MARTIN BOUNDARY POINTS OF A JOHN DOMAIN AND UNIONS OF CONVEX SETS

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Abstract. We show that a John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. The number of minimal Martin boundary points is estimated in terms of the John constant. In particular, if the John constant is bigger than $\sqrt{3}/2$, then there are at most two minimal Martin boundary points at each Euclidean boundary point. For a class of John domains represented as the union of convex sets we give a sufficient condition for the Martin boundary and the Euclidean boundary to coincide.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Let $\delta_D(x) = \operatorname{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ and John center at x_0 if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that

$$\delta_D(y) \geq c_J \ell(\gamma(x, y)) \quad \text{for all } y \in \gamma, \tag{1.1}$$

where $\gamma(x, y)$ is the subarc of γ from x to y and $\ell(\gamma(x, y))$ is the length of $\gamma(x, y)$. It is easy to see that a smooth domain is a John domain with John constant $c_J = 1$. We may say that the bigger c_J is, the smoother D is.

Since the main concern of this paper is the boundary behavior of functions in D, we may replace x_0 by a compact subset K_0 of D. We call such a domain a *general John domain with general John center K_0 and general John constant c_J*. Obviously, a John domain is a general John domain and vice versa. Note that a general John constant is improved, i.e., a John domain with John center at x_0 and John constant c_J can be regarded as a general John domain with general John constant $c'_J \geq c_J$ by replacing x_0 by a larger compact set K_0. Several general John domains have been studied in connection with the Martin boundary, e.g. Denjoy domains (Benedicks [10]), Lipschitz Denjoy domains (Ancona [6, 7] and Chevallier [11]), sectorial domains (Cranston-Salisbury [12]), quasi-sectorial domains (Lömker [18]), the connected union of a family of open balls with the same radius (Ancona [5]) and so on. The general John constants for these domains can be estimated by the geometrical assumption on the domains. For example, the general John constant $c_J = 1$ for a Denjoy domain.

Let $G(x, y)$ be the Green kernel for D. A Martin kernel at $\xi \in \partial D$ (with reference point x_0) is a limit of the ratio $G(x, y_j)/G(x_0, y_j)$ with $y_j \to \xi$. The totality of Martin kernels gives an ideal boundary of D, referred to as the Martin boundary of D. We identify a Martin kernel and an ideal boundary point; a limit of the ratio $G(x, y_j)/G(x_0, y_j)$ with $y_j \to \xi$ is called a Marin

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boundary point at ξ as well. We say that a positive harmonic function h is *minimal* if every positive harmonic function less than or equal to h coincides with a constant multiple of h. If a Martin kernel is a minimal harmonic function, then we call it a minimal Martin kernel or a minimal Martin boundary point. In general, the Martin boundary need not be homeomorphic to the Euclidean boundary. There may be even infinitely many minimal Martin boundary points at a Euclidean boundary point (Martin [19]).

The purpose of this paper is to show that every John domain has finitely many minimal Martin boundary points at each Euclidean boundary point. Moreover, the number of minimal Martin boundary points is estimated in terms of the John constant.

Theorem 1.1. *Let D be a general John domain with general John constant c_J.*

(i) *The number of minimal Martin boundary points at every Euclidean boundary point $\xi \in \partial D$ is bounded by a constant depending only on the general John constant c_J.*
(ii) *If $c_J > \sqrt{3}/2$, then there are at most two minimal Martin boundary points at every Euclidean boundary point $\xi \in \partial D$*

Remark 1.1. Let D be a sectorial domain whose boundary near the origin lies on three equally distributed rays leaving the origin. Then D is a general John domain with John constant $\sin(\pi/3) = \sqrt{3}/2$. There may be three different minimal Martin boundary points at the origin. See Figure 1.1. This simple example shows that the bound $c_J > \sqrt{3}/2$ in Theorem 1.1 is sharp. Note that the same bound $c_J > \sqrt{3}/2$ also applies to the higher dimensional case.

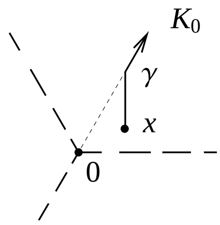

Figure 1.1. The bound $c_J > \sqrt{3}/2$ in Theorem 1.1 is sharp.

Remark 1.2. Theorem 1.1 generalizes some parts of [10], [6, 7], [11], [12] and [18]. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point (via Kelvin transform for [10]). Such a criterion seems to be very difficult for a general John domain, since the boundary may disperse at every point (See e.g. [3, Figure 3 b]).

One might think that the number of minimal Martin boundary points at a Euclidean boundary point would be equal to 1 provided the John constant c_J is sufficiently close to 1. This is not the case in view of Benedicks' work on a Denjoy domain ([10]). The best upper bound obtained from the John constant c_J is at least two as given in Theorem 1.1. Our second purpose is to find a certain class of John domains whose boundary points have one minimal Martin boundary point.

We shall need some other information different from the John constant c_J. Ancona [5, Théorème] gave a condition for the union of a family of open balls with the same radius to have one minimal Martin boundary point at each Euclidean boundary point. By $B(x, r)$ we denote the open ball with center at x and radius r. For a pair of distinct points x and y let $[x, y]$ be the (open) line segment connecting x and y. For $0 < \theta < \pi$ we denote by $\Gamma_\theta(x, y)$ the open circular cone $\{z \in \mathbb{R}^n : \angle zxy < \theta\}$ with vertex at x, axis $[x, y]$ and aperture θ. Ancona says that a domain D is *admissible* if

(A1) D is the union of a family of open balls with the same radius ρ_0.

(A2) Let $\xi \in \partial D$. If D includes two open balls B_1 and B_2 with radius ρ_0 tangential to each other at ξ, then D includes a truncated circular cone $\Gamma_\theta(\xi, y) \cap B(\xi, r)$ for some $\theta > 0$, $r > 0$ and y in the hyperplane tangent to B_i at ξ. See Figure 1.2.

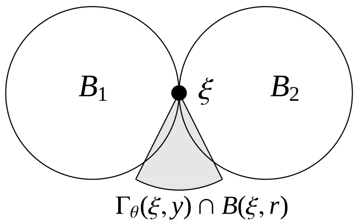

FIGURE 1.2. Condition (A2).

Theorem A (Ancona)**.** *Let D be a bounded admissible domain. Then every Euclidean boundary point of D has one Martin boundary point and it is minimal. Moreover, the Martin boundary of D is homeomorphic to the Euclidean boundary.*

Let us generalize both (A1) and (A2). Clearly, (A1) implies that D is a general John domain with general John constant 1. We would like to consider general convex sets rather than balls with the same radius. They need not be congruent. Observe that Ancona's condition (A2) implies that two balls B_1 and B_2 are *connected* by a truncated cone $\Gamma_\theta(\xi, y) \cap B(\xi, r)$. If $0 < \theta' \le \theta$, then we have

$$\bigcup_{\substack{y \in D, \\ \Gamma_{\theta'}(\xi,y) \cap B(\xi,r') \subset D}} \Gamma_{\theta'}(\xi, y) \cap B(\xi, r') \text{ is connected,}$$

provided $r' > 0$ is sufficiently small. In view of this observation, we generalize (A1) and (A2) as follows. Let $A_0 \ge 1$ and $\rho_0 > 0$. We consider a bounded domain D such that

(I) D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda \in \Lambda}$ such that $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$;

(II) for each $\xi \in \partial D$, there are positive constants $\theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 \le \rho_0 \cos\theta_1$ such that

$$C(\xi) = \bigcup_{\substack{y \in D, \\ \Gamma_{\theta_1}(\xi,y) \cap B(\xi,2\rho_1) \subset D}} \Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1) \text{ is connected.}$$

See Figure 1.3.

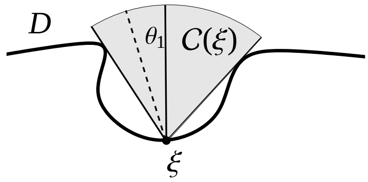

FIGURE 1.3. Condition (II).

Theorem 1.2. *Let D be a bounded domain satisfying (I) and (II). Then every Euclidean boundary point of D has one Martin boundary point and it is minimal. Moreover the Martin boundary of D is homeomorphic to the Euclidean boundary.*

Remark 1.3. Ancona's admissible domains satisfy (I) and (II) of Theorem 1.2. The argument of Ancona depends on the special properties of a ball. His crucial lemma ([5, Lemme 1]) relies on the reflection with respect to a hyperplane, and is applied to a ball by the Kelvin transform ([5, Corollarie 2]). This approach is not applicable to our domains.

Remark 1.4. A Denjoy domain can be represented as the union of a family of open balls with the same radius. A Lipschitz Denjoy domain, a sectorial domain and a quasi-sectorial domain can be represented as the union of a family of open convex sets C_λ satisfying (I). However, they cannot be represented as the union of a family of open balls with the same radius. Our Theorem 1.2 is applicable to these domains.

Remark 1.5. Condition (II) is local in the following sense: Suppose D is the union of a family of open convex sets $\{C_\lambda\}_{\lambda\in\Lambda}$ satisfying (I). If a particular point $\xi \in \partial D$ satisfies (II), then there is one Martin boundary point at ξ and it is minimal.

Remark 1.6. Note that $0 < \theta_1 < \pi/2$ by $0 < \rho_1 \le \rho_0 \cos\theta_1$. The bounds $\theta_1 \le \sin^{-1}(1/A_0)$ and $\rho_1 \le \rho_0 \cos\theta_1$ are sharp. See Hirata [17]. Under these assumptions, there exists a truncated circular cone $\Gamma_{\theta_1}(\xi, y) \cap B(\xi, 2\rho_1)$ included in D.

Both Theorems 1.1 and 1.2 are based on a common geometrical notion, *a system of local reference points*. In Section 2, we shall introduce a quasihyperbolic metric and define a system of local reference points. Then we shall observe that Theorems 1.1 and 1.2 are decomposed into three propositions, namely, Propositions 2.1, 2.2 and 2.3. The first two propositions are purely geometric and will be proved in the same section. Proposition 2.3 involves many potential theoretic arguments. Among them, a Carleson type estimate (Lemma 5.1 in Section 5) for bounded positive harmonic functions vanishing on a portion of the boundary will be crucial. This estimate will be deduced from a Domar's type theorem (Domar [13]) for nonnegative subharmonic functions, as was employed by Benedicks [10] and Chevallier [11]. Domar's argument is applicable to nonlinear equations in a metric measure space ([4]).

By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \ge 1$ such that $A^{-1} f_1 \le f_2 \le A f_1$. The constant A will be called the constant of comparison. We write $B(x, r)$ and $S(x, r)$ for the open ball and the sphere of center at x and radius r, respectively.

2. Local reference points

2.1. **Restatements of Theorems 1.1 and 1.2.** We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D. We say that D satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D. \tag{2.1}$$

A domain satisfying the quasihyperbolic boundary condition is called a Hölder domain by Smith-Stegenga [20, 21]. It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [16, Lemma 3.11]). We need more precise estimates.

Definition 2.1. Let N be a positive integer and $0 < \eta < 1$. We say that $\xi \in \partial D$ has a system of local reference points of order N with factor η if there exist $R_\xi > 0$ and $A_\xi > 1$ with the following property: for each positive $R < R_\xi$ there are N points $y_1 = y_1(R), \dots, y_N = y_N(R) \in D \cap S(\xi, R)$ such that $A_\xi^{-1} R \le \delta_D(y_i) \le R$ for $i = 1, \dots, N$ and

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i)\} \le A_\xi \log \frac{R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)},$$

where $D_R = D \cap B(\xi, \eta^{-3} R)$. If η is not so important, we simply say that $\xi \in \partial D$ has a system of local reference points of order N.

The proofs of Theorems 1.1 and 1.2 can be decomposed into the following three propositions. The first and the second are purely geometric; the third is potential theoretic.

Proposition 2.1. *Let D be a general John domain with John constant c_J. Then every $\xi \in \partial D$ has a system of local reference points of order N with $N \le N(c_J, n) < \infty$. Moreover, if the John constant $c_J > \sqrt{3}/2$, then we can let $N \le 2$ by choosing a suitable factor* $0 < \eta < 1$.

Proposition 2.2. *Let D be a bounded domain satisfying (I) and (II). Then every $\xi \in \partial D$ has a system of local reference points of order* 1.

Remark 2.1. In Proposition 2.1, the constants R_ξ and A_ξ in Definition 2.1 can be taken uniformly for $\xi \in \partial D$, whereas they may depend on ξ in Proposition 2.2.

By $\mathcal{H}_\xi$ we denote the family of all kernel functions at ξ normalized at the John center x_0, i.e., the set of all positive harmonic functions h on D such that $h(x_0) = 1$, $h = 0$ q.e. on ∂D and h is bounded on $D \setminus B(\xi, r)$ for each $r > 0$. Here we say that a property holds q.e. (quasi everywhere) if it holds outside a polar set. A Martin kernel at ξ (with reference point x_0) is a limit of the ratio $G(x, y_j)/G(x_0, y_j)$ of Green functions with $y_j \to \xi$. Suppose $y_j \subset D \cap B(\xi, r/2)$. Then the (global) boundary Harnack principle for a John domain (Bass and Burdzy [9]) implies that the $G(\cdot, y_j)/G(x_0, y_j)$ is bounded on $D \setminus B(\xi, r)$, and so is a Martin kernel at ξ. Obviously, a Martin kernel at ξ is a positive harmonic function vanishing q.e. on ∂D with value 1 at x_0, so that it belongs to $\mathcal{H}_\xi$. Thus Theorems 1.1 and 1.2 will follow from Propositions 2.1, 2.2 and the following:

Proposition 2.3. *Let D be a general John domain. Suppose $\xi \in \partial D$ has a system of local reference points of order N.*

(i) *The number of minimal functions in $\mathcal{H}_\xi$ is bounded by a constant depending only on N.*
(ii) *If $N \le 2$, then there are at most N minimal functions in $\mathcal{H}_\xi$. Moreover, if $N = 1$, then $\mathcal{H}_\xi$ is a singleton and consists of a minimal function.*

2.2. **Proof of Proposition 2.1.** For the proof of the second assertion in Proposition 2.1, we prepare an elementary geometrical observation.

Lemma 2.1. *Let e_1, e_2 and e_3 be points on the unit sphere $S(0,1)$. Then*

$$\max \min_{i \neq j} |e_i - e_j| = \sqrt{3},$$

where the maximum is taken over all positions of e_1, e_2 and e_3.

Proof. This is a well-known fact (Fejes [14]). For the convenience sake of the reader we provide a proof. We can easily prove the lemma for $n = 2$. Let $n \geq 3$. We observe from the compactness of $S(0,1)$ that the maximum d is taken by some points e_1, e_2 and e_3 on $S(0,1)$. There is a unique 2-dimensional plane Π containing e_1, e_2 and e_3, since three distinct points on $S(0,1)$ cannot be collinear. Observe that $S(0,1) \cap \Pi$ is a circle with radius at most 1. Since e_1, e_2 and e_3 are points on this circle, it follows from the case $n = 2$ that $d \leq \sqrt{3}$. The lemma follows. □

Proof of Proposition 2.1. We prove the proposition with $R_\xi = \delta_D(K_0)$. Let $\xi \in \partial D$ and $0 < R < \delta_D(K_0)$. Let us prove the first assertion with $\eta = 1/2$. Take $x \in D \cap \overline{B(\xi, R/2)}$. By definition there is a rectifiable curve γ starting from x and terminating at K_0 such that (1.1) holds. Then the first hit $y(x)$ of $S(\xi, R)$ along γ satisfies $2^{-1}c_J R \leq \delta_D(y(x)) \leq R$ and $k_{D_R}(x, y(x)) \leq A \log \dfrac{R}{\delta_D(x)}$. We associate $y(x)$ with x, although it may not be unique.

Consider, in general, the family of balls $B(y, 4^{-1}c_J R)$ with $y \in S(\xi, R)$. These balls are included in $B(\xi, (4^{-1}c_J + 1)R)$, so that at most $N(c_J, n)$ balls among them can be mutually disjoint. Hence we find N points $x_1, \dots, x_N \in D \cap \overline{B(\xi, R/2)}$ with $N \leq N(c_J, n)$ such that $\{B(y_1, 4^{-1}c_J R), \dots, B(y_N, 4^{-1}c_J R)\}$ is maximal, where $y_j = y(x_j) \in D \cap S(\xi, R)$ is the point associate with x_j as above. This means that if $x \in D \cap \overline{B(\xi, R/2)}$, then $B(y(x), 4^{-1}c_J R)$ intersects some of $B(y_1, 4^{-1}c_J R), \dots, B(y_N, 4^{-1}c_J R)$, say $B(y_i, 4^{-1}c_J R)$. Since $B(y(x), 4^{-1}c_J R) \cap B(y_i, 4^{-1}c_J R) \neq \emptyset$ and $B(y(x), 2^{-1}c_J R) \cup B(y_i, 2^{-1}c_J R) \subset D_R$, it follows that $k_{D_R}(y(x), y_i) \leq A'$. Hence

$$k_{D_R}(x, y_i) \leq k_{D_R}(x, y(x)) + k_{D_R}(y(x), y_i) \leq A \log \frac{R}{\delta_D(x)} + A'.$$

Repeating some points, say $y_1 = y(x_1)$, if necessary, we may assume that this property holds with N independent of R and $N \leq N(c_J, n)$. Thus the first assertion follows.

For the proof of the second assertion, let $\sqrt{3}/2 < b' < b < c_J$ and $\eta = 1 - b/c_J > 0$. Let us prove that ξ has a system of local reference points of order at most 2 with factor η. Let $0 < R < \delta_D(K_0)$. Suppose $x \in D \cap \overline{B(\xi, \eta R)}$. In the same way as in the proof of the first assertion, we find $y(x) \in S(\xi, R)$ such that $k_{D_R}(x, y(x)) \leq A \log \dfrac{R}{\delta_D(x)}$ and

$$\delta_D(y(x)) \geq c_J(1 - \eta)R = bR > b'R > \frac{\sqrt{3}}{2}R.$$

Lemma 2.1 says that at most two disjoint balls of radius $b'R$ can be placed so that their centers lie on the sphere $S(\xi, R)$. Hence we can choose $x_1, x_2 \in D \cap \overline{B(\xi, \eta R)}$ such that $B(y(x), b'R)$ intersects $B(y_i, b'R)$ for some $i = 1, 2$, where $y_i = y(x_i)$. Since $B(y(x), b'R) \cap B(y_i, b'R) \neq \emptyset$ and $B(y(x), bR) \cup B(y_i, bR) \subset D_R$, it follows that $k_{D_R}(y(x), y_i) \leq A$. Hence the proposition follows. □

Remark 2.2. In case $c_J \leq \sqrt{3}/2$, we may have an estimate of N better than the above proof, by considering a lemma similar to Lemma 2.1.

2.3. **Proof of Proposition 2.2.** In this subsection, we assume, by translation and dilation, that $\xi = 0$ and $\rho_1 = 1$ for simplicity. The aperture $\theta_1 \le \sin^{-1}(1/A_0)$ is fixed and we write $\Gamma(x, y)$ for $\Gamma_{\theta_1}(x, y)$. Note that $1 = \rho_1 \le \rho_0 \cos\theta_1$, so that $0 < \theta_1 < \pi/2$ and $\rho_0 \ge \sec\theta_1$. Let C_λ be a convex set appearing in (I) and let $B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0)$. If $x \in \overline{C_\lambda} \setminus B(z_\lambda, \rho_0)$, then

$$\Gamma(x, z_\lambda) \cap B(x, 2) \subset \mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0)) \subset C_\lambda, \tag{2.2}$$

where $\mathrm{co}(\{x\} \cup B(z_\lambda, \rho_0))$ is the convex hull of $\{x\} \cup B(z_\lambda, \rho_0)$. Let

$$\mathcal{Y} = \{y \in S(0, 1) : \Gamma(0, y) \cap B(0, 2) \subset D\}.$$

We first show that $\mathcal{Y} \ne \emptyset$ and that the point 0 can be accessible along a ray issuing from the origin toward a point in $\mathcal{Y}$.

Lemma 2.2. *There is a positive constant $R_0 < 1$ such that if $C_\lambda \cap B(0, R_0) \ne \emptyset$, then $C_\lambda \cap \mathcal{Y} \ne \emptyset$. In particular, $\mathcal{Y} \ne \emptyset$.*

Proof. Suppose to the contrary, there is a sequence C_{λ_j} with $\mathrm{dist}(0, C_{\lambda_j}) \to 0$ and $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$. Let z_{λ_j} be such that $B(z_{\lambda_j}, \rho_0) \subset C_{\lambda_j} \subset B(z_{\lambda_j}, A_0\rho_0)$. Taking a subsequence, if necessary, we may assume that z_{λ_j} converges, say to z_0. We claim

$$\Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}. \tag{2.3}$$

We find $x_{\lambda_j} \in \partial C_{\lambda_j}$ with $x_{\lambda_j} \to 0$. Take $x \in \Gamma(0, z_0) \cap B(0, 2)$. Then $\angle x0z_0 < \theta_1$ and $|x| < 2$ by definition. If j is sufficiently large, then $\angle x x_{\lambda_j} z_{\lambda_j} < \theta_1$ and $|x - x_{\lambda_j}| < 2$ by continuity, so that

$$x \in \Gamma(x_{\lambda_j}, z_{\lambda_j}) \cap B(x_{\lambda_j}, 2) \subset \mathrm{co}(\{x_{\lambda_j}\} \cup B(z_{\lambda_j}, \rho_0)) \subset C_{\lambda_j},$$

by (2.2). Thus (2.3) follows. Now, by definition, $y_0 = z_0/|z_0| \in \mathcal{Y}$ and $y_0 \in \Gamma(0, z_0) \cap B(0, 2) \subset \bigcup_j C_{\lambda_j}$. This contradicts $C_{\lambda_j} \cap \mathcal{Y} = \emptyset$. The lemma follows. □

Observe that if C is a convex set, then the distance function $\delta_C(x) = \mathrm{dist}(x, \partial C)$ is a concave function on $\overline{C}$, i.e.,

$$\delta_C(z) \ge \frac{|z - y|}{|x - y|}\delta_C(x) + \frac{|x - z|}{|x - y|}\delta_C(y) \quad \text{for } z \in [x, y], \tag{2.4}$$

whenever $x \ne y \in \overline{C}$. This fact will be used in the following lemma.

Lemma 2.3. *Let $0 < R_0 < 1$ be as in Lemma 2.2. Suppose $0 < R < \min\{R_0, 3^{-1}\sin\theta_1\}$. If $C_\lambda \cap B(0, R) \ne \emptyset$ and $y \in C_\lambda \cap \mathcal{Y}$, then there exists a point $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin\theta_1)$ such that*

$$\delta_{C_\lambda \cap \Gamma(0,y)}(w) \ge \frac{\sin\theta_1}{4}R.$$

Proof. Take $x \in C_\lambda \cap B(0, R)$. Then $[x, y] \subset C_\lambda$. Observe that there is a point $w_1 \in [x, y] \cap \overline{\Gamma(0, y)}$ with $|w_1| \le R/\sin\theta_1$. In fact, if $x \in \overline{\Gamma(0, y)}$, then $w_1 = x$ satisfies the condition. Otherwise, let w_1 be the intersection of $[x, y]$ and $\partial\Gamma(0, y)$. By elementary geometry

$$R > \mathrm{dist}(x, [0, y]) \ge \mathrm{dist}(w_1, [0, y]) = |w_1| \sin\theta_1,$$

so that $|w_1| \le R/\sin\theta_1$. Since $|w_1 - y| \ge 1 - R/\sin\theta_1$ and $3R/\sin\theta_1 < 1$, we find a point $w_2 \in [w_1, y] \subset C_\lambda \cap \Gamma(0, y)$ with $|w_1 - w_2| = R/\sin\theta_1$. By (2.4) with $C = \Gamma(0, y)$ we obtain

$$\delta_{\Gamma(0,y)}(w_2) \ge \frac{|w_1 - w_2|}{|w_1 - y|}\delta_{\Gamma(0,y)}(y) \ge \frac{R/\sin\theta_1}{R/\sin\theta_1 + 1}\sin\theta_1 > \frac{R}{2}.$$

Moreover $|w_2| \le 2R/\sin\theta_1$. Since $|w_2 - z_\lambda| \ge \rho_0 - 2R/\sin\theta_1 > R$ by $3R/\sin\theta_1 < 1 \le \rho_0$, we can take a point $w \in [w_2, z_\lambda] \subset C_\lambda$ such that $|w - w_2| = R/4$. Then it follows from (2.4) with $C = C_\lambda$ that

$$\delta_{C_\lambda}(w) \ge \frac{|w - w_2|}{|z_\lambda - w_2|}\delta_{C_\lambda}(z_\lambda) \ge \frac{R/4}{A_0\rho_0}\rho_0 \ge \frac{\sin\theta_1}{4}R.$$

Hence

$$\delta_{\Gamma(0,y)\cap C_\lambda}(w) \ge \min\left\{\frac{R}{2} - \frac{R}{4}, \frac{\sin\theta_1}{4}R\right\} = \frac{\sin\theta_1}{4}R.$$

Moreover,

$$|w| \le |w - w_2| + |w_2 - w_1| + |w_1| \le \frac{R}{4} + \frac{R}{\sin\theta_1} + \frac{R}{\sin\theta_1} < \frac{3R}{\sin\theta_1}.$$

Thus the lemma is proved. □

Proof of Proposition 2.2. Let $0 < R_0 < 1$ be as in Lemma 2.2 and let $0 < \eta^3 < 6^{-1}\sin\theta_1$. Suppose $0 < R < \min\{R_0, 3^{-1}\sin\theta_1\}$. By Lemma 2.2 we fix $y_0 \in \mathcal{Y}$ and write $y_R = Ry_0$. It is sufficient to show that

$$k_{D_R}(x, y_R) \le A\log\frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap \overline{B(0, \eta R)}, \tag{2.5}$$

where A is independent of x and R. Take $x \in D \cap \overline{B(0, \eta R)}$. Then there is a convex set C_λ containing x and there is $y \in C_\lambda \cap \mathcal{Y}$ by Lemma 2.2. By Lemma 2.3 we find a point $w \in C_\lambda \cap \Gamma(0, y) \cap B(0, 3R/\sin\theta_1)$ such that $\delta_{C_\lambda\cap\Gamma(0,y)}(w) \ge 4^{-1}R\sin\theta_1$. Since

$$\delta_{D_R}(z) \ge \delta_{C_\lambda}(z) \ge \frac{|x - z|}{|x - w|}\delta_{C_\lambda}(w) \ge \frac{\sin^2\theta_1}{16}|x - z| \quad \text{for } z \in [x, w]$$

by $[x, w] \subset B(0, 2^{-1}\eta^{-3}R)$ and (2.4), it follows that

$$k_{D_R}(x, w) \le \int_{[x,w]} \frac{ds(z)}{\delta_{D_R}(z)} \le A\log\frac{R}{\delta_D(x)} + A.$$

Since

$$\delta_{D_R}(z) \ge \delta_{\Gamma(0,y)}(z) \ge \frac{|w - z|}{|w - Ry|}\delta_{\Gamma(0,y)}(Ry) \ge \frac{\sin^2\theta_1}{4}|x - z| \quad \text{for } z \in [w, Ry],$$

it also follows that

$$k_{D_R}(w, Ry) \le \int_{[w,Ry]} \frac{ds(z)}{\delta_{D_R}(z)} \le A\log\frac{R}{\delta_D(x)} + A.$$

Note that $C(0) \cap S(0,1)$ is connected by the assumption (II). In view of $\operatorname{dist}(\mathcal{Y}, S(0,1) \setminus C(0)) \ge \sin\theta_1$ and $C(0) \subset D$, we see that $k_{D_R}(Ry, y_R) \le A$, with A independent of R, y and y_R. Thus (2.5) follows from the triangle inequality. □

3. Refinement of Domar's theorem

Domar [13, Theorem 2] gave a criterion for the boundedness of a subharmonic function majorized by a positive function. We need its quantitative refinement, i.e., the dependency of the bound is given explicitly.

Lemma 3.1. *Let u be a nonnegative subharmonic function on a bounded domain Ω. Suppose there is $\varepsilon > 0$ such that*

$$I = \int_\Omega (\log^+ u)^{n-1+\varepsilon} dx < \infty.$$

Then

$$u(x) \le \exp(2 + AI^{1/\varepsilon} \delta_\Omega(x)^{-n/\varepsilon}), \tag{3.1}$$

where A is a positive constant depending only on ε and the dimension n.

For the proof we prepare the following.

Lemma 3.2. *Let u be a nonnegative subharmonic function on $B(x, R)$. Suppose $u(x) \ge t > 0$ and*

$$R \ge L_n |\{y \in B(x, R) : e^{-1}t < u(y) \le et\}|^{1/n}, \tag{3.2}$$

where $L_n = (e^2/v_n)^{1/n}$ and v_n is the volume of the unit ball. Then there exists a point $x' \in B(x, R)$ with $u(x') > et$.

Proof. Observe that (3.2) is equivalent to

$$\frac{|\{y \in B(x, R) : e^{-1}t < u(y) \le et\}|}{|B(x, R)|} \le \frac{1}{e^2}.$$

Suppose $u \le et$ on $B(x, R)$. Then the mean value property of subharmonic functions yields

$$\begin{aligned} t \le u(x) &\le \frac{1}{|B(x, R)|} \int_{B(x,R)} u(y) dy \\ &= \frac{1}{|B(x, R)|} \left(\int_{B(x,R) \cap \{u \le e^{-1}t\}} u dy + \int_{B(x,R) \cap \{u > e^{-1}t\}} u dy \right) \\ &\le e^{-1}t + \frac{1}{e^2} et < t. \end{aligned}$$

This is a contradiction. □

Proof of Lemma 3.1. Since the right hand side of (3.1) is not less than e^2, it is sufficient to show that

$$\delta_\Omega(x) \le AI^{1/n} (\log u(x))^{-\varepsilon/n}, \quad \text{whenever } u(x) > e^2. \tag{3.3}$$

Fix $x_1 \in \Omega$ with $u(x_1) > e^2$ and let us prove (3.3) with $x = x_1$. Let

$$R_j = L_n |\{y \in \Omega : e^{j-2} u(x_1) < u(y) \le e^j u(x_1)\}|^{1/n} \quad \text{for } j \ge 1.$$

We choose a sequence $\{x_j\}$ as follows: If $\delta_\Omega(x_1) < R_1$, then we stop. If $\delta_\Omega(x_1) \ge R_1$, then $B(x_1, R_1) \subset \Omega$, so that there exists $x_2 \in B(x_1, R_1)$ such that $u(x_2) > eu(x_1)$ by Lemma 3.2. Next we consider $\delta_\Omega(x_2)$. If $\delta_\Omega(x_2) < R_2$, then we stop. If $\delta_\Omega(x_2) \ge R_2$, then $B(x_2, R_2) \subset \Omega$, so that there exists $x_3 \in B(x_2, R_2)$ such that $u(x_3) > e^2 u(x_1)$ by Lemma 3.2. Repeat this procedure to obtain a finite or infinite sequence $\{x_j\}$. We claim

$$\delta_\Omega(x_1) \le 2 \sum_{j=1}^\infty R_j. \tag{3.4}$$

Suppose first $\{x_j\}$ is finite. If $\delta_\Omega(x_1) < R_1$, then (3.4) trivially holds. If $\delta_\Omega(x_1) \ge R_1$, then we have an integer $J \ge 2$ such that

$$\begin{aligned}&\delta_\Omega(x_1) \ge R_1, \dots, \delta_\Omega(x_{J-1}) \ge R_{J-1}, \delta_\Omega(x_J) < R_J,\\ &x_2 \in B(x_1, R_1), x_3 \in B(x_2, R_2), \dots, x_J \in B(x_{J-1}, R_{J-1}).\end{aligned}$$

Hence we have

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) < R_1 + \cdots + R_{J-1} + R_J,$$

so that (3.4) follows. Suppose next $\{x_j\}$ is infinite. Since $u(x_j) > e^j u(x_1) \to \infty$, it follows from the local boundedness of a subharmonic function that x_j goes to the boundary. Hence, there is an integer $J \ge 2$ such that $\delta_\Omega(x_J) \le \frac{1}{2}\delta_\Omega(x_1)$. Then

$$\delta_\Omega(x_1) \le |x_1 - x_2| + \cdots + |x_{J-1} - x_J| + \delta_\Omega(x_J) \le R_1 + \cdots + R_{J-1} + \frac{1}{2}\delta_\Omega(x_1),$$

so that (3.4) follows. In view of (3.4) we observe that (3.3) follows from

$$\sum_{j=1}^{\infty} R_j \le A I^{1/n} (\log u(x_1))^{-\varepsilon/n}. \tag{3.5}$$

To show (3.5), let j_1 be the integer such that $e^{j_1} < u(x_1) \le e^{j_1+1}$. Then $j_1 \ge 2$ and

$$R_j \le L_n |\{y \in \Omega : e^{j_1+j-2} < u(y) \le e^{j_1+j+1}\}|^{1/n}.$$

Since the family of intervals $\{(e^{j_1+j-2}, e^{j_1+j+1}]\}_j$ overlaps at most 3 times, it follows from Hölder's inequality that

$$\begin{aligned}\sum_{j=1}^{\infty} R_j &\le 3L_n \sum_{j=j_1}^{\infty} |\{y \in \Omega : e^{j-1} < u(y) \le e^j\}|^{1/n}\\ &\le 3L_n \left(\sum_{j=j_1}^{\infty} \frac{1}{j^{(n-1+\varepsilon)/(n-1)}}\right)^{(n-1)/n} \left(\sum_{j=j_1}^{\infty} j^{n-1+\varepsilon} |\{y \in \Omega : e^{j-1} < u(y) \le e^j\}|\right)^{1/n}\\ &\le A j_1^{-\varepsilon/n} \left(\int_\Omega (\log^+ u)^{n-1+\varepsilon} dy\right)^{1/n}\\ &\le A(\log u(x_1))^{-\varepsilon/n} I^{1/n}.\end{aligned}$$

Thus (3.5) follows. The lemma is proved. □

4. Integrability of negative power of the distance function

Inspired by Smith and Stegenga [20, Theorem 4] we have proved that for a bounded John domain there is a positive constant τ such that

$$\int_D \delta_D(x)^{-\tau} dx < \infty$$

([1, Lemma 5]). We need its local version.

Lemma 4.1. *Let D be a general John domain with John constant c_J and generalized John center K_0. Then there are positive constants τ and A depending on c_J such that*

$$\int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)}\right)^\tau dx \le AR^n$$

for each $\xi \in \partial D$ *and* $0 < R < \delta_D(K_0)$.

Proof. Let

$$V_j = \{x \in D \cap B(\xi, R + (1 + c_J^{-1})2^{1-j}R) : 2^{-j-1}R \le \delta_D(x) < 2^{-j}R\}$$

for $j \ge 0$. For a moment we fix $x \in \bigcup_{i=j+1}^{\infty} V_i$. By definition there is a rectifiable curve γ connecting x and K_0 with (1.1). Hence we find $y \in \gamma$ such that $\delta_D(y) = 2^{-j}R \ge c_J|x-y|$. In other words $x \in \overline{B(y, c_J^{-1}2^{-j}R)}$. We observe

$$|B(y, 5c_J^{-1}2^{-j}R)| \le A|V_j \cap B(y, c_J^{-1}2^{-j}R)|. \tag{4.1}$$

In fact, take $y^* \in \partial D$ such that $|y - y^*| = 2^{-j}R$, and then take $y' \in [y, y^*]$ with $\delta_D(y') = \frac{1}{2}(2^{-j}R + 2^{-j-1}R)$. An elementary geometrical observation gives $B(y', 2^{-j-2}R) \subset V_j \cap B(y, c_J^{-1}2^{-j}R)$, so that (4.1) follows.

Now the covering lemma yields a sequence $\{y_k\}$ such that

$$\bigcup_{i=j+1}^{\infty} V_i \subset \bigcup_k \overline{B(y_k, 5c_J^{-1}2^{-j}R)}$$

and $\{\overline{B(y_k, c_J^{-1}2^{-j}R)}\}_k$ are disjoint. Hence

$$\sum_{i=j+1}^{\infty} |V_i| = \left|\bigcup_{i=j+1}^{\infty} V_i\right| \le \sum_k |B(y_k, 5c_J^{-1}2^{-j}R)| \le A_1 \sum_k |V_j \cap B(y_k, c_J^{-1}2^{-j}R)| \le A_1|V_j|$$

by (4.1). Let $1 < t < 1 + A_1^{-1}$. In the same way as in [1, Lemma 5] we have

$$\sum_{j=0}^{\infty} t^j |V_j| \le \frac{t}{1 - (t-1)A_1} \sum_{j=0}^{\infty} |V_j| \le A|B(\xi, R + (1 + c_J^{-1})2R)| \le AR^n.$$

Since $t^j < (R/\delta_D(x))^\tau \le t^{j+1}$ on V_j with $\tau = \log t / \log 2 > 0$, it follows that

$$\int_{D \cap B(\xi,R)} \left(\frac{R}{\delta_D(x)}\right)^\tau dx \le \sum_{j=0}^{\infty} t^{j+1}|V_j| \le AR^n.$$

Thus the lemma follows. □

5. Growth of positive harmonic functions

In this section we shall show Proposition 2.3 (i) by investigating the growth of $h \in \mathcal{H}_\xi$. Throughout the section we let D be a general John domain and let $\xi \in \partial D$ be fixed. We say that $x, y \in D$ are connected by a Harnack chain $\{B(x_j, \frac{1}{2}\delta_D(x_j))\}_{j=1}^k$ if $x \in B(x_1, \frac{1}{2}\delta_D(x_1))$, $y \in B(y_k, \frac{1}{2}\delta_D(y_k))$, and $B(x_j, \frac{1}{2}\delta_D(x_j)) \cap B(x_{j+1}, \frac{1}{2}\delta_D(x_{j+1})) \ne \emptyset$ for $j = 1, \dots, k-1$. The number k is called the length of the Harnack chain. We observe that the shortest length of the Harnack chain connecting x and y is comparable to $k_D(x, y)$. Therefore, the Harnack inequality yields that there is a constant $A_2 > 1$ depending only on n such that

$$\exp(-A_2(k_D(x,y) + 1)) \le \frac{h(x)}{h(y)} \le \exp(A_2(k_D(x,y) + 1)) \tag{5.1}$$

for every positive harmonic function h on D. If D is a John domain with John constant c_J and John center x_0, then we have from (2.1)

$$\frac{h(x)}{h(x_0)} \le A_3 \left(\frac{\delta_D(x_0)}{\delta_D(x)}\right)^{\lambda} \tag{5.2}$$

with λ and $A_3 > 0$ depending only on the John constant c_J. If D is a general John domain with John constant c_J and John center K_0, then (5.2) holds with the same λ and another A_3 depending only on c_J, x_0 and K_0.

Let Ω be an open set intersecting ∂D. Let h be a bounded positive harmonic function in $D \cap \Omega$ vanishing q.e. on $\partial D \cap \Omega$. We extend h to $\Omega \setminus D$ by 0 outside D and denote by h^* its upper regularization. Then we observe that h^* is a nonnegative subharmonic function on Ω ([8, Theorem 5.2.1]). We shall apply the refinement of Domar's theorem (Lemma 3.1) to the subharmonic function h^* to obtain a Carleson type estimate.

Lemma 5.1. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Suppose h is a positive harmonic function in $D \cap B(\xi, \eta^{-3}R)$ vanishing q.e. on $\partial D \cap B(\xi, \eta^{-3}R)$. If h is bounded in $D \cap B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$, then*

$$h \le A \sum_{i=1}^{N} h(y_i) \quad \text{on } D \cap S(\xi, \eta^2 R), \tag{5.3}$$

where A is independent of h and R.

Proof. Let $0 < R < R_\xi$. Then we find $y_1, \dots, y_N \in D \cap S(\xi, R)$ with $\delta_D(y_i) \approx R$ such that

$$\min_{i=1,\dots,N} \{k_{D_R}(x, y_i)\} \le A \log \frac{R}{\delta_D(x)} + A \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}.$$

By (5.1) we find a constant $A_4 > 1$ such that

$$h(x) \le A_4 \left(\frac{R}{\delta_D(x)}\right)^{\lambda} \sum_{i=1}^{N} h(y_i) \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}. \tag{5.4}$$

Let us apply Lemma 3.1 to $\varepsilon = 1$, $u = h^*/(A_4 \sum_{i=1}^{N} h(y_i))$ and $\Omega = B(\xi, \eta R) \setminus \overline{B(\xi, \eta^3 R)}$. Let $\tau > 0$ be as in Lemma 4.1. Apply the elementary inequality:

$$(\log t)^n \le \left(\frac{n}{\tau}\right)^n t^\tau \quad \text{for } t \ge 1$$

to $t = R/\delta_D(x) \ge 1$ for $x \in \Omega$. Then

$$\left[\log^+ \left(\frac{R}{\delta_D(x)}\right)\right]^n \le A \left(\frac{R}{\delta_D(x)}\right)^{\tau},$$

so that it follows from (5.4) and Lemma 4.1 that

$$I = \int_\Omega (\log^+ u)^n dx \le A \int_{D \cap B(\xi, R)} \left(\frac{R}{\delta_D(x)}\right)^{\tau} dx \le AR^n.$$

Hence, Lemma 3.1 yields that $u \le \exp(2 + AIR^{-n}) \le A$ on $S(\xi, \eta^2 R)$, i.e., (5.3) holds. □

Let us apply Lemma 5.1 to a kernel function $h \in \mathcal{H}_\xi$ to obtain the following growth estimate.

Lemma 5.2. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Let $h \in \mathcal{H}_\xi$. Then*

$$h(x) \le A|x-\xi|^{-\lambda} \quad \text{for } x \in D,$$

where $\lambda > 0$ is as in (5.2) *and A is independent of R, x and h.*

Proof. By Lemma 5.1 we have (5.3). Since h is bounded apart from a neighborhood of ξ, the maximum principle gives

$$h(x) \le A \sum_{i=1}^{N} h(y_i) \quad \text{for } x \in D \setminus B(\xi, \eta^2 R).$$

Apply (5.2) to each $y_i \in D \cap S(\xi, R)$ with $\delta_D(y_i) \approx R$. Then obtain $h(y_i) \le AR^{-\lambda}$. This, together with the above estimate, yields $h(x) \le A|x-\xi|^{-\lambda}$ for $x \in D$. The lemma is proved. □

Here we record another application of Lemma 5.1, as this will be useful later.

Lemma 5.3. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Let h be a bounded positive harmonic function on $D \cap B(\xi, \eta^{-3}R)$ vanishing q.e. on $\partial D \cap B(\xi, \eta^{-3}R)$. Then*

$$h \le A \sum_{i=1}^{N} h(y_i) \quad \text{on } D \cap \overline{B(\xi, \eta^2 R)},$$

where A is independent of R and h.

Proof. We have (5.3). Apply the maximum principle to $D \cap B(\xi, \eta^2 R)$. □

The following lemma is well-known.

Lemma 5.4. *Suppose there exist a positive integer M and a positive constant A with the following property: if $h_0, \dots, h_M \in \mathcal{H}_\xi$, then there is j such that*

$$h_j \le A \sum_{i \ne j} h_i \quad \text{on } D.$$

Then $\mathcal{H}_\xi$ has at most M minimal harmonic functions.

Proof of Proposition 2.3 for $N \ge 3$. Let $h_j \in \mathcal{H}_\xi$ for $j = 0, \dots, M$. Let h_j^* be the upper regularization of the extension of h_j to $\mathbb{R}^n \setminus \{\xi\}$ as before Lemma 5.1 and let H_j be the Kelvin transform of h_j^* with respect to $S(\xi, 1)$, i.e.,

$$H_j(x) = |x-\xi|^{2-n} h_j^*(\xi + |x-\xi|^{-2}(x-\xi)).$$

Observe that H_j is a nonnegative subharmonic function on $\mathbb{R}^n$ which is positive and harmonic on the Kelvin image D^* of D and is equal to 0 q.e. outside D^*. Moreover, Lemma 5.2 shows

$$H_j(x) \le A|x-\xi|^{2-n+\lambda}.$$

Thus H_j is of order at most $2-n+\lambda$. As in Benedicks [10, Theorem 2], we let

$$w = \max_{j=0,\dots,M} \{H_j - \sum_{i \ne j} H_i\}$$

and let w^+ be the upper regularization of $\max\{w, 0\}$. Then w^+ is a nonnegative subharmonic function on $\mathbb{R}^n$ of order at most $2 - n + \lambda$. If none of $\{x : H_j(x) > \sum_{i\neq j} H_i(x)\}$ is empty, then w^+ has $M + 1$ tracts. Hence, [15, Theorem 3] yields

$$2 - n + \lambda \geq \frac{1}{2}\log\left(\frac{M+1}{4}\right) + \frac{3}{2} \quad \text{if } M \geq 3.$$

Hence, if $M > 4\exp(1 - 2n + 2\lambda) - 1$, then $\{x : H_j(x) > \sum_{i\neq j} H_i(x)\} = \emptyset$ for some $j = 0, \dots, M$. This means that $H_j \leq \sum_{i\neq j} H_i$ on D^*, so that

$$h_j \leq \sum_{i\neq j} h_i \quad \text{on } D.$$

Hence Lemma 5.4 implies that $\mathcal{H}_\xi$ has at most M minimal harmonic functions, or equivalently there are at most M minimal Martin boundary points at ξ. Thus the number of minimal Martin boundary points at ξ is bounded by $4\exp(1 - 2n + 2\lambda)$. □

Remark 5.1. The above proof gives a coarse estimate of the number of minimal harmonic functions of $\mathcal{H}_\xi$ in terms of λ depending on the John constant c_J. More delicate arguments will be needed for a sharp estimate.

6. Weak boundary Harnack principle

In this section we shall prove Proposition 2.3 for $N \leq 2$. Throughout the section we let D be a general John domain and fix $\xi \in \partial D$. Since most arguments are valid for any $N \geq 1$, except for (6.5), we shall state the results for general N. Proposition 2.3 will be derived from a certain estimate of the Green function. There is a difference of the behavior of the Green function G for D between the cases $n = 2$ and $n \geq 3$, i.e., if $n \geq 3$ and $R > 0$ is small, then

$$G(x, y) \approx R^{2-n} \quad \text{for } x \in S(y, \frac{1}{2}\delta_D(y)) \text{ with } \delta_D(y) \approx R;$$

if $n = 2$, then this estimate does not necessarily hold. To avoid this difficulty we consider the Green function G_R for the intersection $\widetilde{D}_R = D \cap B(\xi, A_5 R)$ with sufficiently large $A_5 > \eta^{-3}$. Then we have for any $n \geq 2$,

$$G_R(x, y) \approx R^{2-n} \quad \text{for } x \in S(y, \frac{1}{2}\delta_D(y)) \text{ with } \delta_D(y) \approx R, \tag{6.1}$$

where the constant of comparison depends only on D and A_5.

By $\omega(x, E, U)$ we denote the harmonic measure of E for an open set U evaluated at x. The box argument in [2, Lemma 2] (see [9] for the original form) gives the following estimate of the harmonic measure.

Lemma 6.1. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \dots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. If $x \in D \cap \overline{B(\xi, \eta^3 R)}$, then*

$$\omega(x, D \cap S(\xi, \eta^2 R), D \cap B(\xi, \eta^2 R)) \leq A R^{n-2} \sum_{i=1}^{N} G_R(x, y_i), \tag{6.2}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Proof. Let us begin with an estimate of harmonic measure in a John domain. For $0 < r < \delta_D(K_0)$ let $U(r) = \{x \in D : \delta_D(x) < r\}$. Then each point $x \in U(r)$ can be connected to K_0 by a curve such that (1.1) holds. Hence, $B(x, A_6 r) \setminus U(r)$ includes a ball with radius r, provided A_6 is large. This implies that

$$\omega(x, U(r) \cap S(x, A_6 r), U(r) \cap B(x, A_6 r)) \le 1 - \varepsilon_0 \quad \text{for } x \in U(r)$$

with $0 < \varepsilon_0 < 1$ depending only on A_6 and the dimension. Let $R \ge r$ and repeat this argument with the maximum principle. Then there exist positive constants A_7 and A_8 such that

$$\omega(x, U(r) \cap S(x, R), U(r) \cap B(x, R)) \le \exp(A_7 - A_8 R/r). \tag{6.3}$$

See [2, Lemma 1] for details.

Let $0 < R < R_\xi$. For each $x \in D \cap \overline{B(\xi, \eta R)}$ there is a local reference point $y(x) \in \{y_1, \dots, y_N\}$ such that

$$k_{D_R}(x, y(x)) \le A_\xi \log \frac{R}{\delta_D(x)} + A_\xi$$

by definition. Let $y'(x) \in S(y(x), \frac{1}{2}\delta_D(y(x)))$. Then we observe that $k_{D_R \setminus \{y(x)\}}(x, y'(x)) \le A_\xi \log(R/\delta_D(x)) + A_\xi$. Letting $u(x) = R^{n-2} \sum_{i=1}^N G_R(x, y_i)$, we obtain from (5.1) and (6.1) that

$$u(x) \ge A \left(\frac{\delta_D(x)}{R} \right)^\lambda \quad \text{for } x \in D \cap \overline{B(\xi, \eta R)}$$

with some $\lambda > 0$ depending only on n, c_J, R_ξ and A_ξ. Let $D_j = \{x \in D : \exp(-2^{j+1}) \le u(x) < \exp(-2^j)\}$ and $U_j = \{x \in D : u(x) < \exp(-2^j)\}$. Then we see that

$$U_j \cap B(\xi, \eta R) \subset \left\{ x \in D : \delta_D(x) < AR \exp\left(-\frac{2^j}{\lambda} \right) \right\}.$$

Define a decreasing sequence R_j by $R_0 = \eta^2 R$ and

$$R_j = \left(\eta^2 - \frac{6(\eta^2 - \eta^3)}{\pi^2} \sum_{k=1}^j \frac{1}{k^2} \right) R \quad \text{for } j \ge 1.$$

Let $\omega_0 = \omega(\cdot, D \cap S(\xi, \eta^2 R), D \cap B(\xi, \eta^2 R))$ and put

$$d_j = \begin{cases} \displaystyle\sup_{x \in D_j \cap B(\xi, R_j)} \frac{\omega_0(x)}{u(x)} & \text{if } D_j \cap B(\xi, R_j) \ne \emptyset, \\ 0 & \text{if } D_j \cap B(\xi, R_j) = \emptyset. \end{cases}$$

It is sufficient to show that d_j is bounded by a constant independent of R and j, since $R_j > \eta^3 R$ for all $j \ge 0$. Apply the maximum principle to $U_j \cap B(\xi, R_{j-1})$ to obtain

$$\omega_0(x) \le \omega(x, U_j \cap S(\xi, R_{j-1}), U_j \cap B(\xi, R_{j-1})) + d_{j-1} u(x).$$

Divide the both sides by $u(x)$ and take the supremum over $D_j \cap B(\xi, R_j)$. Then (6.3) yields

$$d_j \le \exp\left(2^{j+1} + A_7 - A_8 \frac{R_{j-1} - R_j}{AR \exp(-2^j/\lambda)} \right) + d_{j-1},$$

provided j is so large, say $j \ge j_0$, that

$$\frac{R_{j-1} - R_j}{AR \exp(-2^j/\lambda)} = \frac{6(\eta^2 - \eta^3)}{\pi^2} \frac{\exp(2^j/\lambda)}{Aj^2} \ge 1.$$

Hence, for $j \geq j_0$,

$$d_j \leq d_{j_0-1} + \sum_{j=j_0}^{\infty} \exp\left(2^{j+1} + A_7 - A_8 \frac{6(\eta^2 - \eta^3)}{\pi^2} \frac{\exp(2^j/\lambda)}{A j^2}\right) < \infty.$$

For $j \leq j_0$ we have $d_j \leq \exp(2^{j+1}) \leq \exp(2^{j_0+1})$. Hence we obtain $\sup_{j\geq 0} d_j < \infty$. Thus (6.2) follows. □

Lemma 6.2. *Let $\xi \in \partial D$ have a system of local reference points $y_1, \ldots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. If $x \in D \cap \overline{B(\xi, \eta^3 R)}$ and $y \in D \cap S(\xi, \eta^{-3} R)$, then*

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^{N} G_R(x, y_i) \sum_{j=1}^{N} G_R(y_j, y), \tag{6.4}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Proof. Apply Lemma 5.3 to $h(x) = G_R(x, y)$ with $y \in D \cap S(\xi, \eta^{-3} R)$. Then

$$G_R(x, y) \leq A \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in D \cap S(\xi, \eta^2 R).$$

Hence (6.2) yields

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^{N} G_R(x, y_i) \sum_{j=1}^{N} h(y_j) \quad \text{for } x \in D \cap \overline{B(\xi, \eta^3 R)}$$

by the maximum principle. The lemma follows. □

For further arguments we need the following improvement of (6.4): If $x \in D \cap S(\xi, \eta^9 R)$ and $y \in D \cap S(\xi, \eta^{-3} R)$, then

$$G_R(x, y) \leq AR^{n-2} \sum_{i=1}^{N} G_R(x, y_i) G_R(y_i, y) \tag{6.5}$$

where A depends only on n, c_J, R_ξ and A_ξ. Note that the cross terms $G_R(x, y_i) G_R(y_j, y)$ ($i \neq j$) disappear from the right hand side of (6.4).

If $N = 1$, then (6.5) is nothing but (6.4). If $N \leq 2$, then Ancona's ingenious trick [6, Théorème 7.3] gives (6.5) from (6.4). However, the proof is rather complicated and we postpone the proof to the next section. The remaining arguments are rather easy and hold for arbitrary $N \geq 1$, provided (6.5) holds. Let us show the weak boundary Harnack principle defined by Ancona [6, Définition 2.3].

Lemma 6.3 (Weak Boundary Harnack Principle)**.** *Let $\xi \in \partial D$ have a system of local reference points $y_1, \ldots, y_N \in D \cap S(\xi, R)$ of order N with factor η for $0 < R < R_\xi$. Moreover, suppose* (6.5) *holds. Let $h_0, h_1, \ldots, h_N \in \mathcal{H}_\xi$. Then*

$$h_0(x) \leq A \sum_{i=1}^{N} \frac{h_0(y_i)}{h_i(y_i)} h_i(x) \quad \textit{for } x \in D \setminus B(\xi, \eta^9 R). \tag{6.6}$$

where A depends only on n, c_J, R_ξ and A_ξ.

Proof. In (6.5) we replace the roles of x and y and write z for y. By dilation and changing A_5 we obtain from the symmetry of the Green function that if $x \in D \cap S(\xi, \eta^9 R)$ and $z \in D \cap S(\xi, \eta^{21} R)$, then

$$G_R(x,z) \le AR^{n-2} \sum_{i=1}^{N} G_R(x,z_i) G_R(z_i,z),$$

where $z_1, \dots, z_N \in D \cap S(\xi, \eta^{12} R)$ are local reference points. Moreover, for each z_i we find a local reference point $y_{j(i)} \in D \cap S(\xi, R)$ such that $k_{\widetilde{D}_R \setminus \{x,z\}}(z_i, y_{j(i)}) \le A$. In view of (5.1), we have $G_R(x,z_i) \approx G_R(x, y_{j(i)})$ and $G_R(z_i,z) \approx G_R(y_{j(i)}, z)$, whenever $x \in D \cap S(\xi, \eta^9 R)$ and $z \in D \cap S(\xi, \eta^{21} R)$. Hence we obtain that if $x \in D \cap S(\xi, \eta^9 R)$ and $z \in D \cap S(\xi, \eta^{21} R)$, then

$$G_R(x,z) \le AR^{n-2} \sum_{i=1}^{N} G_R(x,y_i) G_R(y_i,z). \tag{6.7}$$

Let $r = \eta^{-3} R$ and $\rho = \eta^{21} R$. Observe that the regularized reduced function $\widehat{R}_{h_0}^{D \cap (S(\xi,r) \cup S(\xi,\rho))}$ with respect to $\widetilde{D}_R$ is a Green potential of measures μ concentrated on $D \cap S(\xi, r)$ and ν on $D \cap S(\xi, \rho)$ such that $\widehat{R}_{h_0}^{D \cap (S(\xi,r) \cup S(\xi,\rho))} = h_0$ on $D \cap B(\xi, r) \setminus \overline{B(\xi, \rho)}$. It follows from (6.5) and (6.7) that for $x \in D \cap S(\xi, \eta^9 R)$,

$$\begin{aligned}
h_0(x) &= \int_{D \cap S(\xi,r)} G_R(x,y) d\mu(y) + \int_{D \cap S(\xi,\rho)} G_R(x,z) d\nu(z) \\
&\le AR^{n-2} \sum_{i=1}^{N} \left(\int_{D \cap S(\xi,r)} G_R(x,y_i) G_R(y_i,y) d\mu(y) + \int_{D \cap S(\xi,\rho)} G_R(x,y_i) G_R(y_i,z) d\nu(z) \right) \\
&= AR^{n-2} \sum_{i=1}^{N} G_R(x,y_i) h_0(y_i).
\end{aligned}$$

Let $\varepsilon = 1 - \eta^9$. Observe from (6.1) and the Harnack inequality that $h_i(y_i) R^{n-2} G_R(x, y_i) \approx h_i(x)$ for $x \in S(y_i, \varepsilon \delta_D(y_i))$, and so is for $x \in D \cap S(\xi, \eta^9 R) \subset D \setminus B(y_i, \varepsilon \delta_D(y_i))$ by the maximum principle. Hence (6.6) follows for $x \in D \setminus B(\xi, \eta^9 R)$ by the maximum principle. □

Proof of Proposition 2.3 (ii) for $N \le 2$. Obviously (6.5) holds for $N = 1$; (6.5) holds for $N = 2$, as we shall show in the next section. Hence Lemma 6.3 is applicable. Varying R in Lemma 6.3, we obtain relationships among kernel functions in $\mathcal{H}_\xi$, which yield Proposition 2.3. This procedure is the same as in Ancona [6, Théoremè 2.5] and we omit the details. □

Remark 6.1. We do not know whether the weak boundary Harnack principle holds for $N \ge 3$. In special cases, such as a sectorial domain whose boundary lies on N rays leaving ξ, we can apply the weak boundary Harnack principle repeatedly to subdomains containing just one ray and conclude the weak boundary Harnack principle for the sectorial domain itself (cf. Cranston and Salisbury [12, p. 36]).

7. Proof of (6.5)

In this section we shall prove the following:

Lemma 7.1. *Let $\xi \in \partial D$ have a system of local reference points $y_1, y_2 \in D \cap S(\xi, R)$ of order* 2 *with factor η for $0 < R < R_\xi$. If $x \in D \cap S(\xi, \eta^9 R)$ and $y \in D \cap S(\xi, \eta^{-3} R)$, then* (6.5) *holds.*

We employ Ancona's trick [6, Théorème 7.3]. Since our setting is slightly different from Ancona's, we provide a proof for the sake of the reader's convenience.

Proof. Besides the local reference points $y_1, y_2 \in D \cap S(\xi, R)$, we take local reference points $y_1^*, y_2^* \in D \cap S(\xi, \eta^6 R)$ with

$$\min_{i=1,2}\{k_{D\cap B(\xi,\eta^3 R)}(x, y_i^*)\} \le A_\xi \log \frac{\eta^6 R}{\delta_D(x)} + A_\xi \quad \text{for } x \in D \cap \overline{B(\xi, \eta^7 R)}.$$

Then

$$\min_{j=1,2}\{k_{D_R}(y_i^*, y_j)\} \le A_\xi \log \frac{R}{\delta_D(y_i^*)} + A_\xi \le A_\xi.$$

So, we may assume either

$$k_{D_R}(y_1^*, y_1) \le A \text{ and } k_{D_R}(y_2^*, y_1) \le A, \tag{7.1}$$

or

$$k_{D_R}(y_1^*, y_1) \le A \text{ and } k_{D_R}(y_2^*, y_2) \le A, \tag{7.2}$$

by replacing the roles of y_1 and y_2, if necessary.

First consider the case when (7.1) holds. Suppose $x \in D \cap S(\xi, \eta^9 R)$. Then (5.1) and (6.4) for y_1^*, y_2^* yield

$$G_R(x, y) \le AR^{n-2} \sum_{i,j} G_R(x, y_i^*) G_R(y_j^*, y) \le AR^{n-2} G_R(x, y_1) G_R(y_1, y)$$

for $y \in D \cap S(\xi, \eta^3 R)$, and hence for $y \in D \cap S(\xi, \eta^{-3} R)$ by the maximum principle. Hence the lemma follows in this case.

Next consider the case when (7.2) holds. Let $\Phi = \{z \in \widetilde{D}_R : G_R(z, y_1) \ge G_R(z, y_2)\}$. If either $x, y \in \Phi$ or $x, y \in \widetilde{D}_R \setminus \Phi$, then (6.5) follows from (6.4). Let us consider the remaining cases. If necessary, exchanging the roles of y_1 and y_2, we may assume that $x \in \Phi \cap S(\xi, \eta^9 R)$ and $y \in (\widetilde{D}_R \setminus \Phi) \cap S(\xi, \eta^{-3} R)$. Let $E = \Phi \setminus B(\xi, \eta^3 R)$ and consider the regularized reduced function $\widehat{R}^E_{G_R(\cdot,y)}$ with respect to $\widetilde{D}_R$. This function is represented as the Green potential of a measure μ concentrated on ∂E. For a moment let $z \in E$. Then we have from (6.4) for y_1^*, y_2^* and the maximum principle

$$G_R(x, z) \le AR^{n-2} \sum_{i,j} G_R(x, y_i^*) G_R(y_j^*, z). \tag{7.3}$$

It is easy to see from (7.2) that $k_{D_R\setminus\{x\}}(y_i^*, y_i) \le A$, so that $G_R(x, y_i^*) \le AG_R(x, y_i)$ for $i = 1, 2$ by (5.1). We also have $G_R(y_j^*, z) \le AG_R(y_j, z)$ for $j = 1, 2$. In fact, if $z \in B(y_j, \frac{1-\eta^6}{2}\delta_D(y_j))$, then $G_R(y_j, z) \approx |y_j - z|^{2-n} \ge AR^{2-n} \ge AG_R(y_j^*, z)$; if $z \in \widetilde{D}_R \setminus B(y_j, \frac{1-\eta^6}{2}\delta_D(y_j))$, then (7.2) gives $k_{D_R\setminus\{z\}}(y_j^*, y_j) \le A$, and hence $G_R(y_j^*, z) \approx G_R(y_j, z)$ by (5.1). Hence (7.3) becomes

$$G_R(x, z) \le AR^{n-2} \sum_{i,j} G_R(x, y_i) G_R(y_j, z) \le AR^{n-2} G_R(x, y_1) G_R(y_1, z)$$

by the definition of Φ. Therefore

$$\widehat{R}^E_{G_R(\cdot,y)}(x) \le AR^{n-2}G_R(x,y_1)\int_E G_R(y_1,z)d\mu(z)$$
$$= AR^{n-2}G_R(x,y_1)\widehat{R}^E_{G_R(\cdot,y)}(y_1) \le AR^{n-2}G_R(x,y_1)G_R(y_1,y). \tag{7.4}$$

Let $v_y = G_R(\cdot,y) - \widehat{R}^E_{G_R(\cdot,y)}$. Then

$$v_y = 0 \quad \text{q.e. on } E = \Phi \setminus B(\xi,\eta^3 R). \tag{7.5}$$

By (6.4) we have

$$v_y(z) \le G_R(z,y) \le AR^{n-2}G_R(z,y_2)G_R(y_2,y) \quad \text{for } z \in D \cap \partial\Phi \cap \overline{B(\xi,\eta^3 R)}. \tag{7.6}$$

Observe that

$$D \cap \partial(\Phi \cap B(\xi,\eta^3 R)) \subset (\Phi \setminus B(\xi,\eta^3 R)) \cup (D \cap \partial\Phi \cap \overline{B(\xi,\eta^3 R)}).$$

Hence (7.5), (7.6) and the maximum principle yield

$$v_y \le AR^{n-2}G_R(\cdot,y_2)G_R(y_2,y) \quad \text{on } \Phi \cap \overline{B(\xi,\eta^3 R)}.$$

This, together with (7.4), implies

$$G_R(x,y) \le AR^{n-2}(G_R(x,y_1)G_R(y_1,y) + G_R(x,y_2)G_R(y_2,y)).$$

The proof is complete. □

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