Masks for Hadamard transform optics, and weighing designs

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This paper gives a brief survey of the design of masks for Hadamard spectrometers and image scanners. Three different criteria are described for judging a mask, as well as techniques for choosing masks that are not too far from the optimum.

I. Introduction

The use of masks to improve the performance of spectrometers and image scanners has been proposed by a number of authors,[1-8] and several papers exist describing experiments performed with such instruments.[8-12]

Briefly, such an instrument consists of four essential components: an optical separator, an encoding mask, a detector, and a processor (Fig. 1).

The separator might be nothing more than a lens that produces a focused image at the mask, thereby separating light arriving from different spatial elements of a scene. It might equally well be a dispersing system that separates different spectral components of a beam and focuses them onto different locations on the mask.

A particular location on the mask either transmits light to the detector, absorbs the light, or reflects it toward a reference detector. In this way, the intensity of an element of the separated beam is modulated; its intensity is, respectively, multiplied by +1, 0, or −1, since the readings that are recorded are the differences in the intensity of light reaching the main detector and reference detector.

If there is only one detector, the modulation consists of +1's and 0's only, and the recorded intensity is just the intensity of radiation transmitted by the mask.

When m intensity values are to be determined, at least m different detector readings corresponding to m different mask arrays are required.

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Received 16 April 1975.

Three important questions in designing such an instrument are: (a) How should the mask be chosen? (b) How much does this improve the accuracy of the measurements? (c) How close is this to the optimum mask design?

Such questions have been studied for many years in statistics under the name of weighting designs. Up to now this seems to have escaped the notice of workers in optics, but a considerable body of literature exists (see, for example, Raghavarao[13] or Banerjee[14]).

Even so, the specific answers to (a), (b), and (c) are not readily available in this literature, and so we shall give here a short survey of what is presently known about the answers. The expert in statistics will find little that is new; however, we do give the first complete proof we have seen that the average mean square error for the best mask of 0's and 1's is about four times that for the best mask of −1's, 0's, and +1's.

II. Weighing Designs and Masks

Yates[15] seems to have been the first to point out that by weighing several objects together instead of separately it may be possible to determine the individual weights more accurately.

For example, suppose four objects are to be weighed, using a balance that makes an error e each time it is used. Assume that e is a random variable with mean zero and variance σ^2.

First, suppose the objects are weighed separately. If the unknown weights are x_1, x_2, x_3, x_4, the measurements are y_1, y_2, y_3, y_4, and the errors made by the balance are e_1, e_2, e_3, e_4, the four weighings give four equations:

$$y_1 = x_1 + e_1, \quad y_2 = x_2 + e_2, \quad y_3 = x_3 + e_3, \quad y_4 = x_4 + e_4.$$

The best estimates of the unknown weights are the measurements themselves:

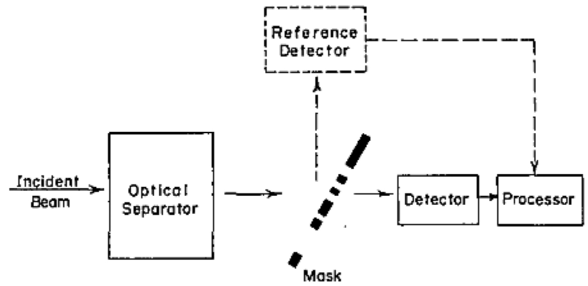

Fig. 1. The prime components used in Hadamard transform optics.

$$\hat{x}_1 = y_1 = x_1 + e_1, \hat{x}_2 = y_2 = x_2 + e_2, \ldots.$$

These are unbiased estimates:

$$E\hat{x}_1 = x_1, E\hat{x}_2 = x_2, \ldots \quad (E \text{ denotes expected value}),$$

with variance or mean square error

$$E(\hat{x}_1 - x_1)^2 = Ee_1^2 = \sigma^2, \ldots.$$

On the other hand, suppose the balance is a chemical balance with two pans, and the four weighings are made as follows:

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 + x_4 + e_1, \\ y_2 &= x_1 - x_2 + x_3 - x_4 + e_2, \\ y_3 &= x_1 + x_2 - x_3 - x_4 + e_3, \\ y_4 &= x_1 - x_2 - x_3 + x_4 + e_4. \end{aligned} \tag{1}$$

This means that in the first weighing all four objects are placed in the left-hand pan, and in the other weighings two objects are in the left pan and two in the right. Since the coefficient matrix on the right is a Hadamard matrix, it is easy to solve for x_1, x_2, x_3, x_4. Thus the best estimate for x_1 (see below for a justification of this) is

$$\begin{aligned} \hat{x}_1 &= \frac{1}{4}(y_1 + y_2 + y_3 + y_4) \\ &= x_1 + \frac{1}{4}(e_1 + e_2 + e_3 + e_4). \end{aligned}$$

The variance of ce, where c is a constant, is c^2 times the variance of e, and the variance of a sum of independent random variables is the sum of the individual variances. Therefore the variance of $\hat{x}_1$ (and also of $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$) is $4\sigma^2/16 = \sigma^2/4$.

Weighing the objects together has reduced the mean square error by a factor of 4.

Finally, suppose the balance is a spring balance with only one pan, so only coefficients 0 and 1 can be used. A good method of weighing the four objects is

$$\begin{aligned} y_1 &= \phantom{x_1 + {}} x_2 + x_3 + x_4 + e_1, \\ y_2 &= x_1 + x_2 \phantom{{} + x_3 + x_4} + e_2, \\ y_3 &= x_1 \phantom{{} + x_2} + x_3 \phantom{{} + x_4} + e_2, \\ y_4 &= x_1 \phantom{{} + x_2 + x_3} + x_4 + e_4. \end{aligned} \tag{2}$$

In this case the variances of $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$ are $4\sigma^2/9$, $7\sigma^2/9$, $7\sigma^2/9$, $7\sigma^2/9$, respectively, a smaller improvement than in the previous case.

In general, if there are p unknowns $x_1, \ldots, x_p$, and N measurements $y_1, \ldots, y_N$ are made, involving errors $e_1, \ldots, e_N$, we have

$$y_i = w_{i1}x_1 + \ldots + w_{ip}x_p + e_i, \quad i = 1, \ldots, N,$$

or in matrix form

$$y = Wx + e, \tag{3}$$

where $y = (y_1, \ldots, y_N)^T$, $x = (x_1, \ldots, x_p)^T$, $e = (e_1, \ldots, e_N)^T$, and T denotes transpose. A particular choice for the $N \times p$ coefficient matrix $W = (w_{ij})$ is called a weighing design.

The connection with Hadamard transform optics is straightforward. In this type of optical system the x_i's represent individual spatial and/or spectral elements whose intensities are to be determined. In contrast to scanning instruments that would measure the intensities one at a time, the Hadamard transform optical system measures (i.e., weighs) the intensities of several x_i's simultaneously.

The two types of weighing design—the chemical balance design (with coefficients w_{i1} that may be -1, 0, or $+1$) and the spring balance design (in which the coefficients must be 0 or 1)—are realized in optical systems by masks W that use reflected, absorbed, or transmitted light in the first case or simply open or closed slots in the second case.[4] In spite of the name Hadamard transform optics, the example of Eq. (2) shows that the matrices W should not be restricted to those obtained from Hadamard matrices. (We shall refer to W indiscriminantly as a mask, matrix, or weighing design.)

Weighing designs are also applicable to other problems of measurements (such as lengths, voltages, resistances, concentrations of chemicals, etc.) in which the measure of several objects is the sum (or a linear combination) of the individual measurements.[14,16]

In Secs. III and IV we shall discuss the best choice for a weighing design, subject to the following assumptions:

(1) The errors e_i are uncorrelated (see Banerjee[17] for the general case);

(2) The errors e_i are independent of the amount being weighed. In the weighing problem this assumes that the objects are light, and in the optical scheme this is an assumption of linear behavior of the photodetector. (Raghavarao *et al.*[18] discuss a more general case.)

(3) The number of measurements (N) is equal to the number of unknowns (p), and the matrix W is invertible. Then the best (linear, unbiased) estimate for x from Eq. (3) is[4,19]

$$\hat{x} = W^{-1}y = x + W^{-1}e.$$

(In the general case, when W does not have an inverse, the best estimate is

$$\hat{x} = W^{+}y,$$

where W^+ is the Moore-Penrose generalized inverse of W.[20,21] If the rank of W is less than p, this is called a singular weighing design and may occur, for example, if an experiment is interrupted; e.g., suppose that a Hadamard transform spectrometer is used for an astronomical observation. Just before the end of the planned run of N measurements a patch of cloud passes over the dome. What information can be salvaged from the measurements actually made? Such questions are discussed by Raghavarao[22] and Banerjee.[23])

Finally we mention that Kiefer,[24] Kiefer and Wolfowitz,[25] and Fedorov[19] study more general problems of the optimum design of experiments, including the use of randomized weighing designs.

III. Matrices with Entries −1, 0, +1

From now on we assume that (1), (2), (3) hold. Then how should one choose the matrix, or mask, W? The mean square error of the estimate of the ith unknown x_i is

$$\epsilon_i = E(\hat{x}_i - x_i)^2.$$

Ideally one would like to minimize simultaneously $\epsilon_1, \ldots, \epsilon_N$. The most important result is due to Hotelling,[26] who showed that for any choice of mask W with $|w_{ij}| \leq 1$, the ϵ_i are bounded by $\epsilon_i \geq (\sigma^2/N)$, and that it is possible to have $\epsilon_i = (\sigma^2/N)$ for all $i = 1, \ldots, N$ if and only if a Hadamard matrix H_N of order N exists (by taking $W = H_N$).

So if a Hadamard matrix of order N exists, taking $W = H_N$ gives the best possible weighing design. This design reduces the mean square error by a factor of N compared to the mean square error (σ^2) of a single weighing. This result to some extent justifies the name Hadamard transform spectroscopy. Equation (1) was constructed with $W = H_4$.

We note that the intensity values to be determined can correspond to any set of spatial elements. Whether these elements are arrayed in a linear display such as in grating spectrometry, or in a two-dimensional array, as in imaging, is unimportant. The Hadamard matrix provides optimum encoding no matter what shape the array may be.

A Hadamard matrix of order N is an $N \times N$ matrix H_N of +1's and −1's that satisfies

$$H_N H_N^T = NI_N, \quad I_N = N \times N \text{ unit matrix}.$$

These matrices are thought to exist if and only if N = 1, 2, or a multiple of 4. Numerous constructions are known, and a plentiful supply of Hadamard matrices are available.[27–30] So if N is a multiple of 4 and masks with entries ±1 can be used, the problem of the best choice of mask is solved.

What if N is not a multiple of 4, or if only masks of 0's and 1's can be used? Then it is not possible to simultaneously minimize $\epsilon_1, \ldots, \epsilon_N$, and some other criterion must be used. Three different measurements of efficiency have been proposed:

An A-optimal[31] weighing design is one that minimizes the average mean square error, i.e., minimizes

$$\epsilon = \frac{1}{N}(\epsilon_1 + \ldots + \epsilon_N) = \frac{\sigma^2}{4}\, Tr\,(W^T W)^{-1},$$

where Tr denotes the trace of a matrix.

A D-optimal[32] design is one that maximizes the magnitude of the determinant of W $|\det(W)|$. This is equivalent to minimizing the generalized variance of the errors $\hat{x}_i - x_i$, which is $\sigma^2 \det(W^T W)^{-1}$. A D-optimal design minimizes the volume of the region in which the estimate $\hat{x}$ is expected to lie.

An E-optimal[33] design is one that maximizes the smallest eigenvalue $\lambda_{\min}$ of $W^T W$. To justify this, suppose one needed to determine a linear combination of the x_i's, say $\phi = c_1 x_1 + \ldots + c_N x_N$, where $c_1^2 + \ldots + c_N^2 = 1$. An E-optimal design minimizes the maximum mean square error of the best estimate $\hat{\phi}$ for all choices of the c_i's.

These criteria do not always agree. Probably A-optimality is the most important, provided the individual ϵ_i's are roughly equal.

It is reassuring that a Hadamard design $W = H_N$ is A-, D-, and E-optimal. In fact if $W = H_N$, we have

$$W = H_N\colon \epsilon = \epsilon_i = \frac{\sigma^2}{N}, \quad |\det H_N| = N^{N/2}, \quad \text{and } \lambda_{\min} = N, \tag{4}$$

while for any other W we have

$$\epsilon > \frac{\sigma^2}{N}, \; |\det W| < N^{N/2}, \text{ and } \lambda_{\min} < N.$$

The rest of this section describes masks that can be used when N is not a multiple of 4.

A. Masks from Conference Matrices

These are similar to Hadamard matrices but with a slightly different defining equation. They also give rise to good weighing designs. A conference matrix C_N of order N is an $N \times N$ matrix with diagonal entries 0 and other entries +1 or −1, which satisfies

$$C_N C_N^T = (N-1) I_N.$$

The name arises from the use of such matrices in the design of networks having the same attenuation between every pair of terminals (see Belevitch[34–36]).

N must be even for C_N to exist. But if N is a multiple of 4 these are inferior to Hadamard matrices, so we shall assume N has the form $4t + 2$. In this case, by suitably multiplying rows and columns by −1, C_N can be put in the form

$$C_N = \begin{bmatrix} 0 & 1 & 1 \ldots 1 \\ 1 & & \\ 1 & & B_{N-1} \\ \vdots & & \\ 1 & & \end{bmatrix} \tag{5}$$

where $B_{N-1} = (b_{ij})$ is a symmetric matrix.[37] Several constructions for conference matrices are known.[29,30,37,38] The most useful for our purpose is Paley's construction[39]: Let $N = 4t + 2 = p + 1$, where p is an odd prime, and set $b_{ij} = 0$ if $i = j$, $b_{ij} = 1$ if $j - i$ is a square (modulo p), and $b_{ij} = -1$ if $j - i$ is not a square (modulo p). The resulting matrix (5) is a conference matrix. For example, if $p = 5$, the squares modulo 5 are $1^2 = 1, 2^2 = 4$, and we obtain

$$C_6 = \begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{matrix}, \text{ where } - \text{ stands for } -1.$$

Note that this construction gives a matrix B_{N-1} that is a circulant.

The same construction works if p is replaced by any odd prime power p^m; the rows and columns of B_{N-1} are labeled with the elements of the Galois field $GF(p^m)$; $b_{ij} = 1$ if $j - 1$ is a square in $GF(p^m)$; etc. Now B_{N-1} is not a circulant. Galois fields are finite fields (see, e.g., Raghavarao[13]).

The construction gives symmetric, circulant, conference matrices of orders 6, 14, 18, 30, 38, 42, 54, 62, . . . , and symmetric conference matrices of orders 10, 26, 50, ... The matrices C_{10}, C_{14}, C_{18} are given in full by Raghavarao.[40]

Choosing the mask $W = C_N$ we obtain a weighing design[40] that has the parameters

$$W = C_N\colon \epsilon = \epsilon_i = \frac{\sigma^2}{N-1}, \quad |\det C_N| = (N-1)^{N/2}, \quad \lambda_{\min} = N - 1. \tag{6}$$

These are only slightly inferior to the parameters of a Hadamard matrix, Eq. (4). We mention in passing that for $N = 4t + 1$ the mask $W = B_N + I_N$ has $\epsilon = \epsilon_i = (2\sigma^2)/(N-1)$.

B. Masks from Symmetric Block Designs

An (N, k, λ) symmetric block design consists of a collection of subsets (called blocks) of size k taken from a set of N objects, such that any two blocks have exactly λ objects in common. [N, k, and λ are related by $k(k-1) = \lambda(N-1)$.]

For example, the blocks of a (7,3,1) design are

124, 235, 346, 457, 561, 672, 713.

From one symmetric block design we can always get another by taking as blocks the complements of the original blocks, that is, the objects not in the original blocks. This is an $(N, k' = N - k, \lambda' = N - 2k + \lambda)$ symmetric design. The complement of the preceding example is a (7,4,2) design with blocks

3567, 4671, 5712, 6123, 7234, 1345, 2456.

Block designs are frequently used in statistics, and many methods of construction are available.[41] (The statistical literature uses v instead of N for the number of objects.)

A symmetric block design is conveniently described by its incidence matrix (which explains the connection with masks). This is an $N \times N$ matrix $A_N = (a_{ij})$, where $a_{ij} = 1$ if the ith block contains the jth object, $= 0$ if not. The incidence matrix of the preceding (7,4,2) design is

$$A_7 = S_7 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\!\!\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}. \tag{7}$$

Generalizing this example, we can obtain symmetric block designs from Hadamard matrices as follows. By suitably multiplying rows and columns by -1, H_{N+1} can be put in the form

$$H_{N+1} = \left[\begin{array}{c|c} 1 & 1 \quad 1 \ldots 1 \\ \hline 1 & \\ 1 & G_N \\ \vdots & \\ 1 & \end{array}\right]. \tag{8}$$

If 1's are replaced by 0's and -1's by 1's, G_N is changed into the incidence matrix S_N of an $(N = 4t - 1, k = 2t, \lambda = t)$ symmetric block design. Equation (7) shows S_7.

Now suppose A_N is the incidence matrix of any (N,k,λ) symmetric block design. A_N satisfies the equations

$$A_N A_N^T = A_N^T A_N = (k - \lambda) I_N + \lambda J_N, \quad A_N J_N = J_N A_N = k J_N.$$

We may use A_N itself as a mask of 0's and 1's—see Sec. IV. To get a mask of +1's and -1's, set $D_N = J_N - 2A_N$, or in other words change 0's to +1's and 1's to -1's. Then choosing the mask $W = D_N$ we obtain a weighing design with the parameters

$$W = D_N\colon \epsilon = \epsilon_i = \frac{\sigma^2[N^2 - N - 4(N-2)n]}{4n[N^2 - 4(N-1)n]}, \quad |\det D_N| = (4n)^{(N-1)/2}[N^2 - 4(N-1)n]^{1/2}, \quad \lambda_{\min} = \min[4n, N^2 - 4(N-1)n], \tag{9}$$

where $n = k - \lambda$. The complementary symmetric design gives a mask with the same parameters.

To illustrate this, the $(N = 4t - 1, 2t, t)$ symmetric block design that we obtained from H_{4t} gives a weighing design with mask $W = G_N$ and parameters

$$W = G_N\colon \epsilon = \epsilon_i = \frac{2\sigma^2}{N+1}, \quad |\det G_N| = (N+1)^{(N-1)/2}, \quad \lambda_{\min} = 1. \tag{10}$$

The mean square error ϵ here is about twice that for H_N [Eq. (4)].

In some cases better masks exist. For example, as found by Rao,[42] when $N = 4t - 1 > 3$, an $[N = (d^2 + 3)/4, k = (N + d)/2, \lambda = (N + 2d + 3)/4]$ symmetric block design gives a mask $W = Q_N$ (say), which has parameters [from Eq. (9)]

$$W = Q_N\colon \epsilon = \epsilon_i = \frac{(4N-6)\sigma^2}{(4N-3)(N-3)}, \quad |\det Q_N| = (N-3)^{(N-1)/2}(4N-3)^{1/2}, \quad \lambda_{\min} = N - 3. \tag{11}$$

These parameters are better than Eq. (10). Unfortunately, at present the only known designs of this family are Q_7 and Q_{31}. This is typical: the best symmetric block designs for our purposes have k roughly equal to $\frac{1}{2}N$ and give masks with very good parameters, but not many examples are known.

A similar situation holds for $N = 4t + 1$. In this case as Raghavarao[40] has shown, an $[N = (d^2 + 1)/2, k = (N + d)/2, \lambda = (N + 2d + 1)/4]$ symmetric block design gives a mask $W = P_N$ (say), which has parameters

$$W = P_n: \epsilon = \epsilon_i = \frac{2\sigma^2}{2N - 1},$$
$$|\det P_N| = (N-1)^{(N-1)/2}(2N-1)^{1/2}, \lambda_{\min} = N - 1. \quad (12)$$

Again, at present only P_5, P_{13}, and P_{25} are known to exist. The corresponding matrices are given in Ref. 40.

C. Masks with $WW^T = \alpha I + \beta J$

All the masks given so far [except for Eq. (2)] have the property that the mean square errors of the unknowns, the ϵ_i's, are equal. In much statistical literature it is assumed that the weighing design satisfies the conditions

$$\epsilon_i = E(\hat{x}_i - x_i)^2 = \epsilon \text{ (independent of } i)$$
and $\quad (13)$
$$E(\hat{x}_i - x_i)(\hat{x}_j - x_j) = \eta \text{ (independent of } i,j, \text{ for } i \neq j).$$

This is equivalent to assuming that W_N satisfies $W_N W_N{}^T = \alpha I_N + \beta J_N$ for suitable α and β. With these assumptions, Raghavaro[13,40] has shown that P_N is A-, D-, and E-optimal, that C_N is A- and E-optimal, and Rao[42] has shown that Q_N is A- and E-optimal for $N > 3$.

However, as the weighing designs (2), (17), and (20) show, weighing designs that do not satisfy Eq. (13) may have a lower average mean square error, or a larger determinant or $\lambda_{\min}$, than those that do.

D. Masks with the Largest Determinant

Let us consider the problem of finding a D-optimal mask W_N, that is, an $N \times N$ matrix of -1's, 0's, and $+1$'s with the largest possible determinant. Let $g(N)$ denote this largest determinant. By expanding the determinant about any column, it follows that the largest determinant can always be attained by a matrix containing only -1's and $+1$'s (and no zero's).

The following bounds on $g(N)$ are known[43]:

$$g(N) \leq \begin{cases} N^{N/2} & \text{if } N = 4t; \quad (14) \\ (N-1)^{(N-1)/2}(2N-1)^{1/2} & \text{if } N \text{ is odd}; \quad (15) \\ 2(N-2)^{(N-2)/2}(N-1) & \text{if } N = 4t + 2. \quad (16) \end{cases}$$

We have already seen [Eq. (4)] that if $N = 4t$ we can achieve $g(N) = N^{N/2}$ by a Hadamard matrix. For many values of $N = 4t + 2$, $g(N) = 2(N-2)^{(N-2)/2}(N-1)$ can be achieved by double circulant matrices of the form

$$W_N = \begin{bmatrix} X & Y \\ -Y^T & X^T \end{bmatrix} = E_N \text{ (say)}, \quad (17)$$

where X and Y are $(N/2) \times (N/2)$ circulant matrices chosen so that

$$E_N{}^T E_N = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}, \quad Z = \left(\frac{1}{2}N - 2\right) I_{N/2} + 2J_{N/2}.$$

The parameters of this weighing design are

$$W_N = E_N: \epsilon = \epsilon_i = \frac{\sigma^2}{N-1},$$
$$|\det E_N| = 2(N-2)^{(N-2)/2}(N-1), \lambda_{\min} = N - 2. \quad (18)$$

For example, a D-optimal mask with $N = 6$ is

$$E_6 = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & 1 & - \\ \hline 1 & - & - & 1 & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \end{array}\right]. \quad (19)$$

Ehlich[44] and Yang[45,46] have given such matrices for $N = 2, 6, 10, 14, 18, 26, 30, 38, 42, 46, 50, 54, 62, 66$. It is interesting to compare Eqs. (18) with the parameters (6) of the C_N matrices. They both appear to exist for the same values of N. E_N has the same mean square error, slightly larger determinant (larger by a factor of about 1.21 when N is large), but smaller $\lambda_{\min}$.

The matrices P_5, P_{13}, P_{25} achieve Eq. (15) and so are also D-optimal. The exact value of $g(N)$ is known[47] for $N \leq 14$.

For large N, a good method of obtaining a matrix with a large determinant is to take the next largest Hadamard matrix and prune it to size.[48]

E. Small Masks

To end this section we give three matrices found by Mood,[16] which show that the best weighing designs can be complicated, even for small N, and need not satisfy Eqs. (13). For $N = 3$, there are three matrices with the largest determinant, 4, namely:

$$\text{(a)} \begin{bmatrix} 0 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}; \quad \text{(c)} \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix}. \quad (20)$$

The corresponding parameters are

	$(\epsilon_1, \epsilon_2, \epsilon_3)/\sigma^2$	ϵ/σ^2	det	$\lambda_{\min}$
(a)	$\left(\frac{1}{2}, \frac{3}{8}, \frac{3}{8}\right)$	$\frac{5}{12}$	4	2;
(b)	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	4	1;
(c)	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$	4	1.

Mood also gives D-optimal designs for $N = 5$, 6, and 7.

IV. Matrices with Entries 0 and 1

A matrix (or mask) W with entries 0 and 1 is usually less expensive to build than a mask with entries -1, 0, and $+1$. However, the price that must be paid for this is an increase by a factor of about 4 in the mean square error ϵ. This result is derived in the Appendix and can be observed by comparing the designs of this section with those of Sec. III.

If W_{N+1} is an $(N + 1) \times (N + 1)$ matrix with entries -1 and $+1$ only, there is a standard way to get an $N \times N$ matrix X_N of 0's and 1's. By suitably multiplying rows and columns of W_{N+1} by -1, make the first row and column equal to $+1$. Then deleting this first row and column and changing $+1$'s to 0's and -1's to 1's we obtain X_N. For example,

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \text{ gives } X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The determinants of W_{N+1} and X_N are related by

$$\det W_{N+1} = (-2)^N \det X_N. \tag{21}$$

A. Masks with the Largest Determinant

Because of this transformation from W_{N+1} to X_N, the problem of finding the largest determinant of any $N \times N$ matrix of 0's and 1's is equivalent to the problem of finding the largest determinant of any $(N + 1) \times (N + 1)$ matrix of -1's and $+1$'s. A solution to one problem gives a solution to the other.[49] If $f(N)$ denotes the value of the largest (0,1) determinant, Eq. (21) implies

$$f(N) = (1/2^N)g(N + 1).$$

Therefore all the results about $g(N)$ given in Sec. III apply to $f(N)$. In particular,

$$f(N) \leq \begin{cases} 2^{-N}{}_{(N+1)}(N + 1)/2 & \text{if } N = 4t - 1, \quad (22) \\ 2^{-N}N^{N/2}(2N + 1)^{1/2} & \text{if } N \text{ is even,} \quad (23) \\ 2^{-N+1}N(N - 1)^{(N-1)}/2 & \text{if } N - 4t + 1. \quad (24) \end{cases}$$

The matrices E_{N+1} of Eq. (17) produce D-optimal (0,1) matrices $\bar{E}_N$ (say) for $N = 4t + 1$, with

$$|\det \bar{E}_N| = 2^{-N+1}N(N - 1)^{(N-1)/2}. \tag{25}$$

(There seems to be no simple formula for ϵ or $\lambda_{\min}$.) For example,

$$\bar{E}_5 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

B. Masks from Hadamard Matrices

Similarly a Hadamard matrix H_{N+1} produces the (0,1) matrix S_N, where $N = 4t - 1$, as in the paragraph following Eq. (8). This weighing design has the parameters

$$W = S_N: \epsilon = \epsilon_i = \frac{4N\sigma^2}{(N + 1)^2},$$
$$|\det S_N| = 2^{-N}(N + 1)^{(N+1)/2}, \lambda_{\min} = \frac{1}{4}(N + 1), \tag{26}$$

and is D-optimal (and presumably also A- and E-optimal[50]).

C. Masks from Symmetric Block Designs[51–54]

Now we may use the incidence matrix A_N itself as the mask. This has the parameters

$$W = A_N: \epsilon = \epsilon_i = \frac{k(N - 2) + 1}{k^2(N - k)}\sigma^2;$$
$$|\det A_N| = k(k - \lambda)^{(N-1)/2}; \lambda_{\min} = k - \lambda. \tag{27}$$

Example[55]: For $N = 13$ there are three symmetric block designs, all with circulant incidence matrices. The parameters of the corresponding masks are as follows:

	(N, k, λ)	1st row of A_N	ϵ/σ^2	ϵ/σ^2	$\lambda_{\min}$
(a)	(13,4,1)	1101000001000	0.314	2916	3
(b)	(13,9,6)	1111101110010	0.309	6561	3
(c)	(13,12,11)	1111111111110	0.924	12	1.

Note that (a) and (b) are complementary symmetric block designs. Thus in the (0,1) case complementing can change the parameters. Design (b) is clearly the best of the three, although it has a smaller determinant than the D-optimal design $\bar{E}_{13}$, which has determinant 9477.

D. Small Masks

Finally, we give the two D-optimal (0,1) matrices for $N = 4$ (Mood[16]):

$$\text{(a)} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Matrix (b) was used in Eq. (2). The corresponding parameters are

	$(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)/\sigma^2$	ϵ/σ^2	det	$\lambda_{\min}$
(a)	$\left(\frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right)$	0.778	3	3,
(b)	$\left(\frac{4}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right)$	0.694	3	1.56.

E. Summary

Table I gives a summary of these results, giving the name of the mask W and the equation where its parameters can be found. These parameters are the average mean square error ϵ (the most important), the determinant of the mask, and the smallest eigenvalue $\lambda_{\min}$ of W^TW. The standards by which to judge these parameters are those of the Hadamard mask [Eq. (4)]. For masks of -1's, 0's, and $+1$'s, Hadamard masks H_N are the best, while the binary

Table I. Summary of Masks, Giving Name, Size ($N \times N$), Equation where Parameters can be Found, and Values of N for which Defined

	$N = 4t$	$N = 4t + 1$	$N = 4t + 2$	$N = 4t + 3$
Masks of −1's,0's,+1's	H_N,Eq(4): all N(?)	D_N,Eq(9): all N $B_N + I_N$: many N P_N,Eq(12): N = 5,13,25	C_N,Eq(6): many N E_N,Eq(18): many N	D_N,Eq(9): all N G_N,Eq(10): all N(?) Q_N,Eq(11): N = 7,31
Masks of 0's,1's	A_N,Eq(27): all N	$\bar{E}_N$,Eq(25): many N	A_N,Eq(27): all N	S_N,Eq(26): all N(?)

Hadamard masks S_N are the best masks of 0's and 1's.

V. Discussion

In Sec. II we briefly mentioned singular designs. To date users of Hadamard optics have not made use of instruments operating in this mode, but a whole new area of optical development may open up in this direction.

Suppose that the image of a scene is to be obtained in the presence of natural or deliberate interference that comes in the form of noise spikes that make individual intensity measurements completely useless. These measurements can be clearly identified because they are much larger (noisier) than expected and can therefore be eliminated; but the information they were meant to gather is lost.

If we operated in a normal scanning mode in which we studied the scene element by element, we would irretrievably lose all information about some elements. But the use of Hadamard optics can permit error corrections so that information about each pictorial element is obtained despite the noise.

We can, for example, make use of an $N \times p$ encoding mask with N rows to encode a scene with $p = N - n$ elements, where n' is the number of measurements expected to be compromised through noise. The n noisy measurements can then be removed, and the brightness distribution of the p picture elements calculated on the basis of the p relatively noisefree data points.

This type of procedure of course is reminiscent of, and makes available to optics, much of the body of knowledge that constitutes coding theory—the study of the transmission of messages through a noisy medium. This is effectively done by adding redundancy to the message in such a way as to permit error correction. In the same way, the redundancy built into the Hadamard mask in the example given above would be used for error correction. This entire topic is too large to treat here in detail, but it provides a natural future for Hadamard transform optics.

Similar considerations can also enter the design of an optical system in which the number of unknowns to be determined p does not correspond to the rank of any Hadamard matrix. In that case the optimum procedure may be the construction of a system that could in principle solve for N unknowns, $N > p$, where N is the rank of a known Hadamard matrix.

Thanks are due to L. A. Shepp for suggesting the use of the arithmetic-mean geometric-mean inequality in the Appendix and to C. L. Mallows for helpful discussions. The work of the second author has been supported by NASA grant NGR 33-010-210 and AFCRL contract F19628-74-C-0110.

Appendix

Theorem: For any (0,1) matrix W_N, the average mean square error $\epsilon \to 4\sigma^2/N$ as $N \to \infty$.

Proof: Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $(W_N{}^T W_N)^{-1}$. Then

$$\begin{aligned}
\epsilon &= \frac{\sigma^2}{N}\, Tr(W_N{}^T W_N)^{-1} = \frac{\sigma^2}{N} \sum_{i=1}^{N} \lambda_i \\
&\geq \sigma^2 (\lambda_1 \ldots \lambda_N)^{1/N} \text{ by the arithmetic-mean geometric-mean inequality,} \\
&= \sigma^2 (\det W_N{}^T W_N)^{-1/N} = \sigma^2 (\det W_N)^{-2/N} \\
&\geq \sigma^2 4 (N+1)^{-(N+1)/N} \text{ by Eqs. (22), (23), (24)} \\
&\to 4\sigma^2/N \text{ as } N \to \infty. \qquad \text{Q.E.D.}
\end{aligned}$$

All previous proofs of this result seem to assume either that Eq. (13) holds, or else[5,6] that there is one object that is present in every weighing, i.e., that the spring balance is biased.

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