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MARTINGALES ON RIEMANNIAN MANIFOLDS AND THE NONLINEAR HEAT EQUATION

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Solutions to the nonlinear heat equation for maps between Riemannian manifolds are studied by determining starting points for martingales on Riemannian manifolds with prescribed terminal values. Monotonicity properties of the Riemannian quadratic variation for these martingales allow to explain blow-up of the heat flow in finite time. Moreover, the probabilistic construction of martingales with given terminal state is discussed, and partial regularity results for the heat flow are established.

1 Harmonic mappings and deformation by heat flow

Let (M,g) and (N,h) be Riemannian manifolds of dimensions m and n, let $f\colon M \to N$ be a smooth map. We consider the two fundamental forms of f.

(a) (*First fundamental form of* f) The pullback of the metric h under f gives a bilinear form $f^*h \in \Gamma(T^*M \otimes T^*M)$ which is defined by

$$(f^*h)_x(u,v) = h_{f(x)}(df_x u, df_x v) \quad \text{for } u,v \in T_xM.$$

(b) (*Second fundamental form of* f) With respect to the Levi-Civita-connections on M and N one considers $\nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$ defined as covariant derivative of $df \in \Gamma(T^*M \otimes f^*TN)$.

By taking traces (with respect to the Riemannian metrics on M and N) we get

(i) $\|df\|^2 = \operatorname{trace} f^*h \in C^\infty(M)$, the energy density of f, and

(ii) $\tau(f) = \operatorname{trace} \nabla df \in \Gamma(f^*TN)$, the tension field of f.

Smooth maps $f\colon M \to N$ with vanishing tension field $\tau(f)$ are called harmonic. In local coordinates the harmonic map equation is written as

$$\tau^k(f) = \Delta_M f^k + ({}^N\Gamma^k \circ f)\,(df, df) = 0\,, \quad k = 1, \ldots, n \tag{1.1}$$

where Δ_M is the Laplace-Beltrami operator on (M,g). The analytical difficulties in the study of harmonic maps arise from the nonlinearity in (1.1) which reflects the fact that the map f takes its values in a curved Riemannian manifold (N,h).

Given some smooth initial map $u_0\colon M \to N$, we are interested in its development under the heat flow

$$\tfrac{\partial}{\partial t}u = \tfrac{1}{2}\tau(u)\,, \quad u|_{t=0} = u_0\,. \tag{1.2}$$

If M is not compact we assume that the energy $E(u_0)$ of u_0 is finite, and moreover, that $\|du_0\|^2$ is bounded on M. Note that harmonic maps $f\colon M \to N$ are stationary points of the energy functional

$$E(f) = \int_M \|df\|^2 \, d\,\mathrm{vol} \tag{1.3}$$

with respect to compactly supported variations of f. On the other hand, solutions of (1.2) satisfy the energy inequality

$$E\big(u(t,\,\cdot\,)\big) + \int_0^t \int_M \Big\|\frac{\partial u}{\partial s}\Big\|^2 (s,x)\,\mathrm{vol}(dx)\,ds \le E(u_0)\,. \tag{1.4}$$

From (1.4) it looks reasonable to expect that the flow defined by (1.2) will come to a rest, as $t \to \infty$, producing a harmonic map.

If M and N are compact, there is a famous global existence result in this direction, due to Eells and Sampson [6]. Suppose the sectional curvature Riem^N of N is non-positive, then for any $u_0 \in C^\infty(M,N)$ the heat equation (1.2) admits a unique, global, smooth solution $u\colon [0,\infty[\, \times M \to N$. Moreover, as $t \to \infty$, the maps $u(t,\,\cdot\,)$ converge smoothly to a harmonic map $u_\infty \in C^\infty(M,N)$ homotopic to u_0. Thus deformation by heat flow allows to determine harmonic representatives in each homotopy class.

If the curvature assumption $\mathrm{Riem}^N \le 0$ is dropped the situation turns out to be much more complicated. Equation (1.2) may blow up in finite time by topological reasons [2,1] in the sense that for some $T > 0$ and some $x_0 \in M$,

$$\limsup_{t \nearrow T} \sup_{x \in B_\varepsilon(x_0)} \|d\,u(t,x)\|^2 = \infty \tag{1.5}$$

for any $\varepsilon > 0$; here $B_\varepsilon(x_0)$ is the geodesic ball about x_0 of radius ε. One main problem is to determine conditions creating singularities out of smooth initial data in finite time. In addition, we would like to understand such results in probabilistic terms.

It is well-known how to use probability, i.e., the theory of Brownian motion, in the linear case of harmonic functions. For $N = \mathbb{R}^n$, the unique solution of the heat equation (1.2) is given by

$$u(t,x) = \mathbb{E}[u_0 \circ X_t^x] \tag{1.6}$$

where X^x is $\mathrm{BM}(M,g)$ started at x, provided that Brownian motion on (M,g) has infinite lifetime.

Obviously, formula (1.6) is meaningless in the general case of curved targets N; taking expectations is by definition a linear operation ruling out straightforward generalizations of (1.6) to the nonlinear heat equation. Nevertheless, we would like to give an interpretation of the harmonic map heat flow in terms of appropriate "expectations" of the N-valued random variables $u_0 \circ X_t^x$. There are several approaches to define expectations (means) for manifold-valued random variables[11,9]. It turns out that the correct replacement of the linear expectation operator is given by a quite sophisticated nonlinear concept, namely the starting points of ∇-martingales with the given random variable as prescribed terminal value[16]. It has interesting consequences that global geometry enters the description via the concept of martingales on Riemannian manifolds.

2 Probabilistic Description of the Harmonic Map Heat Flow

In the linear case, $N = \mathbb{R}^n$, $u(t,x) = \mathbb{E}[u_0 \circ X_t^x]$ represents the solution of the heat equation, moreover, for each $(t,x) \in \mathbb{R}_+ \times M$,

$$Y_s = \mathbb{E}^{\mathcal{F}_s}[u_0 \circ X_t^x] = (P_{t-s}u_0)(X_s^x), \quad 0 \le s \le t,$$

defines a (uniformly integrable) martingale with starting point $Y_0 = u(t,x)$ and terminal value $u_0 \circ X_t^x$. The observation that $u(t,x)$ may be seen as expectation of $u_0 \circ X_t^x$ in the sense that there is a martingale, starting at $u(t,x)$ and ending up at $u_0 \circ X_t^x$, carries over to the general case of non-trivial target manifolds[16].

Theorem 2.1 *Let u: $[0,T[\times M \to N$ be a smooth solution of the heat equation (1.2). Then, for $(t,x) \in [0,T[\times M$, the N-valued process*

$$Y_s = u(t-s, X_s^x), \quad 0 \le s \le t, \tag{2.1}$$

is an H^2-martingale on (N,h) with $Y_0 = u(t,x)$ and $Y_t = u_0 \circ X_t^x$.

Martingales on (N,h) are taken with respect to the Levi-Civita-connection[8]; recall that Y is in the Hardy class H^2 if its Riemannian quadratic variation satisfies

$$\mathbb{E}\int_0^t h(dY, dY) < \infty. \tag{2.2}$$

The underlying probability space is (without restriction of generality) the standard m-dimensional Wiener space $\big(C(\mathbb{R}_+, \mathbb{R}^m), \mathcal{F}, \mathbb{P}\big)$ with its natural Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. With respect to this filtration, the Brownian motions X^x on (M,g) are constructed in the usual way[7,10] by solving an SDE on the orthonormal frame bundle $O(M) \xrightarrow{\pi} M$

$$dU = \sum_{i=1}^m L_i(U) * dB^i, \quad U_0 = e \in \pi^{-1}\{x\}, \tag{2.3}$$

and projecting U down to M, i.e., $X^x = \pi \circ U$. Here B is BM($\mathbb{R}^m$); in other words, X^x is the stochastic development on M of a flat BM B in $\mathbb{R}^m \cong T_x M$ (identified via the frame e). Note that any manifold-valued semimartingale is a stochastic development of a flat semimartingale, and vice versa. In our situation, the anti-development Z on $\mathbb{R}^n \cong T_{f(x)}N$ of the N-valued semimartingale $\tilde{X} = f \circ X^x$, defined as image of X^x under a map $f \in C^\infty(M, N)$, is given by

$$dZ = \tilde{/\!/}_{t,0}\, df\, /\!/_{0,t}\, dB + \tfrac{1}{2}\, \tilde{/\!/}_{t,0}\, \tau(f) \circ X^x\, dt\,, \tag{2.4}$$

where $/\!/_{0,t}$ and $\tilde{/\!/}_{0,t}$ are the parallel transports along the paths of X, resp. $\tilde{X}$. Specifically, the Riemannian quadratic variation of $\tilde{X}$ measures the energy of f along the paths of X^x, i.e.,

$$h(d\tilde{X}, d\tilde{X}) = \left(\|df\|^2 \circ X^x \right) dt\,. \tag{2.5}$$

Applied to the N-valued semimartingale $Y_s = u(t-s, X_s^x)$ in (2.1), we get for its anti-development Z in $T_{u(t,x)}N$ (modulo differentials of local martingales)

$$dZ \overset{\mathrm{m}}{=} \tilde{/\!/}_{s,0} \big(-\partial_t u + \tfrac{1}{2}\tau(u) \big)(t-s, X_s^x)\, ds = 0\,. \tag{2.6}$$

Recall that martingales are characterized as stochastic developments of continuous local martingales [8]. Thus, if u solves the heat equation (1.2), then $Y_s = u(t-s, X_s^x)$ is a martingale on N. In addition, its Riemannian quadratic variation is given by

$$h(dY, dY) = \|du(t-s,\,\cdot\,)\|^2 (X_s^x)\, ds\,. \tag{2.7}$$

3 Monotonicity Properties and Blow-up in Finite Time

In this section we use the description of the heat flow in terms of manifold-valued martingales to derive development of singularities in finite time; detailed proofs of our results can be found in Thalmaier [16]. For the sake of simplicity, we restrict ourselves to the special case $M = \mathbb{R}^m$; from now on N is supposed to be compact. Let u: $[0, T[\times \mathbb{R}^m \to N$ be a solution of (1.2) such that

$$E(u_0) < \infty, \text{ and } \|du_0\|^2 \text{ is bounded on } \mathbb{R}^m. \tag{3.1}$$

As explained above, for each $(t, x) \in [0, T[\times \mathbb{R}^m$, there is an H^2-martingale $Y^{(t,x)} \equiv (Y_s^{(t,x)})_{0 \le s \le t}$ with starting point $Y_0^{(t,x)} = u(t, x)$ and terminal value $Y_t^{(t,x)} = u_0 \circ X_t^x$ a.s. The Riemannian quadratic variations of these martingales satisfy a specific monotonicity property. This monotonicity is basically a consequence of Brownian scaling and the appropriate parabolic version of the famous monotonicity conditions in the theory of harmonic maps [14,15].

Theorem 3.1 (Monotonicity Formula) *Let* $u\colon [0,T[\,\times \mathbb{R}^m \to N$ *be a solution of the heat equation* (1.2) *such that condition* (3.1) *is fulfilled. Then, for each* $(t,x) \in [0,T[\,\times \mathbb{R}^m$ *and each* $\alpha \in \,]0,1[$,

$$\Phi\colon r \mapsto \mathbb{E}\int_{\alpha r}^{r} h(dY^{(t,x)}, dY^{(t,x)}) \tag{3.2}$$

defines a non-decreasing function Φ *on* $]0,t]$.

The crucial observation is that smallness of $\Phi(r)$ can be turned into a priori estimates for the heat flow.

Theorem 3.2 *There exists a constant* $\varepsilon_0 = \varepsilon_0(m,N) > 0$ *depending only on* m *and* N *such that for any solution* $u\colon [0,T[\,\times \mathbb{R}^m \to N$ *of the heat equation satisfying condition* (3.1) *the following is true: If*

$$\Phi(r) = \mathbb{E}\int_{r/2}^{r} h(dY^{(t,x)}, dY^{(t,x)}) < \varepsilon_0 \tag{3.3}$$

for some $(t,x) \in [0,T[\,\times \mathbb{R}^m$ *and some* r *such that* $0 < r \le t < T$, *then* $\|du\|^2(t,x) \le C$ *with a constant* $C = C\big(r,m,N,E(u_0)\big)$.

The a priori estimates of Theorem 3.2 can be adapted to obtain a global existence result for solutions of the heat equation, which immediately leads to blow-up in finite time in certain cases of homotopically nontrivial initial data u_0. If the energy of u_0 is sufficiently small, then the deformation under the heat flow necessarily goes towards a constant map; singularities develop in finite time if this is impossible from topological reasons[1,2].

Theorem 3.3 *For any* $T > 0$ *there exists a constant* $\varepsilon_1 = \varepsilon_1(m,N,T)$ *depending only on* m, N *such that any solution* $u\colon [0,T[\,\times \mathbb{R}^m \to N$ *of the heat equation with* $\|du_0\|^2$ *bounded on* $\mathbb{R}^m$ *and* $E(u_0) < \varepsilon_1$ *can be extended to a global (smooth) solution* $u\colon [0,\infty[\,\times \mathbb{R}^m \to N$ *which converges to a constant harmonic map* u_∞ *as* $t \to \infty$.

Corollary *Let* $T > 0$, *and take* $\varepsilon_1 = \varepsilon_1(m,N,T)$ *as in Theorem* 3.3. *Then, for homotopically nontrivial* $u_0\colon \mathbb{R}^m \to N$ *with* $\|du_0\|^2 \in L^\infty$ *and* $E(u_0) < \varepsilon_1$, *solutions of the heat equation* (1.2) *blow up before time* T. *Moreover, the blow-up time* $T^* = T^*(u_0)$ *approaches* 0 *as* $E(u_0)$ *decreases to* 0.

Proof. Otherwise, by Theorem 3.3, there would exist a global solution to (1.2) inducing a homotopy $u_0 \simeq u_\infty \equiv \text{const}$, in contradiction to the assumption that u_0 is homotopically nontrivial. $\square$

4 Backward Stochastic Differential Equations

As explained in section 2, associated to the nonlinear heat equation is the following reachability problem for N-valued martingales.

Problem *Given $f\colon M \to N$ and $\xi^{(t,x)} = f \circ X_t^x$ for some $(t,x) \in \mathbb{R}_+ \times M$, where X^x is* $\mathrm{BM}(M,g)$ *as constructed above. Find an N-valued H^2-martingale $Y = (Y_s^{(t,x)})_{0\le s\le t}$, adapted to the standard m-dimensional Brownian filtration on $C(\mathbb{R}_+, \mathbb{R}^m)$ with $m = \dim M$, such that $Y_t = \xi^{(t,x)}$ a.s.*

By Itô's formula [8], an N-valued semimartingale Y obeys the composition rule

$$d(\varphi \circ Y) = d\varphi(\tilde{U} dZ) + \tfrac{1}{2}\nabla d\varphi(dY, dY) \tag{4.1}$$

for $\varphi \in C^\infty(N)$; here $\tilde{U}$ is a horizontal lift of Y to the orthonormal frame bundle $O(N)$ over N (uniquely determined by a choice of $\tilde{U}_0$ over Y_0) and Z is the $\mathbb{R}^n$-valued anti-development of Y which may be expressed as Stratonovich integral $Z = \int_{\tilde{U}} \vartheta$ of the canonical connection 1-form ϑ on $O(M)$ along $\tilde{U}$ [7,10]. The requirement that Y provides a martingale on (N,h) means by definition that Z is a local martingale on $\mathbb{R}^n$. In this case, by Itô's representation theorem, there exists a predictable $\mathbb{R}^m \otimes \mathbb{R}^n$-valued process representing Z as $Z_t = \int_0^t H dB$ where B is canonical Brownian motion on m-dimensional Wiener space with respect to which the M-valued BM X^x is defined; see (2.3). Substituting $dZ = HdB$ into (4.1) gives

$$d(\varphi \circ Y) = d\varphi(CdB) + \tfrac{1}{2}\sum_{i=1}^m \nabla d\varphi(Ce_i, Ce_i)\, dt \tag{4.2}$$

for any $\varphi \in C^\infty(N)$ where $C := \tilde{U}H$ is an $\mathbb{R}^m \otimes TN$-valued semimartingale above Y, i.e., $C_s \in \mathbb{R}^m \otimes T_{Y_s}N$ a.s.

Thus, given an N-valued random variable of the type $\xi^{(t,x)} = f \circ X_t^x$, the problem is to find a semimartingale C taking its values in the vector bundle $\mathbb{R}^m \otimes TN$ over N, adapted to the m-dimensional Brownian filtration, such that with $Y := \pi \circ C$ for each $\varphi \in C^\infty(N)$ the following equation holds for any $0 \le s \le t$:

$$\varphi \circ \xi^{(t,x)} = \varphi \circ Y_s + \int_s^t d\varphi(CdB) + \tfrac{1}{2}\sum_{i=1}^m \int_s^t \nabla d\varphi(Ce_i, Ce_i)\, dr\,. \tag{4.3}$$

Of course, it is enough to assure (4.3) for the coordinate functions $\varphi = \varphi_j$ $(1 \le j \le \ell)$ of an embedding $\iota\colon N \hookrightarrow \mathbb{R}^\ell$. In terms of a classical solution to (1.2) the process C is easily determined as

$$C_s = du(t-s, X_s^x)\, U_s \tag{4.4}$$

where U is horizontal Brownian motion on $O(M)$ such that $\pi \circ U = X^x$, as defined by equation (2.3).

Equation (4.3) defines a backward SDE and is easily reduced to an equation of the type studied by Pardoux-Peng[12]. For instance, according to their set-up, given a (differentiable) map

$$F \colon M \times \mathbb{R}^\ell \times (\mathbb{R}^m \otimes \mathbb{R}^\ell) \to \mathbb{R}^\ell\,,$$

and an $\mathcal{F}_t$-measurable random variable $\xi \in L^2(\mathbb{P};\mathbb{R}^\ell)$ for some $t > 0$, one may consider the problem of finding continuous adapted $\mathbb{R}^\ell$-, resp. $\mathbb{R}^m \otimes \mathbb{R}^\ell$-valued processes $(Y_s)_{0\le s\le t}$ and $(C_s)_{0\le s\le t}$ such that

$$\begin{cases} dY = C\,dB + F(X^x, Y, C)\,ds \\ Y_t = \xi \end{cases} \tag{4.5}$$

(the filtration is again the Brownian filtration of the m-dimensional Brownian motion B); note that in integrated form (4.5) reads as

$$\xi = Y_s + \int_s^t C\,dB + \int_s^t F(X^x, Y, C)\,dr\,, \quad 0 \le s \le t\,. \tag{4.6}$$

We observe that for the "linear" case $F \equiv 0$, a solution (Y,C) to (4.5) is given by

$$Y_s = \mathbb{E}^{\mathcal{F}_s}[\xi] = \mathbb{E}[\xi] + \int_0^s C\,dB, \tag{4.7}$$

where the matrix process C is determined by Y via Itô's representation theorem. Among other things, Pardoux-Peng[12] show that under a global Lipschitz condition for F, there exists a unique pair (Y,C) of square-integrable continuous adapted processes solving (4.5), or equivalently (4.6).

As is well-known[5,16], the main difficulty in applying the theory of backward SDE to the construction of martingales on manifolds with prescribed terminal state comes from the fact that (4.3) fails to satisfy a global Lipschitz condition: Due to the geometric nature of the problem the drift term on the right-hand side depends quadratically on C which implies that equation (4.3) is not covered by the existence and uniqueness results of Pardoux-Peng. However, this fact should not be seen as shortcoming of the theory of backward stochastic differential equations; it precisely reflects the nontrivial interplay of local analysis and global geometry captured in the nonlinear heat flow.

Obviously there is no easy way around the mentioned difficulties of constructing martingales on manifolds with given terminal value[4,5,13,16]. In the following section we suggest an approximation scheme to construct appropriate martingales for terminal values of the form $\xi^{(t,x)} = f \circ X_t^x$ where t may be arbitrarily large[17]. From now on both M and N are supposed to be compact.

5 Penalty Approximation for Martingales

Since the target N is compact, we may assume that (N,h) is isometrically embedded into $\mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$; by composing with the embedding $N \hookrightarrow \mathbb{R}^\ell$, maps $M \to N$ will be considered as maps into $\mathbb{R}^\ell$. The main problem with harmonic maps $f\colon M \to N$ taking their values in a curved submanifold $N \subset \mathbb{R}^\ell$ is now the nonlinear constraint $f(M) \subset N$.

A possible way to deal with this difficulty is to relax the constraint $f(M) \subset N$ but to penalize its violation. Applied to the case of the heat equation, we may construct $\mathbb{R}^\ell$-valued processes which approximate the desired N-valued martingale closer and closer. More accurately, we construct an approximating sequence of $\mathbb{R}^\ell$-valued semimartingales by solving backward SDEs on $\mathbb{R}^\ell$, each satisfying a global Lipschitz condition, and use compactness arguments to find an N-valued martingale with the prescribed terminal value. There are several ways to achieve this; we follow the so-called penalty approximation [3]. The method is most easily explained in the context of the variational problem for harmonic maps. Roughly speaking, instead of working with the standard Dirichlet form

$$E(f) = \int_M \|df\|^2 \, d\,\mathrm{vol} \tag{5.1}$$

for maps $f\colon M \to \mathbb{R}^\ell$ with $f(M) \subset N$, we drop the target constraint and regard all maps $f\colon M \to \mathbb{R}^\ell$ as admissible, but penalize violations of the constraint $f(M) \subset N$ proportional to the distance from the submanifold N. More precisely, let $V(N)$ be a tubular neighborhood of N of radius 3δ in the flat ambient space $\mathbb{R}^\ell$, diffeomorphic to $N \times B_{3\delta}(0)$ where $B_{3\delta}(0)$ is the ball in $\mathbb{R}^{\ell-n}$ of radius 3δ. Thus, elements in $V(N)$ are represented as (y,v) with $y \in N$ and $v \in T_yN^\perp$; both the projection $\pi\colon (y,v) \mapsto y$ onto N and $\mathrm{dist}^2(\,\cdot\,,N)\colon (y,v) \mapsto \|v\|^2$ are C^∞ on $V(N)$. Further, choose a differentiable real function χ on $\mathbb{R}_+$ with $\chi' \geq 0$ such that $\chi(r^2) = r^2$ for $r \leq \delta$ and $\chi(r^2) = 4\delta^2$ for $r \geq 2\delta$. Now, instead of (5.1) consider the variational integral

$$E_\varepsilon(f) = \int_M \Big[\|df\|^2 + \frac{2}{\varepsilon}\,\chi\big(\mathrm{dist}^2(f,N)\big)\Big]\, d\,\mathrm{vol}\,, \quad \varepsilon > 0\,, \tag{5.2}$$

for functions $f\colon M \to \mathbb{R}^\ell$. The idea is that, when minimizing (5.2) for small values of ε, the term $(2/\varepsilon)\int_M \chi\big(\mathrm{dist}^2(f,N)\big)\,d\,\mathrm{vol}$ will force the unconstrained functions f to approach the submanifold N, since violations of $f(M) \subset N$ are penalized more severely, as $\varepsilon \searrow 0$. Instead of $\tau(f) = 0$, the Euler-Lagrange equation associated with the functional (5.2) reads as

$$\Delta_M f - (2/\varepsilon)\,(\mathrm{grad}\,\phi)(f) = 0 \tag{5.3}$$

where $\phi := (1/2)\,\chi\big(\mathrm{dist}^2(\,\cdot\,,N)\big)$. Hence, in replacement of the heat equation (1.2), we consider for $\varepsilon > 0$ the following evolution equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u - \dfrac{1}{2}\Delta_M u + \dfrac{1}{\varepsilon}\,(\mathrm{grad}\,\phi)(u) = 0 \quad \text{on } [0,\infty[\,\times M \\ u|_{t=0} = u_0\,. \end{cases} \tag{5.4}$$

Note that $\mathrm{grad}\,\phi$ is a smooth vector field on $\mathbb{R}^\ell$ with $\mathrm{grad}\,\phi = 0$ outside the tubular neighborhood $V(N)$ and $(\mathrm{grad}\,\phi)_z = \chi'\big(\mathrm{dist}^2(z,N)\big)\ \mathrm{dist}(z,N)\,A_z$ where $A_z \in T_{\pi(z)}N^\perp$ for $z \in V(N)$. It is well-known that (5.4) has a unique global smooth solution $u^{(\varepsilon)}\colon \mathbb{R}_+ \times M \to \mathbb{R}^\ell$ for each $\varepsilon > 0$.

The nonlinear equation (5.4) is easily represented in stochastic terms. Fix $\varepsilon > 0$ and $(t,x) \in \mathbb{R}_+ \times M$. Let again B be the canonical Wiener process on the underlying m-dimensional Wiener space $\big(\Omega = C(\mathbb{R}_+,\mathbb{R}^m), \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t\in\mathbb{R}_+}\big)$, and let X^x be $\mathrm{BM}(M,g)$ with starting point x, constructed via stochastic development of B as in (2.3). For $\xi^{(t,x)} = u_0 \circ X_t^x$, consider the backward SDE problem of finding adapted processes $Y_s^{(\varepsilon)}$, $C_s^{(\varepsilon)}$ $(0 \le s \le t)$ with values in $\mathbb{R}^\ell$, resp. $\mathbb{R}^m \otimes \mathbb{R}^\ell$, such that

$$\begin{cases} dY^{(\varepsilon)} = C^{(\varepsilon)}\,dB + \frac{1}{\varepsilon}\,(\mathrm{grad}\,\phi)(Y^{(\varepsilon)})\,ds \\ \quad Y_t^{(\varepsilon)} = \xi^{(t,x)}\,. \end{cases} \tag{5.5}$$

Obviously, (5.5) satisfies a global Lipschitz condition, and hence for each $\varepsilon > 0$, according to Pardoux-Peng [12], there is a unique pair $(Y^{(\varepsilon)}, C^{(\varepsilon)})$ of continuous $(\mathcal{F}_s)$-adapted processes, square integrable over $\Omega \times [0,t]$ with respect to $\mathbb{P} \otimes ds$, providing a solution to (5.5). Note that

$$Y_s^{(\varepsilon)} = \mathbb{E}^{\mathcal{F}_s}\Big[\xi^{(t,x)} - \frac{1}{\varepsilon}\int_s^t (\mathrm{grad}\,\phi)(Y_r^{(\varepsilon)})\,dr\Big], \tag{5.6}$$

and specifically,

$$Y_0^{(\varepsilon)} = \mathbb{E}\Big[\xi^{(t,x)} - \frac{1}{\varepsilon}\int_0^t (\mathrm{grad}\,\phi)(Y_r^{(\varepsilon)})\,dr\Big]. \tag{5.7}$$

Moreover, observe that (5.6) does not involve $C^{(\varepsilon)}$; using Itô's representation theorem, $C^{(\varepsilon)}$ is recovered from $Y^{(\varepsilon)}$ via

$$\begin{aligned} Y_s^{(\varepsilon)} - \frac{1}{\varepsilon}\int_0^s (\mathrm{grad}\,\phi)(Y_r^{(\varepsilon)})\,dr &= \mathbb{E}^{\mathcal{F}_s}\Big[\xi^{(t,x)} - \frac{1}{\varepsilon}\int_0^t (\mathrm{grad}\,\phi)(Y_r^{(\varepsilon)})\,dr\Big] \\ &= Y_0^{(\varepsilon)} + \int_0^s C^{(\varepsilon)}\,dB\,. \end{aligned} \tag{5.8}$$

It is easily checked by means of Itô's formula that, in terms of the solution $u^{(\varepsilon)}\colon \mathbb{R}_+ \times M \to \mathbb{R}^\ell$ to (5.4), the unique pair $(Y^{(\varepsilon)}, C^{(\varepsilon)})$ of square integrable processes solving (5.5) is given by

$$Y_s^{(\varepsilon)} = u^{(\varepsilon)}(t-s, X_s^x)\,, \quad C_s^{(\varepsilon)} = du^{(\varepsilon)}(t-s, X_s^x)\, U_s\,. \tag{5.9}$$

On the other hand, given $(Y^{(\varepsilon)}, C^{(\varepsilon)})$, the solution to (5.4) is determined by $u^{(\varepsilon)}(t,x) = Y_0^{(\varepsilon)}$; in addition $du^{(\varepsilon)}(t,x) = C_0^{(\varepsilon)}\, U_0^{-1}$.

Our goal is to construct the desired N-valued martingale $Y^{(t,x)}$ with terminal value $Y_t^{(t,x)} = \xi^{(t,x)}$ from the $Y^{(\varepsilon)}$ as $\varepsilon \searrow 0$ suitably. The main tool to achieve this is formulated in the following lemma [17].

Lemma 5.1 *Let $(t,x) \in \mathbb{R}_+ \times M$ and $s_0 \in\,]0,t]$. For $\varepsilon > 0$ let $(Y^{(\varepsilon)}, C^{(\varepsilon)})$ be the unique square-integrable pair solving (5.5). There is a constant $c = c\big(s_0, E(u_0)\big)$ depending only on s_0 and the energy $E(u_0)$ of u_0 such that*

$$\sup_{s_0 \le s \le t} \Big[\mathbb{E}\,\|C_s^{(\varepsilon)}\|^2 + \frac{1}{\varepsilon}\,\mathbb{E}\chi\big(\mathrm{dist}^2(Y_s^{(\varepsilon)}, N)\big)\Big] \le c\,. \tag{5.10}$$

Using compactness arguments relying on Lemma 5.1, we are able to show [17] that for each $s_0 > 0$

$$Y^{(\varepsilon_n)} \Rightarrow Y \quad \text{in } L^2\big([s_0, t] \times \Omega\big)\,,$$

as $\varepsilon_n \searrow 0$ appropriately, where $(Y_s)_{s_0 \le s \le t}$ is an N-valued martingale. Hence, for any $(t,x) \in \mathbb{R}_+ \times M$, we can find a martingale $Y_s^{(t,x)}$ on (N,h), defined for $0 < s \le t$, such that $Y_t^{(t,x)} = u_0 \circ X_t^x$ a.s. Note that $Y^{(t,x)}$ may not necessarily have a starting point. The obvious question is what can be said about the "singularity set"

$$\Sigma := \Big\{(t,x) \in \mathbb{R}_+ \times M : \lim_{s \searrow 0} Y_s^{(t,x)} \text{ does not exist a.s.}\Big\}. \tag{5.11}$$

Moreover, we like to consider $u(t,x) := Y_0^{(t,x)}$ for $(t,x) \notin \Sigma$ and to clarify in which sense u provides a solution of the heat equation.

6 Partial Regularity

As seen in section 3, solutions to the heat equation (1.2) may blow up in finite time. Note that after the first singularity has appeared, equation (1.2) is no longer well-defined as a classical PDE; one has to consider distributional solutions. Existence of global weak solutions to (1.2) has been established by Chen-Struwe [3].

In the stochastic description the starting points of martingales on (N,h) with terminal value $u_0 \circ X_t^x$ correspond to solutions of the heat equation (1.2). The construction of appropriate martingales should be seen as a way of pulling out randomness of the variables $u_0 \circ X_t^x$, in order to reduce it to a constant point. There may be topological obstructions to do this: The paths of the martingales $Y_s^{(t,x)}$ constructed above may start oscillating as $s \searrow 0$ in such a way that

$$\mathbb{E}\int_s^t h(dY^{(t,x)}, dY^{(t,x)}) \to \infty$$

as $s \searrow 0$. In other words, the martingales behave as started at $t = -\infty$ on their intrinsic time scale. The following theorem summarizes our main results in this direction.

Theorem 6.1 (Main Theorem) *Let (M,g) and (N,h) be compact Riemannian manifolds and $u_0 \in C^\infty(M,N)$. For $x \in M$ let X^x be* BM(M,g) *started at x, adapted to the standard m-dimensional Brownian filtration. Then, for each point $(t,x) \in \mathbb{R}_+ \times M$, there is a martingale $Y^{(t,x)} = (Y_s^{(t,x)} : 0 < s \le t)$ on (N,h) with terminal value $Y_t^{(t,x)} = u_0 \circ X_t^x$ such that for*

$$\Sigma = \Big\{(t,x) \in \mathbb{R}_+ \times M : \lim_{s \searrow 0} Y_s^{(t,x)} \textit{ does not exist a.s.}\Big\}$$

the following statements hold:

(i) $\Sigma = \Big\{(t,x) \in \mathbb{R}_+ \times M : \lim_{r \searrow 0} \mathbb{E}\int_{r/2}^r h(dY^{(t,x)}, dY^{(t,x)}) > 0\Big\}.$

(ii) *Σ is closed, and $u(t,x) := Y_0^{(t,x)}$ is smooth for $(t,x) \notin \Sigma$.*

(iii) *If $t > 0$, then $\Sigma \cap \big(\{t\} \times M\big)$ has finite $(m-2)$-dimensional Hausdorff measure.*

(iv) *$u(t,x)$ for $(t,x) \notin \Sigma$ defined by* (ii) *extends to a global distributional solution of* (1.2).

(v) *For all $(t,x) \in \mathbb{R}_+ \times M$, we have $Y_s^{(t,x)} = u(t-s, X_s^x), \quad 0 < s \le t$.*

(vi) *u coincides with the classical solution on $[0,T^*[\times M$ where $T^* \in \,]0,\infty]$ is the first blow-up time for the classical solution to* (1.2).

Details of the proof will appear elsewhere [17]. It looks plausible to conjecture that (s, X_s^x) does not hit the singularity set Σ a.s. if not started there.

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