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Mass and Spin of Exact Solutions of the Poincaré Gauge Theory

Peter BAECKLER, Ralf HECHT, Friedrich W. HEHL
and Takeshi SHIRAFUJI*

Institute for Theoretical Physics, University of Cologne
D-5000 Köln 41
**Physics Department, Saitama University, Urawa 338*

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We calculate the mass and spin of exact solutions of the Poincaré gauge theory, which asymptotically go over to a de Sitter space of constant curvature. Using certain energy-momentum and spin complexes in suitable frames, we find that the total mass of a spherically symmetric solution is just equal to the mass parameter of the solution, whereas the total spin vanishes.

Recently, in the framework of the Poincaré gauge theory (PGT),[1)] the question of the total energy and spin of an isolated system has been discussed in some detail, assuming that the spacetime around the system is asymptotically flat.[2)] In the present paper we would like to relax the assumption of asymptotic flatness, and asymptotically only require a spacetime of constant curvature, because exact solutions of the PGT typically approach a de Sitter space for increasing radial coordinate r.[3)]

The underlying spacetime of the PGT is a Riemann-Cartan spacetime with torsion and curvatures:

$$F_{ij}{}^{\alpha}:=2(\partial_{[i}e_{j]}{}^{\alpha}+\Gamma_{[i|\beta}{}^{\alpha}e_{|j]}{}^{\beta})\,, \tag{1}$$

$$F_{ij\alpha}{}^{\beta}:=2(\partial_{[i}\Gamma_{j]\alpha}{}^{\beta}+\Gamma_{[i|\varepsilon}{}^{\beta}\Gamma_{|j]\alpha}{}^{\varepsilon})\,, \tag{2}$$

where $e_i{}^{\alpha}$ describe the orthonormal tetrads and $\Gamma_i^{\alpha\beta}=-\Gamma_i^{\beta\alpha}$ the Lorentz connection. Throughout this paper we use the same notations as in Ref. 4): For example, i, j =world indices, and α, β=Lorentz indices. We write the total Lagrangian as

$$\mathcal{L}_{\text{tot}}=\mathcal{V}(e_i{}^{\alpha}, F_{ij}{}^{\alpha}, F_{ij\alpha}{}^{\beta})+\mathcal{L}(\Psi, D_{\alpha}\Psi)\,, \tag{3}$$

where the first term is the gauge Lagrangian and the second is the Lagrangian for matter fields collectively denoted by Ψ. Here D_{α} represents the covariant derivative.

We introduce the field momenta by

$$\mathcal{H}_{\alpha}{}^{ij}=2\partial\mathcal{V}/\partial F_{ji}{}^{\alpha}\,, \qquad \mathcal{H}_{\alpha\beta}{}^{ij}=2\partial\mathcal{V}/\partial F_{ji}{}^{\alpha\beta}\,, \tag{4}$$

then the field equations are written as[4)]

$$D_j\mathcal{H}_{\alpha}{}^{ij}-\mathcal{E}_{\alpha}{}^{i}=e\Sigma_{\alpha}{}^{i}\,, \tag{5}$$

$$D_j\mathcal{H}_{\alpha\beta}{}^{ij}-\mathcal{E}_{\alpha\beta}{}^{i}=e\tau_{\alpha\beta}{}^{i}\,, \tag{6}$$

where $e:=\det e_i{}^{\alpha}$ and

$$\mathcal{E}_{\alpha}{}^{i}:=e^{i}{}_{\alpha}\mathcal{V}-F_{\alpha j}{}^{\gamma}\mathcal{H}_{\gamma}{}^{ji}-F_{\alpha j}{}^{\gamma\delta}\mathcal{H}_{\gamma\delta}{}^{ji}\,, \tag{7}$$

$$\mathcal{E}_{\alpha\beta}{}^{i} := \mathcal{H}_{[\beta\alpha]}{}^{i} . \tag{8}$$

Here $e\Sigma_\alpha{}^i := \delta\mathcal{L}/\delta e_i{}^\alpha$ and $e\tau_{\alpha\beta}{}^i := \delta\mathcal{L}/\delta\Gamma_i{}^{\alpha\beta}$ are the material energy-momentum and spin tensors, respectively. Since the field momenta of (4) are antisymmetric in i and j, one can define, provided $\mathcal{H}_\alpha{}^{ij} \neq 0$ (this excludes the Einstein-Cartan theory from our considerations) and $\mathcal{H}_{\alpha\beta}{}^{ij} \neq 0$, the energy complex $\check{\epsilon}_\alpha{}^i$ and the spin complex $\check{\epsilon}_{\alpha\beta}{}^i$ by

$$\check{\epsilon}_\alpha{}^i := \partial_j \mathcal{H}_\alpha{}^{ij} , \qquad \check{\epsilon}_{\alpha\beta}{}^i := \partial_j \mathcal{H}_{\alpha\beta}{}^{ij} , \tag{9}$$

which ensure the ordinary conservation laws,

$$\partial_i \check{\epsilon}_\alpha{}^i = 0 , \qquad \partial_i \check{\epsilon}_{\alpha\beta}{}^i = 0 . \tag{10}$$

Accordingly, the field momenta turn out to be the superpotentials of the PGT. If the field equations (5) and (6) are fulfilled, we find[5)]

$$\check{\epsilon}_\alpha{}^i = \mathcal{E}_\alpha{}^i + e\Sigma_\alpha{}^i + \Gamma_{j\alpha}{}^\gamma \mathcal{H}_\gamma{}^{ij} , \tag{11}$$

$$\check{\epsilon}_{\alpha\beta}{}^i = \mathcal{E}_{\alpha\beta}{}^i + e\tau_{\alpha\beta}{}^i + 2\Gamma_{j[\alpha}{}^\gamma \mathcal{H}_{|\gamma|\beta]}{}^{ij} . \tag{12}$$

The total energy-momentum and spin are defined by

$$P_\alpha := \int_\sigma \check{\epsilon}_\alpha{}^i d\sigma_i , \tag{13}$$

$$S_{\alpha\beta} := \int_\sigma \check{\epsilon}_{\alpha\beta}{}^i d\sigma_i \tag{14}$$

with σ being a spacelike surface. If we can drop the integral over the timelike hypersurface at infinity, the conservation laws of (10) ensure that P_α and $S_{\alpha\beta}$ are conserved. In particular, if we take x^0=const, we can rewrite (13) and (14) as

$$P_\alpha = \oint \mathcal{H}_\alpha{}^{0a} dS_a , \tag{15}$$

$$S_{\alpha\beta} = \oint \mathcal{H}_{\alpha\beta}{}^{0a} dS_a , \tag{16}$$

where $a=1, 2, 3$ and the integrals are taken over the 2-surface of a large 3-dimensional spherical volume of radius R. Moreover, we know from general relativity that the integrals have to be evaluated with respect to asymptotic Cartesian tetrads. Here, in our case of an asymptotic de Sitter spacetime, we have to require *conformally Cartesian tetrads*.

The energy complex of Ref. 2), $\tilde{T}_k{}^i = \partial_j \tilde{\Phi}_k{}^{ij}$ with $\partial_i \tilde{T}_k{}^i = 0$, carries two holonomic indices. Its superpotential $\tilde{\Phi}_k{}^{ij}$ is related to our $\mathcal{H}_\alpha{}^{ij}$ by[2)]

$$\tilde{\Phi}_k{}^{ij} = e_k{}^\alpha \mathcal{H}_\alpha{}^{ij} + \Gamma_k{}^{\alpha\beta} \mathcal{H}_{\alpha\beta}{}^{ij} . \tag{17}$$

As for the spin complexes, they coincide. In the context of Ref. 2), the total energy-momentum reads $\tilde{P}_k := \int \tilde{T}_k{}^i d\sigma_i = \oint \tilde{\Phi}_k{}^{0a} dS_a$. The total energy-momentum P_α of (13) is different from $\tilde{P}_k$. We expect, however, that both definitions —— perhaps up to a contribution from the constant de Sitter background —— lead to the same result, provided we use the conformally Cartesian tetrads and coordinates, respectively. In this paper, for computational ease, we will concentrate on the complexes as given in (9).

All the considerations up to now do not depend on a specific form of the gauge Lagrangian. Now we will choose a Lagrangian and study the energy and spin of an exact solution for a specific PGT. For simplicity, we take the purely quadratic Lagrangian,[4)]

$$\mathcal{V}/e=(1/4l^2)(F^{ij}{}_{\alpha}F_{ji}{}^{\alpha}+2F^{i\beta}{}_{\beta}F_{i\gamma}{}^{\gamma})+(1/4\varkappa)F^{ij}{}_{\alpha\beta}F_{ji}{}^{\alpha\beta}\,, \tag{18}$$

which can be considered to represent the generic features of the PGT. Here we have l=Planck length, $\varkappa$=dimensionless coupling constant for the Lorentz gauge bosons. We use units with $c=\hbar=1$ and the Minkowski metric $g_{\alpha\beta}=\mathrm{diag}(-+++)$. As for the material part, we assume only a Maxwell field with the energy-momentum tensor

$$\Sigma_{\alpha}{}^{i}=F_{\alpha j}F^{ij}-(1/4)e^{i}{}_{\alpha}F_{jk}F^{jk}\,, \tag{19}$$

and vanishing spin tensor. Here F_{ij} represents the electromagnetic field strength, satisfying the Maxwell equations,

$$\partial_j(eF^{ij})=0\,, \qquad \partial_{[i}F_{jk]}=0\,. \tag{20}$$

Equations (5) and (6), together with (20), form a system of coupled field equations, the simplest solution of which with dynamic torsion is the spherically symmetric Baeckler solution,[6)] or, if electrically charged, the Baeckler-Lee solution.[7)] The solution is appreciably simplified,[5)] if we use a suitably rotated Schwarzschild (rSS) basis, which reads in Schwarzschild coordinates (t, r, θ, φ) as follows:

$$\begin{aligned}
&e^{\hat{t}}=(1/2)(2\psi(r)+1)dt+(1/2)(1-1/2\psi(r))dr\,,\\
&e^{\hat{r}}=(1/2)(2\psi(r)-1)dt+(1/2)(1+1/2\psi(r))dr\,,\\
&e^{\hat{\theta}}=rd\theta\,,\\
&e^{\hat{\varphi}}=r\sin\theta d\varphi
\end{aligned} \tag{21}$$

with

$$2\psi(r)=1-2(Mr-Q^2)/r^2+(\varkappa/4l^2)r^2\,. \tag{22}$$

Here $m=8\pi M/l^2$ and Q are the gravitational mass and charge, respectively. The tetrad (21) belongs to a Schwarzschild-de Sitter (or Kottler) line element,

$$ds^2=-2\psi(r)dt^2+(1/2\psi(r))dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{23}$$

The torsion of the Baeckler-Lee solution, referred to the tetrad (21), is given in a matrix form by

$$F_A{}^{\gamma}=\frac{M}{r^2}\begin{pmatrix} -1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix} -\frac{Q^2}{r^3}\begin{pmatrix} -2 & -2 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix}, \tag{24}$$

where $\gamma=\hat{t}, \hat{r}, \hat{\theta}, \hat{\varphi}$, and $A, C=\hat{t}\hat{r}, \hat{t}\hat{\theta}, \hat{t}\hat{\varphi}, \hat{\theta}\hat{\varphi}, \hat{\varphi}\hat{r}, \hat{r}\hat{\theta}$. Observe that the mass gives rise to a Coulomb-like behavior of the torsion. The only nonvanishing component of the electromagnetic field strength turns out to be $F_{\hat{t}\hat{r}}=E_{\hat{r}}=2Q/r^2$. The Riemann-Cartan curvature is

$$F_A{}^C=\frac{\varkappa}{4l^2}\left(\mathbf{1}_6+\frac{Mr-Q^2}{r^2}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & -1 \end{pmatrix}\right). \tag{25}$$

The two field momenta are given by

$$\mathcal{H}_{\gamma}{}^{A}=\frac{e}{l^2}\left(\frac{2M}{r^2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} -\frac{Q^2}{r^3}\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & -1 & \cdot \end{pmatrix}\right) \tag{26}$$

and

$$\mathcal{H}_A{}^C=-(e/\varkappa)F_A{}^C\,. \tag{27}$$

The correctness of this solution has been checked with the help of the EXCALC-package of the computer algebra system REDUCE.[8)]

The "naive" Schwarzschild (SS) tetrad,

$$\boldsymbol{\omega}^{\hat{t}}=\sqrt{2\psi}\,dt\,, \qquad \boldsymbol{\omega}^{\hat{r}}=dr/\sqrt{2\psi}\,, \qquad \boldsymbol{\omega}^{\hat{\theta}}=r d\theta\,, \qquad \boldsymbol{\omega}^{\hat{\varphi}}=r\sin\theta d\varphi \tag{28}$$

can be rotated into the rSS-tetrad (21) by employing a suitable boost $\boldsymbol{e}^{\alpha}=\Lambda^{\alpha}{}_{\beta}(x)\boldsymbol{\omega}^{\beta}$. The rSS-tetrad has a distinctive property: The connection $\overset{\mathrm{rSS}}{\Gamma}_{\gamma\alpha\beta}$ of the solution referred to $\boldsymbol{e}^{\alpha}$ and expressed in terms of polar coordinates, is conformally flat, that is, it takes its flat values for vanishing "cosmological" constant $\varkappa/4l^2$.

Isotropic coordinates[9)] $(t, \rho, \theta, \varphi)$ will allow us to find, by an analogous procedure as above, an anholonomic connection which is *conformally Cartesian*, that is, it vanishes up to a term proportional to $\varkappa/4l^2$. The metric (23) in isotropic coordinates reads

$$ds^2=-e^{2\lambda(\rho)}dt^2+e^{2\mu(\rho)}(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\varphi^2)\,. \tag{29}$$

The attached "naive" isotropic (IS) tetrad can be determined from the SS-tetrad by applying the spatial rotation

$$\Pi^{\alpha}{}_{\beta}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ 0 & \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ 0 & \cos\theta & -\sin\theta & 0 \end{bmatrix}, \tag{30}$$

i.e., $\overset{\mathrm{IS}}{e}{}^{\alpha}=\Pi^{\alpha}{}_{\beta}\boldsymbol{\omega}^{\beta}$. Again, we have to boost this tetrad with $\Lambda^{\alpha}{}_{\beta}(x)$ in order to find the rotated isotropic (rIS) tetrad. Starting from the rSS-tetrad, however, we have

$$\overset{\mathrm{rIS}}{e}{}^{\alpha}=\Pi^{\alpha}{}_{\beta}(x)e^{\beta}\,. \tag{31}$$

The corresponding anholonomic connection $\overset{\mathrm{rIS}}{\Gamma}_{\gamma\alpha\beta}$ will be proportional to $\varkappa/4l^2$, and it is conformally Cartesian. The desired field momenta, starting from (26) and (27), turn out to be

$$\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\sigma}{}^{ij}=\Pi_{\sigma}{}^{\gamma}\mathcal{H}_{\gamma}{}^{ij}=\Pi_{\sigma}{}^{\gamma}e^{i}{}_{\alpha}e^{j}{}_{\beta}\mathcal{H}_{\gamma}{}^{\alpha\beta} \tag{32}$$

and

$$\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\sigma\tau}{}^{ij}=\Pi_{\sigma}{}^{\alpha}\Pi_{\tau}{}^{\beta}\mathcal{H}_{\alpha\beta}{}^{ij}=\Pi_{\sigma}{}^{\alpha}\Pi_{\tau}{}^{\beta}e^{i}{}_{\gamma}e^{j}{}_{\delta}\mathcal{H}_{\alpha\beta}{}^{\gamma\delta}\,. \tag{33}$$

The calculation of non-vanishing components of the field momenta, which enter the integrals (15) and (16), yields

$$\begin{aligned}
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{0}}{}^{01}&=-(2/l^2)\sin\theta\cdot(M-Q^2/r)\,,\\
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{1}}{}^{01}&=(2/l^2)\sin^2\theta\cos\varphi\cdot(M-Q^2/r)\,,\\
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{2}}{}^{01}&=(2/l^2)\sin^2\theta\sin\varphi\cdot(M-Q^2/r)\,,\\
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{3}}{}^{01}&=(2/l^2)\sin\theta\cos\varphi\cdot(M-Q^2/r)
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{0}\hat{1}}{}^{01}&=(1/2l^2)r^2\sin^2\theta\cos\varphi\,,\\
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{0}\hat{2}}{}^{01}&=(1/2l^2)r^2\sin^2\theta\sin\varphi\,,\\
\overset{\mathrm{rIS}}{\mathcal{H}}{}_{\hat{0}\hat{3}}{}^{01}&=(1/2l^2)r^2\sin\theta\cos\theta\,.
\end{aligned} \tag{35}$$

Observe that these quantities are expressed in terms of Schwarzschild coordinates. Straightforwardly we find

$$\begin{aligned}
P_{\hat{0}}&=-(2/l^2)\lim_{r\to\infty}\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\varphi(M-Q^2/r)\\
&=-(8\pi/l^2)M=-m\,,
\end{aligned} \tag{36}$$

$$P_1=P_2=P_3=0\,,$$

and all

$$S_{\alpha\beta}=0\,. \tag{37}$$

This is not an unexpected result. Note that in the limit of vanishing curvature, i.e., of $\varkappa\to 0$, our solution degenerates into the Reissner-Nordström solution of a gravitational theory with absolute parallelism and a corresponding torsion square Lagrangian.[10] Our result that the total energy coincides with the gravitational mass m is untouched by this limiting transition.

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