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Mass of a Galaxy and Dissipative Process in the Hot Universe

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Decay of the acoustic motions in an early stage of the hot universe is studied in detail. Considering a growth of density perturbation by gravitation and its decay by dissipative processes, i.e. viscosity and thermal conductivity, we can get a gross feature of the size spectrum of density inhomogeneity at the stage of recombination of the cosmic plasma. The size spectrum has a flat maximum between $10^{12-13}\,M_{\odot}$ and $10^{17-18}\,M_{\odot}$. After the time of decoupling, the density perturbation evolves into a galaxy or a cluster of galaxies by the hydrodynamical instability. The maximum mass formed by this mechanism is given as $10^{14-15}\,M_{\odot}$, assuming the density perturbation $\delta\rho/\rho \simeq 10^{-1}$.

§ 1. Introduction

It is an important problem to explain why a galaxy and a cluster of galaxy have characteristic masses such as $10^{10}\sim10^{15}M_{\odot}$. This strong inhomogeneity in matter distribution of the universe is generally considered to result from the evolution of small departures in the strictly homogeneous universe. To explain the characteristic mass of these objects, therefore, we must consider a size spectrum of weak inhomogeneity in the early stage of the universe. In this paper, we give a discussion about the size spectrum in relation to dissipative processes in the hot universe model proposed by Gamow.[1)]

The evolution of the weak inhomogeneity in density, velocity and metric has been studied by Lifshitz[2)] and many other authors.[2a)] In their treatment, cosmic fluid is regarded as an ideal fluid, and dissipative processes such as viscosity or thermal conduction are neglected. However, Misner[3)] took notice of an important role of the dissipative process at the early stage of the hot universe. Silk[4)] studied a decay of acoustic motions by thermal conduction. Based on this theory, Silk[4)] and Harrison[5)] could find a reasonable value as the minimum mass of the density inhomogeneity. Following the turbulence hypothesis for galaxy formation proposed by Ozernoi and Chernin,[6)] we estimated the minimum mass from a dissipation time of vortical motions and the maximum mass from a growth time of density perturbation.[7)] We also studied heating of the matter resulting from the dissipation of turbulent motions.[8)] Recently, Peebles and Yu[9)] studied, in more detail, the size spectrum of density inhomogeneity. In this paper, we consider the evolution of size spectrum and the heating due to the dissipation of acoustic motions. Our discussions of the size spectrum are similar to those by Peebles and Yu,[9] but may be more instructive than theirs though their treatment on the interaction between radiation and matter is more complete than our treatment by diffusion approximation.

In § 2, we treat a decay of acoustic motions in radiative gas based on the relativistic hydrodynamics with dissipation, a formulation of which is given in the Appendix. In § 3, we consider an applicability of the fluid approximation of the radiation in the hot universe model. In § 4, a gross feature of the size spectrum and its evolution is discussed. In § 5, the heating of matter by the dissipation of acoustic motions is discussed and, in § 6, mass range of the astronomical objects resulting from the inhomogeneity is given.

§ 2. Dissipation of acoustic motions in radiative gas

In the early stage of the hot universe model, radiation energy density is very large in comparison with matter energy density. In this section, we clarify the dissipation of acoustic motions at this stage.

The set of hydrodynamic equations with dissipation is given by Eqs. (A·5), (A·6), (A·9) and (A·10) in the Appendix. We approximate them by neglecting gravitation and assuming non-relativistic velocity. Further, we linearize these equations assuming small departures from the static and homogeneous medium as follows:*)

$$\frac{\partial \varepsilon_1}{\partial t}+(\varepsilon+p)\nabla\cdot\boldsymbol{u}+\nabla\cdot\boldsymbol{q}=0\,, \tag{2·1}$$

$$\frac{(\varepsilon+p)}{c^2}\frac{\partial \boldsymbol{u}}{\partial t}+\nabla p_1+\frac{1}{c^2}\frac{\partial \boldsymbol{q}}{\partial t}-\eta\left(\nabla^2\boldsymbol{u}+\frac{1}{3}\nabla\nabla\cdot\boldsymbol{u}\right)=0\,, \tag{2·2}$$

$$\boldsymbol{q}=-K\left(\nabla T_1+T\frac{1}{c^2}\frac{\partial \boldsymbol{u}}{\partial t}\right), \tag{2·3}$$

$$\frac{\partial \rho_1}{\partial t}+\rho\nabla\cdot\boldsymbol{u}=0\,, \tag{2·4}$$

where the subscript 1 denotes the perturbed quantities. The third term on the left-hand side of Eq. (2·2) and the second term on the right-hand side of Eq. (2·3) represent a relativistic effect of thermal conduction. To complete the set of equations, we require the equations of state;

$$\varepsilon=\rho c^2+bT^4, \qquad p=\tfrac{1}{3}bT^4. \tag{2·5}$$

A coefficient of viscosity[10] η and a coefficient of thermal conductivity K due to the scattering of photons are given as

*) The velocity u^α is related to the four velocity U^α in the Appendix as $u^\alpha=cU^\alpha$.

$$\eta=\frac{4}{15}\frac{cbT^4}{\kappa\rho}, \qquad K=\frac{4}{3}\frac{cbT^3}{\kappa\rho}, \tag{2·6}$$

where κ is the opacity of electron scattering.

Using Eqs. (2·1) and (2·4), Eqs. (2·2) and (2·3) are rewritten as

$$\frac{\partial^2\xi}{dt^2}-c_s^2\nabla^2\xi+\left(1-\frac{3c_s^2}{c^2}\right)\frac{\partial\nabla\cdot\tilde{q}}{\partial t}-\frac{4\eta}{3(\varepsilon+p)}\nabla^2\left(\frac{\partial\xi}{\partial t}+\nabla\cdot\tilde{q}\right)=0, \tag{2·7}$$

$$\nabla\cdot\tilde{q}=-\frac{KT}{4bT^4}\left(\nabla^2\xi-\frac{3}{c^2}\frac{\partial^2\xi}{\partial t^2}\right), \tag{2·8}$$

where $\xi=T_1/T$, $\tilde{q}=q/4bT^4$ and the sound velocity c_s is given as

$$c_s=c\sqrt{\frac{4}{3}\frac{p}{\varepsilon+p}}. \tag{2·9}$$

Taking the perturbation of wave number k, i.e. $\nabla^2\xi=-k^2\xi$, Eqs. (2·7) and (2·8) give

$$\frac{\partial^2\xi}{\partial t^2}+c_s^2k^2\xi+2t_d^{-1}\frac{\partial\xi}{\partial t}+0\left(\frac{\xi}{t_d^2}\right)=0, \tag{2·10}$$

where t_d is the decay time of acoustic motions defined as follows:

$$t_d^{-1}=t_{dv}^{-1}+t_{dc}^{-1}, \tag{2·11}$$

$$t_{dv}=\frac{5}{2}\left(\frac{c}{c_s}\right)^2 t_{\text{diff}}, \tag{2·12}$$

$$t_{dc}=6\left\{1-3\left(\frac{c_s}{c}\right)^2\right\}^{-2} t_{\text{diff}} \tag{2·13}$$

and

$$t_{\text{diff}}=\frac{\kappa\rho}{k^2c}, \tag{2·14}$$

t_{dv} and t_{dc} denoting the decay times by viscosity and by conductivity, respectively, and t_{diff} being the diffusion time of a photon.

Comparing t_{dv} and t_{dc}, we can say that the dissipation is mainly due to the viscosity in the state of $bT^4\gg\rho c^2$ and due to the conductivity in the state of $\rho c^2\gg bT^4$. If we express the total decay time t_d as

$$t_d=\alpha t_{\text{diff}}, \tag{2·15}$$

α is nearly constant for any states, i.e. $\alpha=15/2$ for $bT^4\gg\rho c^2$ and $\alpha=6$ for $\rho c^2\gg bT^4$.

Now, we notice that the dissipation by conductivity is suppressed as c_s tends to $c/\sqrt{3}$.*) The suppression factor $\{1-3(c_s/c)^2\}^{-2}$ in Eq. (2·13) exactly corre-

*) See the footnote on page 373.

sponds to the two relativistic terms in Eqs. (2·2) and (2·3). As seen from Eqs. (2·7) and (2·8), these terms suppress the effect of heat flow and the heat flow itself by the factor $\{1-3(c_s/c)^2\}^{-1}$ respectively. Then, if one of the relativistic terms is neglected, $t_{dc}=6\{1-3(c_s/c)^2\}^{-1}t_{\rm diff}$, which just corresponds to the decay time by the conduction obtained by Silk.[4] This difference arises because Silk neglected incorrectly the relativistic effect of the conduction in Eq. (5) in his paper,[4] which corresponds to Eq. (2·2) in our equations.

§ 3. Decay of acoustic and vortical motions in the expanding hot universe

Now, we give a set of equations governing the evolution of weak inhomogeneity in the expanding hot universe. Considering a weak gravitational field due to a perturbed distribution of matter superposed on the Friedman metric, we take a metric such as

$$ds^2=-(1+h_{00})c^2dt^2+(a^2\delta_{\alpha\beta}+h_{\alpha\beta})dx^\alpha dx^\beta+h_{0\alpha}cdtdx^\alpha,$$

where a denotes the expansion factor and h_{ab} denotes the perturbed metric tensors. Using the above metric under the coordinate conditions[11] $h_{0\alpha}=0$ and $h_\alpha{}^\alpha=0$ and approximating for non-relativistic velocity, the linearized equations are given, corresponding to Eqs. (2·1)~(2·14), as

$$\frac{1}{a^3}\frac{\partial a^3\varepsilon_1}{\partial t}+(\varepsilon+p)\nabla\cdot\boldsymbol{u}+p_1\frac{1}{a^3}\frac{da^3}{dt}+\nabla\cdot\boldsymbol{q}=0\,, \tag{3·1}$$

$$\frac{1}{c^2}\frac{1}{a^5}\frac{\partial}{\partial t}[a^5(\varepsilon+p)\boldsymbol{u}]+\frac{1}{a^2}\left[\nabla p_1+\frac{\varepsilon+p}{c^2}\nabla\phi\right]$$
$$-\frac{\eta}{a^2}\left[\nabla^2\boldsymbol{u}+\frac{1}{3}\nabla\nabla\cdot\boldsymbol{u}\right]+\frac{1}{a^4}\frac{\partial a^4}{\partial t}\boldsymbol{q}=0\,, \tag{3·2}$$

$$\boldsymbol{q}=-K\left(\frac{1}{a^2}\nabla T_1+T\frac{1}{c^2}\frac{\partial\boldsymbol{u}}{\partial t}+2\frac{\dot a}{a}T\frac{\boldsymbol{u}}{c}+\frac{1}{a^2}T\frac{\nabla\phi}{c^2}\right), \tag{3·3}$$

$$\frac{1}{a^3}\frac{\partial a^3\rho_1}{\partial t}+\rho\nabla\cdot\boldsymbol{u}=0\,, \tag{3·4}$$

where the gravitational potential $\phi=c^2h_{00}/2$ is given by

$$\frac{1}{a^2}\nabla^2\phi+3\frac{\dot a}{a}\frac{\partial\phi}{\partial t}+6\frac{\ddot a}{a}\phi=4\pi G(\varepsilon_1+3p_1)/c^2. \tag{3·5}$$

For the acoustic motions such as $\nabla\cdot\boldsymbol{u}\neq 0$, the above equations reduce to

*) For the general equation of state, t_{dc} becomes as

$$t_{dc}=\frac{T\varepsilon_T}{k^2K}\frac{\varepsilon_T}{p_T}\frac{p_\rho\rho+p_T(\varepsilon+p-\varepsilon_\rho\rho)/\varepsilon_T}{\varepsilon+p-\varepsilon_\rho\rho}\left(1-\frac{\varepsilon_T T}{\varepsilon+p-\varepsilon_\rho\rho}\right)^{-1}\left(1-\frac{\varepsilon_T}{p_T}\frac{c_s^2}{c^2}\right)^{-1},$$

where c_s is the sound velocity, $\varepsilon_T=\partial\varepsilon/\partial T$, $p_\rho=\partial p/\partial\rho$ and so on. This expression shows that the dissipation by the conduction is generally suppressed as c_s reaches the light velocity.

$$\frac{\partial^2\xi}{\partial t^2}+c_s^2\frac{k^2}{a^2}\left(1-\frac{k_J^2}{k^2}\right)\xi+2t_d^{-1}\frac{\partial\xi}{\partial t}+0\left(\frac{\xi}{t_d t},\frac{\xi}{t_d^2},\frac{\xi}{t^2}\right)=0\,, \tag{3·6}$$

where k is a wave number in the comoving coordinate, t is a time since the start of the cosmic expansion, t_d has been given by Eqs. (2·11) ~ (2·13), putting $t_{\text{diff}}=a^2\kappa\rho/(k^2c)$, and Jeans wave number k_J is given as

$$\frac{k_J}{a}\equiv\frac{2\pi}{\lambda_J}=\sqrt{4\pi G\left(\rho+\frac{8bT^4}{3c^2}\right)}\Big/c_s\,, \tag{3·7}$$

λ_J being the Jeans wave length.

Equation (3·6) reduces to Eq. (2·10), if we consider the cases

$$t_d>t_s \quad \text{and} \quad t>t_s\,, \tag{3·8}$$

t_s being a period of sound wave given by

$$t_s=\frac{2\pi a}{c_s k}\,. \tag{3·9}$$

We can easily see that $t>t_s$ implies also $k>k_J$. Under the assumption of Eq. (3·8), Eq. (3·6) is solved as

$$\xi\sim a^{-\gamma}\exp\left(ik\int^t\frac{c_s}{a}dt-\int^t\frac{dt}{t_d}\right), \tag{3·10}$$

where $\gamma=0$ for $bT^4\gg\rho c^2$ and $\gamma=1/4$ for $\rho c^2\gg bT^4$.

For vortical motions such as $\nabla\cdot\boldsymbol{u}=0$, the temperature as well as the density is uniform, and a fundamental equation is[8]

$$\frac{1}{c^2}\frac{1}{a^5}\frac{\partial}{\partial t}\{a^5(\varepsilon+p)\boldsymbol{u}\}+\frac{\varepsilon+p}{c^2}\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\frac{\eta}{a^2}\nabla^2\boldsymbol{u}=0\,. \tag{3·11}$$

Taking the vortical motion with wave number k, i.e. $\nabla^2\boldsymbol{u}=-k^2\boldsymbol{u}$, the decay time by viscosity t_{dt} is given as

$$t_{dt}=\frac{5}{3}\left(\frac{c}{c_s}\right)^2 t_{\text{diff}}\sim t_{dv}\,. \tag{3·12}$$

A change of the vortical velocity with wave number k by the inertia term in Eq. (3·11) takes a time of the order of

$$t_u=\frac{\lambda}{v}\equiv\frac{2\pi}{ku}\,, \tag{3·13}$$

where $\lambda=2\pi a/k$ and $v=au$ are a proper wave length and a proper velocity, respectively. The ratio (t_{dt}/t_u) represents the Reynolds number and, if $(t_{dt}/t_u)\gg1$, the fluid motions may be generally in turbulent motions. Therefore, the decay of velocity cannot be described only in terms of the viscosity. Elsewhere, we have studied the decay of turbulence in the expanding universe.[8]

In Fig. 1, we give a comparison among t, t_s and t_d for the acoustic motions

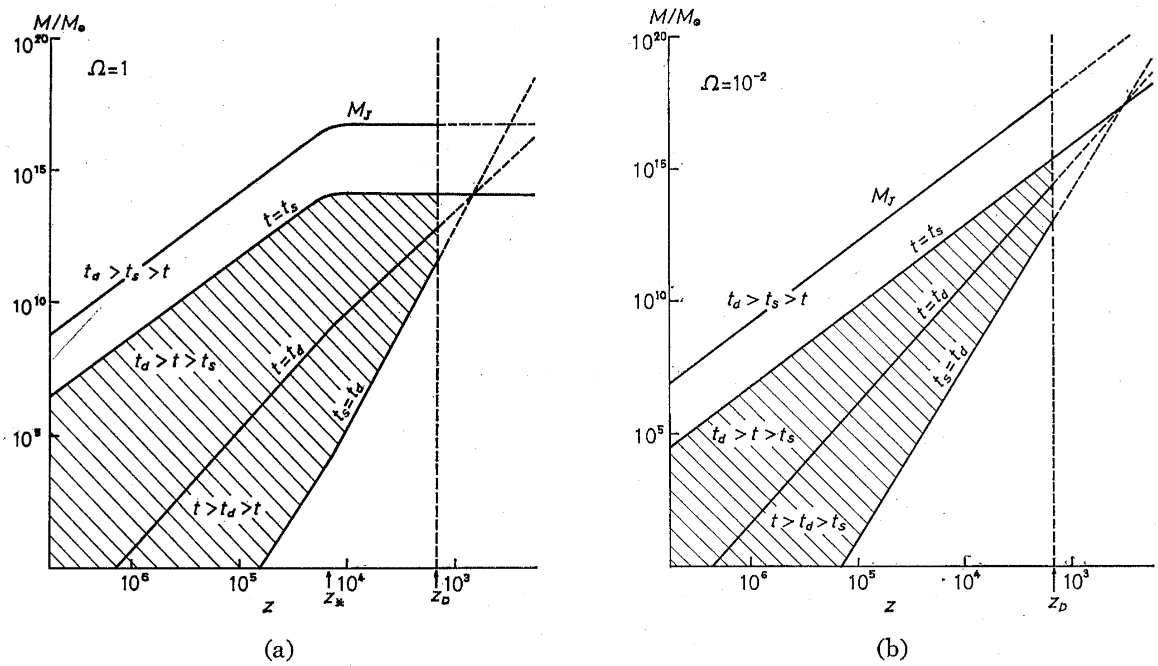

Fig. 1. Comparison among the time scales t, t_s and t_d. The ordinate represents the size of the density inhomogeneity by the mass in solar mass unit and the abscissa does the stage of the expanding universe by the redshift parameter z. z_* and z_D represent the time at which $bT^4=\rho c^2$ and that of the recombination of plasma, respectively. Ω is a parameter to represent the matter density at present as $\rho_0=10^{-29}\Omega$ g/cm^3. M_J is the Jeans mass defined by the radiation pressure.

about the stage of recombination of cosmic plasma.[12] Except t, the time scales are dependent on the size of motions. In stead of k, we use a mass M defined by

$$M=\rho(2\pi a/k)^3. \tag{3·14}$$

The stage of evolution is represented by red-shift parameter defined by $z+1=a_0/a$, where a_0 is the present value. The model of the universe is specified by a parameter $\Omega=\rho_0/(10^{-29}\text{ g/cm}^3)$, where ρ_0 is the average matter density at present.

The hatched region in Figs. 1(a) and (b) represents the sizes of the density inhomogeneity satisfying the condition in Eq. (3·8). The mass $M(t=t_s)$ of the upper boundary of this region is related to Jeans mass, $M_J=\rho\lambda_J^3$, as

$$M_J=10^{29.1}\Omega z^{-3}M_\odot=10^{2.4}M(t=t_s) \quad \text{for } bT^4\gg\rho c^2,$$

$$M_J=10^{16.7}\Omega^{-2}M_\odot=10^{2.6}M(t=t_s) \quad \text{for } \rho c^2\gg bT^4. \tag{3·15}$$

The mass $M(t_s=t_d)$ of the lower boundary is related to a mass within the mean free path, $M_{\text{m.f.p.}}=\rho(\kappa\rho)^{-3}$, as

$$M(t_s=t_d)/M_{\text{m.f.p.}}=10^{2.3\sim2.5}(c/c_s)^2. \tag{3·16}$$

For the case of small size such as $M < M_{\mathrm{m.f.p.}}$, the fluid approximation of the radiation is invalid. After the epoch of the recombination of plasma, the fluid approximation becomes also invalid.

In Fig. 2, we show a comparison among t, t_u and t_{dt} for the vortical motions. Such a consideration has already been given by us,[7),8)] but Fig. 2 is given here for comparison with Fig. 1 in the case of acoustical motions. The vortical motion with $M < \rho(vt)^3$ will decay into motions with smaller sizes in contrast to the acoustic motions, in which the motion is frozen for $M > M(t_d = t)$.

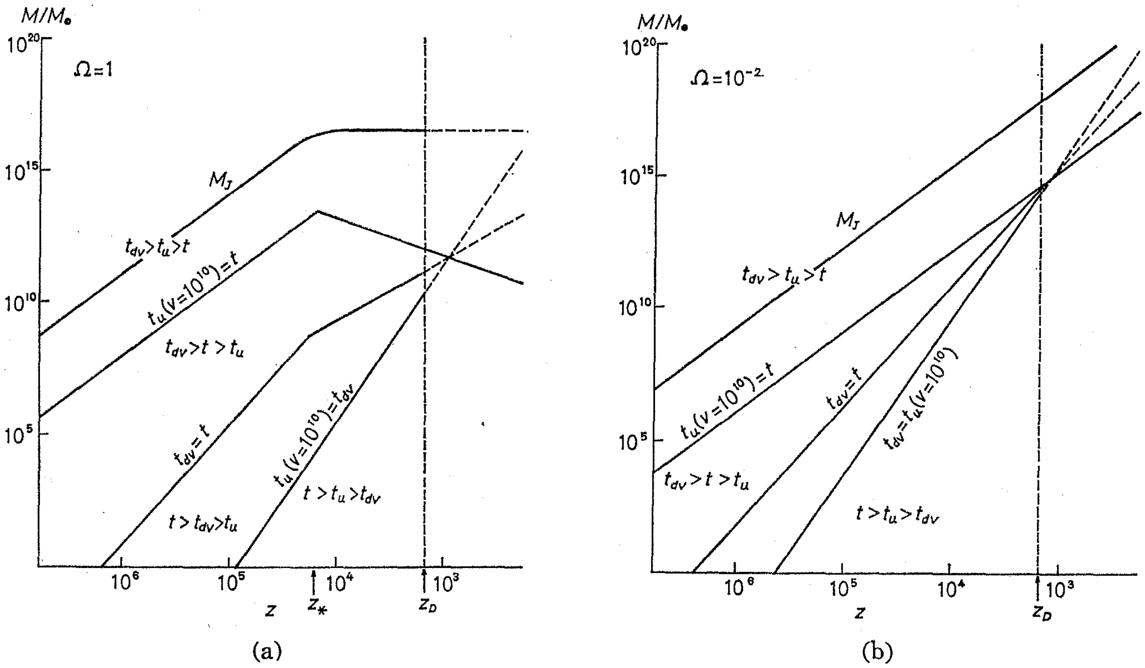

Fig. 2. Comparison among the time scales t, t_u and t_{dv}. In these Figures, t_u is calculated assuming $v = 10^{10}$ cm/sec at the stage $bT^4 \gg \rho c^2$, during which stage the velocity remains constant. The other notations are the same with those in Fig. 1.

§ 4. Size spectrum of density inhomogeneity

We consider the evolution of density inhomogeneity with a fixed mass M. In an early stage, M is larger than Jeans mass M_J, and its amplitude is growing monotonically by the self-gravity. As M_J is increasing with time, M_J soon overcomes M and it begins to oscillate by the pressure of the radiation as a sound wave. Thereafter, its sound wave suffers a strong decay by dissipation after the stage $t = t_d(M)$. From these considerations, a gross feature of the evolution of the size spectrum can be drawn schematically.

(a) *Damping by dissipation*

We consider the Fourier components ξ_k of the perturbed quantity defined as follows:

$$\xi(\boldsymbol{r}, t) = \int \xi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} . \tag{4·1}$$

Equation (3·10) gives

$$|\xi_{\boldsymbol{k}}|^2 \sim |\xi_{\boldsymbol{k}i}|^2 a^{-2\gamma} \exp(-2k^2/k_d^2(t)) \tag{4·2}$$

and

$$k_d^{-2} = k^{-2} \int^t \frac{dt}{t_d} = \beta \frac{ct}{a^2 \kappa \rho} , \tag{4·3}$$

where $|\xi_{\boldsymbol{k}i}|^2$ denotes the value at $k=k_J$ and $\beta=4/45$ for $bT^4 \gg \rho c^2$, $\beta=1/10$ for $\rho c^2 \gg bT^4$.

Denoting the spatial average by $\langle\ \rangle$, we define a size spectrum $F(k)$ as follows:

$$\langle|\xi|^2\rangle \propto \int |\xi_{\boldsymbol{k}}|^2 k^3 \frac{dk}{k} \propto \int F(k) \frac{dk}{k} . \tag{4·4}$$

Denoting the size by M instead of k, Eqs. (4·4), (4·2) and (3·14) give

$$F(M) \sim |\xi_{Mi}|^2 \frac{M^{-1}}{a^{2\gamma}} \exp\left[-2\left(\frac{M_d}{M}\right)^{2/3}\right] \tag{4·5}$$

and

$$\begin{aligned} M_d(z) &= \rho\left(\frac{2\pi a}{k_d}\right)^3 \\ &= \frac{10^{13.8}}{\Omega^{1/2}}\left(\frac{10^3}{z+1}\right)^{4.5} M_{\odot} \qquad \text{for } bT^4 \gg \rho c^2 \\ &= \frac{10^{13.4}}{\Omega^{5/4}}\left(\frac{10^3}{z+1}\right)^{3.75} M_{\odot} \qquad \text{for } \rho c^2 \gg bT^4. \end{aligned} \tag{4·6}$$

M_d relates to the mass $M(t=t_d)$ in Fig. 1 as

$$M_d/M(t=t_d) = (2\pi)^3 (\alpha\beta)^{3/2}.$$

(b) *Growing by gravitation*

As seen from Fig. 1, the dissipation is ineffective in the course of growing by gravitation. Therefore, the theory of adiabatic gravitational instability is applicable. In reference to the results of this theory, we put a gross feature of evolution of density inhomogeneity as follows,[2),2a)] in the state $bT^4 \gg \rho c^2$

$$\begin{aligned} |\xi_{\boldsymbol{k}}| &\propto a^2 && \text{for } k<k_J \\ &\propto \text{Constant} && \text{for } k>k_J ,^{*)} \end{aligned} \tag{4·7a}$$

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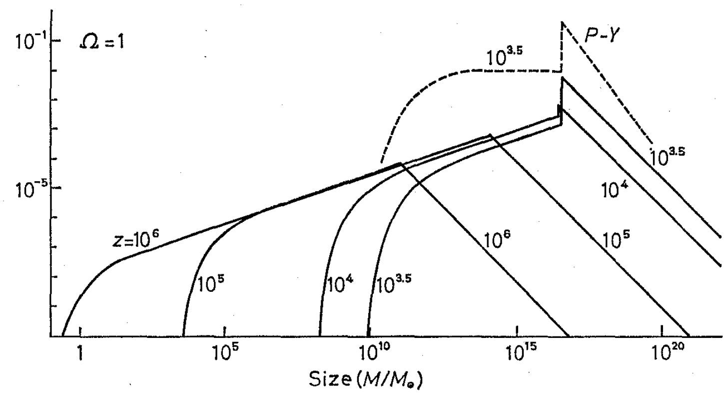
*) In reality, the amplitude is oscillating with time in the case of $k>k_J$ and the periodic size spectrum is obtained as shown by Sunyaev and Zeldovich[13)] and Peebles and Yu.[9)] However, we have not taken into consideration this shortly periodic oscillation, because we mean by a size of the inhomogeneity a size of the localized inhomogeneity rather than a wave length of the exactly periodic inhomogeneity. To describe the amplitude of the localized inhomogeneity, we should use an average of amplitude over an appreciable range of wave number; by this procedure the periodic nature of the size spectrum will disappear.

in the state of $\rho c^2 \gg bT^4$

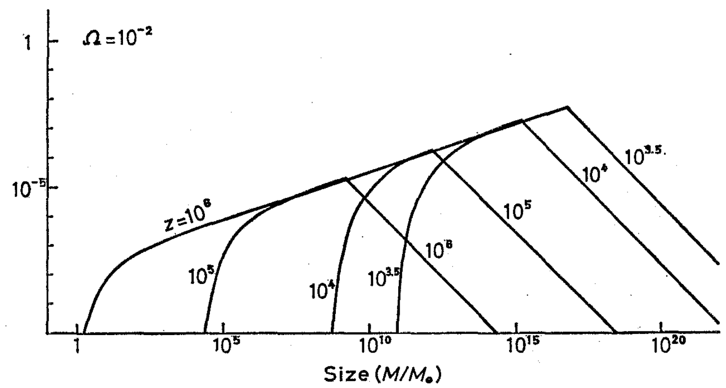
$$|\xi_k| \propto a \qquad \text{for } k<k_J$$
$$\propto a^{-1/4} \qquad \text{for } k>k_J\,.^{*)} \tag{4·7b}$$

The growth time of the adiabatic perturbation is independent of k, but an epoch when $k>k_J$ is attained is dependent on k. For example, the scale factor is proportional to k^{-1} at the time when $k=k_J$ is attained, in the stage $bT^4 \gg \rho c^2$, and a shape of size spectrum for $k>k_J$ is modified as

$$F(k) \propto |\xi_{ki}|^2 k^3 \times \frac{1}{k^4}\,, \tag{4·8}$$



(a)



(b)

Fig. 3. Diagram to demonstrate the evolution of size spectrum. The abscissa represents the size of density inhomogeneity by the mass in units of solar masses, and the ordinate does the amplitude of the inhomogeneity in an arbitrary unit. The size spectra at the stages $z=10^6$, 10^5, 10^4, $10^{3.5}$ are drawn, assuming the initial spectrum as the white noise, i.e. $F(k) \propto k^3$. The dotted curve denoted by P-Y is the shape of the spectrum taken from the initial spectrum such as $F(k) \propto k^2$, proposed by Peebles and Yu.[9)]

*) See the footnote on page 377.

using Eq. (4·7a). Such considerations about other sizes and other stages enable us to obtain the evolution of the size spectrum as shown in Fig. 3. As the initial spectrum, we have assumed a white noise spectrum, i.e. $|\xi_k|^2=$const. For comparison, an example for the case $|\xi_k|^2 \propto k$ proposed by Peebles and Yu[9] is also given. A characteristic feature is that these spectra have a flat maximum between k_J and k_d.

§ 5. Heating by dissipation

A damping of the inhomogeneous motions by dissipation accompanies a production of heat. As to the heating by turbulent vortical motions, we have studied eleswhere,[8] and, in this section, we study the heating by acoustic motions.

(a) *Heating rate*

Equation (A·8) is approximated for the case of sound wave as

$$\rho\frac{\partial S}{\partial t}+\nabla\cdot\left(\frac{\boldsymbol{q}}{T}\right)=\frac{\epsilon_d}{T} \tag{5·1}$$

and

$$\epsilon_d=\eta\left(\frac{\partial u^\beta}{\partial x^\alpha}+\frac{\partial u^\alpha}{\partial x^\beta}-\frac{2}{3}\delta_\alpha{}^\beta\nabla\cdot\boldsymbol{u}\right)\frac{\partial u^\beta}{\partial x^\alpha}+\frac{a^2}{KT}\boldsymbol{q}^2. \tag{5·2}$$

Averaging over space, we have, writing $\langle T\rangle$ by T,

$$\rho\frac{\partial\langle S\rangle}{\partial t}=\frac{\langle\epsilon_d\rangle}{T} \tag{5·3}$$

and

$$\langle\epsilon_d\rangle=\frac{4}{27}\left\{1+\left(1-\frac{3c_s^2}{c^2}\right)^2\frac{c^2}{c_s^2}\right\}\frac{bT^4c\bar{k}^2}{\kappa\rho a^2}\frac{v^2}{c^2}, \tag{5·4}$$

where

$$\bar{k}^2=\int|\xi_k|^2k^4dk\Big/\int|\xi_k|^2k^2dk\,.$$

To derive Eq. (5·4), we have assumed an isotropic inhomogeneity and used relations such as

$$\left\langle\left(\frac{T_1}{T}\right)^2\right\rangle=\frac{1}{27}\frac{v^2}{c_s^2}\,,\qquad \langle k_\alpha^2\rangle=\langle k_\beta^2\rangle=k^2/3\,,$$

$$\langle(k_\alpha v_\beta)^2\rangle=\langle k_\alpha^2\rangle\langle v_\beta^2\rangle \quad \text{and so on}, \tag{5·5}$$

$\boldsymbol{v}$ being a proper velocity defined by $\boldsymbol{v}=a\boldsymbol{u}$. If the spectrum is a white noise, $\bar{k}^2=3k_d^2/2$, and Eq. (5·4) becomes using Eq. (4·3)

$$\langle\epsilon_d\rangle \simeq \frac{5}{2}\frac{bT^4}{c^2}\frac{v^2}{t} \qquad \text{for } bT^4 \gg \rho c^2$$

$$\simeq 5\rho\frac{v^2}{t} \qquad \text{for } \rho c^2 \gg bT^4. \tag{5·6}$$

If the spectrum is the one given in Fig. 3, $\bar{k}^2$ is nearly equal to $k_d k_J$ and $\langle\epsilon_d\rangle$ is given by multiplying Eq. (5·6) by (k_J/k_d).

Apparently, $\langle\epsilon_d\rangle$ diverges as $t\to 0$. However, it comes from an artificial assumption that the initial spectrum is infinitely large at $k\to\infty$. If we put a cutoff k_m for large k, the heating rate in the stage $t<t_m$ becomes

$$\langle\epsilon_d\rangle \simeq \frac{bT^4}{c^2}\frac{v^2}{t_m} \qquad \text{for } bT^4 \gg \rho c^2,$$

where t_m is given by $t_m/(a^2(t_m)\rho(t_m)) = \kappa/(\beta k_m^2 c)$. Therefore, the heating is rather in-effective in the early stage such as $t<t_m$.

(b) *Temperature of matter*

In the hot universe model, a thermal equilibrium between matter and radiation is not attained generally after the annihilation of electron-positron pairs. As the relaxation time of radiative equilibrium is large, we assume the dissipated energy to heat the matter. Under this assumption, the entropy of matter S_m is determined by

$$\rho\frac{d\langle S_m\rangle}{dt} = \frac{\langle\epsilon_d\rangle - \langle\epsilon_c\rangle}{T_m}, \tag{5·7}$$

where T_m is temperature of matter and ϵ_c is cooling rate of matter. The main source of the cooling is due to Compton effect,[14] and it is given as

$$\epsilon_c = \frac{4bT^4\sigma_T\rho}{m_e m_H c}k(T_m - T_r), \tag{5·8}^{*)}$$

where T_m, σ_T, m_e, m_H and k are radiation temperature, Thomson cross section, the electron mass, the proton mass and Boltzmann constant.

From Eqs. (5·7) and (5·8), we have

$$\frac{dT_m}{dt} = -2\frac{\dot{a}}{a}T_m - \frac{8}{3}\frac{bT^4\sigma}{m_e c}(T_m - T_r) + \frac{2}{3}\frac{m_H}{k}\frac{\langle\epsilon_d\rangle}{\rho}. \tag{5·9}$$

If the condition $t \gg m_e c/(bT^4\sigma_T)$ is satisfied, the above equation is solved as[8]

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*) The expression (5·8) is correct, if the radiation spectrum is the Planck distribution. In more general cases, ϵ_c may be smaller than Eq. (5·8). Such an appreciable modification of the spectrum is possible only when an amount of energy of order $bT_r^3T_m$ is supplied to the radiation. However, an available energy in our case is smaller than the radiation energy, and Eq. (5·8) may remain correct in our problem.

$$T_m - T_r \simeq \frac{m_e c}{4bT^4\sigma_T} \frac{m_H\langle\epsilon_d\rangle}{\rho\kappa}. \tag{5·10}$$

Substituting Eq. (5·6), we have, writing $v_9 = v/10^9$ cm/sec,

$$T_m - T_r \simeq \frac{10^{2.8}}{\Omega}\left(\frac{10^3}{z+1}\right) v_9{}^2\ {}^\circ\mathrm{K} \qquad \text{for } bT^4 \gg \rho c^2$$

$$\simeq 10^{5.0}\Omega^{1/2}\left(\frac{10^3}{z+1}\right)^2 v_9{}^2\ {}^\circ\mathrm{K} \qquad \text{for } \rho c^2 \gg bT^4. \tag{5·11}$$

The heating is more effective in a later stage. The last epoch where our theory is applicable is the epoch of recombination of plasma at $z = z_D (\equiv 1.5 \cdot 10^3)$. Therefore, we have the maximum temperature, considering $v < c_s$, as

$$T_m - T_r < 10^{5.8}/\Omega^{1/2}\ {}^\circ\mathrm{K} \qquad \text{for } \Omega > 10^{-1.1}$$

and

$$T_m - T_r < 10^{5.2}/\Omega\ {}^\circ\mathrm{K} \qquad \text{for } \Omega < 10^{-1.1}. \tag{5·12}$$

If T_m is maintained to be larger than $10^4\ {}^\circ$K by the heating, the matter will remain in an ionized state by collisional ionization.[15] Such a delay of the recombination might occur if $v > c_s/10$ from Eq. (5·12). In the case of the spectrum as shown in Fig. 3, the maximum temperature of Eq. (5·12) is further reduced by a factor k_d/k_J and it seems impossible to maintain $T_m > 10^4\ {}^\circ$K after $z = z_D$.

§ 6. Formation of bound systems and their masses

In the earlier sections, our consideration has been restricted to the stage before the recombination of plasma. However, the gravitationally bound systems such as galaxies or galaxy clusters are considered to be formed at the stage when the pressure of the primeval cosmic radiation was decoupled with the motion of matter.[16] By the recombination of plasma, the decoupling of the radiation is suddenly realized. At this epoch, the sound velocity decreases from that by the radiation pressure c_s into that by the matter pressure $c_{sm} = \sqrt{5p_m/3\rho}$, p_m being the matter pressure, and the fluidal velocity of matter changes from subsonic into supersonic one if $c_s > v > c_{sm}$.[6] The hydrodynamic instability will arise during the stage of this sudden decrease of the sound velocity in matter, and the strong density inhomogeneity will appear, which evolves into the gravitationally bound systems.[6),7),17]

The criterion of the hydrodynamic instability is given at the time of decoupling as

$$t_u < t \tag{6·1}$$

and

$$c_s > v > c_{sm} . \tag{6·2}$$

Equation (6·1) puts a restriction on the size of the formed objects. The maximum mass $M_{\max}$ has been discussed by us for the vortical motions[7] and by Chernin for the acoustic motions.[17] Equation (6·2) puts a restriction about the amplitude of inhomogeneity before the decoupling.

For the acoustic motions, the condition $v > c_{sm}$ at z_D can be written as

$$\frac{\rho_1}{\rho} > \sqrt{\Omega}\, 10^{-4.1} \text{ and } 10^{-4.6} \text{ for } \Omega > 10^{-1.1} \text{ and } \Omega < 10^{-1.1} \text{ respectively}. \tag{6·3}$$

As the size spectrum has a sharp cutoff at the small mass, the minimum mass $M_{\min}$ is determined. From these discussions, we get the relations

$$M_{\max} \simeq M_J(z_D) \left(\frac{\rho_1}{\rho}\right)^3, \qquad M_{\min} \simeq \frac{1}{10} M_d(z_D) \tag{6·4}$$

for the acoustic motions, and

$$M_{\max} \simeq M_J(z_D) \left(\frac{v}{c_s}\right)^3, \qquad M_{\min} \simeq \frac{1}{10} M_d(z_D) \left(\frac{c_s}{c}\right)^3 \tag{6·5}$$

for the vortical motions, where

$$M_J(z_D) = 10^{16.7} \Omega^{-2}, \quad M_d(z_D) = 10^{12.7} \Omega^{-5/4} \quad \text{for } \Omega > 10^{-1.1}, \tag{6·6}$$

$$M_J(z_D) = 10^{19.6} \Omega, \quad M_d(z_D) = 10^{13.1} \Omega^{-1/2} \quad \text{for } \Omega < 10^{-1.1}, \tag{6·7}$$

in solar mass unit. The factor 1/10 of $M_{\min}$ in Eqs. (6·4) and (6·5) is not a well-founded value. If the degree of inhomogeneity at z_D is $\rho_1/\rho \simeq 10^{-1} \sim 10^{-2}$ or $v/c_s \simeq 10^{-1} \sim 10^{-2}$, the above relations give a reasonable value as the mass of a galaxy or a cluster of galaxies. Apparently these masses is larger than the mass of a galaxy, but it is plausible that the loosely bound pre-galactic cloud has a larger mass than that of the compactly bound current galaxy. However, we do not give any discussion on the problem of whether the pre-galactic cloud is formed by the above stated manner in a galaxy or in a cluster of galaxies.

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Appendix

Formulation of relativistic hydrodynamics with dissipation has been given by Eckart[18] and many other authors.[19] We summarize this formalism to derive the equation of motion for radiative gas.

Including viscosity and thermal conduction, the energy-momentum tensor is written as

$$T^{ab}=\varepsilon U^a U^b+p h^{ab}+\tau^{ab}+\frac{1}{c}(q^a U^b+q^b U^a), \quad (A\cdot 1)$$

where ε is energy density, p pressure, $h^{ab}=g^{ab}+U^a U^b$, q^a heat flow vector, U^a four-velocity such as $U^a U_a=1$ and τ^{ab} viscous tensor defined as

$$\tau^{ab}=\eta\{h^{ac}h^{bd}(U_{c;d}+U_{d;c})-\tfrac{2}{3}h^{ab}h^{cd}U_{c;d}\}, \quad (A\cdot 2)$$

where η is a coefficient of viscocity. Divergence of the energy-momentum tensor can be written as

$$T^{ab}{}_{;b}\equiv\Phi U^a+\Psi^a=0, \quad (A\cdot 3)$$

where Φ and Ψ^a are such a scalar and a vector given later. If Ψ^a satisfies such a condition as

$$\Psi^a U_a=0 \quad (A\cdot 4)$$

we get from Eq. (A·3)

$$\Phi=D\varepsilon+(\varepsilon+p)U^b_{;b}+U^b_{;c}\tau_b{}^c+\frac{1}{c}(q^b_{;b}+q_b DU^b)=0, \quad (A\cdot 5)$$

$$\Psi^a=(\varepsilon+p)DU^a+h^{ab}(p_{,b}+\tau^c_{b;c})+\frac{1}{c}\{h^{ab}Dq_b+U^a_{;b}q^b+U^b_{;b}q^a\}=0, \quad (A\cdot 6)$$

where $D\varepsilon\equiv U^a\varepsilon_{,a}$, $DU^a\equiv U^b U^a_{;b}$ and so on. As seen from Eq. (A·6), Eq. (A·4) requires that

$$q^a U_a=0. \quad (A\cdot 7)$$

Next, we rewrite Eq. (A·5) using the specific entropy S as

$$\rho DS+\left(\frac{q^a}{cT}\right)_{;a}=\frac{1}{T}\tau_b{}^c U^b{}_{;c}-\frac{q^a}{T^2 c}(T_{,a}+TDU_a), \quad (A\cdot 8)$$

where ρ and T are mass density and temperature respectively. The second law of thermodynamics requires that the right-hand side of Eq. (A·8) is always positive. The first term representing the heat production by viscosity can be proved positive. Requireing Eq. (A·7) and the positive definiteness of the second term of the right-hand side of Eq. (A·8), we can put an appropriate expression for q^a as

$$q^a=-Kh^{ab}(T_{,b}+TDU_b), \quad (A\cdot 9)$$

where K is thermal conductivity. This generalization of Fourier's law to relativistic thermodynamics was firstly given by Eckart.[18]

Thus, we have get the basic equations in Eqs. (A·5), (A·6) and (A·9).

The full set of equaions is completed by adding them the equation of state and the conservation of particle number such as

$$\rho D\frac{1}{\rho}=U^b{}_{;b}\,. \tag{A·10}$$

Here, we give a brief comment on a relation between the expression of heat flow, Eq. (A·9), and the thermal equilibrium. If there is not heat flow in some region, it implies that a thermal equilibrium is attained between this region and the ambient region. If we put $q^\alpha=0$ in the comoving coordinate (the Greek letter denoting $\alpha=1, 2, 3$), Eq. (A·9) gives the temperature gradient in thermal equilibrium as

$$\frac{\partial T\sqrt{-g_{00}}}{\partial x^\alpha}=0\,, \tag{A·11}$$

where we have assumed $g_{0\alpha}=0$ and used a relation

$$DU_\alpha=\frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^\alpha}\,.$$

Equation (A·11) has been known from other considerations as a temperature gradient at thermal equilibrium.[20] Thus, Eckart's expression for q^α is proved to be consistent with the condition of thermal equilibrium.

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