

“MASS CONCENTRATION” PHENOMENON FOR THE  
 NONLINEAR SCHRÖDINGER EQUATION WITH  
 THE CRITICAL POWER NONLINEARITY

II

BY HAYATO NAWA

§0. Introduction and main results.

This paper is a sequel to the previous one [20]. We continue the study of the blow-up problem for the nonlinear Schrödinger equation:

$$(Cp) \quad \begin{cases} \text{(NLS)} & 2i \frac{\partial u}{\partial t} + \Delta u + F(u) = 0, \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^N, \\ \text{(IV)} & u(0, x) = u_0(x), \quad x \in \mathbf{R}^N, \end{cases}$$

where  $i = \sqrt{-1}$ ,  $u_0 \in H^1 = H^1(\mathbf{R}^N)$ ,  $\Delta$  is the Laplace operator on  $\mathbf{R}^N$  and  $F$  is a complex valued function satisfying, at least, the following assumptions:

$$(F.1) \quad F(0) = 0,$$

$$(F.2) \quad F \in C(C; C),$$

$$(F.3) \quad |F(z) - F(w)| \leq M(1 + |z|^{4/N} + |w|^{4/N})|z - w|, \quad z, w \in C,$$

for some positive constant  $M$ .

Typical examples of  $F$  are

$$(NF) \quad F(u) = |u|^{p-1}u + \chi |u|^{q-1}u, \quad \chi \in \mathbf{R}, \quad 1 \leq q < p \leq 1 + 4/N.$$

Here, we list several basic notations which will be used throughout this paper.

$\mu$ : Lebesgue measure on  $\mathbf{R}^N$ ,

$$B(y; R) = \{x \in \mathbf{R}^N; |x - y| \leq R\},$$

$[f > \varepsilon] = \{x \in \mathbf{R}^N; f(x) > \varepsilon\}$  or the characteristic function of this set,

$$\partial_t = \partial/\partial t, \quad \nabla = (\partial_1, \partial_2, \dots, \partial_N), \quad \partial_j = \partial/\partial x_j,$$

$L^\alpha = L^\alpha(\mathbf{R}^N)$  denotes the space of  $\alpha$ -summable functions on  $\mathbf{R}^N$  with the norm  $\|\cdot\|_\alpha$ ,

$$\|\cdot\| = \|\cdot\|_2,$$

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$W^{s,\alpha} = W^{s,\alpha}(\mathbf{R}^N)$  represents the standard Sobolev space of order  $s$  and exponent  $\alpha$  on  $\mathbf{R}^N$ ,

$$H^s = W^{s,2},$$

$$\Sigma = \{v \in H^1; \|v\|^2 + \|\nabla v\|^2 + \|xv\|^2 < +\infty\},$$

$$\langle \cdot, \cdot \rangle = L^2\text{-inner product},$$

$\mathcal{S} = \mathcal{S}(\mathbf{R}^N)$ : the Schwartz space of rapidly decreasing  $C^\infty$ -functions,

$\mathcal{S}' = \mathcal{S}'(\mathbf{R}^N)$ : the dual of  $\mathcal{S}$ ,

$$\sigma = 2 + 4/N,$$

$$2^* = 2N/(N-2) \text{ if } N \geq 3, \quad 2^* = \infty \text{ if } N=1 \text{ and } N=2,$$

$$E(v) = \|\nabla v\|^2 - \frac{2}{\sigma} \|v\|_\sigma^\sigma.$$

We regard (NLS) as an abstract evolution equation in  $H^{-1} = H^{1'}$ , and we say that  $u$  is a solution to (Cp) on  $[0, T)$  if and only if  $u$  satisfies the integral equation

$$(0.1) \quad u(t) = U(t)u_0 + \frac{i}{2} S(t; F(u)) \quad \text{in } L^2$$

for any  $t \in [0, T)$ , where  $U(t)$ ,  $S(t; \cdot)$  are linear operators given by

$$(0.2) \quad U(t) = \exp\left(\frac{i}{2} t \Delta\right) \quad (\text{free propagator}),$$

$$(0.3) \quad S(t; v) = \int_0^t U(t-\tau)v(\tau) d\tau,$$

respectively. For the precise definitions and properties of these operators, see Kato [10] and Yajima [30]. The integral in (0.3) is understood to be the Bochner integral in  $H^{-1}$ .

In the particular case of  $F(z) = |z|^{p-1}z$  with  $1 < p < 2^* - 1$  (where  $2^* = 2N/(N-2)$  if  $N > 2$ , otherwise  $2^* = +\infty$ ), it is well known that for  $p \geq 1 + 4/N$  there are singular solutions of (Cp) for certain initial data (see Glassy [9] and M. Tsutsumi [25]). That is, there are some solutions  $u(t)$  of (Cp) such that

$$u(\cdot) \in C([0, T); H^1) \quad \text{and} \quad \lim_{t \rightarrow T} \|\nabla u(t)\| = +\infty.$$

However, the formation of singularities in blow-up solutions for the critical case  $p = 1 + 4/N$  seems to be quite different from that of blow-up solutions for the supercritical case  $1 + 4/N < p < 2^* - 1$ . In the critical case there are blow-up solutions which lose their  $L^2$ -continuity because of the so-called "mass concentration" phenomenon (Weinstein [28], Nawa and M. Tsutsumi [21] and Merle and Y. Tsutsumi [17]), while Merle [15] suggests that in the supercritical case every blow-up solution has a strong limit in  $L^2$  at the blow-up time.

In the case of  $F(u) \sim |u|^{4/N}u$  as  $|u| \rightarrow +\infty$ , Merle and Y. Tsutsumi [17] show that no blow-up solution to (Cp) has a strong limit in  $L^2$  as  $t \rightarrow T$  ( $T$  is the blow-up time), and that  $L^2$ -concentration occurs at the origin for all the

radially symmetric blow-up solutions to (Cp), when  $N \geq 2$ . In [20] we also investigate the “mass concentration” phenomenon, and proved the following theorems.

**THEOREM A.** *Assume that  $F$  satisfies (F.1)-(F.3). Then for any  $u_0 \in H^1$  there exist a positive number  $T$  (maximal existence time, i. e. blow-up time) and a unique  $H^1$ -solution  $u(t)$  to (Cp) such that*

$$u, \nabla u \in C([0, T]; L^2) \cap L^{2+4/N}_{loc}(0, T; L^{2+4/N})$$

and  $u(t)$  satisfies (0.1). Assume further that  $T < +\infty$ , so that  $\lim_{t \rightarrow T} \|\nabla u(t)\| = +\infty$  (Blow-up). Then  $u(t)$  does not have a strong limit in  $L^2$  as  $t \rightarrow T$ .

**THEOREM B.** *Let  $F$  be (NF) with  $p = \sigma - 1 = 1 + 4/N$ . Suppose that the solution  $u(t)$  to (Cp) blows up at  $t = T \in (0, \infty]$ , i. e.,  $\lim_{t \rightarrow T} \|\nabla u(t)\| = \lim_{t \rightarrow T} \|u(t)\|_\sigma = \infty$ . Set*

$$(B.1) \quad \lambda(t) = 1 / \|u(t)\|_\sigma^{p/2}.$$

$$(B.2) \quad S_\lambda u(t, x) = \lambda^{N/2} u(t, \lambda x).$$

$$(B.3) \quad A \equiv \sup_{R > 0} \liminf_{t \uparrow T} \left\{ \sup_{y \in \mathbb{R}^N} \int_{B(y; R)} |S_{\lambda(t)} u(t, x)|^2 dx \right\}.$$

(1) *If  $\|u_0\|^2 = A$ , then for any  $\varepsilon > 0$ , there are a constant  $K > 0$  and a function  $y(t) \in C([0, T]; \mathbb{R}^N)$  such that*

$$(B.4) \quad \liminf_{t \uparrow T} \int_{B(K)} |S_{\lambda(t)} u(t, x + y(t))|^2 dx > (1 - \varepsilon) A.$$

*If we impose the condition  $u_0 \in \Sigma$ ,  $T < \infty$  and  $\chi \leq 0$ , then we have*

$$(B.5) \quad \sup_{t \in (0, T)} |y(t)\lambda(t)| < +\infty,$$

$$(B.6) \quad \liminf_{t \uparrow T} \int_{B_t} |u(t, x)|^2 dx > (1 - \varepsilon) A,$$

where  $B_t = B(y(t)\lambda(t); K\lambda(t))$ .

(2) *If  $u_0$  is radially symmetric and  $N \geq 2$ , we have (B.4) and (B.6) with  $y(t) \equiv 0$  and  $A = \|Q\|^2$ , where  $Q$  is a ground state (non trivial minimal  $L^2$  norm solution of*

$$(B.7) \quad \Delta Q - Q + |Q|^{4/N} Q = 0, \quad Q \in H^1.$$

(For the equation (B.7), see e.g. [1], [3], [22] and [27].)

(3) *If  $\|u_0\| = \|Q\|$ , we have the results of (1) with  $A = \|Q\|^2$  in (B.4) and (B.6).*

Here we note that:

(1)  $\lambda(t) \rightarrow 0$  as  $t \rightarrow T$ .

(2) We do not assume the radial symmetricity of solutions to (Cp).

Recently, Y. Tsutsumi [26] has investigated the rate of  $L^2$ -concentration for radially symmetric blow-up solutions to (Cp) with  $N \geq 2$  and  $F(u) \sim |u|^{4/N}u$  as  $|u| \rightarrow +\infty$ , and showed that for any  $\varepsilon > 0$ , there exists a  $K > 0$  such that

$$(0.4) \quad \liminf_{t \rightarrow T} \|u(t); L^2(|x| < K(T-t)^{1/2})\| \geq (1-\varepsilon)\|Q\|,$$

where  $Q$  is a ground state (minimal  $L^2$  norm) solution of (B.7).

In this paper we have the following theorem, which is an improvement of Theorem B and (0.4).

**THEOREM C.** *Let  $F$  be (NF) with  $p = \sigma - 1 = 1 + 4/N$ . Suppose that the solution  $u(t)$  to (Cp) blows up at  $t = T \in (0, \infty]$ , i. e.,  $\lim_{t \rightarrow T} \|\nabla u(t)\| = \lim_{t \rightarrow T} \|u(t)\|_\sigma = +\infty$ . Let  $(t_n)_n$  be any sequence such that  $t_n \rightarrow T$  as  $n \rightarrow \infty$ . Set*

$$(C.1) \quad \lambda_n \equiv \lambda(t_n) = 1/\|u(t_n)\|_\sigma^{p/2},$$

$$(C.2) \quad S_\lambda u(t, x) = \lambda^{N/2} u(t, \lambda x).$$

*Then there exists a subsequence of  $(t_n)_n$  (we still denote it by  $(t_n)_n$ ) which satisfies the following properties: one can find a sequence  $(y_n)_n$  in  $\mathbf{R}^N$  such that, for any  $\varepsilon > 0$ , there is a positive constant  $K$ ;*

$$(C.3) \quad \liminf_{n \rightarrow \infty} \int_{B(K)} |S_{\lambda_n} u(t_n, x + y_n)|^2 dx \geq (1-\varepsilon)\|Q\|^2.$$

*If we impose the condition  $u_0 \in \Sigma$ ,  $\chi \leq 0$  and  $T < \infty$ , then we have*

$$(C.4) \quad \sup_{n \in \mathbf{N}} |y_n \lambda_n| < +\infty,$$

$$(C.5) \quad \liminf_{n \rightarrow \infty} \int_{B_n} |u(t_n, x)|^2 dx \geq (1-\varepsilon)\|Q\|^2,$$

where  $B_n = B(y_n \lambda_n; K \lambda_n)$ .

Our proof of Theorem C depends heartily on the following

**PROPOSITION D.** *Let  $F$  be (NF) with  $p = \sigma - 1 = 1 + 4/N$ . Suppose that the solution  $u(t)$  to (Cp) blows up at  $t = T \in (0, \infty]$ , i. e.,  $\lim_{t \rightarrow T} \|\nabla u(t)\| = \lim_{t \rightarrow T} \|u(t)\|_\sigma = +\infty$ . Let  $(t_n)_n$  be any sequence such that  $t_n \rightarrow T$  as  $n \rightarrow \infty$ . Set*

$$(D.1) \quad \lambda_n \equiv \lambda(t_n) = 1/\|u(t_n)\|_\sigma^{p/2},$$

$$(D.2) \quad u_n(t, x) \equiv S_{\lambda_n} u(t, x) = \lambda_n^{N/2} u(t, \lambda_n x).$$

*Then there exists a subsequence of  $(t_n)_n$  (we still denote it by  $(t_n)_n$ ) which satisfies the following properties: one can find  $L \in \mathbf{N} \cup \{\infty\}$  and sequences  $(y^j_n)_n$  in  $\mathbf{R}^N$  for  $1 \leq j \leq L$  such that*

- (D.3)  $\lim_{n \rightarrow \infty} |y_n^j - y_n^k| = \infty \quad (j \neq k),$
- (D.4)  $f_n^1 \equiv u_n(t_n, x + y_n^1) \longrightarrow f^1 \quad \text{weakly in } H^1,$
- (D.5)  $f_n^j \equiv (f_n^{j-1} - f^{j-1})(t_n, \cdot + y_n^j) \longrightarrow f^j \quad (j \geq 2) \text{ weakly in } H^1$
- (D.6)  $\lim_{n \rightarrow \infty} \{E(f_n^j) - E(f_n^j - f^j)\} = E(f^j),$
- (D.7)  $\lim_{j \rightarrow L} \lim_{n \rightarrow \infty} \|f_n^j - f^j\|_\sigma = 0 \quad (L = +\infty),$
- (D.7)'  $\lim_{n \rightarrow \infty} \|f_n^L - f^L\|_\sigma = 0 \quad (L < +\infty),$
- (D.8)  $\lim_{j \rightarrow L} \lim_{n \rightarrow \infty} \left\{ \sup_{y \in \mathbf{R}^N} \int_{B(y; R)} |(f_n^j - f^j)(x)|^2 dx \right\} = 0 \quad \text{if } L = +\infty,$
- (D.8)'  $\lim_{n \rightarrow \infty} \left\{ \sup_{y \in \mathbf{R}^N} \int_{B(y; R)} |(f_n^L - f^L)(x)|^2 dx \right\} = 0 \quad \text{if } L < +\infty,$

where  $R$  is any positive constant.

In view of this proposition, we can understand the assertion of Theorem B (2) and (3).

**COROLLARY E.** (1) *If  $u_0$  is radially symmetric and  $N \geq 2$ , we have  $L=1$  with  $y_n^1 \equiv 0$  in Proposition D, so that we have Theorem B (2).*

(2) *If  $\|u_0\| = \|Q\|$ , we have  $L=1$  in Proposition D, so that we have Theorem B (3).*

*Remarks.* 1. Roughly speaking, Proposition D suggests that  $u$  behaves like;

$$(0.5) \quad |u(t, x)|^2 \longrightarrow \sum_{j=1}^L \|f^j\|^2 \delta(x - a_j) + r(x) \quad \text{in } \mathcal{S}'$$

as  $t \rightarrow T$ , where  $a_j = \lim_{n \rightarrow \infty} \lambda_n y_n^j$ ,  $\delta(x - a)$  is a Dirac mass at  $a \in \mathbf{R}^N$  and  $r(x)$  is a remainder term (it may be a function). We note that  $L$  may be infinite. It could happen that  $a_j = a_k$  ( $j \neq k$ ). However we see from this theorem that "mass concentration" occurs at some points.

In [21], the author and M. Tsutsumi characterized the initial datum (in  $\Sigma$ ) leading to the solution which develops the singularity like Dirac mass  $\delta$ :

$$(0.6) \quad \lim_{t \rightarrow T} \|(x - a)u(t)\| = 0 \quad \text{for some } a \in \mathbf{R}^N,$$

so that

$$(0.7) \quad |u(t, x)|^2 \longrightarrow \|u_0\|^2 \delta(x - a) \quad \text{in } \mathcal{S}'$$

as  $t \rightarrow T$  ([21; Theorem 1]). Thus we know there are many blow-up solutions satisfying

$$(0.8) \quad \lim_{t \rightarrow T} \|(x - a)u(t)\| > 0$$

for each  $a \in \mathbf{R}^N$  ([21; Corollary 1.1]). So  $L \geq 2$  or  $r \neq 0$  occurs in (0.5), which also explains how the blow-up solution  $u$  performs (0.8).

Recently Merle [16] has constructed blow-up solutions which concentrates their “ $L^2$  mass” at exactly  $m$  points  $\{a_1, a_2, \dots, a_m\}$  such that

$$(0.9) \quad \lim_{t \rightarrow T} \|u(t, x); L^2_{loc}(\mathbf{R}^N \setminus \{a_1, a_2, \dots, a_m\})\| = 0.$$

This example corresponds to (0.5) with  $L=m$  and  $r=0$ .

On the other hand, some numerical computations suggest that there are blow-up solutions which behave like (0.5) with  $L=1$  and  $r$  being some function (see e.g. [24] and [27]).

2. The spatial dilation operator  $S_\lambda$  was introduced by Weinstein for the first time in [28]. We note that our scaling function  $\lambda$  is different from the one in [28]. Our choice of scaling function  $\lambda$  simplifies our calculations in §3, and we can treat more general nonlinearities than those in [28].

3. The proof of Theorem C is inspired by the method of concentration-compactness due to Lions [12, 13] and the argument performed in Weinstein [28]. We, however, repeatedly use the same compactness device as in Lieb [11] and Brézis and Lieb [3] to decompose  $(u_n)_n$  iteratively into several parts (possibly infinite parts) with the help of Brezis and Lieb’s lemma [4]. It is worth while to note that the case  $L=1$  in Theorem C (1) does not always correspond to the terminology “tightness” in the method of concentration-compactness for the concentration function of  $|u_n|^2$ : If  $L=1$ , it could happen that

$$(1.0) \quad \lim_{n \rightarrow \infty} \|u_n(t_n, \cdot + y_n^1) - f^1\| \neq 0,$$

although it holds that

$$(1.1) \quad \lim_{n \rightarrow \infty} \|u_n(t_n, \cdot + y_n^1) - f^1\|_\sigma = 0.$$

Thus  $L=1$  is not equivalent to (0.7).

4. Weinstein [32] proved the similar result to Theorem C. However he treated only the single power case and in his paper there is only a proof for the radially symmetric case.

### §1. Preliminaries.

In this section we collect several well-known facts about solutions to (Cp) and those to (0.5), and recall the weak compactness result due to Lieb [11], a related lemma from [6] (see also [3]) and Bézis and Lieb’s lemma [4], which will be crucial for the proof of Theorem C.

We use the notation  $\sigma = 2 + 4/N$ .

LEMMA 1.1. *Let*

$$(1.1) \quad I \equiv \inf \{ \|\nabla v\|^2 \|v\|^{4/N} / \|v\|^\sigma; v \in H^1 \text{ and } v \neq 0 \}.$$

The infimum  $I$  is attained at a function  $Q$  with the following properties:

- (1)  $Q$  is positive and radially symmetric.
- (2)  $Q \in H^1(\mathbf{R}^N) \cap C^2(\mathbf{R}^N)$  and satisfies

$$(1.2) \quad E(Q) \equiv \|\nabla Q\|^2 - (2/\sigma)\|Q\|_\sigma^\sigma = 0.$$

(3)  $Q$  is a solution to (0.5) of minimal  $L^2$  norm (the ground state). In addition,

$$(1.3) \quad I = 2\|Q\|^{\sigma-2}/\sigma.$$

(4)  $Q$  is a solution to the following variational problem.

$$(1.4) \quad \text{minimize } \|v\|; E(v) \leq 0 \text{ and } v \in H^1/\{0\}.$$

(5) Let  $S'$  be the best constant for the interpolation estimate:

$$(1.5) \quad \|v\|_\sigma^\sigma \leq S' \|\nabla v\|^2 \|v\|^{\sigma-2}, \quad N \geq 1.$$

Then  $S' = 1/I$ .

For Lemma 1.1, see Weinstein [28], Berestycki and Lions [1] and Strauss [22] (see also [19] for part (4)).

LEMMA 1.2. (1) Assume that  $F$  satisfies (F.1)–(F.3). For any  $u_0 \in H^1$ , there exist a positive number  $T$  and a unique solution

$$(1.6) \quad u(\cdot) \in C([0, T]; H^1) \cap L_{\text{loc}}^\sigma(0, T; L^\sigma)$$

to (Cp) satisfying (0.1) with the alternatives; either  $T = +\infty$  or  $T < +\infty$  and

$$(1.7) \quad \lim_{t \rightarrow T} \|\nabla u(t)\| = \lim_{t \rightarrow T} \|u(t)\|_\sigma = +\infty.$$

(2) In addition to (F.1)–(F.3), assume that  $F$  satisfies

$$(F.4) \quad \text{Im } F(z)\bar{z} = 0, \quad z \in \mathbf{C},$$

$$(F.5) \quad \text{there exists } G \in C^2(\mathbf{C}; \mathbf{R}) \text{ such that } F = \frac{\partial G}{\partial \bar{z}}.$$

Then the above solution  $u$  satisfies:

$$(1.8) \quad \|u(t)\| = \|u_0\|,$$

$$(1.9) \quad \begin{aligned} H(u(t)) &\equiv \|u(t)\|^2 - \langle G(u(t)), 1 \rangle \\ &= H(u_0), \end{aligned}$$

for  $t \in [0, T)$ . If  $u_0 \in \Sigma$ , then  $u \in C([0, T); \Sigma)$  and satisfies

$$\begin{aligned}
 (1.10) \quad \|xu(t)\|^2 &= \|xu_0\|^2 + 2t \operatorname{Im} \langle x \cdot \nabla u_0, u_0 \rangle + t^2 H(u_0) \\
 &\quad - N \int_0^t (t-\tau) \langle F(u), u \rangle(\tau) d\tau \\
 &\quad + (N+2) \int_0^t (t-\tau) \langle G(u), 1 \rangle(\tau) d\tau.
 \end{aligned}$$

(3) If  $u_0 \in H^1$  and  $\|u_0\| < \|Q\|$ , then the solution  $u$  exists globally in time.

We can find the proof of part (1) and (2) in Kato [10] and Ginibre and Velo [7]. See also the proof of Lemma 2.4 in the previous paper [20]. For the identity (1.10), see e.g., Glassy [9] and M. Tsutsumi [25]. One can find the proof of part (3) in Weinstein [27].

LEMMA 1.3 (Frölich, Lieb and Loss [6]). Let  $1 < \alpha < \beta < \gamma$  and let  $g$  be a measurable function on  $\mathbf{R}^N$  such that, for some positive constants  $C_\alpha, C_\beta, C_\gamma$ ,

- (i)  $\|g\|_\alpha \leq C_\alpha$ ,
- (ii)  $\|g\|_\beta \geq C_\beta > 0$ ,
- (iii)  $\|g\|_\gamma \leq C_\gamma$ .

Then  $\mu(\{|g| > \eta\}) > C$  for some  $\eta, C > 0$  depending on  $\alpha, \beta, \gamma, C_\alpha, C_\beta, C_\gamma$ , but not on  $g$ .

LEMMA 1.4 (Lieb [11]). (1) Let  $1 \leq \alpha < \infty$  and let  $v$  be a function such that  $v \in L^1_{\text{loc}}, \nabla v \in L^\alpha, \|\nabla v\|_\alpha \leq A$  and  $\mu(\{|v| > \eta\}) \geq C$  for some positive constants  $A, \eta, C$ . Then, there exists a shift  $T_y v(x) = v(x+y)$  such that, for some constant  $\delta = \delta(A, C, \eta)$ ,  $\mu(B \cap \{|T_y v| > \eta/2\}) > \delta$ , where  $B = B(1)$ .

(2) Let  $1 < \alpha < \infty$  and let  $(f_n)_n$  be a uniformly bounded sequence of functions in  $W^{1,\alpha}(\mathbf{R}^N)$  with the property that  $\mu(\{|f_n| > \eta\}) \geq C$  for some  $\eta, C > 0$ . Then there exists a sequence  $(y_n)_n$  in  $\mathbf{R}^N$ ,  $\phi_n(x) \equiv f_n(x+y_n)$ , such that, for some subsequence  $\{n_j\}$ ,  $\phi_{n_j} \rightarrow \phi$  weakly in  $W^{1,\alpha}(\mathbf{R}^N)$  and  $\phi \neq 0$ .

We note that part (2) is a direct consequence of the Banach-Alaoglu theorem and part (1).

LEMMA 1.5 (Brézis and Lieb [4]). Let  $0 < \alpha < \infty$  and let  $(f_n)_n$  be a uniformly bounded sequence in  $L^\alpha$ . Suppose that  $f_n \rightarrow f$  a.e. in  $\mathbf{R}^N$ . (By Fatou's Lemma  $f \in L^\alpha$ .) Then,

$$(1.11) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} (|f_n(x)|^\alpha - |f_n(x) - f(x)|^\alpha - |f(x)|^\alpha) dx = 0.$$

Our proof of Theorem C depends heartily on the above two lemmas, which enable us to derive Proposition D.



§ 2. Proofs of main results.

The main purpose of this section is to prove Theorem C. For simplicity, we suppose  $N \geq 3$  (in the case of  $N \leq 2$ , we need a slight modification in (2.7) below).

Let the nonlinearity  $F$  be  $(NF)$ , which satisfies (F.1)-(F.5). Suppose that the solution  $u(t)$  to  $(Cp)$  blows up at time  $t=T \in (0, \infty]$ , i.e.,  $\lim_{t \rightarrow T} \|\nabla u(t)\| = \infty$ . From Lemma 1.2,  $u(t)$  satisfies the mass conservation law (1.8) and the energy conservation law (1.9) for  $t \in [0, T)$ . We note that, in this case,

$$(2.1) \quad G(u) = \frac{2\chi}{q+1} |u|^{q+1} + \frac{2}{\sigma} |u|^\sigma.$$

*Proof of Theorem C.*

We recall that

$$(2.2) \quad \lambda \equiv \lambda(t) = 1 / \|u(t)\|_\sigma^{q/2},$$

$$(2.3) \quad u_\lambda \equiv S_\lambda u(t, x) = \lambda^{N/2} u(t, \lambda x).$$

One can see that

$$(2.4) \quad \|u_\lambda\| = \|u\| = \|u_0\|,$$

$$(2.5) \quad \|u_\lambda\|_\sigma = 1.$$

Moreover we have that

$$(2.6) \quad \begin{aligned} E(u_\lambda) &= \lambda^2 E(u(t)) \\ &= \lambda^2 \left\{ H(u_0) + \frac{2\chi}{q+1} \|u(t)\|_{q+1}^{q+1} \right\} \longrightarrow 0 \quad (t \rightarrow T). \end{aligned}$$

since, by Hölder's inequality, it holds that

$$\begin{aligned} \lambda^2 \|u(t)\|_{q+1}^{q+1} &\leq \lambda^2 \|u(t)\|^{1-a} \|u(t)\|_\sigma^a \\ &= \|u_0\|^{1-a} (\lambda^2 \|u(t)\|_\sigma^a), \end{aligned}$$

where  $a = (N/4)(q-1)\sigma < \sigma$  (because  $q-1 < 4/N$ ). From (2.5), (2.6) and Sobolev's inequality one has

$$(2.7) \quad \|u_\lambda\|_{2^*} \leq S \|\nabla u_\lambda\| \leq S$$

for sufficiently small  $\lambda$ , where  $S$  is the Sobolev best constant.

By (2.4), (2.5) and (2.7) we have, for some constants  $\eta, C > 0$ ,

$$(2.8) \quad \mu(\{|u_\lambda(t)| > \eta\}) > C$$

with the help of Lemma 1.3.

For any sequence  $(t_n)_n$  such that  $t_n \rightarrow T$  ( $n \rightarrow \infty$ ), we use the notations:

$$(2.9) \quad \lambda_n \equiv \lambda(t_n),$$

$$(2.10) \quad u_n(t_n, x) \equiv \lambda_n^{N/2} u(t_n, \lambda_n x).$$

We shall prove Proposition D.

*Proof of Proposition D.* By (2.8) and Lemma 1.4 (2), we can shift each  $u_n$  so that

$$(2.11) \quad f_n^1 \equiv u_n(t_n, x + y_n^1) \longrightarrow f^1 \neq 0 \quad \text{weakly in } H^1.$$

This is valid only for a subsequence. We shall however often extract subsequence without explicitly mentioning this fact.

From (2.1), one has

$$(2.12) \quad f_n^1 \longrightarrow f^1 \quad \text{in } L_{\text{loc}}^2,$$

so that

$$(2.13) \quad f_n^1 \longrightarrow f^1 \quad \text{a. e. in } \mathbf{R}^N.$$

Hence we have, by Lemma 1.5,

$$(2.14) \quad \lim_{n \rightarrow \infty} (\|f_n^1\|_\sigma^\sigma - \|f_n^1 - f^1\|_\sigma^\sigma) = \|f^1\|_\sigma^\sigma,$$

and by the weak convergence of  $\nabla f_n^1$  and the uniqueness of the limit,

$$(2.15) \quad \lim_{n \rightarrow \infty} (\|\nabla f_n^1\|^2 - \|\nabla f_n^1 - \nabla f^1\|^2) = \|\nabla f^1\|^2.$$

Combining (2.14) and (2.15), we get

$$(2.16) \quad \lim_{n \rightarrow \infty} \{E(f_n^1) - E(f_n^1 - f^1)\} = E(f^1).$$

We deduce from (2.5), (2.10) and (2.14) that following two limit exist and equalities hold;

$$(2.17) \quad \lim_{n \rightarrow \infty} \|f_n^1 - f^1\|_\sigma^\sigma = 1 - \|f^1\|_\sigma^\sigma,$$

$$(2.18) \quad \lim_{n \rightarrow \infty} E(f_n^1 - f^1) = -E(f^1).$$

Suppose  $\lim_{n \rightarrow \infty} \|f_n^1 - f^1\|_\sigma \neq 0$ . Then one can verify that  $f_n^1 - f^1$  satisfies the assumptions in Lemma 1.3 with  $\alpha=2$ ,  $\beta=\sigma$  and  $\gamma=2^*$ . So at this stage, we consider  $f_n^1 - f^1$ , and repeat the above argument. There exists a sequence  $(y_n^2)_n$  in  $\mathbf{R}^N$  such that

$$(2.19) \quad f_n^2 \equiv (f_n^1 - f^1)(t_n, \cdot + y_n^2) \longrightarrow f^2 \neq 0 \quad \text{weakly in } H^1,$$

$$(2.20) \quad \lim_{n \rightarrow \infty} \{E(f_n^2) - E(f_n^2 - f^2)\} = E(f^2).$$

We note that  $\lim_{n \rightarrow \infty} |y_n^1 - y_n^2| = \infty$ , since  $f_n^1 - f^1 \rightarrow 0$  weakly in  $H^1$ . By (2.17)-(2.20), one can also verify that

$$(2.21) \quad \lim_{n \rightarrow \infty} \|f_n^2 - f^2\|_\sigma^2 = 1 - \|f^2\|_\sigma^2 - \|f^1\|_\sigma^2,$$

$$(2.22) \quad \lim_{n \rightarrow \infty} E(f_n^2 - f^2) = -E(f^1) - E(f^2),$$

since we have that

$$\lim_{n \rightarrow \infty} \|f_n^1 - f^1\|_\sigma = \lim_{n \rightarrow \infty} \|f_n^2\|_\sigma,$$

$$\lim_{n \rightarrow \infty} E(f_n^1 - f^1) = \lim_{n \rightarrow \infty} E(f_n^2),$$

by the translation invariance of the norm  $\|\cdot\|_\sigma$  and the functional  $E(\cdot)$ .

Repeating this procedure, we obtain sequences  $(y_n^j)_n$  in  $\mathbf{R}^N$  for  $1 \leq j$  such that

$$\lim_{n \rightarrow \infty} |y_n^j - y_n^k| = \infty \quad (j \neq k)$$

and corresponding functions

$$f_n^j \equiv (f_n^{j-1} - f^{j-1})(t_n, \cdot + y_n^j) \longrightarrow f^j \quad \text{weakly in } H^1,$$

where  $f_n^j$  satisfies

$$(2.23) \quad \lim_{n \rightarrow \infty} (\|f_n^j\|_\sigma^2 - \|f_n^j - f^j\|_\sigma^2) = \|f^j\|_\sigma^2,$$

$$(2.24) \quad \lim_{n \rightarrow \infty} (\|\nabla f_n^j\|^2 - \|\nabla f_n^j - \nabla f^j\|^2) = \|\nabla f^j\|^2,$$

so that we have

$$(2.25) \quad \lim_{n \rightarrow \infty} \{E(f_n^j) - E(f_n^j - f^j)\} = E(f^j).$$

We also obtain by induction that

$$(2.26) \quad \lim_{n \rightarrow \infty} \|f_n^j - f^j\|_\sigma^2 = 1 - \sum_{k=1}^j \|f^k\|_\sigma^2,$$

$$(2.27) \quad \lim_{n \rightarrow \infty} E(f_n^j - f^j) = - \sum_{k=1}^j E(f^k),$$

(D.8) and (D.8)' immediately follow from (D.7) and (D.7)'. (D.7)' is obvious by the construction of  $f^j$ 's. Therefore it remains to prove (D.7). Suppose the contrary that there exists a constant  $C_0 > 0$  and  $J \in \mathbf{N}$  such that

$$(2.28) \quad \lim_{n \rightarrow \infty} \|f_n^j - f^j\|_\sigma^2 > C_0$$

for any  $j \geq J$ . Thus there is a constant  $C_1 > 0$  such that

$$(2.29) \quad \|f^{j+1}\|_\sigma^2 > C_1$$

for any  $j \geq J$ , since the size of  $\|f^{j+1}\|$ , essentially depends on the lower bound

of  $\|f_n^j - f^j\|_\sigma$  by Lemma 1.3, Lemma 1.4 and the construction of  $f^j$ . We choose and  $k \in \mathbb{N}$  (specified later). Using the formula (2.26) for  $j \in \{J, J+1, \dots; J+k\}$ , we have by (2.28) and (2.29) that

$$\begin{aligned} 1 - C_0 &> \lim_{n \rightarrow \infty} \{ \|f_n^j - f^j\|_\sigma^\sigma - \|f_n^{j+k} - f^{j+k}\|_\sigma^\sigma \} \\ &= \sum_{j=1}^k \|f^{j+j}\|_\sigma^\sigma \\ &> k C_1. \end{aligned}$$

Thus we reach a contradiction, if we take  $k$  as  $k C_1 \geq 1 - C_0$ .

*Remark 2.1.* Proposition D asserts that  $u_n$  behaves like a superposition of several parts  $u_n^1, u_n^2, u_n^3, \dots, u_n^L$  ( $L$  may be infinite) as  $n \rightarrow \infty$ . The above argument is somewhat related to those used in Lions [14], Brézis and Coron [2] and Struwe [23].

We now distinguish two cases:

*Case I*  $L = \infty$  and  $E(f^j) > 0$  for any  $j \in \mathbb{N}$ ,

*Case II*  $L < \infty$  or  $E(f^k) \leq 0$  for some  $k \in \mathbb{N}$ .

We shall establish that Case I cannot occur.

Suppose that we are in Case I. We recall (2.27) and define the sequence  $(E_j)_j$  by

$$(2.30) \quad E_j \equiv \lim_{n \rightarrow \infty} \{ -E(f_n^j - f^j) \} = \sum_{k=1}^j E(f^k).$$

Hence the sequence  $(E_j)_j$  is positive and increasing, since  $E(f^k) > 0$  for any  $k \in \mathbb{N}$ . On the other hand we have by the definition of  $E$  that

$$(2.31) \quad \begin{aligned} (0 <) E_j &= \lim_{n \rightarrow \infty} \{ -E(f_n^j - f^j) \} \\ &\leq (2/\sigma) \lim_{n \rightarrow \infty} \|f_n^j - f^j\|_\sigma^\sigma. \end{aligned}$$

Thus  $(E_j)_j$  has a subsequence decreasing to 0 by (D.7) in Proposition D. Therefore we reach a contradiction and Case I is excluded.

Hence the only case which occurs is Case II. Since  $L < \infty$  implies that

$$\sum_{j=1}^L E(f^j) \leq 0,$$

we have  $E(f^k) \leq 0$  for some  $k \in \mathbb{N}$  in this case. Thus we get, by Lemma 1.1(4),

$$(2.32) \quad \|f^k\| \geq \|Q\|.$$

We also have (2.11) and (2.13) for all  $j \in \{1, 2, \dots, L\}$ , so it follows from Lemma 1.5 that

$$(2.33) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{R}^N} | |f_n^j(x)|^2 - |f_n^j(x) - f^j(x)|^2 - |f^j(x)|^2 | dx = 0.$$

Hence, for any sequence  $(y_n)_n$  in  $\mathbf{R}^N$ , it holds (2.33) with  $f_n^j$  and  $f^j$  replaced by  $f_n^j(t_n, x + y_n)$  and  $f^j(t_n, x + y_n)$ , respectively. So we have that, for any domain  $\Omega \subset \mathbf{R}^N$ ,

$$(2.34) \quad \lim_{n \rightarrow \infty} \int_{x \in y_n + \Omega} | |f_n^j(x)|^2 - |f_n^j(x) - f^j(x)|^2 - |f^j(x)|^2 | dx = 0.$$

Therefore, for  $f^k$  and any  $K > 0$ , we have by (2.34) with  $j \in \{1, 2, \dots, k\}$  and  $\Omega = B(K)$  that

$$(2.35) \quad \int_{B(K)} |f^k|^2 dx \leq \int_{B(K)} |\phi_n|^2 dx - \sum_{j=1}^{k-1} \int_{B(K)} |\phi^j|^2 dx + o(1),$$

where  $o(1)$  is a quantity converging to 0 as  $n \rightarrow \infty$  and

$$(2.36) \quad \phi_n = u_n \left( t_n, x + \sum_{m=1}^k y_n^m \right),$$

$$(2.37) \quad \phi^j = f^j \left( x + \sum_{m=j+1}^k y_n^m \right).$$

The main conclusion of Theorem C thus follows from (2.35), since one can see that, for any  $\varepsilon > 0$ ,

$$(2.38) \quad \|\Omega\|^2 - \varepsilon \leq \int_{B(K)} |f^k|^2 dx$$

for sufficiently large  $K > 0$ . In the case of  $u_0 \in \Sigma$ ,  $\chi \leq 0$  and  $T < +\infty$  (1.10), implies the boundedness of  $\|xu(t)\|$ . Therefore we obtain (C.4) and (C.5), since we have by Chebychev's inequality;

$$(2.39) \quad \int_{|x| > R} |u(t, x)|^2 dx \leq \frac{1}{R^2} \|xu(t)\|^2.$$

Therefore we conclude the proof of Theorem C.

*Proof of Corollary E.*

We recall (2.11). In view of (D.7)' and (D.8)' in Proposition D, it is enough to prove that we have

$$(2.40) \quad f_n^1 \equiv u_n(t_n, x + y_n^1) \longrightarrow f^1 \neq 0 \quad (n \rightarrow \infty).$$

strongly in  $L^\sigma$  or  $L^2$  for some  $f \in H^1$ .

*Proof of (1).* We will show that (2.40) with  $y_n^1 \equiv 0$  holds true in the strong topology of  $L^\sigma$ . If the initial datum  $u_0$  is radially symmetric, so is the solution

to  $(Cp)$  in  $C([0, T]; H^1)$ . Thus each  $u_n$  is also radially symmetric. Since  $(u_n)_n$  is a bounded sequence in  $H^1$  by (2.4), (2.7) and (2.10), we have (2.40) with  $y_n^1 \equiv 0$  in the strong topology of  $L$  for a subsequence (we still denote it by the same letter) by a radial compactness lemma due to Strauss [22] (see also [1]). We note that  $f^1 \neq 0$  by (2.5) and (2.10).

*Proof of (2).* We will show that (2.40) holds true in the strong topology in  $L^2$ . Now we suppose  $\|f^1\| < \|Q\|$  where  $Q$  is the ground state solution to (0.5). Then one has  $E(f) > 0$  by (1.4) in Lemma 1.1. This together with (2.18) implies that

$$(2.41) \quad \lim_{n \rightarrow \infty} \|f_n^1 - f^1\| \geq \|Q\|$$

by (1.4) again. We note that the limit in the left hand side of (2.41) exists, since the weak convergence of  $f_n^1 \rightarrow f$  in  $L^2$  together with the fact  $\|f_n^1\| = \|u_0\| = \|Q\|$  implies that

$$(2.42) \quad \lim_{n \rightarrow \infty} \|f_n^1 - f^1\|^2 = \|Q\|^2 - \|f^1\|^2.$$

Here we reach a contradiction, since (2.41) and (2.42) yield  $\|Q\| < \|Q\|$ . Therefore we have  $\|f^1\| = \|Q\|$ , so that we obtain

$$(2.43) \quad \lim_{n \rightarrow \infty} \|f_n^1 - f^1\| = 0.$$

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*Note added in proof.* After completing this work, we found that we could improve the proof of Theorem C to refine Theorem B (1) with the result that we have  $A \geq \|Q\|^2$  for any blow-up solution. This will appear elsewhere.

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DEPARTMENT OF MATHEMATICS  
FACULTY OF SCIENCE  
TOKYO INSTITUTE OF TECHNOLOGY  
OH-OKAYAMA, MEGURO  
TOKYO 152, JAPAN