infty}\frac{\left(\frac{x}{\lambda_1}\right)^{\beta_1-1}e^{\left(\frac{x}{\lambda_1}\right)^{\beta_1}}\alpha_1^{\left(1-\mathfrak{E}(x;\beta_1,\lambda_1,\delta_1)\right)}\left(\alpha_2^{\left(1-\mathfrak{E}(x;\beta_2,\lambda_2,\delta_2)\right)}-1\right)}{\left[\theta_1+\frac{(1-\theta_1)}{(\alpha_1-1)}\left(\alpha_1^{\left(1-\mathfrak{E}(x;\beta_1,\lambda_1,\delta_1)\right)}-1\right)\right]^2\left[\theta_2+\frac{(1-\theta_2)}{(\alpha_2-1)}\left(\alpha_1^{\left(1-\mathfrak{E}(x;\beta_2,\lambda_2,\delta_2)\right)}-1\right)\right]}dx. \qquad (14)$$

Table 2. *Some numerical values of reliability stress-strength model for different cases*

Actual value										$\mathfrak{y}$		
α_1	β_1	θ_1	λ_1	δ_1	α_2	β_2	θ_2	λ_2	δ_2	0.5	1.5	3
$\mathfrak{y}$	1.5	1.5	1.5	2	1.2	1.2	1.2	1.2	2	0.5826	0.6692	0.7190
1.5	1.5	$\mathfrak{y}$	1.5	2	1.2	1.2	1.2	1.2	2	0.4920	0.6692	0.7659
1.5	1.5	1.5	$\mathfrak{y}$	2	1.2	1.2	1.2	1.2	2	0.3565	0.6692	0.7673
1.5	1.5	1.5	1.5	$\mathfrak{y}$	1.2	1.2	1.2	1.2	2	0.8985	0.7329	0.5697
1.5	1.5	1.5	1.5	2	1.2	1.2	1.2	$\mathfrak{y}$	2	0.7942	0.6398	0.5624
1.5	1.5	1.5	1.5	2	1.2	1.2	1.2	1.2	$\mathfrak{y}$	0.2880	0.5887	0.7686
1.5	1.5	1.5	1.5	2	1.2	$\mathfrak{y}$	1.2	1.2	2	0.7865	0.6354	0.5250
3	1.5	1.5	1.5	2	2.5	1.2	$\mathfrak{y}$	1.2	3	0.8590	0.7515	0.6668
$\mathfrak{y}$	1.5	1.5	1.5	2	2.5	1.2	1.2	1.2	3	0.8334	0.7783	0.7420
3	1.5	1.5	1.5	0.7	2.5	1.8	$\mathfrak{y}$	0.2	3	0.9851	0.9793	0.9757
3	1.5	$\mathfrak{y}$	1.5	0.7	2.5	1.8	1.5	0.2	3	0.9409	0.9793	0.9895
3	$\mathfrak{y}$	1.5	1.5	0.7	2.5	1.8	1.5	0.2	3	0.8185	0.9793	0.7565
3	1.5	1.5	1.5	$\mathfrak{y}$	2.5	1.8	1.5	0.2	3	0.9853	0.9550	0.9080
3	1.5	1.5	1.5	0.7	2.5	1.8	1.5	0.2	$\mathfrak{y}$	0.9576	0.9717	0.9793

The reliability stress-strength for the MOAPEW model in Equation (4) will be calculated numerically. Table 2 show some numerical values of the reliability stress-strength model for different cases.

4. STATISTICAL PROPERTIES OF THE MOAPEW

In this part, we look at some statistical properties of the MOAPEW distribution, including the quantile, variance, and mode.

4.1. Generate Sample

We can obtain a quantile function of the MOAPEW distribution by inverting the cdf Equation (10) as shown below:

$$t_u = \lambda\left\{\ln\left(1-\frac{1}{\lambda\delta}\ln\left(1-\frac{ln\left((\theta\alpha-1)u+1\right)-ln\left(1+u(\theta-1)\right)}{\ln(\alpha)}\right)\right)\right\}^{1/\beta};0<u<1. \quad (15)$$

From Equation (15), we can obtain the median (M) or the 2nd-quartile of MOAPEW distribution when $u = 0.5$ as following

$$M = \lambda\left\{\ln\left(1-\frac{1}{\lambda\delta}\ln\left(1-\frac{ln(\theta\alpha+1)-ln(\theta+1)}{\ln(\alpha)}\right)\right)\right\}^{1/\beta}. \quad (16)$$

If $u = 0.25$ and $u = 0.75$, respectively, we can obtain the first and third quartiles (q_1 *and* q_3) of the MOAPEW distribution. We can obtain the Galton skewness (Sk) from Equations (15) and (16), 1st, 2nd, 3td-quartiles of MOAPEW model, that is named as Bowley's skewness, that is written as:

$$Sk = \frac{q_3 - 2M + q_1}{q_3 - q_1},$$

also, kurtosis (Ku) is written as:

$$Ku = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{q_3 - q_1}.$$

4.2. Mode of MOAPEW

The log of MOAPEW density is taken from Equation (9), by

$$\ln\left(f(t,\Psi)\right) \propto (\beta-1)\ln(t) + \left(\frac{t}{\lambda}\right)^{\beta} - \lambda\delta e^{\left(\frac{t}{\lambda}\right)^{\beta}} - \lambda\delta\left(1-e^{\left(\frac{t}{\lambda}\right)^{\beta}}\right)\ln(\alpha) - 2\ln\Big[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}-1\right)\Big], \quad (17)$$

then $\frac{d\ln\left(f(t,\Psi)\right)}{dt} = 0$, we get

$$\frac{(\beta-1)}{t} + \frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1} + e^{\left(\frac{t}{\lambda}\right)^{\beta}}\left[\delta\beta\left(\frac{t}{\lambda}\right)^{\beta-1}(\ln(\alpha)-1)\right] + \frac{2\beta\delta\ln(\alpha)e^{\left(\frac{t}{\lambda}\right)^{\beta}}\left(\frac{t}{\lambda}\right)^{\beta-1}\mathfrak{E}(t;\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}}{\theta\frac{\alpha-1}{\theta-1}-\left(\alpha^{1-\mathfrak{E}(t;\beta,\lambda,\delta)}-1\right)} = 0. \quad (18)$$

The mean, median, skewness, variance (Var), and kurtosis for MOAPEW distribution using different parameter values are determined using the R program and shown in Table 3. The findings indicate that for fixed α, β, and λ, the mean, variance, and median for delta increases are increasing, while for δ increases, skewness and kurtosis are decreasing. With fixed δ, β, and λ, the mean, variance, and median for α increases are increasing, while for α increases, skewness and kurtosis are decreasing when β less than 1, but when β is more than 1, skewness and kurtosis are increasing. See Figure 1, which confirms the result of Table 3: the MOAPEW has left-skewed, reversed-J shaped, right-skewed, and symmetrical shapes.

Table 3. *The mean, var, skewness, M, and kurtosis of the MOAPEW model using different parameter values*

				$\alpha = 0.5$					$\alpha = 3$				
β	θ	λ	δ	Mean	Var	M	SK	Ku	Mean	Var	M	SK	Ku
0.5	0.5	0.5	0.5	0.1319	0.0529	0.0355	0.5817	1.6703	0.2398	0.0958	0.1180	0.4059	1.0340
			1.5	0.4854	0.4357	0.2113	0.4580	1.1670	0.8189	0.6874	0.5581	0.2618	0.6570
			3	0.9571	1.2310	0.5416	0.3573	0.8845	1.5342	1.7775	1.2148	0.1731	0.4521
		3	0.5	0.0502	0.0143	0.0074	0.6642	2.2837	0.0992	0.0311	0.0295	0.5380	1.5545
			1.5	0.2979	0.3582	0.0605	0.6278	1.9630	0.5632	0.7025	0.2221	0.4749	1.2693
			3	0.7913	1.9050	0.2129	0.5817	1.6703	1.4386	3.4500	0.7079	0.4059	1.0340
	1.5	0.5	0.5	0.2733	0.1098	0.1479	0.3685	0.9254	0.4415	0.1658	0.3341	0.1947	0.5174
			1.5	0.9139	0.7602	0.6649	0.2243	0.5683	1.3688	0.9901	1.2284	0.0758	0.2333
			3	1.6898	1.9245	1.4020	0.1402	0.3759	2.4149	2.2976	2.3121	0.0192	0.0891
		3	0.5	0.1160	0.0375	0.0388	0.5109	1.4306	0.2060	0.0696	0.1098	0.3572	0.9545
			1.5	0.6494	0.8228	0.2855	0.4415	1.1513	1.0954	1.3571	0.7204	0.2706	0.7063
			3	1.6396	3.9544	0.8875	0.3685	0.9254	2.6490	5.9684	2.0047	0.1947	0.5174
1.5	0.5	0.5	0.5	0.2360	0.0243	0.2070	0.1450	0.3570	0.2717	0.0462	0.2242	0.1640	0.4181
			1.5	0.3943	0.0481	0.3752	0.0592	0.1522	0.5696	0.1346	0.5315	0.0576	0.1703
			3	0.5176	0.0643	0.5135	0.0014	0.0177	0.8326	0.2175	0.8187	-0.0005	0.0291
		3	1	0.5093	0.1588	0.4051	0.2133	0.5540	0.3381	0.1153	0.2301	0.2752	0.7178
			2	0.9758	0.4899	0.8166	0.1819	0.4571	0.9325	0.6788	0.7064	0.2203	0.5597
			3	1.4161	0.8740	1.2420	0.1450	0.3570	1.6301	1.6641	1.3450	0.1640	0.4181

5. CASES FOR ESTIMATING PARAMETERS

This section discusses the estimation of the parameter of the MOAPEW distribution in the presence of censoring Type I and Type II using Bayesian, MPS and MLE estimation methods in detail.

5.1. MLE Based Censored Samples

Type I and Type II, MLE functions are described as follows:

$$L(\Psi) = \frac{n!}{(n-r)!}\left(1 - F(\wp;\Psi)\right)^{n-r} \prod_{i=1}^{r} f(x_{\mathrm{i:n}};\Psi), \tag{19}$$

while in Type I censored $\wp = T$ and in Type II censored $\wp = x_{r:n}$. More specifics can be found here. See [22]. For more examples, see [23-25].

In Type, I censored,remove un-failed units from a test at a pre-specified time, where they may be tested simultaneously or in sequence. The data consists of the observations $x_{1:n} < x_{2:n} < \cdots < \wp$ and the information that $(n-r)$ items survive beyond the time of termination T, where r is the uncensored item number. Assuming that n observation is located in life testing for MOAPEW distribution, and at a certain time T, the test is finished before the failure of all n observation. At random, the number of failures r is determined. In Type II censoring, a lifetime test is halted within a certain number of failures, whereby n and r are fixed and predictable, but T is unpredictable. The loglikelihood function is calculated using the Equation below of the MOAPEW distribution based on Type I and Type II censoring:

$$l(\Psi)=r\,ln\left(\frac{ln(\alpha)\,\delta\beta}{(\alpha-1)}\right)+(\beta-1)\sum_{i=1}^{r}ln\left(\frac{t_{i:n}}{\lambda}\right)-\sum_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^{\beta}+\sum_{i=1}^{r}\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)$$
$$+ln(\alpha)\sum_{i=1}^{r}\left(1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\right)-2\sum_{i=1}^{r}ln\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right] \quad (20)$$
$$+n\ln(\theta)-(n-r)\left[\ln(\alpha-1)+ln\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{(1-\mathfrak{E}(\wp;\beta,\lambda,\delta))}-1\right)\right)\right]$$
$$+(n-r)ln\left[\alpha-\alpha^{(1-\mathfrak{E}(\wp;\beta,\lambda,\delta))}\right].$$

Equation (20) can be explicitly maximized using the R program by an mle2 function in order to solve the non-linear log-likelihood functions that is derived by differentiating Equation (20) (with respect to the distribution parameters Ψ) and equate it to zero.

5.2. MPS Based on Censoring Sample

This is the general form of MPS with Type I censored and Type II censored.

$$S(\Psi)=\frac{n!}{(n-r)!}\left(1-F(\wp;\Psi)\right)^{n-r}\prod_{i=1}^{r+1}\left(D_{i:n}(t;\Psi)\right), \quad (21)$$

since $D_{i:n}(t;\Psi)=\begin{cases}F(t_{1:n},\Psi)\\ F(t_{i:n},\Psi)-F(t_{(i-1):n},\Psi);\ i=2\ldots r,\\ 1-F(t_{r:n},\Psi)\end{cases}$ while in Type I censored $\wp=T$ and in Type II censored $\wp=x_{r:n}$. For more details, see [25, 26]. The natural log of maximum product spacing function with its general form for both types of censoring is as follows

$$\ln S(\Psi)=(n-r)\left[\ln\left(\frac{\theta}{\alpha-1}\right)+\ln\left(\alpha^{1-\mathfrak{E}(\wp;\beta,\lambda,\delta)}-1\right)-ln\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(\wp;\beta,\lambda,\delta)}-1\right)\right)\right]$$
$$+\ln\left(\alpha^{1-\mathfrak{E}(t_{1:n};\beta,\lambda,\delta)}-1\right)-\ln(\alpha-1)-\ln\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{1:n};\beta,\lambda,\delta)}-1\right)\right)+\ln\left(\frac{\theta}{(\alpha-1)}\right)$$
$$+\ln\left[\alpha-\alpha^{(1-\mathfrak{E}(t_{r:n};\beta,\lambda,\delta))}\right]-\ln\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{(1-\mathfrak{E}(t_{r:n};\beta,\lambda,\delta))}-1\right)\right] \quad (22)$$
$$+\sum_{i=2}^{r}\left[\frac{\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1}{(\alpha-1)\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)}-\frac{\alpha^{1-\mathfrak{E}(t_{i-1:n};\beta,\lambda,\delta)}-1}{(\alpha-1)\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i-1:n};\beta,\lambda,\delta)}-1\right)\right)}\right].$$

With regard to the unknown parameters, the partial derivatives of MPS can not be explicitly determined under the censored sample, so numerical algorithms such as the Conjugate Gradients method are used to count the MPS of Ψ.

5.3. Analysis of Bayesian Estimation

The Bayesian estimation of unknown parameters is considered $\Psi=(\alpha,\beta,\theta,\lambda,\delta)$ based on Types I and II censoring scheme. Bayesian estimate of an assumption that the RV Ψ have an independent identically distributed *(iid)* $Gamma(\Gamma)$ prior, these are $\alpha\sim Gamma(a_1,b_1)$, $\beta\sim Gamma(a_2,b_2)$, $\theta\sim Gamma(a_3,b_3)$, $\lambda\sim Gamma(a_4,b_4)$ and $\delta\sim Gamma(a_5,b_5)$. The joint pdf prior of Ψ should be documented as

$$g(\Psi)\propto\ \alpha^{a_1-1}e^{-\frac{\alpha}{b_1}}\beta^{a_2-1}e^{-\frac{\beta}{b_2}}\theta^{a_3-1}e^{-\frac{\theta}{b_3}}\lambda^{a_4-1}e^{-\frac{\lambda}{b_4}}\delta^{a_5-1}e^{-\frac{\delta}{b_5}};$$
$$g(\Psi)\propto\prod_{j=1}^{5}\Psi_j^{a_j-1}e^{-\frac{\Psi_j}{b_j}};\ \ j=1,2,\ldots 5, \quad (23)$$

where all the hyperparameters a_j and b_j; $j=1,2,\ldots 5$ are known and non-negative. According to [27], the hyper-parameters are chosen to balance the experimenter's prior belief in terms of the prior gamma distribution. By equating variance and mean of $\widehat{\Psi}_j^{\,i}$ to the mean and variance of the prior results (Gamma priors), where $i = 1, 2, \ldots k$ and k is the number of samples available from the MOAPEW distribution.

$$\frac{1}{k}\sum_{i=1}^{k}\widehat{\Psi}_j^{\,i} = \frac{a_j}{b_j} \quad \& \quad \frac{1}{k-1}\sum_{i=1}^{k}\left(\widehat{\Psi}_j^{\,i} - \frac{1}{k}\sum_{j=1}^{k}\widehat{\Psi}_j^{\,i}\right)^2 = \frac{a_j}{b_j^{\,2}}.$$

Based on the following likelihood function

$$L(\Psi) = \left(\frac{\theta\delta\beta\ln(\alpha)}{(\alpha-1)}\right)^r e^{\sum_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^\beta}\left(\frac{\theta\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{(\alpha-1)\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r}\prod_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^{\beta-1}\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)^2}\right).$$

Acoording to Liklihood function in previous equation and jount prior in Equation (23), we get the MOAPEW joint posterior in its general form:

$$\pi(\Psi|t) = \mathbb{C}\prod_{j=1}^{5}\left(\Psi_j^{a_j-1}e^{-\frac{\Psi_j}{b_j}}\right)\left(\frac{\theta\delta\beta\ln(\alpha)}{(\alpha-1)}\right)^r e^{\sum_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^\beta}\prod_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^{\beta-1}$$
$$\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)^2}\right)\left(\frac{\theta\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{(\alpha-1)\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r}, \quad (24)$$

since $\mathbb{C}$ is normalizing constant.

Markov-Chain Monte-Carlo (MCMC) is known to be a sampling technique operated by a computer. In Bayesian inference, MCMC is explicitly useful as a result of concentrating on the posterior distributions, which are often difficult to deal with via analytic examination, so MCMC helps the user to approximate aspects that can not be explicitly computed of posterior distributions.

We can express the conditional posterior densities of Ψ as shown below:

$$\pi_1^*(\alpha|\beta,\theta,\lambda,\delta,t) \propto \alpha^{a_1-1}e^{-\frac{\alpha}{b_1}}[\ln(\alpha)]^r\left(\frac{[\alpha-\varphi(\wp,\alpha,\beta,\lambda)]}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}(\varphi(\wp,\alpha,\beta,\lambda)-1)\right]}\right)^{n-r}$$
$$\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)^2}\right), \quad (25)$$

$$\pi_2^*(\beta|\alpha,\theta,\lambda,\delta,t) \propto \beta^{a_2+r-1}e^{-\frac{\beta}{b_2}}e^{\sum_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^\beta}\left(\frac{\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r}\prod_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^{\beta-1}$$
$$\prod_{i=1}^{r}\left(\frac{\left(\frac{x_i}{\beta}\right)^{\lambda-1}\varphi(x_{i:n},\alpha,\beta,\lambda)}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}(\varphi(x_{i:n},\alpha,\beta,\lambda)-1)\right]^2}\right)\left(\frac{[\alpha-\varphi(\wp,\alpha,\beta,\lambda)]}{(\alpha-1)\left[\theta+\frac{(1-\theta)}{(\alpha-1)}(\varphi(\wp,\alpha,\beta,\lambda)-1)\right]}\right)^{n-r} \quad (26)$$

$$\pi_2^*(\theta|\alpha,\beta,\lambda,\delta,t) \propto \theta^{a_3+n-1}e^{-\frac{\theta}{b_3}}\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)^2}\right)\left(\frac{\theta\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r}, \quad (27)$$

$$\pi_2^*(\lambda|\alpha,\beta,\lambda,\delta,t) \propto \lambda^{a_4-1}e^{-\frac{\lambda}{b_4}}e^{\sum_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^\beta}\prod_{i=1}^{r}\left(\frac{t_{i:n}}{\lambda}\right)^{\beta-1}\left(\frac{\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r} \quad (28)$$

$$\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1)\right)^{2}}\right)$$

and

$$\pi_2^*(\delta|\alpha,\beta,\theta,\lambda,t) \propto \delta^{a_5-1}\, e^{-\frac{\delta}{b_5}}\delta^{r}\left(\frac{\left[\alpha-\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}\right]}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{\left(1-\mathfrak{E}(\wp;\beta,\lambda,\delta)\right)}-1\right)\right]}\right)^{n-r}\prod_{i=1}^{r}\left(\frac{\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\mathfrak{E}(t_{i:n};\beta,\lambda,\delta)}-1\right)\right)^{2}}\right), \quad (29)$$

Therefore, we use the Metropolis-Hastings Algorithm to produce these distributions via this method ([28] with the normal proposal). Readers may refer to [29, 30] and [25] for more information on the implementation of the Metropolis-Hasting algorithm.

6. MONTE-CARLO STUDY OF SIMULATION

Throughout this portion, a Monte-Carlo simulation is used to estimate the MOAPEW distribution parameters based on Type I and II censored, employing MLE, MPS, and Bayesian methods and utilizing R packages and the steps outlined below.

In the simulation algorithm, Monte Carlo experiments were carried out using the quantile (15) under the following data created the form of MOAPEW distribution, where *x* is distributed as MOAPEW distribution for various parameters $\Psi = (\alpha, \beta, \theta, \lambda, \delta)$.

case 1: $\alpha = 0.75$; $\beta = 1.5$; $\theta = 0.5$; $\lambda = 0.75$; $\delta = 0.75$, **case** 2: $\alpha = 3$; $\beta = 1.5$; $\theta = 0.5$; $\lambda = 1.5$; $\delta = 3$ and **case** 3: $\alpha = 3$; $\beta = 1.5$; $\theta = 0.5$; $\lambda = 0.75$; $\delta = 3$, for different samples sizes $n = 30, 70\ and\ 150$, and for a variety of censored sample schemes, wherein Type II scheme as following: when *n* is 30, *r* is $(20, 25)$, when *n* is 70, *r* is $(50, 60)$, and when *n* is 150, *r* is $(120, 140)$. While in the Type I scheme, we used different times. Equations (20) and (21) may be used to calculate parameter estimation. optim function in R packages is used to estimate the parameters, and we can find the Bayesian parameter estimation by using Equations (25)-(29) and 10,000 iterations.

The best approach may be characterized as a scheme that minimizes bias and mean squared error (MSE), where $MSE = Mean(\Psi - \widehat{\Psi})^2$ and $Bias = \Psi - \widehat{\Psi}$, where $\widehat{\Psi}$ is the estimated value of Ψ.

By Tables 4-6, we can conclusions the following:

1. As n is raised in all of the calculations, the bias and MSE decrease.
2. As the number of failures (r) rises in the Type II censored sample, the significance of the Bias and MSE for MOAPEW distribution parameters declines.
3. By raising the time (T) for censored Type I samples, the value of Bias and MSE decreases for the parameters of MOAPEW distribution.
4. Bayesian estimates have much more relative efficiency, unlike MLE and MPS with most MOAPEW distribution parameters.

Table 4. *Estimated values of parametrs for censored sample in case 1*

$\alpha = 0.75;\ \beta = 1.5;\ \theta = 0.5;\ \lambda = 0.75;\ \delta = 0.75$								
			MLE		MPS		Bayesian	
N	Scheme		Bias	MSE	Bias	MSE	Bias	MSE
30	Type II r=20	$\hat{\alpha}$	0.0819	0.0769	0.0296	0.0834	0.0513	0.0675
		$\hat{\beta}$	0.0853	0.4048	-0.2226	0.3714	0.1176	0.1233
		$\hat{\theta}$	0.0012	0.2532	0.1853	0.2389	0.1593	0.1721
		$\hat{\lambda}$	0.0873	0.2589	-0.0284	0.3205	0.2952	0.2189
		$\hat{\delta}$	-0.1431	0.2571	0.0052	0.1271	-0.1218	0.2219
	Type II r=25	$\hat{\alpha}$	0.0654	0.0295	0.0875	0.0318	0.0492	0.0205
		$\hat{\beta}$	0.1085	0.3124	-0.1548	0.3529	0.0755	0.1018
		$\hat{\theta}$	0.0544	0.1832	0.1408	0.2093	0.2140	0.1578
		$\hat{\lambda}$	0.0045	0.1638	-0.0891	0.2456	0.2313	0.1594
		$\hat{\delta}$	-0.0812	0.0700	-0.1717	0.0740	-0.0711	0.0699
	complete r=30	$\hat{\alpha}$	0.0584	0.0310	0.0620	0.0407	0.0454	0.0251
		$\hat{\beta}$	0.0911	0.3014	-0.2071	0.3585	0.1094	0.0946
		$\hat{\theta}$	0.0435	0.1305	0.1471	0.1772	0.0390	0.1033
		$\hat{\lambda}$	-0.0613	0.0934	-0.0496	0.1871	0.2122	0.0240
		$\hat{\delta}$	-0.0895	0.0390	-0.1309	0.0556	-0.0402	0.0319
	Type I T=1	$\hat{\alpha}$	0.0119	0.0659	0.0117	0.0829	0.0134	0.0588
		$\hat{\beta}$	0.1221	0.3115	-0.1695	0.3620	0.0468	0.0991
		$\hat{\theta}$	0.0489	0.2303	0.1921	0.2870	0.1232	0.1518
		$\hat{\lambda}$	0.1093	0.1915	0.0417	0.2907	0.3078	0.1523
		$\hat{\delta}$	-0.0074	0.1982	-0.0003	0.2044	0.1713	0.1930
70	Type II r=50	$\hat{\alpha}$	0.0336	0.0306	0.0306	0.0369	0.0250	0.0217
		$\hat{\beta}$	0.0833	0.1955	-0.0522	0.1749	0.0731	0.0997
		$\hat{\theta}$	0.0406	0.1251	0.0420	0.1226	0.0401	0.1200
		$\hat{\lambda}$	0.0369	0.1420	-0.0596	0.1832	0.2391	0.1458
		$\hat{\delta}$	-0.0325	0.0369	0.0393	0.0321	-0.0347	0.0279
	Type II r=60	$\hat{\alpha}$	0.0167	0.0302	-0.0273	0.0283	0.1317	0.0219
		$\hat{\beta}$	0.0185	0.1864	-0.1467	0.1988	-0.0253	0.0559
		$\hat{\theta}$	-0.0010	0.0775	0.1026	0.0902	0.0318	0.0645
		$\hat{\lambda}$	0.0136	0.1189	-0.0665	0.1156	0.1371	0.0902
		$\hat{\delta}$	-0.0446	0.0365	-0.0842	0.0304	-0.0310	0.0214
	complete r=70	$\hat{\alpha}$	0.0571	0.0179	0.0547	0.0204	0.0562	0.0170
		$\hat{\beta}$	0.0118	0.1417	-0.1847	0.1920	0.0049	0.0537
		$\hat{\theta}$	0.0337	0.0696	0.1034	0.0851	0.0302	0.0618
		$\hat{\lambda}$	-0.0253	0.0874	-0.0177	0.0811	0.0216	0.0781
		$\hat{\delta}$	-0.0623	0.0279	-0.1196	0.0447	-0.0246	0.0201
	Type I T=1	$\hat{\alpha}$	-0.0056	0.0601	-0.0135	0.0859	0.0120	0.0583
		$\hat{\beta}$	0.0288	0.1808	-0.1380	0.2181	-0.0060	0.0342
		$\hat{\theta}$	0.1123	0.1959	0.2568	0.3055	0.1355	0.0805
		$\hat{\lambda}$	0.0441	0.0949	-0.0003	0.1302	0.1793	0.0578
		$\hat{\delta}$	0.0432	0.1574	0.0742	0.1711	0.1228	0.0741
	Type I T=1.2	$\hat{\alpha}$	-0.0044	0.0590	-0.0099	0.0845	0.0121	0.0480
		$\hat{\beta}$	0.0290	0.1781	-0.1374	0.2087	-0.0048	0.0304
		$\hat{\theta}$	0.1105	0.1896	0.2513	0.3013	0.1329	0.0781
		$\hat{\lambda}$	0.0426	0.0895	-0.0013	0.1293	0.1709	0.0580
		$\hat{\delta}$	0.0395	0.1458	0.0719	0.1697	0.1238	0.0742

Table 4. *Estimated values of parameters for censored sample in case 1*

			MLE		MPS		Bayesian	
n	Scheme		Bias	MSE	Bias	Bias	MSE	Bias
150	Type II r=120	$\hat{\alpha}$	0.0116	0.0558	0.0132	0.0434	0.0201	0.0419
		$\hat{\beta}$	-0.0026	0.0773	-0.0796	0.0793	0.0204	0.0453
		$\hat{\theta}$	0.0490	0.1863	0.1720	0.1500	0.1081	0.0983
		$\hat{\lambda}$	-0.0037	0.0623	0.0227	0.0562	0.1022	0.0609
		$\hat{\delta}$	-0.0012	0.3268	0.1511	0.2028	0.2025	0.2213
	Type II r=140	$\hat{\alpha}$	-0.0236	0.0328	-0.0554	0.0516	-0.0196	0.0315
		$\hat{\beta}$	0.0435	0.0795	-0.0531	0.0882	0.0257	0.0485
		$\hat{\theta}$	0.0122	0.0618	0.1084	0.1215	0.0841	0.0594
		$\hat{\lambda}$	0.0060	0.0338	-0.0159	0.0519	0.0125	0.0314
		$\hat{\delta}$	-0.0283	0.0350	-0.0046	0.0708	-0.0156	0.0319
	complete r=150	$\hat{\alpha}$	0.0199	0.0123	0.0131	0.0170	0.0173	0.0108
		$\hat{\beta}$	-0.0127	0.0877	-0.1163	0.1040	-0.0830	0.0431
		$\hat{\theta}$	0.0180	0.0429	0.0684	0.0542	0.2179	0.0401
		$\hat{\lambda}$	0.0027	0.0612	-0.0616	0.0669	0.1009	0.0607
		$\hat{\delta}$	-0.0378	0.0126	-0.0673	0.0182	-0.0144	0.0101
	Type I T=1	$\hat{\alpha}$	-0.0347	0.0473	-0.0625	0.0660	0.1000	0.1391
		$\hat{\beta}$	0.0449	0.0829	-0.0527	0.0971	0.0220	0.0292
		$\hat{\theta}$	0.0325	0.1854	0.2663	0.4204	0.0992	0.0657
		$\hat{\lambda}$	-0.0152	0.0564	0.0528	0.1020	0.1685	0.0509
		$\hat{\delta}$	-0.0430	0.2340	0.2289	0.5224	0.1130	0.0711
	Type I T=1.2	$\hat{\alpha}$	0.0182	0.0458	0.0341	0.1092	0.0663	0.1177
		$\hat{\beta}$	0.0118	0.0802	-0.0864	0.0922	0.0168	0.0287
		$\hat{\theta}$	0.0667	0.1148	0.1768	0.1754	0.0664	0.0511
		$\hat{\lambda}$	0.0251	0.0409	0.0137	0.0640	0.0873	0.0282
		$\hat{\delta}$	0.0304	0.0875	0.0749	0.1356	0.0764	0.0527

Table 5. *Estimated values of parameters for censored sample in case 2*

			MLE		MPS		Bayesian	
n	Scheme		Bias	MSE	Bias	MSE	Bias	MSE
150	Type II r=120	$\hat{\alpha}$	-0.0719	0.3969	0.1023	0.0634	0.0834	0.0417
		$\hat{\beta}$	0.0950	0.1582	-0.0541	0.0968	0.0314	0.0354
		$\hat{\theta}$	-0.1176	0.5020	0.2301	0.5087	0.2138	0.4060
		$\hat{\lambda}$	-0.1472	0.2648	-0.0084	0.1282	0.1307	0.0648
		$\hat{\delta}$	-0.3999	0.4572	-0.0766	0.3291	0.1606	0.1231
	Type II r=140	$\hat{\alpha}$	-0.1094	0.1752	0.0247	0.1896	0.0620	0.0317
		$\hat{\beta}$	-0.0158	0.1366	-0.0922	0.1012	0.0131	0.0336
		$\hat{\theta}$	0.0327	0.4315	0.2999	0.5113	0.0607	0.0725
		$\hat{\lambda}$	-0.1418	0.2062	-0.0347	0.2116	0.4821	0.2004
		$\hat{\delta}$	-0.2593	0.4507	-0.1602	0.4037	-0.0982	0.0938
	complete r=150	$\hat{\alpha}$	-0.0720	0.1140	0.0666	0.1672	0.0711	0.0305
		$\hat{\beta}$	-0.0108	0.0910	-0.1198	0.0999	-0.0421	0.0254
		$\hat{\theta}$	0.1160	0.2325	0.3427	0.5749	0.2200	0.0712
		$\hat{\lambda}$	-0.0450	0.2041	-0.0042	0.2225	0.3974	0.2002
		$\hat{\delta}$	-0.1903	0.2202	-0.1230	0.2292	-0.1989	0.0912
	Type I T=1	$\hat{\alpha}$	-0.0651	0.1501	0.0758	0.1639	0.0711	0.0466
		$\hat{\beta}$	0.0100	0.0922	-0.0899	0.1017	0.0077	0.0363
		$\hat{\theta}$	0.0859	0.2843	0.2986	0.5870	0.0585	0.0815
		$\hat{\lambda}$	-0.0389	0.1773	0.0760	0.2089	0.0419	0.1299
		$\hat{\delta}$	-0.2191	0.5268	-0.0796	0.4147	-0.0442	0.1056
	Type I T=1.2	$\hat{\alpha}$	-0.0370	0.1247	0.1200	0.1197	0.0893	0.0442
		$\hat{\beta}$	0.0136	0.0875	-0.0964	0.0978	0.0122	0.0458
		$\hat{\theta}$	0.0625	0.2317	0.3119	0.5442	0.0925	0.1171
		$\hat{\lambda}$	-0.0666	0.1690	0.1166	0.1621	0.0931	0.1219

		$\hat{\delta}$	-0.2131	0.5065	0.0413	0.4307	0.0822	0.1087

Table 5. *Estimated values of parameters for censored sample in case 2*

$\alpha = 3;\ \beta = 1.5;\ \theta = 0.5;\ \lambda = 1.5;\ \delta = 3$								
			MLE		MPS		Bayesian	
n	Scheme		Bias	MSE	Bias	MSE	Bias	MSE
30	Type II r=20	$\hat{\alpha}$	-0.2477	0.5606	0.1213	0.1746	0.0910	0.0323
		$\hat{\beta}$	0.2958	0.6409	-0.0957	0.4518	0.1839	0.1173
		$\hat{\theta}$	-0.2565	1.6561	0.3466	1.3502	0.2230	0.2167
		$\hat{\lambda}$	-0.3383	0.5371	-0.1238	0.6612	0.2715	0.1795
		$\hat{\delta}$	-0.8491	2.3278	-0.4603	0.9737	-0.0372	0.0767
	Type II r=25	$\hat{\alpha}$	-0.2144	0.3028	-0.0042	0.2677	0.0901	0.0300
		$\hat{\beta}$	0.1185	0.3663	-0.2113	0.3791	0.0969	0.0844
		$\hat{\theta}$	0.3077	0.7967	0.3072	0.7664	0.1706	0.1419
		$\hat{\lambda}$	-0.1852	0.7220	-0.1472	0.9273	0.3812	0.2887
		$\hat{\delta}$	-0.5476	0.8841	-0.5515	0.7720	-0.0791	0.0782
	complete r=30	$\hat{\alpha}$	-0.1453	0.1686	0.0561	0.1516	0.0823	0.0298
		$\hat{\beta}$	0.0775	0.2740	-0.2178	0.3188	0.0472	0.0588
		$\hat{\theta}$	0.2052	0.5995	0.2062	0.5847	0.2882	0.5278
		$\hat{\lambda}$	-0.1825	0.6219	-0.0977	0.8303	0.1781	0.1227
		$\hat{\delta}$	-0.4827	0.5941	-0.4541	0.5700	0.0896	0.0771
	Type I T=1	$\hat{\alpha}$	-0.1498	0.3437	0.1301	0.3988	0.0862	0.0325
		$\hat{\beta}$	0.0913	0.2920	-0.2125	0.3362	0.0618	0.0733
		$\hat{\theta}$	0.1908	0.8304	0.7330	2.1075	0.1282	0.1398
		$\hat{\lambda}$	-0.1078	0.5590	0.1225	0.8624	0.3440	0.3053
		$\hat{\delta}$	-0.5166	0.8719	-0.3601	0.7832	-0.0132	0.0826
70	Type II r=50	$\hat{\alpha}$	-0.1321	0.1550	0.1200	0.1481	0.0794	0.0283
		$\hat{\beta}$	0.0692	0.2652	-0.0690	0.2369	0.0435	0.0548
		$\hat{\theta}$	-0.2042	0.5848	0.1322	0.5616	0.2526	0.4838
		$\hat{\lambda}$	-0.1730	0.6048	-0.0970	0.5173	0.1730	0.1100
		$\hat{\delta}$	-0.4765	0.5708	-0.3727	0.5818	0.0946	0.0768
	Type II r=60	$\hat{\alpha}$	-0.0684	0.1227	0.0612	0.1220	0.0957	0.0396
		$\hat{\beta}$	0.0164	0.1634	-0.0119	0.1623	0.0479	0.0452
		$\hat{\theta}$	0.2696	0.5822	0.2652	0.5395	0.2687	0.1820
		$\hat{\lambda}$	-0.0708	0.4620	-0.0702	0.4593	0.3098	0.2178
		$\hat{\delta}$	-0.3081	0.3187	-0.3013	0.3042	-0.0901	0.0863
	complete r=70	$\hat{\alpha}$	-0.0610	0.1208	0.0595	0.1215	0.0861	0.0347
		$\hat{\beta}$	0.0120	0.1605	-0.0117	0.1602	-0.0062	0.0419
		$\hat{\theta}$	0.1941	0.5079	0.1955	0.5014	0.2045	0.1595
		$\hat{\lambda}$	-0.0697	0.4019	-0.0300	0.4005	0.4595	0.2136
		$\hat{\delta}$	-0.3053	0.3051	-0.2812	0.3020	-0.1285	0.0859
	Type I T=1	$\hat{\alpha}$	-0.1283	0.2821	0.1275	0.3560	0.0799	0.0298
		$\hat{\beta}$	0.0515	0.2054	-0.1112	0.2005	0.0426	0.0548
		$\hat{\theta}$	0.0532	0.5773	0.0494	0.5333	0.0971	0.1187
		$\hat{\lambda}$	-0.1117	0.4610	0.1658	0.6260	0.3154	0.2577
		$\hat{\delta}$	-0.3499	0.3263	-0.1964	0.2843	0.0488	0.0994
	Type I T=1.2	$\hat{\alpha}$	-0.0893	0.1682	0.0956	0.1746	0.0851	0.0365
		$\hat{\beta}$	0.0264	0.1720	-0.1524	0.1980	0.0410	0.0622
		$\hat{\theta}$	0.1240	0.4738	0.0474	0.5043	0.0761	0.1015
		$\hat{\lambda}$	-0.0715	0.3607	0.1266	0.4911	0.4136	0.3068
		$\hat{\delta}$	-0.3437	0.9679	-0.1415	0.6635	-0.0360	0.0899

Table 6. *Estimated values of parametrs for censored sample in case 3*

$\alpha=3;\ \beta=1.5;\ \theta=0.5;\ \lambda=0.75;\ \delta=3$								
			MLE		MPS		Bayesian	
n	Scheme		Bias	MSE	Bias	MSE	Bias	MSE
30	Type II r=20	$\hat{\alpha}$	-0.1454	0.2322	0.0651	0.0933	0.0849	0.0314
		$\hat{\beta}$	0.1719	0.3433	-0.1287	0.3464	0.0626	0.1165
		$\hat{\theta}$	0.0693	0.4201	0.3767	0.4168	0.1465	0.1580
		$\hat{\lambda}$	-0.4935	0.2488	-0.5304	0.2391	0.1890	0.1317
		$\hat{\delta}$	-0.5330	1.0772	-0.1468	0.3317	0.1412	0.0921
	Type II r=25	$\hat{\alpha}$	-0.1631	0.0849	-0.0672	0.0801	0.0841	0.0305
		$\hat{\beta}$	0.1358	0.3099	-0.2210	0.3049	0.0484	0.1106
		$\hat{\theta}$	0.2237	0.2841	0.4657	0.2855	0.1231	0.1482
		$\hat{\lambda}$	-0.3989	0.1710	-0.4435	0.1702	0.1640	0.1152
		$\hat{\delta}$	-0.4656	0.4191	-0.3954	0.3557	0.1688	0.1097
	complete r=30	$\hat{\alpha}$	-0.0789	0.0281	-0.0271	0.0200	0.0763	0.0268
		$\hat{\beta}$	0.1580	0.2400	-0.1971	0.2370	-0.0007	0.1154
		$\hat{\theta}$	0.2796	0.2538	0.4505	0.2421	0.0812	0.1349
		$\hat{\lambda}$	-0.0965	0.0808	-0.0007	0.0803	0.0816	0.0712
		$\hat{\delta}$	-0.1808	0.1101	-0.1890	0.1002	0.0921	0.1031
	Type I T=0.6	$\hat{\alpha}$	-0.0540	0.2204	0.0513	0.2127	0.0823	0.0296
		$\hat{\beta}$	0.1427	0.4065	-0.1372	0.4002	0.0092	0.0427
		$\hat{\theta}$	0.1372	0.4528	0.1251	0.4143	0.1472	0.1058
		$\hat{\lambda}$	0.0614	0.2198	0.0611	0.2390	0.0521	0.0872
		$\hat{\delta}$	-0.2281	0.6187	-0.0748	0.6042	0.0108	0.0124
70	Type II r=50	$\hat{\alpha}$	-0.1006	0.1936	0.0781	0.1925	0.0803	0.0314
		$\hat{\beta}$	0.0428	0.1674	-0.1726	0.1622	0.0205	0.0739
		$\hat{\theta}$	0.0376	0.2933	0.3926	0.2994	0.0228	0.1284
		$\hat{\lambda}$	-0.0735	0.0571	-0.0284	0.0479	0.0479	0.0517
		$\hat{\delta}$	-0.4129	1.0565	-0.0918	0.4370	0.1908	0.1325
	Type II r=60	$\hat{\alpha}$	-0.0886	0.1604	0.0645	0.1267	0.0797	0.0308
		$\hat{\beta}$	0.0232	0.1672	-0.1679	0.1620	-0.0117	0.0532
		$\hat{\theta}$	0.1305	0.2787	0.4199	0.2660	0.1181	0.1247
		$\hat{\lambda}$	-0.0365	0.0486	-0.0002	0.0426	0.0316	0.0419
		$\hat{\delta}$	-0.2282	0.3482	-0.0917	0.2114	0.0569	0.1003
	complete r=70	$\hat{\alpha}$	-0.0627	0.0932	0.0748	0.0646	0.0611	0.0359
		$\hat{\beta}$	0.0144	0.1582	-0.0133	0.1588	0.0251	0.0518
		$\hat{\theta}$	0.1267	0.2317	0.1288	0.2241	0.1293	0.2043
		$\hat{\lambda}$	0.0250	0.0419	-0.0068	0.0418	0.0220	0.0394
		$\hat{\delta}$	-0.2135	0.1773	-0.0144	0.1721	0.0306	0.0989
	Type I T=0.4	$\hat{\alpha}$	-0.0487	0.2007	0.1208	0.1992	0.1051	0.0391
		$\hat{\beta}$	0.0522	0.1966	-0.1260	0.1923	0.0007	0.0310
		$\hat{\theta}$	0.1188	0.4571	0.4318	0.3960	0.2374	0.1368
		$\hat{\lambda}$	0.1553	0.2576	0.1561	0.2432	0.1872	0.0556
		$\hat{\delta}$	-0.1259	0.5400	0.0422	0.5112	-0.0291	0.0146
	Type I T=0.6	$\hat{\alpha}$	-0.0614	0.1227	0.0830	0.1241	0.0819	0.0366
		$\hat{\beta}$	0.0685	0.1865	-0.1192	0.1821	-0.0326	0.0342
		$\hat{\theta}$	0.0590	0.2916	0.3066	0.2858	0.1369	0.0869
		$\hat{\lambda}$	0.0557	0.1315	0.1111	0.1313	0.2303	0.0488
		$\hat{\delta}$	-0.1770	0.2474	-0.0820	0.2418	0.0111	0.0139

Table 6. *Estimated values of parametrs for censored sample in case 3*

$\alpha = 3;\ \beta = 1.5;\ \theta = 0.5;\ \lambda = 0.75;\ \delta = 3$								
			MLE		MPS		Bayesian	
n	Scheme		Bias	MSE	Bias	MSE	Bias	MSE
150		$\hat{\beta}$	0.0145	0.0973	-0.0884	0.0928	0.0179	0.0437
		$\hat{\theta}$	0.0439	0.1812	0.2321	0.3585	0.0932	0.0890
		$\hat{\lambda}$	-0.0169	0.0341	0.0492	0.1298	0.0292	0.0214
		$\hat{\delta}$	-0.0948	0.4395	0.0774	0.5166	0.1988	0.1622
	Type II r=140	$\hat{\alpha}$	-0.0531	0.1263	0.0919	0.0553	0.0839	0.0400
		$\hat{\beta}$	0.0183	0.0910	-0.0754	0.0909	-0.0026	0.0444
		$\hat{\theta}$	0.0154	0.1726	0.2847	0.3463	0.0927	0.0947
		$\hat{\lambda}$	-0.0358	0.0335	0.0156	0.0781	0.0320	0.0275
		$\hat{\delta}$	-0.2015	0.3676	-0.0226	0.1917	0.1141	0.1474
	complete r=150	$\hat{\alpha}$	-0.0463	0.1076	0.0651	0.1587	0.0776	0.0370
		$\hat{\beta}$	0.0119	0.0960	-0.0997	0.1053	-0.0517	0.0370
		$\hat{\theta}$	0.0594	0.1615	0.2290	0.3334	0.2524	0.1559
		$\hat{\lambda}$	0.0189	0.0675	0.0112	0.0461	0.0927	0.0382
		$\hat{\delta}$	-0.1156	0.2052	-0.0251	0.2765	-0.0272	0.1191
	Type I T=0.4	$\hat{\alpha}$	0.0628	0.4705	0.1824	0.2605	0.0977	0.0390
		$\hat{\beta}$	0.0355	0.0907	-0.0885	0.1117	-0.0236	0.0204
		$\hat{\theta}$	0.0598	0.1548	0.2373	0.3457	0.2508	0.1177
		$\hat{\lambda}$	0.0631	0.2016	0.2627	0.4367	0.1531	0.0380
		$\hat{\delta}$	0.0606	0.6304	0.2131	0.5186	-0.0389	0.0180
	Type I T=0.6	$\hat{\alpha}$	-0.0087	0.1101	0.1124	0.1651	0.0837	0.0383
		$\hat{\beta}$	0.0407	0.0991	-0.0737	0.1052	-0.0162	0.0275
		$\hat{\theta}$	0.0790	0.1114	0.0698	0.1148	0.0949	0.0614
		$\hat{\lambda}$	0.0471	0.0516	0.0679	0.0546	0.2318	0.0475
		$\hat{\delta}$	-0.0660	0.1866	0.0411	0.2466	0.0160	0.0208

7. DATA ANALYSIS

In this section, two real data sets are devoted to demonstrating the MOAPEW distribution's fitness.

Firstly: The first data discussed by [8] which they intoduced alpha power Weibull (APW), The details are the strengths of 1.5 cm glass fibres, calculated at the National Physical Laboratory, England. The data are as follows: these refers to the fatigue times of 6061 T6, aluminum coupons. The data consists of 101 units with maximum stress per cycle 31,000 psi. in [31] used these data to illustrate the applications of their distribution. The data set is as follows: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27,1.39, 1.49, 1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62,1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84,1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.8. In [32] used this data to fit the Weibull-Lomax (WL) distribution.

Table 7. *Different criteria for the first data set by using the estimated parameters*

	MOAPEW	APEW	APW	EW	w	MOAPE	WL	MOW	MOEW
α	1.0077	14.5185	10.9377			251.6759	0.0159		
β	2.9775	3.9357	4.5126	0.7192	5.8099	559.8229	3.5387	16.9520	2.4863
θ	18.9700					5.2738	5.9572	3.2222	0.6148
λ	3.5644	2.5322	1.4394	0.0845	1.6260		5.9796	1.1202	0.0549
δ	9.4846	3.7099		0.0025					0.0094
KS	**0.0997**	0.1225	0.1233	0.1278	0.1537	0.1552	0.1426	0.9696	0.7483
P-Value	**0.5578**	0.3011	0.2939	0.2549	0.1018	0.0961	0.1543	0.0000	0.0000
W*	**0.1056**	0.1635	0.1770	0.1620	0.2481	0.2949	0.1993	2.5310	1.2323
A*	**0.6151**	0.9293	0.9922	0.9515	1.3821	1.6332	1.1314	12.5564	6.7064

From the above table, we can deduce that the MOAPEW model posses the highest p-value and the lowest distance of Kolmogorov Smirnov (KS) value in contrast with all other models used here to fit the current data, that means that the new model fits the data better than the MOAPE, WL, MOW, APW, MOEW, APEW and W models. Cramér-von Mises (W*) and Anderson-Darling (A*) values for the eight models are given for the first data set in Table 7.

By referring to the results in Table 7, we find that the MOAPEW distribution has the smallest value for KS, W*, and A* values. Based on Figure 4, it is clear that the MOAPEW model fits the first data.

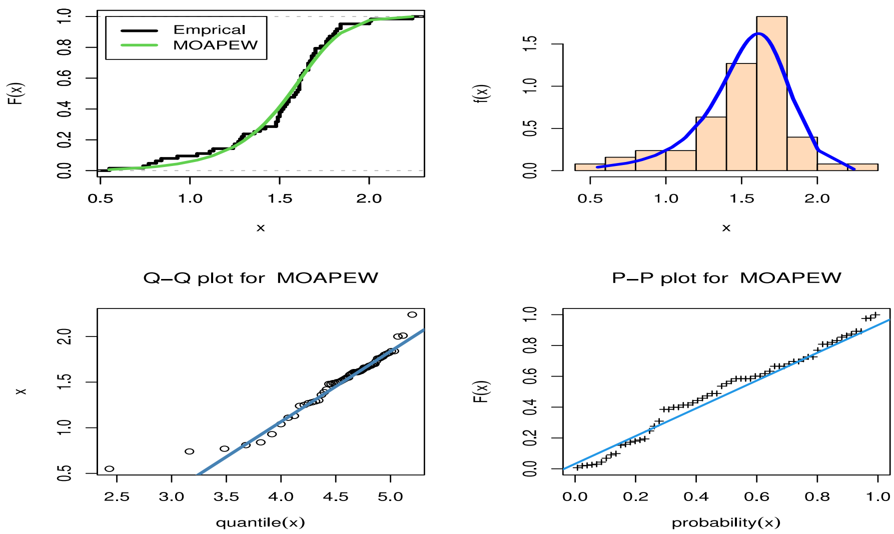

Figure 4. *Different graphs, for the first data set*

Secondly: For illustrative purposes, we present an examination using strength knowledge collection originally recorded by [33]. For single carbon fibers and impregnated 1000-carbon fiber tows, it defines the strength measured in GPA, see [34]. The collection of data is as follows: 0.562, 0.564, 0.729, 1.216, 1.474, 1.632, 1.816, 2.020, 2.317, 1.247, 1.490, 1.676, 1.824, 2.023, 2.334, 1.256, 1.503, 1.684, 1.836, 2.050, 2.340, 0.802, 1.271, 1.520, 1.685, 1.879, 2.059, 2.346, 0.950, 1.277, 1.522, 1.728, 1.883, 2.068, 2.378, 1.053, 1.305, 1.524, 1.740, 1.892, 2.071, 2.483, 1.111, 1.348, 1.551, 1.764, 1.934, 2.130, 2.835, 1.115, 1.313, 1.551, 1.761, 1.898, 2.098, 2.683, 1.194, 1.390, 1.609, 1.785, 1.947, 2.204, 2.835, 1.208, 1.429, 1.632, 1.804, 1.976, 2.262.

Table 8. *Different criteria for the second data set, by using its estimated parameters*

	MOAPEW	APEW	APW	EW	MOAPE	W	WL	MOEW	MOW
α	79.0161	24.5012	0.4072	3.8376	55.9197	-	3.5612	-	-
β	0.2613	2.5206	4.1740	8.8161	3.5152	1.8802	4.2201	3.5970	3.9858
θ	131.9896	-	-	42.5144	96.3132	-	0.7179	1.0554	0.8423
λ	0.0142	8.7229	2.0103	-	-	3.8438	1.6130	3.9862	1.9302
δ	14.0324	10.9663	-	-	-	-	-	3.6889	-
KS	**0.0422**	0.0469	0.0423	0.0469	0.0459	0.0462	0.0443	0.1136	0.4724
P-Value	**0.9997**	0.9991	0.9997	0.9981	0.9987	0.9985	0.9993	0.3354	0.0000
W*	**0.0157**	0.0163	0.0210	0.0211	0.0173	0.0209	0.0205	0.0433	0.5694
A*	**0.1277**	0.1282	0.1828	0.1859	0.1339	0.1848	0.1819	0.3525	1.4185

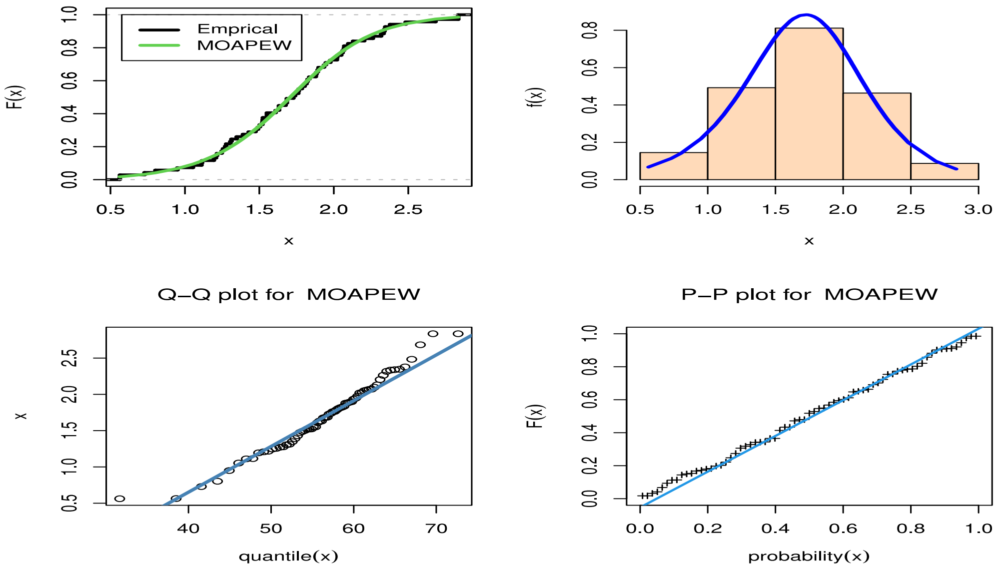

Figure 5. *Different graphs, for the second data set*

The MOAPEW model is a superior model for fitting the data among all its competitors, such as the MOAPE, WL, MOW, APW, MOEW, APEW, and W models, where this have the lowest values for KS, W*, and A* than another seven models see Tables 7 and 8. The relative histogram and fitted of MOAPEW distribution of both data set are discussed in Figures 4 and 5, respectively. Also, we present the Q-Q and P-P plot for the first and second data, and respectively, This helps one to contrast the empirical data with the MOAPEW distribution.

8. CONCLUSION

In this paper, we propose a new five-parameter distribution, called the Marshall-Olkin alpha power extended Weibull (MOAPEW), which extends from the extended Weibull distribution. MOAPEW's distribution is inspired by the wide use of the Weibull model in life testing and offers more flexibility to evaluate lifetime results. Some statistical characteristics such as survival, danger, reverse hazard, stress-strength reliability measure, quantile, median, and mode have been obtained from the MOAPEW distribution. The estimation of MOAPEW distribution parameters are derived from MLE, MPS, and Bayesian. The estimation methods are used to estimate the MOAPEW distribution parameters on the basis of Type I and Type II censor samples, and the result of the simulation is used to evaluate the model's performance. The Bayesian calculations for most MOAPEW distribution parameters have more relative efficiency than MLE and MPS. The two actual data show that the proposed MOAPEW distribution significantly outperforms the MOAPE distribution, WL, MOW, APW, MOEW, APEW, and Weibull distributions. We suggest studying Renyi entropy for the proposed distribution in future studies, such as [35-37]. We suggest studying a more flexible censored sample for the proposed distribution in future studies, such as [38, 39]. We suggest researching multivariate analysis, such as [40, 41] for the proposed distribution.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

REFERENCES

[1] Marshall, A. W., Olkin, I., “A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families”, Biometrika, 84(3): 641-652, (1997).

[2] Ghitany, M. E., Al-Hussaini, E. K., Al-Jarallah, R. A., "Marshall–Olkin extended Weibull distribution and its application to censored data", Journal of Applied Statistics, 32(10): 1025-1034, (2005).

[3] Ghitany, M. E., Al-Awadhi, F. A., Alkhalfan, L. A., "Marshall–Olkin extended Lomax distribution and its application to censored data", Communications in Statistics-Theory and Methods, 36(10): 1855-1866, (2007).

[4] Alice, T., Jose, K. K., "Marshall-Olkin logistic processes", STARS Int Journal, 6(1): 1-11, (2005).

[5] Okasha, H. M., Kayid, M., "A new family of Marshall–Olkin extended generalized linear exponential distribution", Journal of Computational and Applied Mathematics, 296: 576-592, (2016).

[6] Ahmad, H. H., Almetwally, E., "Marshall-Olkin Generalized Pareto Distribution: Bayesian and Non-Bayesian Estimation", Pakistan Journal of Statistics and Operation Research, 16(1): 21-33, (2020).

[7] Mahdavi, A., Kundu, D., "A new method for generating distributions with an application to exponential distribution", Communications in Statistics-Theory and Methods, 46(13): 6543-6557, (2017).

[8] Nassar, M., Alzaatreh, A., Mead, M., Abo-Kasem, O., "Alpha power Weibull distribution: Properties and applications", Communications in Statistics-Theory and Methods, 46(20): 10236-10252, (2017).

[9] Elbatal, I., Ahmad, Z., Elgarhy, B. M., Almarashi, A. M., "A New Alpha Power Transformed Family of Distributions: Properties and Applications to the Weibull Model", Journal of Nonlinear Science and Applications, 12(1): 1-20, (2018).

[10] Dey, S., Ghosh, I., Kumar, D., "Alpha-power transformed Lindley distribution: properties and associated inference with application to earthquake data", Annals of Data Science, 6(4): 623-650, (2018).

[11] Dey, S., Nassar, M., Kumar, D., "Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function", Journal of Computational and Applied Mathematics, 348:130-145, (2019).

[12] Hassan, A. S., Elgarhy, M., Mohamd, R. E., Alrajhi, S., "On the Alpha Power Transformed Power Lindley Distribution", Journal of Probability and Statistics, 2019.

[13] Basheer, A. M., "Alpha power inverse Weibull distribution with reliability application", Journal of Taibah University for Science, 13(1): 423-432, (2019).

[14] Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., Afify, A. Z., "The Marshall–Olkin alpha power family of distributions with applications", Journal of Computational and Applied Mathematics, 351: 41-53, (2019).

[15] Almetwally, E. M., Ahmad, H. A. H., "A new generalization of the Pareto distribution and its applications", Statistics in Transition New Series, 21(5): 61-84, (2020).

[16] Basheer, A. M., "Marshall–Olkin alpha power inverse exponential distribution: properties and applications", Annals of data science, 1-13, (2019).

[17] Almetwally, E. M., Sabry, M. A., Alharbi, R., Alnagar, D., Mubarak, S. A., Hafez, E. H., "Marshall–Olkin Alpha Power Weibull Distribution: Different Methods of Estimation Based on Type-I and Type-II Censoring", Complexity, (2021). https://doi.org/10.1155/2021/5533799

[18] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., "Some New Logistic Differential Models: Properties and Applications", LAP LAMBERT Academic Publishing, (2019).

[19] Basheer, A. M., Almetwally, E. M., Okasha, H. M., "Marshall-Olkin Alpha Power Inverse Weibull Distribution: Non Bayesian and Bayesian Estimations", Journal of Statistics Applications & Probability, To apper, (2021).

[20] Yong, T., "Extended Weibull distributions in reliability engineering (Doctoral dissertation)", ch. 4, (2004).

[21] Balakrishnan, N., Ng, H. K. T., Precedence-type tests and applications, Wiley, Hoboken, (2006).

[22] Balakrishnan, N., "Progressive censoring methodology: an appraisal", Test, (2007).

[23] Almetwaly, E. M., Almongy, H. M., "Estimation of the Generalized Power Weibull Distribution Parameters Using Progressive Censoring Schemes", International Journal of Probability and Statistics, 7(2): 51-61, (2018).

[24] Hassan, A. S., Abd-Allah, M., "On the Inverse Power Lomax Distribution", Annals of Data Science, 6(2): 259-278, (2019).

[25] Almetwally, E. M., Almongy, H. M., "Maximum Product Spacing and Bayesian Method for Parameter Estimation for Generalized Power Weibull Distribution under Censoring Scheme", Journal of Data Science, 17(2): 407-444, (2019).

[26] Almetwally, E. M., Almongy, H. M., "Estimation Methods for the New Weibull-Pareto Distribution: Simulation and Application", Journal of Data Science, 17(3): 610-630, (2019).

[27] Kundu, D., Howlader, H., "Bayesian inference and prediction of the inverse Weibull distribution for Type II censored data", Computational Statistics & Data Analysis, 54(6): 1547-1558, (2010).

[28] Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., Teller, E., "Equation of state calculations by fast computing machines", The journal of chemical physics, 21(6): 1087-1092, (1953).

[29] Nassar, M., Abo-Kasem, O., Zhang, C., Dey, S., "Analysis of Weibull Distribution Under Adaptive Type II Progressive Hybrid Censoring Scheme", Journal of the Indian Society for Probability and Statistics, 19(1): 25-65, (2018).

[30] Almetwally, E. M., Almongy, H. M., El sayed Mubarak, A., "Bayesian and Maximum Likelihood Estimation for the Weibull Generalized Exponential Distribution Parameters Using Progressive Censoring Schemes", Pakistan Journal of Statistics and Operation Research, 14(4): 853-868, (2018).

[31] Smith, R. L., Naylor, J., "A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution", Journal of the Royal Statistical Society: Series C (Applied Statistics), 36(3): 358-369, (1987).

[32] Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M. "The Weibull-Lomax distribution: properties and applications", Hacettepe Journal of Mathematics and Statistics, 44(2): 461-480, (2015).

[33] Badar, M. G., Priest, A. M., “Statistical aspects of fiber and bundle strength in hybrids composite”, Progress in Science and Engineering Composites, 1129–1136, (1982).

[34] Dey, S., Dey, T., Kundu, D., “Two-parameter Rayleigh distribution: different methods of estimation”, American Journal of Mathematical and Management Sciences, 33(1): 55-74, (2014).

[35] Ijaz, M., Asim, S. M., “Lomax exponential distribution with an application to real-life data”, PloS one, 14(12): e0225827, (2019).

[36] Hassan, A., Mohamed, R., “Parameter Estimation for Inverted Exponentiated Lomax Distribution with Right Censored Data”, Gazi University Journal of Science, 32(4): 1370-1386, (2019).

[37] Hassan, A., Elshrpieby, E., Mohamed, R., “Odd Generalized Exponential Power Function Distribution: Properties & Applications”, Gazi University Journal of Science, 32(1): 351-370, (2019).

[38] Alshenawy, R., Al-Alwan, A., Almetwally, E. M., Afify, A. Z., Almongy, H. M., “Progressive type II censoring schemes of extended odd Weibull exponential distribution with applications in medicine and engineering”, Mathematics, 8(10): 1679, (2020).

[39] Alshenawy, R., Sabry, M. A., Almetwally, E. M., Almongy, H. M., “Product Spacing of Stress–Strength under Progressive Hybrid Censored for Exponentiated-Gumbel Distribution”, Computers, Materials & Continua, 66(3): 2973-2995, (2021).

[40] Muhammed, H. Z., Almetwally, E. M., “Bayesian and Non-Bayesian Estimation for the Bivariate Inverse Weibull Distribution Under Progressive Type II Censoring”, Annals of Data Science, 1-32, (2020).

[41] El-Morshedy, M., Alhussain, Z. A., Atta, D., Almetwally, E. M., Eliwa, M. S., “Bivariate Burr X generator of distributions: properties and estimation methods with applications to complete and type II censored samples”, Mathematics, 8(2): 264, (2020).