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MASS PROBLEMS ASSOCIATED WITH EFFECTIVELY CLOSED SETS

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Abstract. The study of mass problems and Muchnik degrees was originally motivated by Kolmogorov's non-rigorous 1932 interpretation of intuitionism as a calculus of problems. The purpose of this paper is to summarize recent investigations into the lattice of Muchnik degrees of nonempty effectively closed sets in Euclidean space. Let $\mathcal{E}_{\mathrm{w}}$ be this lattice. We show that $\mathcal{E}_{\mathrm{w}}$ provides an elegant and useful framework for the classification of certain foundationally interesting problems which are algorithmically unsolvable. We exhibit some specific degrees in $\mathcal{E}_{\mathrm{w}}$ which are associated with such problems. In addition, we present some structural results concerning the lattice $\mathcal{E}_{\mathrm{w}}$. One of these results answers a question which arises naturally from the Kolmogorov interpretation. Finally, we show how $\mathcal{E}_{\mathrm{w}}$ can be applied in symbolic dynamics, toward the classification of tiling problems and $\boldsymbol{Z}^d$-subshifts of finite type.

CONTENTS

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1. Introduction.

1.1. Turing degrees. In his ground-breaking 1936 paper [115], Turing proved the existence of mathematical problems which are algorithmically unsolvable. Actually Turing exhibited a specific algorithmically unsolvable problem, known as *the halting problem*. During the years 1950–1970 it was discovered that algorithmically unsolvable problems exist in virtually every branch of mathematics: group theory [80, 1], number theory [24], analysis, combinatorics [5, 82], geometry [73, Appendix], topology, mathematical logic [112], and even elementary calculus [81]. Among the specific, natural, unsolvable problems which were discovered during this period are: the Entscheidungsproblem for logical validity in the predicate calculus, the triviality problem for finitely presented groups, Hilbert's Tenth Problem in number theory [39], the domino problem, the homeomorphism problem for finite simplicial complexes, the diffeomorphism problem for compact manifolds, and the problem of integrability in elementary terms.

In an influential 1954 paper [56], Kleene and Post introduced a scheme for classifying unsolvable mathematical problems. Informally, by a *real* we mean a point in an effectively presented complete separable metric space. For instance, a *real* in our sense could be a real number, or a set of natural numbers, or a sequence of natural numbers, or a set of finite strings of symbols from a fixed finite alphabet, or a point in d-dimensional Euclidean space $[0, 1]^d$ where d is a positive integer, or a (code for a) point in $\mathrm{C}([0, 1]^d)$ or $L_p([0, 1]^d)$ for $1 \leq p < \infty$, or (a code for) an infinite sequence of real numbers, or a (code for a) Borel probability measure on $[0, 1]^d$, etc.

Two reals are said to be *Turing equivalent* if each is computable using the other as a Turing oracle. (We follow the terminology of Rogers [83].) According to Kleene and Post [56], the *Turing degree* of a real is its equivalence class under this equivalence relation. Each of the specific unsolvable problems mentioned in the previous paragraph is a *decision problem* and may therefore be straightforwardly described or "encoded" as a real. (More specifically, each of the mentioned problems amounts to the question of deciding whether a given string of symbols from a fixed finite alphabet belongs to a particular set of such strings. For instance, the triviality problem for finitely presented groups may be described in terms of the set of all finite presentations of the trivial group.) Once this has been done, it can be shown that each of these problems is of the same Turing degree as the halting problem. This Turing degree is denoted $\mathbf{0}'$. Thus the specific Turing degree $\mathbf{0}'$ is extremely useful and important.

Let $\mathcal{D}_{\mathrm{T}}$ be the set of all Turing degrees. For each real x, the Turing degree of x is denoted $\deg_{\mathrm{T}}(x)$. If $\mathbf{a}$ and $\mathbf{b}$ are the Turing degrees of reals x and y respectively, we write $x \leq_{\mathrm{T}} y$ or

$\mathbf{a} \leq \mathbf{b}$ to mean that y is "at least as unsolvable as" x in the following sense: x is computable using y as a Turing oracle. We also write $x <_{\mathrm{T}} y$ or $\mathbf{a} < \mathbf{b}$ to mean that $x \leq_{\mathrm{T}} y$ and $y \not\leq_{\mathrm{T}} x$. Kleene and Post proved that $\leq$ is a partial ordering of $\mathcal{D}_{\mathrm{T}}$ and every finite set of Turing degrees in $\mathcal{D}_{\mathrm{T}}$ has a supremum with respect to $\leq$. They also proved that there are infinitely many Turing degrees which are less than $\mathbf{0}'$, and there are uncountably many other Turing degrees which are incomparable with $\mathbf{0}'$. Thus $\mathcal{D}_{\mathrm{T}}$ has a rich algebraic structure. However, the Turing degrees which are less than $\mathbf{0}'$ or incomparable with $\mathbf{0}'$ have turned out to be almost useless for the classification of specific algorithmically unsolvable problems.

Given a real x, let x' be a real which encodes the halting problem *relative to* x, i.e., with x used as a Turing oracle. If $\mathbf{a}$ is the Turing degree of x, let $\mathbf{a}'$ be the Turing degree of x'. It can be shown that $\mathbf{a}'$ is independent of the choice of x. The operator $\mathbf{a} \mapsto \mathbf{a}'$ from Turing degrees to Turing degrees is known as the *Turing jump* operator. Generalizing Turing's proof of the unsolvability of the halting problem, one may show that $\mathbf{a} < \mathbf{a}'$. In other words, $\mathbf{a}'$ is "more unsolvable than" $\mathbf{a}$. See for instance [83, §13.1]. Inductively we write $\mathbf{a}^{(0)} = \mathbf{a}$ and $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$ for all natural numbers n. Extending this induction into the transfinite, it is possible to define $\mathbf{a}^{(\alpha)}$ where α ranges over a rather large initial segment of the ordinal numbers. We then have $\mathbf{a}^{(\alpha)} < \mathbf{a}^{(\beta)}$ whenever $\alpha < \beta$. See for instance [86, Part A] and [47, 92].

Let $\mathbf{0}$ be the bottom degree in $\mathcal{D}_{\mathrm{T}}$. Thus $\mathbf{0}$ is the Turing degree of any computable real. By repeatedly applying the Turing jump operator, we obtain a tranfinite hierarchy of specific, natural Turing degrees

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$$

where α ranges over a large initial segment of the ordinal numbers including all of the constructibly countable ones [47, 92]. Moreover, this hierarchy of specific, natural Turing degrees has been somewhat useful for classifying unsolvable mathematical problems. See for instance [83, §14.8] and [74]. However, no other specific Turing degrees have been useful in this regard.

Summarizing, one may say that the Kleene/Post program of using Turing degrees to classify unsolvable mathematical problems has met with significant but limited success. The Turing degrees $\mathbf{0}$ and $\mathbf{0}'$ have been extremely useful, and the Turing degrees $\mathbf{0}''$, $\mathbf{0}'''$, $\ldots$, $\mathbf{0}^{(\alpha)}$, $\mathbf{0}^{(\alpha+1)}$, $\ldots$ have been somewhat useful, but the other Turing degrees have not been useful at all.

1.2. Muchnik degrees. In 1955 and 1963 respectively, Medvedev and Muchnik [67, 71] introduced two extended degree structures based on mass problems. Regrettably, these alternative structures were largely ignored outside the Soviet Union for a long time. However, over the past 10 years we have learned that the Muchnik degrees are capable of providing an elegant and useful framework for the classification of foundationally interesting problems. Many of these problems are impossible to classify using Turing degrees, and in such cases the Muchnik degrees have emerged as the appropriate classification tool.

The essential concepts are as follows. Let P be a set of reals. We may view P as a *mass problem*, viz., the problem of "finding" a real which belongs to P. In this sense, a "solution" of the problem P is any real $x \in P$. Accordingly, a mass problem P is said to be (algorithmically) *solvable* if there exists a real $x \in P$ which is Turing computable. Furthermore, a mass problem Q is said to be (algorithmically) *reducible* to a mass problem P if each $x \in P$ can be used as a Turing oracle to compute some $y \in Q$. This is Muchnik's notion of *weak reducibility* [71], denoted $\leq_{\mathrm{w}}$. Thus we have

$$P \geq_{\mathrm{w}} Q \quad \text{if and only if} \quad (\forall x \in P)\,(\exists y \in Q)\,(x \geq_{\mathrm{T}} y)\,.$$

We define a *Muchnik degree* to be an equivalence class of mass problems under weak reducibility. The Muchnik degree of a mass problem P is denoted $\deg_{\mathrm{w}}(P)$.

The partial ordering of all Muchnik degrees under weak reducibility is denoted $\mathcal{D}_{\mathrm{w}}$. It can be shown that $\mathcal{D}_{\mathrm{w}}$ is a *lattice* in the sense of Birkhoff [12, 13, 14], i.e., each finite set of degrees in $\mathcal{D}_{\mathrm{w}}$ has a supremum and an infimum in $\mathcal{D}_{\mathrm{w}}$. The top degree in $\mathcal{D}_{\mathrm{w}}$ is $\infty = \deg_{\mathrm{w}}(\emptyset)$ where $\emptyset$ denotes the empty set. The bottom degree in $\mathcal{D}_{\mathrm{w}}$ is $\mathbf{0} = \deg_{\mathrm{w}}(S)$ where S is any solvable mass problem.

There is an obvious embedding of the Turing degrees into the Muchnik degrees, given by $\deg_{\mathrm{T}}(x) \mapsto \deg_{\mathrm{w}}(\{x\})$. Here $\{x\}$ is the singleton set consisting of the real x. This embedding of $\mathcal{D}_{\mathrm{T}}$ into $\mathcal{D}_{\mathrm{w}}$ is one-to-one and preserves essential algebraic structure including $\mathbf{0}$, $\leq$, $\nleq$, the jump operator, and finite suprema. Accordingly, we identify each Turing degree $\deg_{\mathrm{T}}(x)$ with its corresponding Muchnik degree $\deg_{\mathrm{w}}(\{x\})$.

(The *jump operator* on $\mathcal{D}_{\mathrm{w}}$ may be defined as $\mathbf{p} = \deg_{\mathrm{w}}(P) \mapsto \mathbf{p}' = \deg_{\mathrm{w}}(P')$ where $P' = \{x'\,;\ x \in P\}$. Clearly $\mathbf{p} \leq \mathbf{p}'$, but examples show that $\mathbf{p} < \mathbf{p}'$ is not always the case. For instance, let $\mathbf{p} = \inf_n \mathbf{a}_n$ where $\mathbf{a}_n$ for $n = 0, 1, 2, \ldots$ is a sequence of Turing degrees with the property that $\mathbf{a}'_{n+1} \leq \mathbf{a}_n$ for all n. Then $\mathbf{p} = \mathbf{p}'$. The existence of such sequences of Turing degrees is well known. See for instance [109].)

We end this section by mentioning some foundationally interesting examples of Muchnik degrees which are not Turing degrees.

First, let T be a consistent theory which is axiomatizable and effectively essentially undecidable. (For instance, we could take $T = \mathsf{PA} =$ Peano arithmetic, or $T = \mathsf{ZFC} =$ Zermelo/Fraenkel set theory with the Axiom of Choice, or $T = \mathsf{Q} =$ Robinson arithmetic [112], or $T =$ any consistent axiomatizable extension of one of these.) By Gödel's First Incompleteness Theorem [37] (see also [112]), we know that T is incomplete. Let $\mathrm{C}(T)$ be the problem of finding a *completion* of T, i.e., a complete, consistent theory which includes T. By [112] the problem $\mathrm{C}(T)$ is algorithmically unsolvable, and by [36] it is impossible to assign a Turing degree to $\mathrm{C}(T)$. However, since $\mathrm{C}(T)$ may be viewed as a mass problem, it is clear how to assign a Muchnik degree to $\mathrm{C}(T)$. Indeed, by [36, 88, 89] we know that the Muchnik degree of $\mathrm{C}(T)$ is independent of the choice of T. (Indeed, the recursive homeomorphism type of $\mathrm{C}(T)$ is independent of the choice of T. See also [79], [97, §3] and [96, §6].) Thus we have a particular Muchnik degree, denoted $\mathbf{1}$, which is of obvious foundational interest.

(By [25] we may also characterize $\mathbf{1}$ as the Muchnik degree of the problem of finding a probability measure ν on $\{0,1\}^{\boldsymbol{N}}$ which is *neutral*, i.e., every $x \in \{0,1\}^{\boldsymbol{N}}$ is Martin-Löf random with respect to ν.) From [112, 36] it is known that $\mathbf{0} < \mathbf{1} < \mathbf{0}'$. In other words, the problem of finding a completion of T is unsolvable but not so unsolvable as the halting problem.

Second, consider the problem of finding a real which is *random* in the sense of Martin-Löf [66] (see also [98, 76, 28]). This problem is denoted MLR. Clearly each random real is noncomputable, so the problem MLR is algorithmically unsolvable. Moreover, as in the case of $\mathrm{C}(T)$, there is no way to associate a Turing degree to MLR. On the other hand, there is a Muchnik degree $\mathbf{r}_1$ associated to MLR, and by [49, Theorem 5.3] (see also [96]) we know that this Muchnik degree is strictly less than the Muchnik degree of $\mathrm{C}(T)$. Thus we have

$$\mathbf{0} < \mathbf{r}_1 < \mathbf{1} < \mathbf{0}'$$

where $\mathbf{r}_1 = \deg_{\mathrm{w}}(\mathrm{MLR})$.

Third, let K be any class of isomorphism types of algebraic and/or relational structures. There is an obvious Muchnik degree $\mathbf{s}_K \in \mathcal{D}_{\mathrm{w}}$ associated with K. Namely, let $\mathbf{s}_K = \deg_{\mathrm{w}}(\mathrm{Str}_{\boldsymbol{N}}(K))$ where $\mathrm{Str}_{\boldsymbol{N}}(K)$ is the problem of finding a structure M such that (1) the isomorphism type of M belongs to K, and (2) the universe of M is $\boldsymbol{N} = \{0, 1, 2, \dots\}$ = the set of natural numbers. See for instance [110]. Sometimes $\mathbf{s}_K$ is a Turing degree, but often it is not. For example, let K consist of the single isomorphism type ω_1^{CK} = Church/Kleene ω_1 = the least noncomputable transfinite ordinal number. It can be shown that the Muchnik degree $\mathbf{s}_{\omega_1^{\mathrm{CK}}}$ is not a Turing degree. As another example, let T be any of the subsystems of second-order arithmetic considered in [95], and let $K = \mathrm{Mod}_\omega(T)$ = the class of ω-models of T. Again, the Muchnik degree $\mathbf{s}_{\mathrm{Mod}_\omega(T)}$ is not a Turing degree.

The above examples suggest the possible existence of a great many interesting Muchnik degrees associated with specific unsolvable problems. This possibility has been explored over the past 10 years. Some of the resulting Muchnik degrees and their relationships are exhibited in Figure 1 below.

1.3. Effectively closed sets. The lattice $\mathcal{D}_{\mathrm{w}}$ is very large and complicated. (For example, the cardinality of $\mathcal{D}_{\mathrm{w}}$ is $2^{2^{\aleph_0}}$.) In order to obtain a sublattice of $\mathcal{D}_{\mathrm{w}}$ which is smaller and more manageable, we follow the lead of effective descriptive set theory [40, 65, 69] and consider mass problems which are "effectively definable" in some appropriate sense.

Recall that a mass problem is any set in an effectively presented complete separable metric space. Let X be such a space. A set $U \subseteq X$ is said to be *effectively open* if there exist computable sequences of computable points $a_n \in X$ and computable real numbers $r_n \in \boldsymbol{R}$ with $n = 0, 1, 2, \dots$ such that

$$U = \bigcup_{n=0}^{\infty} B(a_n, r_n)\,.$$

Here we are writing $B(a, r) = \{x \in X\,;\ \mathrm{dist}(a, x) < r\}$ where $\mathrm{dist}(x, y)$ = the distance between two points $x, y \in X$. A set $C \subseteq X$ is said to be *effectively closed* if its complement $X \setminus C$ is effectively open.

For example, if X is d-dimensional Euclidean space $\boldsymbol{R}^d$, we may assume that a_n and r_n are rational, i.e., $a_n \in \boldsymbol{Q}^d$ and $r_n \in \boldsymbol{Q}$. Similarly, if X is the Cantor space $\{0, 1\}^{\boldsymbol{N}}$ or the Baire space $\boldsymbol{N}^{\boldsymbol{N}}$, the effectively open sets in X are of the form $U = \bigcup_{\sigma \in S} N_\sigma$ where S is a recursively enumerable set of finite strings $\sigma \in \{0, 1\}^*$ or $\sigma \in \boldsymbol{N}^*$ respectively. Here we are writing

$$N_\sigma = \{x \in X\,;\ \sigma = x\restriction\{0, \ldots, |\sigma| - 1\}\}$$

where $|\sigma|$ = the length of σ.

Clearly every nonempty effectively open set is of Muchnik degree **0**. However, we shall see that there exist effectively closed sets of infinitely many different Muchnik degrees.

In this paper we consider mainly the case when X is *effectively compact*. For example, $X = [0, 1]^d$ = the d-dimensional unit cube, or $X = \{0, 1\}^{\boldsymbol{N}}$ = the Cantor space, or X = the weak-star unit ball in the dual space of $\mathrm{C}[0, 1]^d$ or of $L_p([0, 1]^d)$ where $1 \leq p < \infty$. It can be shown that each effectively closed set in an effectively compact, complete separable metric space is Muchnik equivalent to an effectively closed set in the Cantor space. Accordingly, we define $\mathcal{E}_{\mathrm{w}}$ to be the sublattice of $\mathcal{D}_{\mathrm{w}}$ consisting of the Muchnik degrees of all nonempty effectively closed sets in the Cantor space.

It is interesting to compare $\mathcal{E}_{\mathrm{w}}$ with $\mathcal{E}_{\mathrm{T}}$, the subsemilattice of $\mathcal{D}_{\mathrm{T}}$ consisting of the recursively enumerable Turing degrees. One knows that $\mathcal{E}_{\mathrm{T}}$ has been studied extensively in many publications including [85, 83, 107, 77, 61]. I have shown [99] that $\mathcal{E}_{\mathrm{w}}$ is analogous to $\mathcal{E}_{\mathrm{T}}$ and contains a naturally isomorphic copy of $\mathcal{E}_{\mathrm{T}}$. But I have also shown [96, 99, 100, 102, 22, 103] that $\mathcal{E}_{\mathrm{w}}$, unlike $\mathcal{E}_{\mathrm{T}}$, contains many specific, natural degrees which are associated with natural, foundationally interesting, unsolvable problems. (In many of these papers I used the notation $\mathcal{P}_{\mathrm{w}}$ instead of $\mathcal{E}_{\mathrm{w}}$. I now say $\mathcal{E}_{\mathrm{w}}$ in order to emphasize the analogy with $\mathcal{E}_{\mathrm{T}}$, the semilattice of recursively enumerable Turing degrees.)

(In the same vein one may compare the lattice of Π^0_1 sets in $\{0, 1\}^{\boldsymbol{N}}$ (see for instance [19] and [118]) with the lattice of (complements of) recursively enumerable subsets in $\boldsymbol{N}$ (see for instance [83, Chapter XII] and [107, Chapters X, XI, XV]). In particular, by [96, §9] and [7, 8, 9] we know that certain "smallness properties" of a nonempty Π^0_1 set $P \subseteq \{0, 1\}^{\boldsymbol{N}}$ imply $\mathbf{0} < \mathbf{p} < \mathbf{1}$ where $\mathbf{p} = \deg_{\mathrm{w}}(P) \in \mathcal{E}_{\mathrm{w}}$. The analogous issue for $\mathcal{E}_{\mathrm{T}}$ remains unresolved [102]. See also the discussion of Post's Program in [107].)

The history of $\mathcal{E}_{\mathrm{w}}$ is that I first defined it in 1999 [93, 94]. At the time I noted that $\mathcal{E}_{\mathrm{w}}$ is a countably infinite sublattice of $\mathcal{D}_{\mathrm{w}}$ and that **1** and **0** are the top and bottom degrees in $\mathcal{E}_{\mathrm{w}}$. I also observed that there is at least one other specific, natural degree in $\mathcal{E}_{\mathrm{w}}$, namely $\mathbf{r}_1$. Moreover $\mathcal{E}_{\mathrm{w}}$ is essentially disjoint from $\mathcal{E}_{\mathrm{T}}$, because the only Turing degree belonging to $\mathcal{E}_{\mathrm{w}}$ is **0**. These observations were implicit in the much earlier work of Gandy/Kreisel/Tait [36], Scott/Tennenbaum [88, 89], Jockusch/Soare [49, 48], and Kučera [60]. My contribution in 1999 was to define the lattice $\mathcal{E}_{\mathrm{w}}$ and to call attention to it as a more fruitful alternative to the much-studied semilattice $\mathcal{E}_{\mathrm{T}}$. Later I discovered the existence of many other specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ as illustrated in Figure 1 below. My embedding of $\mathcal{E}_{\mathrm{T}}$ into $\mathcal{E}_{\mathrm{w}}$ [99] was obtained as a byproduct.

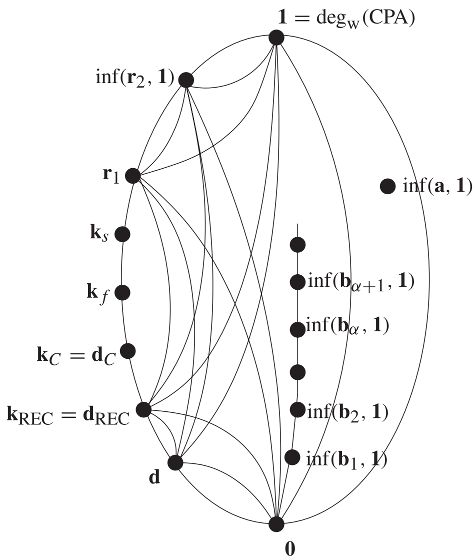

FIGURE 1. A picture of $\mathcal{E}_{\mathrm{w}}$.

This paper is essentially a summary of what I have learned about $\mathcal{E}_{\mathrm{w}}$ over the past 10 years. An obvious reason for undertaking the study of $\mathcal{E}_{\mathrm{w}}$ is that it is the smallest and simplest nontrivial sublattice of $\mathcal{D}_{\mathrm{w}}$ which presents itself in terms of effective descriptive set theory. Beyond this, we shall see that $\mathcal{E}_{\mathrm{w}}$ is a rich and useful structure in its own right.

Here is an outline of the rest of this paper. In Section 2 we exhibit a large variety of specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$. In Section 3 we explore some structural and methodological aspects of $\mathcal{E}_{\mathrm{w}}$. In Section 4 we discuss the original intuitionistic motivation for the study of $\mathcal{D}_{\mathrm{w}}$. In Section 5 we discuss an application of $\mathcal{E}_{\mathrm{w}}$ in the study of tiling problems and symbolic dynamics.

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2. Some specific Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$. Recall from Subsection 1.3 that $\mathcal{E}_{\mathrm{w}}$ is the lattice of Muchnik degrees of nonempty effectively closed sets in the Cantor space. The purpose of this section is to exhibit some specific degrees in $\mathcal{E}_{\mathrm{w}}$ which are associated with specific, natural, algorithmically unsolvable problems. Figure 1 represents the lattice $\mathcal{E}_{\mathrm{w}}$. Each of the black dots except the one labeled $\inf(\mathbf{a}, \mathbf{1})$ represents a specific, natural, Muchnik degree in $\mathcal{E}_{\mathrm{w}}$. We shall now explain each of these black dots.

As noted in Subsections 1.2 and 1.3, the top degree in $\mathcal{E}_{\mathrm{w}}$ is $\mathbf{1} = \deg_{\mathrm{w}}(\mathrm{CPA})$ where CPA is the problem of finding a completion of Peano arithmetic. The bottom degree in $\mathcal{E}_{\mathrm{w}}$ is $\mathbf{0} = \deg_{\mathrm{w}}(S)$ where S is any solvable mass problem. Given two Muchnik degrees $\mathbf{p}$ and $\mathbf{q}$, let $\sup(\mathbf{p}, \mathbf{q})$ and $\inf(\mathbf{p}, \mathbf{q})$ be the Muchnik degrees which are the least upper bound and the greatest lower bound of $\mathbf{p}$ and $\mathbf{q}$ respectively. In [99] I proved that there is a natural one-to-one embedding of the recursively enumerable Turing degrees into $\mathcal{E}_\mathrm{w}$ given by $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1})$. Since the semilattice $\mathcal{E}_\mathrm{T}$ is known to contain infinitely many Turing degrees (see for instance [85]), my embedding of $\mathcal{E}_\mathrm{T}$ into $\mathcal{E}_\mathrm{w}$ implies the existence of infinitely many Muchnik degrees in $\mathcal{E}_\mathrm{w}$. However, since no *specific* recursively enumerable Turing degrees other than $\mathbf{0}'$ and $\mathbf{0}$ are known, my embedding of $\mathcal{E}_\mathrm{T}$ into $\mathcal{E}_\mathrm{w}$ yields no *specific* examples of Muchnik degrees in $\mathcal{E}_\mathrm{w}$ other than $\mathbf{1}$ and $\mathbf{0}$.

2.1. Algorithmic randomness. Historically, the first example of a specific Muchnik degree in $\mathcal{E}_\mathrm{w}$ other than $\mathbf{1}$ and $\mathbf{0}$ was $\mathbf{r}_1 = \deg_\mathrm{w}(\mathrm{MLR})$ = the Muchnik degree of the problem of finding an infinite sequence of 0's and 1's which is random in the sense of Martin-Löf [66].

A related example is as follows. Let $\mathbf{r}_2 = \deg_\mathrm{w}(\mathrm{MLR}_2)$ where

$$\mathrm{MLR}_2 = \{x \in \{0,1\}^N\,;\ x \text{ is Martin-Löf random relative to } 0'\}\,.$$

Here $0'$ denotes the halting problem. One can easily show that $\mathbf{r}_2$ does not belong to $\mathcal{E}_\mathrm{w}$. However, it turns out [96, 99] that $\inf(\mathbf{r}_2, \mathbf{1})$ belongs to $\mathcal{E}_\mathrm{w}$. Moreover, as indicated in Figure 1, we have $\mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1}$. Thus $\inf(\mathbf{r}_2, \mathbf{1})$ is another specific, natural degree in $\mathcal{E}_\mathrm{w}$ which, like $\mathbf{r}_1$, is characterized in terms of algorithmic randomness.

Alternatively, we may characterize $\mathbf{r}_1$ as the maximum Muchnik degree of an effectively closed subset of $\{0,1\}^N$ which is of positive measure. Similarly, we may characterize $\inf(\mathbf{r}_2, \mathbf{1})$ as the maximum Muchnik degree of an effectively closed subset of $\{0,1\}^N$ whose Turing upward closure is of positive measure. See [99, Theorem 3.8].

2.2. Kolmogorov complexity. As already meantioned, $\mathbf{r}_1$ is the Muchnik degree in $\mathcal{E}_\mathrm{w}$ corresponding to Martin-Löf randomness. Subsequently, many other Muchnik degrees in $\mathcal{E}_\mathrm{w}$ of a similar nature were discovered. We shall now develop some of these examples, using the concept of Kolmogorov complexity.

Kolmogorov complexity [28, 76, 62] is a way of measuring the "information content" of a finite mathematical object. The key definitions are as follows. Let $\{0,1\}^*$ be the set of finite sequences of 0's and 1's, i.e., words on the alphabet $\{0,1\}$. We define a *machine* to be a partial recursive function M from $\{0,1\}^*$ to $\{0,1\}^*$. A *universal machine* is a machine U with the property that for all machines M there exists a word $\rho \in \{0,1\}^*$ such that $M(\sigma) = U(\rho^\frown\sigma)$ for all σ in the domain of M. Here $\rho^\frown\sigma$ denotes the *concatenation*, ρ followed by σ, i.e.,

$$\rho^\frown\sigma = \langle \rho(0), \ldots, \rho(|\rho|-1), \sigma(0), \ldots, \sigma(|\sigma|-1)\rangle$$

where $|\sigma|$ = the length of σ. Note that $|\rho^\frown\sigma| = |\rho| + |\sigma|$. It is straightforward to prove the existence of a universal machine. Now let U be a fixed universal machine. By a *description* of $\tau \in \{0,1\}^*$ we mean any $\sigma \in \{0,1\}^*$ such that $U(\sigma) = \tau$. We then define the *complexity* of τ, measured in bits, to be the smallest length of a description of τ. In other words, the complexity of τ is $\mathrm{C}(\tau) = \min\{|\sigma|\,;\ U(\sigma) = \tau\}$. It is straightforward to show that $\mathrm{C}(\tau)$ is, in a sense, asymptotically independent of our choice of a fixed universal machine U. Namely, letting U_1 and U_2 be any two universal machines, and letting $\mathrm{C}_1(\tau)$ and $\mathrm{C}_2(\tau)$ be the complexity of τ as defined in terms of U_1 and U_2 respectively, we have $\exists c\, \forall\tau\, (|\mathrm{C}_1(\tau) - \mathrm{C}_2(\tau)| \leq c)$. In other words, the complexity of τ is well defined up to within an additive constant. An easy argument shows that the complexity of τ is bounded by the length of τ plus a constant, i.e., $\exists c\, \forall \tau\, (\mathrm{C}(\tau) \leq |\tau| + c)$.

For technical reasons it is convenient to consider a "prefix-free" variant of $\mathrm{C}(\tau)$. A machine M is said to be *prefix-free* if the domain of M contains no pair ρ, σ such that ρ is a proper initial segment of σ. A *universal prefix-free machine* is a prefix-free machine U such that for all prefix-free machines M there exists ρ such that $M(\sigma) = U(\rho^\frown\sigma)$ for all σ in the domain of M. The *prefix-free complexity* of τ is defined as $\mathrm{K}(\tau) = \min\{|\sigma|\, ;\, U(\sigma) = \tau\}$ where U is a universal prefix-free machine. As in the case of $\mathrm{C}(\tau)$, it is straightforward to show that $\mathrm{K}(\tau)$ is well defined up to plus or minus a constant. Clearly $\mathrm{C}(\tau)$ and $\mathrm{K}(\tau)$ are closely related. For example, one can show that $\mathrm{C}(\tau) \leq \mathrm{K}(\tau) \leq \mathrm{C}(\tau) + 2\log_2 |\tau|$ up to additive constants.

Now let x be a point in the Cantor space, i.e., an infinite sequence of 0's and 1's. By the *initial segment complexity* of x we mean the asymptotic behavior of the complexity of the finite initial segments $x\restriction\{0, \ldots, n\}$ as n goes to infinity. An interesting theorem of Schnorr (see for instance [98, Theorem 10.3]) says that Martin-Löf randomness can be characterized in terms of initial segment complexity. Specifically, x is random if and only if

$$\exists c \forall n (\mathrm{K}(x\restriction\{0, \ldots, n\}) \geq n - c)\,.$$

In other words, x is random if and only if the initial segment complexity of x is as large as possible.

Unfortunately Schnorr's theorem fails with C in place of K. However, the theorem suggests that initial segment complexity may be useful in uncovering other interesting mass problems. For instance, one may define the *effective Hausdorff dimension* of x as

$$\mathrm{effdim}(x) = \liminf_{n\to\infty} \frac{\mathrm{K}(x\restriction\{0, \ldots, n\})}{n}$$

and here it is immaterial whether we use C or K. Thus $\mathrm{effdim}(x)$ measures the "asymptotic density of information" in x. It is known (see for instance [41]) that effective Hausdorff dimension is closely related to the familiar classical Hausdorff dimension which plays such a large role in fractal geometry [32]. Namely, for any effectively closed set P in the Cantor space $\{0, 1\}^{\mathbb{N}}$, the Hausdorff dimension of P with respect to the standard metric on $\{0, 1\}^{\mathbb{N}}$ is equal to the effective Hausdorff dimension of P, defined as

$$\mathrm{effdim}(P) = \sup\{\mathrm{effdim}(x)\, ;\, x \in P\}\,.$$

Now, given a rational number s in the range $0 \leq s < 1$, it follows from [99, Lemma 3.3] (see also Theorem 3.3.1 below) that the Muchnik degree

$$\mathbf{k}_s = \deg_{\mathrm{w}}(\{x \in \{0, 1\}^{\mathbb{N}}\, ;\, \mathrm{effdim}(x) > s\})$$

belongs to $\mathcal{E}_{\mathrm{w}}$. Moreover, a theorem of Miller [68] may be restated as follows: $s < t$ implies $\mathbf{k}_s < \mathbf{k}_t$. Thus we have an infinite family of specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ which are indexed by the rational numbers s in the interval $0 \leq s < 1$. See also Figure 1.

We have seen that the Muchnik degrees $\mathbf{k}_s$ are defined in terms of effective Hausdorff dimension, which is in turn defined in terms of linear lower bounds on initial segment complexity. We now consider nonlinear lower bounds. Let f be a *convex order function*, i.e., an unbounded computable function $f : \boldsymbol{N} \to [0, \infty)$ such that $f(n) \leq f(n+1) \leq f(n) + 1$ for all n. For example, $f(n)$ could be n or $n/2$ or $n/3$ or $\sqrt{n}$ or $\sqrt[3]{n}$ or $\log n$ or $\log \log n$ or the inverse Ackermann function. By [99, Lemma 3.3] (see also Theorem 3.3.1 below) each of the Muchnik degrees

$$\mathbf{k}_f = \deg_{\mathrm{w}}(\{x \in \{0,1\}^{\boldsymbol{N}} \,;\, \exists c \forall n (\mathrm{K}(x{\upharpoonright}\{0,\ldots,n\}) \geq f(n) - c)\})$$

belongs to $\mathcal{E}_{\mathrm{w}}$. Moreover, Hudelson [43] has generalized Miller's construction [68] to prove that $\mathbf{k}_f < \mathbf{k}_g$ provided $f(n) + 2\log_2 f(n) \leq g(n)$ for all n. Thus we see that the degrees $\mathbf{k}_f$ corresponding to specific, natural, convex order functions f comprise a rich family of specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$. See also Figure 1.

In addition, there are many specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ corresponding to familiar *classes* of recursive functions. Let REC be the class of all total recursive functions, and let C be any reasonably nice subclass of REC. For example C could be the class of polynomial time computable functions, or the class of polynomial space computable functions, or the class of exponential time computable functions, or the class of elementary recursive functions, or the class of primitive recursive functions, or the class of recursive functions at or below level α of the Wainer hierarchy [116] for some particular ordinal $\alpha \leq \varepsilon_0$. Or, C could be REC itself. Our Muchnik degree corresponding to C is

$$\mathbf{k}_C = \deg_{\mathrm{w}}(\{x \in \{0,1\}^{\boldsymbol{N}} \,;\, (\exists h \in C) \forall n (\mathrm{K}(x{\upharpoonright}\{0,\ldots,h(n)\}) \geq n)\})$$

and by [99, Lemma 3.3] (see also Theorem 3.3.1 below) we have $\mathbf{k}_C \in \mathcal{E}_{\mathrm{w}}$. Moreover, Hudelson's theorem [43] (see also [3, Theorems 1.8 and 1.9]) tells us that $\mathbf{k}_{C^*} < \mathbf{k}_C$ provided C^* contains a function which grows significantly faster than all functions in C. (These degrees $\mathbf{k}_C$ are closely related to the degrees $\mathbf{k}_f$ which were defined previously. Namely, to each strictly increasing $h \in C$ we associate a convex order function $h^{-1}(m) =$ the least n such that $h(n) \geq m$. We then have $\mathbf{k}_C = \inf\{\mathbf{k}_{h^{-1}} \,;\, h \in C,\, h \text{ strictly increasing}\}$.) Thus we see that there are many specific degrees in $\mathcal{E}_{\mathrm{w}}$ corresponding to specific subclasses of REC which arise from resource-bounded computational complexity [38] and from proof theory [87, 111, 116]. I first identified these degrees in terms of diagonal nonrecursiveness rather than Kolmogorov complexity [96, §10]. See also Figure 1 and Subsection 2.3 below.

2.3. Diagonal nonrecursiveness. As in [83] let $\varphi_e^{(1)}(n)$ for $e = 1, 2, 3, \ldots$ be a standard enumeration of the 1-place partial recursive functions. A 1-place total function $g : \boldsymbol{N} \to \boldsymbol{N}$ is said to be *diagonally nonrecursive* if $g(n) \neq \varphi_n^{(1)}(n)$ for each n for which $\varphi_n^{(1)}(n)$ is defined. Obviously each diagonally nonrecursive function is nonrecursive. Much more information about diagonally nonrecursive functions can be found in [44, 3, 59, 53]. Let $\mathbf{d} = \deg_{\mathrm{w}}(\mathrm{DNR})$ where

$$\mathrm{DNR} = \{g \in \boldsymbol{N}^{\boldsymbol{N}} \,;\, g \text{ is diagonally nonrecursive}\}\,.$$

By [99, Lemma 3.3] (see also Theorem 3.3.1 below) we have $\mathbf{d} \in \mathcal{E}_{\mathrm{w}}$. Thus $\mathbf{d}$ is yet another example of a specific, natural degree in $\mathcal{E}_{\mathrm{w}}$. See also Figure 1.

The naturalness of $\mathbf{d}$ may be questioned on the grounds that it appears to depend on our choice of a standard enumeration of the partial recursive functions. However, we may respond by noting that the 1-place partial recursive function $\theta(n) \simeq \varphi_n^{(1)}(n)$ is *universal* in the following sense: for all 1-place partial recursive functions $\psi(n)$ there exists a primitive recursive function $p(n)$ such that $\psi(n) \simeq \theta(p(n))$ for all n. (The notation $E_1 \simeq E_2$ means that E_1 and E_2 are both undefined, or E_1 and E_2 are both defined and $E_1 = E_2$. Here E_1 and E_2 are expressions which may or may not be defined.) One can show that $\mathbf{d}$ does not depend on the choice of a universal 1-place partial recursive function. Furthermore, one can construct a universal 1-place partial recursive function $\theta(n)$ which is "linearly universal" in that $p(n)$ may be taken to be linear, i.e., $p(n) = an + b$ for appropriately chosen constants $a, b \in \boldsymbol{N}$. See also the discussion in [96, §10].

In order to obtain additional specific examples of degrees in $\mathcal{E}_{\mathrm{w}}$, recall that REC is the class of total recursive functions. As in Subsection 2.2 let $C = \mathrm{REC}$ or $C =$ any reasonably nice subclass of REC. A 1-place total function $g : \boldsymbol{N} \to \boldsymbol{N}$ is said to be *C-bounded* if there exists $h \in C$ such that $g(n) < h(n)$ for all n. Let $\mathbf{d}_C$ be the Muchnik degree of the problem of finding a diagonally nonrecursive function which is C-bounded. In other words,

$$\mathbf{d}_C = \deg_{\mathrm{w}}(\{g \in \mathrm{DNR}\,;\ (\exists h \in C)\forall n(g(n) < h(n))\})\,.$$

These specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ were first identified in [96, §10]. Moreover, by [53, 55] there is a close relationship between diagonal nonrecursiveness and Kolmogorov complexity, and this leads to the equations $\mathbf{k}_{\mathrm{REC}} = \mathbf{d}_{\mathrm{REC}}$ and $\mathbf{k}_C = \mathbf{d}_C$ as indicated in Figure 1.

2.4. Almost everwhere domination. For total functions $f, g : \boldsymbol{N} \to \boldsymbol{N}$ we say that *f is dominated by g* if $f(n) < g(n)$ for all but finitely many n. A real y is said to be *almost everywhere dominating* [27, 10, 20, 52, 54, 98, 100] if for all reals $x \in \{0,1\}^{\boldsymbol{N}}$ except a set of measure zero, every $f : \boldsymbol{N} \to \boldsymbol{N}$ which is computable using x as a Turing oracle is dominated by some $g : \boldsymbol{N} \to \boldsymbol{N}$ which is computable using y as a Turing oracle. Here we are referring to the fair coin probability measure on $\{0,1\}^{\boldsymbol{N}}$. Let $\mathbf{b}_1 = \deg_{\mathrm{w}}(\mathrm{AED})$ where

$$\mathrm{AED} = \{y\,;\ y \text{ is almost everywhere dominating}\}.$$

One does not expect $\mathbf{b}_1$ to belong to $\mathcal{E}_{\mathrm{w}}$, and indeed it does not. However, it turns out [52, 98] that $\inf(\mathbf{b}_1, \mathbf{1})$ belongs to $\mathcal{E}_{\mathrm{w}}$, so this is another example of a specific, natural degree in $\mathcal{E}_{\mathrm{w}}$. (Our sole reason for viewing $\inf(\mathbf{b}_1, \mathbf{1})$ as a natural degree is that it is the infimum of two other degrees which are obviously natural.) The degree $\inf(\mathbf{b}_1, \mathbf{1})$ is particularly interesting because it is incomparable with other specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ such as $\mathbf{r}_1$ and $\mathbf{d}$. On the other hand, there is a recursively enumerable Turing degree $\mathbf{a}$ such that $\inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{a}, \mathbf{1}) < \mathbf{1}$. For more information on $\inf(\mathbf{b}_1, \mathbf{1})$ and related degrees in $\mathcal{E}_{\mathrm{w}}$, see [100] and [103].

2.5. LR-reducibility and hyperarithmeticity. Given a Turing oracle x, one may relativize the concepts of Martin-Löf randomness and Kolmogorov complexity to x. Let $\mathrm{MLR}^x = \{z \in \{0,1\}^{\boldsymbol{N}}\,;\ z \text{ is Martin-Löf random relative to } x\}$, and let $\mathrm{K}^x(\tau) =$ the prefix-free complexity of τ relative to x. Nies [75] introduced the corresponding reducibility notions, LR-*reducibility* and LK-*reducibility*. Namely, $x \leq_{\mathrm{LR}} y$ if and only if $\mathrm{MLR}^y \subseteq \mathrm{MLR}^x$, and $x \leq_{\mathrm{LK}} y$ if and only if $\exists c\, \forall \tau\, (\mathrm{K}^y(\tau) \leq \mathrm{K}^x(\tau) + c)$. Later Kjos-Hanssen/Miller/Solomon [54] proved that LR-reducibility is equivalent to LK-reducibility. They also used LR-reducibility to give an interesting characterization of almost everywhere domination: $y \in \mathrm{AED}$ if and only if $0' \leq_{\mathrm{LR}} y$. Thus $\mathbf{b}_1 = \deg_{\mathrm{w}}(\{y\,;\, 0' \leq_{\mathrm{LR}} y\})$. See also my exposition in [98, 100]. Here of course $0'$ = the halting problem.

Recently [103] I generalized some of these results concerning almost everywhere domination, from $0'$ to the entire hyperarithmetical hierarchy. For each ordinal $\alpha < \omega_1^{\mathrm{CK}}$ let

$$\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\{y\,;\, 0^{(\alpha)} \leq_{\mathrm{LR}} y\})\,.$$

It turns out [103] that there is a natural embedding of the hyperarithmetical hierarchy into $\mathcal{E}_{\mathrm{w}}$ given by $\mathbf{0}^{(\alpha)} \mapsto \inf(\mathbf{b}_\alpha, \mathbf{1})$ as indicated in Figure 1. See also Example 3.2.1.3 below.

2.6. $\mathcal{E}_{\mathrm{w}}$ and reverse mathematics. *Reverse mathematics* is a program of research in the foundations of mathematics. The purpose of reverse mathematics is to determine the weakest set existence axioms which are needed in order to prove specific core mathematical theorems. In many cases it turns out that the axioms are equivalent to the theorem. The standard reference on reverse mathematics is my book [95]. See also my recent survey paper [104].

Several of the specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ which are depicted in Figure 1 were originally motivated by and correspond closely to various set existence axioms which occur in reverse mathematics. To begin with, the top and bottom degrees **1** and **0** in $\mathcal{E}_{\mathrm{w}}$ correspond to the axiomatic theories WKL_0 and RCA_0 which are known [95] to play an enormous role throughout reverse mathematics. In particular, **1** can be characterized as the Muchnik degree of the problem of finding a countably coded ω-model of WKL_0. See also [97].

Similarly, one can show that $\mathbf{r}_1$ is the Muchnik degree of the problem of finding a countably coded ω-model of WWKL_0. Here WWKL_0 is an axiomatic theory which arises frequently in the reverse mathematics of measure theory. See for instance [119, 16] and [95, §X.1].

In addition, the degrees $\mathbf{b}_\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$ were also inspired by the reverse mathematics of measure theory. Technical results concerning these degrees have been used [27, 103] in order to construct ω-models for some relevant subsystems of second-order arithmetic. The precise relationship between measure theory and the degrees $\mathbf{b}_\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$ is as follows: $0^{(\alpha)} \leq_{\mathrm{LR}} y$ if and only if every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,y}_2$ set of the same measure [103, Corollary 4.12]. Here again we are referring to the fair coin probability measure on $\{0, 1\}^{N}$.

2.7. Summary. To summarize, we have seen that $\mathcal{E}_{\mathrm{w}}$ contains many specific, natural degrees which correspond to foundationally interesting topics. Among these topics are algorithmic randomness, Kolmogorov complexity, effective Hausdorff dimension, resource-bounded computational complexity, subrecursive hierarchies, proof theory, LR-reducibility, hyperarithmeticity, and reverse mathematics.

3. Structural and methodological aspects of $\mathcal{E}_{\mathrm{w}}$.

3.1. Priority arguments. Some structural properties of $\mathcal{E}_{\mathrm{w}}$ are stated in the following theorem.

THEOREM 3.1.1.
1. *$\mathcal{E}_{\mathrm{w}}$ is a countable distributive lattice with* $\mathbf{1}$ *and* $\mathbf{0}$ *as the top and bottom elements.*
2. *Every countable distributive lattice is lattice-embeddable in $\mathcal{E}_{\mathrm{w}}$.*
3. *More generally, given $\mathbf{0} < \mathbf{p} \in \mathcal{E}_{\mathrm{w}}$, every countable distributive lattice is lattice-embeddable into the initial segment of $\mathcal{E}_{\mathrm{w}}$ below* $\mathbf{p}$.
4. *Given $\mathbf{0} < \mathbf{p} \in \mathcal{E}_{\mathrm{w}}$ we can find $\mathbf{p}_1 \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{p}_2 \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{p} = \sup(\mathbf{p}_1, \mathbf{p}_2)$ and $\mathbf{0} < \mathbf{p}_1 < \mathbf{p}$ and $\mathbf{0} < \mathbf{p}_2 < \mathbf{p}$.*

Here items 2 and 3 are from [11] and item 4 is from [6].

As mentioned in Subsection 1.3 above (see also [99]), the study of the lattice $\mathcal{E}_{\mathrm{w}}$ is in some ways analogous to the study of the semilattice $\mathcal{E}_{\mathrm{T}}$ of recursively enumerable Turing degrees. A traditional highlight in the study of $\mathcal{E}_{\mathrm{T}}$ has been the methodology of priority arguments. Over a span of several decades, successively more difficult types of priority arguments including finite injury arguments, infinite injury arguments [84, 85], $0'''$ priority arguments [107, 61], etc., were used to elucidate the structure of $\mathcal{E}_{\mathrm{T}}$. Later it emerged that priority arguments can also be used to study $\mathcal{E}_{\mathrm{w}}$. In particular, each of items 2 through 4 in Theorem 3.1.1 was originally proved by means of a finite injury priority argument. In this vein there is the following generalization of the Sacks Splitting Theorem [107, Theorem VII.3.2] which was proved by essentially the same method.

THEOREM 3.1.2 (Binns Splitting Theorem). *Let $A \subseteq \boldsymbol{N}$ be recursively enumerable, and let $P \subseteq \boldsymbol{N}^{\boldsymbol{N}}$ be effectively closed such that $P \cap \mathrm{REC} = \emptyset$. Then, we can find a pair of recursively enumerable sets A_1, A_2 such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ and no member of P is Turing reducible to A_1 or to A_2.*

Item 4 in Theorem 3.1.1 was originally obtained in [6] as a special case of Theorem 3.1.2 with $A = 0' =$ the halting problem. Namely, let P be such that $\mathbf{p} = \deg_{\mathrm{w}}(P)$ and for $i = 1, 2$ let $\mathbf{p}_i = \inf(\mathbf{p}, \mathbf{a}_i)$ where $\mathbf{a}_i = \deg_{\mathrm{T}}(A_i)$.

REMARK 3.1.3. An open question concerning the structure of $\mathcal{E}_{\mathrm{w}}$ reads as follows.

Given $\mathbf{p}_1 \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{p}_2 \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{p}_1 < \mathbf{p}_2$, can we find $\mathbf{q} \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{p}_1 < \mathbf{q} < \mathbf{p}_2$?

An affirmative answer to this question would constitute an $\mathcal{E}_{\mathrm{w}}$-analog of the Sacks Density Theorem for $\mathcal{E}_{\mathrm{T}}$ [84]. It seems reasonable to conjecture that $\mathbf{q}$ can be constructed using an infinite injury priority argument as in [84].

3.2. Forcing arguments. Forcing constructions have played a large role in the study of $\mathcal{E}_{\mathrm{w}}$. Many of the relationships between specific pairs of degrees in $\mathcal{E}_{\mathrm{w}}$ which are exhibited in Figure 1 have been proved by means of forcing. Here are some examples.

EXAMPLES 3.2.1.

1. The fact that $\mathbf{d} < \mathbf{r}_1$ [96] (see Figure 1) was originally obtained as a consequence of Kumabe's Theorem [59]: there exist diagonally nonrecursive functions which are of minimal Turing degree. The proof of Kumabe's Theorem uses the Kumabe/Lewis technique of *bushy tree forcing* [59, 3].

2. Cholak/Greenberg/Miller [20, §4] introduced an interesting forcing technique in order to construct an almost everywhere dominating real which does not compute a diagonally nonrecursive function. This result is equivalent to saying that $\mathbf{d} \not\leq \mathbf{b}_1$ in Figure 1.

3. In [103, §5] I simplified and generalized the technique of [20, §4] to prove that for any real x we can find a real y such that $x \leq_{\mathrm{LR}} y$ and y does not compute a diagonally nonrecursive function. Consequently we have $\inf(\mathbf{b}_\alpha, \mathbf{d}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{d})$ for each $\alpha < \omega_1^{\mathrm{CK}}$. See also Figure 1.

4. Miller [68] introduced his technique of *forcing with optimal covers* in order to prove that $s < t$ implies $\mathbf{k}_s < \mathbf{k}_t$ in Figure 1.

5. Recently Hudelson [43] modified and generalized Miller's technique in order to prove that $\mathbf{k}_f < \mathbf{k}_g$ whenever f and g are convex order functions satisfying $f(n) + 2\log_2 f(n) \leq g(n)$ for all n. See Figure 1.

REMARK 3.2.2. Even the familiar Kleene/Post/Friedberg technique of forcing with finite conditions and 1-genericity (see [56] and [83, §§13.1, 13.3]) has been very useful in the study of $\mathcal{E}_{\mathrm{w}}$. In [22, §§3,4] we used this method to obtain a natural embedding of the hyperarithmetical hierarchy into $\mathcal{E}_{\mathrm{w}}$. Also, as explained as Subsections 3.4 and 4.2 below, a variant method based on Posner/Robinson [78] has been used in [101] to obtain some interesting structural information concerning $\mathcal{E}_{\mathrm{w}}$.

REMARK 3.2.3. An open structural question concerning $\mathcal{E}_{\mathrm{w}}$ is to compute the Turing degree of $\mathrm{Th}(\mathcal{E}_{\mathrm{w}})$, the first-order theory of $\mathcal{E}_{\mathrm{w}}$. Since the hyperarithmetical hierarchy is naturally embeddable into $\mathcal{E}_{\mathrm{w}}$ [22, 103], it seems reasonable to conjecture that $\mathrm{Th}(\mathcal{E}_{\mathrm{w}})$ should be recursively isomorphic to the ωth Turing jump of the complete Π^1_1 set of integers. This conjecture was first stated in [22].

3.3. The Σ^0_3 Embedding Lemma. As illustrated in Figure 1, many specific, natural degrees in $\mathcal{E}_{\mathrm{w}}$ are of the form $\inf(\mathbf{s}, \mathbf{1})$ where $\mathbf{s}$ is a specific, natural degree which does not belong to $\mathcal{E}_{\mathrm{w}}$. We now present the key theorem which enables us to deal with such degrees. Our theorem is known as the Σ^0_3 Embedding Lemma.

Let X be an effectively presented complete separable metric space. A subset of X is said to be Σ^0_1 if and only if it is effectively open. This class of sets is, of course, the effective analog of the open sets. We now define effective analogs of other classes of sets. For each positive integer k, a set $P \subseteq X$ is said to be Π^0_k if and only if its complement $X \setminus P$ is Σ^0_k. For example, P is Π^0_1 if and only if P is effectively closed. A set $S \subseteq X$ is said to be Σ^0_{k+1} if and only if it is of the form

$$S = \{x \in X\,;\ \exists n((n, x) \in P)\}$$

where $P \subseteq \boldsymbol{N} \times X$ is Π^0_k. Here we are viewing $\boldsymbol{N} \times X$ as an effectively presented complete separable metric space in its own right. Thus $S = \bigcup_{n=0}^{\infty} P_n$ where each $P_n = \{x \in X\,;\,(n, x) \in P\}$ is a Π^0_k set and moreover the sequence P_n for $n = 0, 1, 2, \ldots$ is uniformly Π^0_k.

In particular, the Σ^0_3 sets in X are those of the form

$$S = \{x \in X\,;\,\exists m \forall n((m, n, x) \in U)\}$$

where $U \subseteq \boldsymbol{N} \times \boldsymbol{N} \times X$ is effectively open. Note that the Σ^0_3 sets are the effective analogs of the classical $\mathrm{G}_{\delta\sigma}$ sets [50, Introduction]. For each $k \geq 3$ it can be shown that every Σ^0_k set in X is Muchnik equivalent to a Π^0_{k-1} set in the Baire space $\boldsymbol{N}^{\boldsymbol{N}}$ or in the Cantor space $\{0, 1\}^{\boldsymbol{N}}$. For convenience we shall focus on Σ^0_3 sets in the Baire space, i.e., sets of the form

$$S = \{f \in \boldsymbol{N}^{\boldsymbol{N}}\,;\,\exists k \forall m \exists n R(k, m, n, f)\} \tag{1}$$

where $R \subseteq \boldsymbol{N}^3 \times \boldsymbol{N}^{\boldsymbol{N}}$ is recursive. These sets will play an important role in what follows.

THEOREM 3.3.1 (The Σ^0_3 Embedding Lemma). *Let* $\mathbf{s} = \deg_{\mathrm{w}}(S)$ *where* S *is a* Σ^0_3 *set. Then* $\inf(\mathbf{s}, \mathbf{1})$ *belongs to* $\mathcal{E}_{\mathrm{w}}$.

Theorem 3.3.1 has been extremely useful in showing that various interesting Muchnik degrees belong to $\mathcal{E}_{\mathrm{w}}$. We now present some examples.

EXAMPLES 3.3.2.

1. Recall that a real $x \in \{0, 1\}^{\boldsymbol{N}}$ is said to be 2-*random* if it is Martin-Löf random relative to $0'$ = the halting problem. It is not hard to see that

$$\mathrm{MLR}_2 = \{x \in \{0, 1\}^{\boldsymbol{N}}\,;\,x \text{ is 2-random}\}$$

is Σ^0_3. Therefore, letting $\mathbf{r}_2 = \deg_{\mathrm{w}}(\mathrm{MLR}_2)$ we have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_{\mathrm{w}}$ in view of Theorem 3.3.1. See also Figure 1.

2. Recall from Figure 1 the degrees $\mathbf{r}_1$, $\mathbf{k}_s$, $\mathbf{k}_f$, and $\mathbf{d}$, as well as $\mathbf{k}_C = \mathbf{d}_C$ where C = REC or C = a nice subclass of REC. Each of these was defined in Section 2 as the Muchnik degree of a specific, natural, Σ^0_3 subset of $\{0, 1\}^{\boldsymbol{N}}$ or of $\boldsymbol{N}^{\boldsymbol{N}}$. Moreover, each of these degrees is easily seen to be $\leq \mathbf{1}$. Therefore, by Theorem 3.3.1, each of these degrees belongs to $\mathcal{E}_{\mathrm{w}}$.

3. For each real $x \leq_{\mathrm{T}} 0'$ the singleton set $\{x\}$ is easily seen to be Σ^0_3. Therefore, Theorem 3.3.1 tells us that $\inf(\mathbf{a}, \mathbf{1}) \in \mathcal{E}_{\mathrm{w}}$ where $\mathbf{a} = \deg_{\mathrm{T}}(x) = \deg_{\mathrm{w}}(\{x\})$. This applies in particular if $\mathbf{a}$ is a recursively enumerable Turing degree, as shown in Figure 1. The Arslanov Completeness Criterion [107, Theorem V.5.1] tells us that for all $\mathbf{a} \in \mathcal{E}_{\mathrm{T}}$ the embedding $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1})$ is one-to-one.

4. In [103, Theorem 6.3] I proved that if S is Σ^0_3 then its LR-upward closure

$$S^{\mathrm{LR}} = \{y\,;\,\exists x(x \in S \text{ and } x \leq_{\mathrm{LR}} y)\}$$

is again Σ^0_3. Moreover, it is known (see for instance [86, Part A]) that the singleton set $\{0^{(\alpha)}\}$ is Σ^0_3 for each recursive ordinal $\alpha < \omega_1^{\mathrm{CK}}$. Combining these facts with Theorem 3.3.1 we see that $\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathrm{w}}$ where $\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\{y\,;\,0^{(\alpha)} \leq_{\mathrm{LR}} y\})$. See also Figure 1.

REMARK 3.3.3. In view of Theorem 3.3.1 it seems reasonable to consider a certain sublattice of $\mathcal{D}_{\mathrm{w}}$ which is larger than $\mathcal{E}_{\mathrm{w}}$ but still countable. Namely, let $\mathcal{S}_{\mathrm{w}}$ be the lattice of Muchnik degrees of nonempty Σ^0_3 sets in $\boldsymbol{N}^{\boldsymbol{N}}$, or equivalently, nonempty Π^0_1 sets in $\boldsymbol{N}^{\boldsymbol{N}}$ (see Lemma 3.3.5 below). Trivially $\mathcal{S}_{\mathrm{w}}$ includes $\mathcal{E}_{\mathrm{w}}$ but it also includes much more. In particular, each of the degrees $\mathbf{0}^{(\alpha)}$ and $\mathbf{b}_\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$ belongs to $\mathcal{S}_{\mathrm{w}}$. The structure of $\mathcal{S}_{\mathrm{w}}$ has not been studied extensively, but we can show for instance that $\mathcal{S}_{\mathrm{w}}$ has no top degree. We also have the following result.

COROLLARY 3.3.4. *$\mathcal{E}_{\mathrm{w}}$ is an initial segment of $\mathcal{S}_{\mathrm{w}}$. Specifically, we have*

$$\mathcal{E}_{\mathrm{w}} = \{\mathbf{s} \in \mathcal{S}_{\mathrm{w}} \,;\, \mathbf{s} \leq \mathbf{1}\}\,.$$

PROOF. This is a restatement of Theorem 3.3.1. □

We shall now sketch a proof of Theorem 3.3.1. The theorem was first proved in [99, Lemma 3.3] but the proof given here yields additional useful information. See also Remark 3.3.7 below.

If P and Q are sets of reals, we say that P and Q are *Turing equivalent*, abbreviated $P \equiv_{\mathrm{T}} Q$, if and only if

$$\{\deg_{\mathrm{T}}(x)\,;\, x \in P\} = \{\deg_{\mathrm{T}}(y)\,;\, y \in Q\}\,.$$

Note that $P \equiv_{\mathrm{T}} Q$ implies $\deg_{\mathrm{w}}(P) = \deg_{\mathrm{w}}(Q)$ but not conversely. For $f, g \in \boldsymbol{N}^{\boldsymbol{N}}$ we write $f \oplus g =$ the unique $h \in \boldsymbol{N}^{\boldsymbol{N}}$ such that $h(2n) = f(n)$ and $h(2n+1) = g(n)$ for all n.

LEMMA 3.3.5. *Let S be a Σ^0_3 set in $\boldsymbol{N}^{\boldsymbol{N}}$. Then, we can find a Π^0_1 set Q in $\boldsymbol{N}^{\boldsymbol{N}}$ such that $Q \equiv_{\mathrm{T}} S$.*

PROOF. Since S is Σ^0_3, let $R \subseteq \boldsymbol{N}^3 \times \boldsymbol{N}^{\boldsymbol{N}}$ be a recursive relation such that (1) holds. We then let

$$Q = \{\langle k\rangle^\frown(f \oplus g)\,;\, \forall m\,(g(m) = \text{the least } n \text{ such that } R(k,m,n,f) \text{ holds})\}\,.$$

Clearly Q is a Π^0_1 set in $\boldsymbol{N}^{\boldsymbol{N}}$, and it is easy to check that $Q \equiv_{\mathrm{T}} S$. □

LEMMA 3.3.6. *Let S be a Σ^0_3 set in $\boldsymbol{N}^{\boldsymbol{N}}$. Let P be a nonempty Π^0_1 set in $\{0,1\}^{\boldsymbol{N}}$. Then, we can find a nonempty Π^0_1 set Q in $\{0,1\}^{\boldsymbol{N}}$ such that $Q \equiv_{\mathrm{T}} S \cup P$.*

PROOF. By Lemma 3.3.5 we may safely assume that S is a Π^0_1 set in $\boldsymbol{N}^{\boldsymbol{N}}$. Therefore, let U be a recursive subtree of $\boldsymbol{N}^*$ such that $S = \{\text{paths through } U\}$. In addition, let V be a recursive subtree of $\{0,1\}^*$ such that $P = \{\text{paths through } V\}$. Define W to be the recursive subtree of $\{0,1,2\}^*$ consisting of all strings of the form

$$\sigma_0{}^\frown\langle 2\rangle^\frown\sigma_1{}^\frown\langle 2\rangle^\frown\cdots{}^\frown\langle 2\rangle^\frown\sigma_{n-1}{}^\frown\langle 2\rangle^\frown\sigma_n$$

such that

(a) for each $i \leq n$, $\sigma_i \in V$;
(b) for each $i < n$, σ_i is the leftmost $\sigma \in V$ such that $|\sigma| = |\sigma_i|$;
(c) the string $\langle|\sigma_0|, |\sigma_1|, \ldots, |\sigma_{n-1}|\rangle$ belongs to U.

Letting $Q = \{\text{paths through } W\}$, it is straightforward to verify that $Q \equiv_{\mathrm{T}} S \cup P$. Clearly Q is a Π^0_1 set in $\{0, 1, 2\}^{\boldsymbol{N}}$. Since $\{0, 1, 2\}^{\boldsymbol{N}}$ is effectively homeomorphic to $\{0, 1\}^{\boldsymbol{N}}$, we have our lemma. □

REMARK 3.3.7. Kent and Lewis [51] have studied the lattice of sets of Turing degrees of the form $\{\deg_{\mathrm{T}}(x) ; x \in P\}$ where P is a Π^0_1 subset of $\{0, 1\}^{\boldsymbol{N}}$. Lemma 3.3.6 is of obvious interest in this regard.

PROOF OF THEOREM 3.3.1. Let $P \subseteq \{0, 1\}^{\boldsymbol{N}}$ be Π^0_1 such that $\deg_{\mathrm{w}}(P) = \mathbf{1}$. For instance, we could take $P = \mathrm{CPA}$ or $P = \mathrm{DNR} \cap \{0, 1\}^{\boldsymbol{N}}$. Apply Lemma 3.3.6 to get a Π^0_1 set $Q \subseteq \{0, 1\}^{\boldsymbol{N}}$ such that $Q \equiv_{\mathrm{T}} S \cup P$. Let $\mathbf{q} = \deg_{\mathrm{w}}(Q)$. We then have $\mathbf{q} \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{q} = \deg_{\mathrm{w}}(Q) = \deg_{\mathrm{w}}(S \cup P) = \inf(\mathbf{s}, \mathbf{1})$. □

3.4. A generalization of the Posner/Robinson Theorem. We now prove another lemma which is consequential for the structure of $\mathcal{E}_{\mathrm{w}}$. This is a strengthened version of [101, Lemma 5].

LEMMA 3.4.1. *Let $S \subseteq \boldsymbol{N}^{\boldsymbol{N}}$ be Σ^0_3. Assume that $f, h \in \boldsymbol{N}^{\boldsymbol{N}}$ are such that $S \not\leq_{\mathrm{w}} \{f\}$ and $0 <_{\mathrm{T}} f$ and $f \oplus 0' \leq_{\mathrm{T}} h$. Then, we can find a 1-generic $g \in \boldsymbol{N}^{\boldsymbol{N}}$ such that $S \not\leq_{\mathrm{w}} \{g\}$ and $f \oplus g \equiv_{\mathrm{T}} g' \equiv_{\mathrm{T}} g \oplus 0' \equiv_{\mathrm{T}} h$.*

PROOF. For integers $n \in \boldsymbol{N}$ and strings $\sigma \in \boldsymbol{N}^*$ we write

$$\Phi_n(\sigma) = \langle \varphi^{(1),\sigma}_{n,|\sigma|}(i) \,;\, i < j \rangle$$

where $j =$ the least i such that either $\varphi^{(1),\sigma}_{n,|\sigma|}(i) \uparrow$ or $i \geq |\sigma|$. Note that the mapping $\Phi_n : \boldsymbol{N}^* \to \boldsymbol{N}^*$ is recursive and *monotonic*, i.e., $\sigma \subseteq \tau$ implies $\Phi_n(\sigma) \subseteq \Phi_n(\tau)$. Moreover, for all $g, \widehat{g} \in \boldsymbol{N}^{\boldsymbol{N}}$ we have $g \geq_{\mathrm{T}} \widehat{g}$ if and only if $\exists n\, (\Phi_n(g) = \widehat{g})$. Here we are writing

$$\Phi_n(g) = \bigcup_{l=0}^{\infty} \Phi_n(g \restriction \{0, 1, \dots, l-1\}) \,.$$

Let S, f and h be as in the statement of Lemma 3.4.1. By Lemma 3.3.5 we may safely assume that S is Π^0_1, so let $U \subseteq \boldsymbol{N}^*$ be a recursive tree such that $S = \{\text{paths through } U\}$. Since f is not recursive, we can find a set $A \subseteq \boldsymbol{N}$ such that $f \equiv_{\mathrm{T}} A$ and A is not recursively enumerable. We shall inductively define an increasing sequence of strings $\sigma_k \in \boldsymbol{N}^*$ for $k = 0, 1, 2, \dots$ and then let $g = \bigcup_{k=0}^{\infty} \sigma_k$. In presenting the construction, we shall identify strings with their Gödel numbers.

Stage 0. Let $\sigma_0 = \langle\rangle =$ the empty string.

Stage $3n+1$. Assume inductively that σ_{3n} has been defined. Let $\sigma_{3n+1} = \sigma_{3n}{}^\frown\langle h(n)\rangle$.

Stage $3n+2$. Assume inductively that σ_{3n+1} has been defined. Since A is not recursively enumerable, there exists i such that

$$i \in A \Leftrightarrow \neg\, \exists \sigma\, (\sigma_{3n+1}{}^\frown\langle i\rangle \subseteq \sigma \wedge \Phi_n(\sigma)(0) \downarrow) \,.$$

Using $A \oplus 0'$ as an oracle, find $i_n =$ the least such i. If $i_n \in A$ let $\sigma_{3n+2} = \sigma_{3n+1}{}^\frown\langle i_n\rangle$. If $i_n \notin A$ let $\sigma_{3n+2} =$ the least $\sigma \supseteq \sigma_{3n+1}{}^\frown\langle i_n\rangle$ such that $\Phi_n(\sigma)(0) \downarrow$.

Stage $3n+3$. Assume inductively that σ_{3n+2} has been defined. Let $\sigma_{n,0} = \sigma_{3n+2}$. Suppose that $\sigma_{n,s}$ has been defined. Using $A \oplus 0'$ as an oracle, search for an i such that

$$i \in A \wedge \neg \exists \sigma\, (\sigma_{n,s}{}^\frown\langle i\rangle \subseteq \sigma \wedge \Phi_n(\sigma_{n,s}) \subset \Phi_n(\sigma) \in U)\,. \tag{2}$$

At the same time, using A as an oracle, search for a pair i, σ such that

$$i \notin A \wedge \sigma_{n,s}{}^\frown\langle i\rangle \subseteq \sigma \wedge \Phi_n(\sigma_{n,s}) \subset \Phi_n(\sigma) \in U\,. \tag{3}$$

Since A is not recursively enumerable, at least one of these two searches will eventually succeed. If search (2) succeeds first, let $\sigma_{3n+3} = \sigma_{n,s}{}^\frown\langle i\rangle$. If search (3) succeeds first, let $\sigma_{n,s+1} = \sigma$. In either case let $i_{n,s} = i$.

We claim that for some s, search (2) succeeds first. Otherwise, by performing search (3) for $s = 0, 1, 2, \ldots$ successively, we would obtain infinite increasing sequences of strings

$$\sigma_{n,0} \subset \sigma_{n,1} \subset \cdots \subset \sigma_{n,s} \subset \sigma_{n,s+1} \subset \cdots$$

and

$$\Phi_n(\sigma_{n,0}) \subset \Phi_n(\sigma_{n,1}) \subset \cdots \subset \Phi_n(\sigma_{n,s}) \subset \Phi_n(\sigma_{n,s+1}) \subset \cdots$$

with $\Phi_n(\sigma_{n,s}) \in U$ for all s. Moreover, these sequences of strings would be computable relative to A. Thus, letting $\widehat{f} = \bigcup_{s=0}^{\infty} \Phi_n(\sigma_{n,s})$ we would have $\widehat{f} \in S$ and $\widehat{f} \leq_{\mathrm{T}} A \equiv_{\mathrm{T}} f$, hence $S \leq_{\mathrm{w}} \{f\}$, a contradiction. This proves our claim. From this it follows that σ_{3n+3} is defined.

Clearly the sequence $\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots$ is computable relative to h, because $A \oplus 0' \equiv_{\mathrm{T}} f \oplus 0' \leq_{\mathrm{T}} h$. Moreover, h is computable relative to this sequence, because for all n we have $h(n) = \sigma_{3n+1}(|\sigma_{3n}|)$.

Let $g = \bigcup_{k=0}^{\infty} \sigma_k$. We claim that the sequence $\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots$ is $\leq_{\mathrm{T}} A \oplus g$. Given σ_k we use $A \oplus g$ to compute σ_{k+1} as follows. For $k = 3n$ we have $\sigma_{k+1} = \sigma_k{}^\frown\langle g(|\sigma_k|)\rangle$. For $k = 3n+1$ we have $i_n = g(|\sigma_k|)$ and $\sigma_{k+1} = \sigma_k{}^\frown\langle i_n\rangle$ if $i_n \in A$, otherwise $\sigma_{k+1} =$ the least $\sigma \supseteq \sigma_k{}^\frown\langle i_n\rangle$ such that $\Phi_n(\sigma)(0)\downarrow$. For $k = 3n+2$ we begin with $\sigma_{n,0} = \sigma_k$. Given $\sigma_{n,s}$ we use g to compute $i_{n,s} = g(|\sigma_{n,s}|)$ and then use A to decide whether $i_{n,s} \in A$ or not. If $i_{n,s} \notin A$ we compute $\sigma_{n,s+1} =$ the least σ such that $\sigma_{n,s}{}^\frown\langle i_{n,s}\rangle \subseteq \sigma$ and $\Phi_n(\sigma_{n,s}) \subset \Phi_n(\sigma) \in U$. By the previous claim, we will eventually find an s such that $i_{n,s} \in A$, and then we have $\sigma_{k+1} = \sigma_{n,s}{}^\frown\langle i_{n,s}\rangle$. This proves our claim. Since $A \equiv_{\mathrm{T}} f \leq_{\mathrm{T}} h$ and $g \leq_{\mathrm{T}} h$, it follows that $h \equiv_{\mathrm{T}} A \oplus g \equiv_{\mathrm{T}} f \oplus g$.

Next we claim that the sequence $\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots$ is $\leq_{\mathrm{T}} g \oplus 0'$. Given σ_k we use $g \oplus 0'$ to compute σ_{k+1} as follows. For $k = 3n$ we have $\sigma_{k+1} = \sigma_k{}^\frown\langle g(|\sigma_k|)\rangle$. For $k = 3n+1$ we have $i_n = g(|\sigma_k|)$ and we can then use $0'$ to decide whether there exists σ such that $\sigma_k{}^\frown\langle i_n\rangle \subseteq \sigma$ and $\Phi_n(\sigma)(0)\downarrow$. If such a σ exists, we have $\sigma_{k+1} =$ the least such σ, otherwise $\sigma_{k+1} = \sigma_k{}^\frown\langle i_n\rangle$. For $k = 3n+2$ we have $\sigma_{n,0} = \sigma_k$. Given $\sigma_{n,s}$ we use g to compute $i_{n,s} = g(|\sigma_{n,s}|)$ and then use $0'$ to decide whether there exists σ such that $\sigma_{n,s}{}^\frown\langle i_{n,s}\rangle \subseteq \sigma$ and $\Phi_n(\sigma_{n,s}) \subset \Phi_n(\sigma) \in U$. If such a σ exists, we compute $\sigma_{n,s+1} =$ the least such σ, otherwise $\sigma_{k+1} = \sigma_{n,s}{}^\frown\langle i_{n,s}\rangle$. As before we know that this procedure eventually gives σ_{k+1}. This proves our claim. Thus $h \leq_{\mathrm{T}} g \oplus 0'$. Moreover $g' \leq_{\mathrm{T}} g \oplus 0'$ because for all

n we have $n \in g'$ if and only if $\Phi_n(\sigma_{3n+2})(0)\downarrow$. Since $g \leq_{\mathrm{T}} h$ and $0' \leq_{\mathrm{T}} h$ it follows that $g' \equiv_{\mathrm{T}} h$.

The construction at stage $3n+3$ insures that $S \not\leq_{\mathrm{w}} \{g\}$. Moreover, the construction at stage $3n+2$ insures that g is 1-generic. This completes the proof of Lemma 3.4.1. □

REMARK 3.4.2. The Posner/Robinson Theorem [78] follows from the special case of Lemma 3.4.1 with $S = \emptyset =$ the empty set. Also, Lemma 3.4.1 improves our result in [101, Remark 9] by eliminating the hyperimmunity hypothesis. Other generalizations of the Posner/Robinson Theorem are in [45, 46, 90]

LEMMA 3.4.3. *Let $S \subseteq \boldsymbol{N}^{\boldsymbol{N}}$ be Σ^0_3 such that $S \not\leq_{\mathrm{w}} \{0\}$, i.e., $S \cap \mathrm{REC} = \emptyset$. Let $h \in \boldsymbol{N}^{\boldsymbol{N}}$ be such that $0' \leq_{\mathrm{T}} h$. Then, we can find a 1-generic $g \in \boldsymbol{N}^{\boldsymbol{N}}$ such that $S \not\leq_{\mathrm{w}} \{g\}$ and $g' \equiv_{\mathrm{T}} g \oplus 0' \equiv_{\mathrm{T}} h$.*

PROOF. We proceed as in the proof of Lemma 3.4.1 above. The construction is easier than in Lemma 3.4.1, because we can ignore f. We omit the details. □

As in Remark 3.3.3 let $\mathcal{S}_{\mathrm{w}}$ be the lattice of Muchnik degrees of nonempty Σ^0_3 sets in $\boldsymbol{N}^{\boldsymbol{N}}$.

THEOREM 3.4.4. *Let $\mathbf{s}$ be a Muchnik degree in $\mathcal{S}_{\mathrm{w}}$ such that $\mathbf{0} < \mathbf{s}$. Let $\mathbf{c}$ be a Turing degree such that $\mathbf{0}' \leq \mathbf{c}$. Then, we can find a Turing degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{c}$ and $\mathbf{s} \not\leq \mathbf{a}$. Moreover, given any such Turing degree $\mathbf{a}_1$ we can find another such Turing degree $\mathbf{a}_2$ with the property that $\sup(\mathbf{a}_1, \mathbf{a}_2) = \mathbf{c}$.*

PROOF. These statements are a partial translation of Lemmas 3.4.3 and 3.4.1 into the language of Muchnik degrees and Turing degrees. □

THEOREM 3.4.5. *Let $\mathbf{p}$ be a Muchnik degree in $\mathcal{E}_{\mathrm{w}}$ such that $\mathbf{0} < \mathbf{p}$. Then, we can find a Turing degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ and $\mathbf{p} \not\leq \mathbf{a}$. Moreover, given any such Turing degree $\mathbf{a}_1$ we can find another such Turing degree $\mathbf{a}_2$ with the property that $\sup(\mathbf{a}_1, \mathbf{a}_2) = \mathbf{0}'$. Consequently, letting $\mathbf{p}_i = \inf(\mathbf{p}, \mathbf{a}_i)$ for $i = 1, 2$ we have $\mathbf{p}_i \in \mathcal{E}_{\mathrm{w}}$ and $\mathbf{0} < \mathbf{p}_i < \mathbf{p}$ and $\mathbf{p} = \sup(\mathbf{p}_1, \mathbf{p}_2)$.*

PROOF. Apply Theorem 3.4.4 with $\mathbf{s} = \mathbf{p}$ and $\mathbf{c} = \mathbf{0}'$. From $\mathbf{a}_i < \mathbf{0}'$ plus Theorem 3.3.1 it follows that $\mathbf{p}_i \in \mathcal{E}_{\mathrm{w}}$. From $\mathbf{0} < \mathbf{p}$ and $\mathbf{0} < \mathbf{a}_i$ and $\mathbf{p} \not\leq \mathbf{a}_i$ it follows that $\mathbf{0} < \mathbf{p}_i < \mathbf{p}$. Since $\mathbf{p} \leq \mathbf{1} < \mathbf{0}' = \sup(\mathbf{a}_1, \mathbf{a}_2)$, the distributive law gives $\mathbf{p} = \inf(\mathbf{p}, \sup(\mathbf{a}_1, \mathbf{a}_2)) = \sup(\inf(\mathbf{p}, \mathbf{a}_1), \inf(\mathbf{p}, \mathbf{a}_2)) = \sup(\mathbf{p}_1, \mathbf{p}_2)$. □

REMARK 3.4.6. Theorem 3.4.5 provides another proof of the Splitting Theorem 3.1.1.4 and this alternative proof is in some ways more informative. Also, as we shall see in Section 4, Theorem 3.4.5 answers a question about $\mathcal{E}_{\mathrm{w}}$ which arises naturally from Muchnik's version of Kolmogorov's interpretation of intuitionism.

4. Muchnik degrees and intuitionism. Historically, Muchnik degrees arose from the foundational controversy which was ignited by Brouwer's doctrine of intuitionism. Kolmogorov, in his influential 1932 paper [57, 58], proposed to interpret intuitionism nonrigorously as an *Aufgabenrechnung* (translation: calculus of problems). (This proposal in [57] accounts for the "K" in the so-called BHK-interpretation of intuitionism [114, §§1.3.1, 1.5.3].) In order to rigorously implement Kolmogorov's idea, Medvedev 1955 [67] introduced mass problems, and Muchnik 1963 [71] proved that the lattice $\mathcal{D}_{\mathrm{w}}$ of all Muchnik degrees is *Brouwerian*, i.e., it satisfies Heyting's intuitionistic propositional calculus. My brief account of this history is in [101].

4.1. Interpreting intuitionism in $\mathcal{D}_{\mathrm{w}}$. We shall now briefly describe Kolmogorov's interpretation and Muchnik's implementation of it. The original Kolmogorov idea was to view each intuitionistic proposition as a "problem." An intuitionistic proof of the proposition is then the same thing as a "solution" of the problem. If A and B are problems, let us write $A \vdash B$ to mean that the problem A is "at least as difficult as" the problem B, in the sense that any solution of A would "easily" or "immediately" yield a solution of B. Consequently, A and B are "equivalent" as problems if and only if $A \vdash B$ and $B \vdash A$. We denote this equivalence as $A \equiv B$.

One may combine problems in various ways to obtain new problems. Some of the methods of combination correspond to the intuitionistic propositional connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$, etc. Thus, if A and B are problems, let $A \wedge B$ denote the problem of solving both A and B, let $A \vee B$ denote the problem of solving at least one of A and B, and let $A \Rightarrow B$ denote the "least difficult" problem C such that $A \wedge C \vdash B$. Rephrasing this in terms of "solutions," we see that a solution of $A \wedge B$ should be essentially an ordered pair (x, y) where x and y are solutions of A and B respectively; a solution of $A \vee B$ should consist of a solution of at least one of A and B; and a solution of $A \Rightarrow B$ should be something which allows us to "easily transform" any solution of A into a solution of B. Attempting to state this in another way, let P and Q be the solution sets of A and B respectively, i.e., $P = \{$solutions of $A\}$ and $Q = \{$solutions of $B\}$. Clearly we ought to have something like $P \times Q = \{$solutions of $A \wedge B\}$, and $P \cup Q = \{$solutions of $A \vee B\}$, but the set-theoretic interpretation of $A \Rightarrow B$ is not so clear.

Let us now adopt an instrumentalist viewpoint, according to which any "problem" A is to be identified with its solution set P. On this view, any set P of possible solutions corresponds in turn to a problem, namely, the problem of "finding" an element of P. Muchnik's idea was to identify the possible solutions as Turing oracles. Thus P and Q are sets of Turing oracles, i.e., mass problems, and we interpret $P \vdash Q$ to mean that $P \geq_{\mathrm{w}} Q$, i.e., every solution of P can be used as a Turing oracle to compute some solution of Q. Letting $\mathbf{p} = \deg_{\mathrm{w}}(P)$ and $\mathbf{q} = \deg_{\mathrm{w}}(Q)$, it is easy to check that $\sup(\mathbf{p}, \mathbf{q}) = \deg_{\mathrm{w}}(P \times Q)$ and $\inf(\mathbf{p}, \mathbf{q}) = \deg_{\mathrm{w}}(P \cup Q)$, so we are forced to implement $\wedge$ and $\vee$ respectively as sup and inf in the Muchnik lattice $\mathcal{D}_{\mathrm{w}}$. (For this reason we strongly prefer the notations $\sup(\mathbf{p}, \mathbf{q})$ and $\inf(\mathbf{p}, \mathbf{q})$ for the least upper bound and greatest lower bound operations in $\mathcal{D}_{\mathrm{w}}$. The usual lattice-theoretic notations $\vee$ and $\wedge$ [23] or $+$ and $\times$ [108] are confusing and misleading in the mass problem context.)

Similarly we implement $\vdash$ as $\geq$ and $\equiv$ as $=$ in $\mathcal{D}_{\mathrm{w}}$. In order to implement $\Rightarrow$ in $\mathcal{D}_{\mathrm{w}}$ we need the following theorem.

THEOREM 4.1.1. *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be Muchnik degrees. Then, among all Muchnik degrees* $\mathbf{z}$ *such that* $\sup(\mathbf{p}, \mathbf{z}) \geq \mathbf{q}$ *there is a unique smallest one. We denote this Muchnik degree by* $\mathrm{imp}(\mathbf{p}, \mathbf{q})$.

PROOF. Let P and Q be sets of reals such that $\deg_{\mathrm{w}}(P) = \mathbf{p}$ and $\deg_{\mathrm{w}}(Q) = \mathbf{q}$. Let $\mathrm{imp}(\mathbf{p}, \mathbf{q}) = \deg_{\mathrm{w}}(P \Rightarrow Q)$ where

$$(P \Rightarrow Q) = \{z\,;\ (\forall x \in P)(\exists y \in Q)((x, z) \geq_{\mathrm{T}} y)\}\,.$$

It is straightforward to verify that $\mathrm{imp}(\mathbf{p}, \mathbf{q})$ has the desired property. $\square$

We now have an interpretation of the formulas of propositional calculus in $\mathcal{D}_{\mathrm{w}}$. Namely, if ϕ is a mapping of propositional atoms into $\mathcal{D}_{\mathrm{w}}$, we extend ϕ to propositional formulas as follows: $\phi(A \wedge B) = \sup(\phi(A), \phi(B))$, $\phi(A \vee B) = \inf(\phi(A), \phi(B))$, $\phi(A \Rightarrow B) = \mathrm{imp}(\phi(A), \phi(B))$, and $\phi(\neg A) = \mathrm{imp}(\phi(A), \infty)$ where of course $\infty = \deg_{\mathrm{w}}(\emptyset) =$ the top degree in $\mathcal{D}_{\mathrm{w}}$. Let us define a propositional formula A to be $\mathcal{D}_{\mathrm{w}}$*-valid* if and only if $\phi(A) = \mathbf{0}$ for all ϕ. It is straightforward to show that the axioms of intuitionistic propositional calculus [114, §2.1] are $\mathcal{D}_{\mathrm{w}}$-valid, and that $\mathcal{D}_{\mathrm{w}}$-validity is preserved under the intuitionistic propositional rules of inference. Thus we see that all of the theorems of intuitionistic propositional calculus are $\mathcal{D}_{\mathrm{w}}$-valid. (One can show that the $\mathcal{D}_{\mathrm{w}}$-valid propositional formulas are precisely the theorems of *Jankov logic*, consisting of intuitionistic propositional calculus together with the so-called *weak law of the excluded middle*, $(\neg A) \vee (\neg\neg A)$. See for instance [108].)

The above interpretation of intuitionistic propositional calculus in $\mathcal{D}_{\mathrm{w}}$ can be extended to an interpretation of intuitionistic arithmetic, intuitionistic analysis, and intuitionistic higher-order logic. This is accomplished as follows. Recall from Subsection 1.1 that $\mathcal{D}_{\mathrm{T}}$ is the partial ordering of all Turing degrees. A set $\mathcal{U} \subseteq \mathcal{D}_{\mathrm{T}}$ is said to be *upward closed* if for all $\mathbf{a} \in \mathcal{U}$ and $\mathbf{a} \leq \mathbf{b} \in \mathcal{D}_{\mathrm{T}}$ we have $\mathbf{b} \in \mathcal{U}$. Obviously the upward closed sets in $\mathcal{D}_{\mathrm{T}}$ form a complete and completely distributive lattice under reverse inclusion. Moreover, as noted by Muchnik [71], the upward closed sets in $\mathcal{D}_{\mathrm{T}}$ are the open sets of a topology on $\mathcal{D}_{\mathrm{T}}$.

THEOREM 4.1.2. *The lattice* $\mathcal{D}_{\mathrm{w}}$ *is canonically isomorphic to the lattice of upward closed sets in* $\mathcal{D}_{\mathrm{T}}$ *ordered by reverse inclusion.*

PROOF. Recall from Subsection 1.2 that each Turing degree is identified with a Muchnik degree. Thus $\mathcal{D}_{\mathrm{T}} \subseteq \mathcal{D}_{\mathrm{w}}$. For each $\mathbf{p} \in \mathcal{D}_{\mathrm{w}}$ the corresponding upward closed set in $\mathcal{D}_{\mathrm{T}}$ is

$$\mathcal{U}_{\mathbf{p}} = \{\mathbf{a} \in \mathcal{D}_{\mathrm{T}}\,;\ \mathbf{p} \leq \mathbf{a}\}$$

and all upward closed sets in $\mathcal{D}_{\mathrm{T}}$ are of this form. It is also clear that $\mathbf{p} \leq \mathbf{q}$ if and only if $\mathcal{U}_{\mathbf{p}} \supseteq \mathcal{U}_{\mathbf{q}}$, so we have a canonical isomorphism as required. $\square$

REMARK 4.1.3. For any topological space X, let $\mathrm{Sh}(X)$ be the category of sheaves over X. This category $\mathrm{Sh}(X)$ is the standard example of a *topos*. See for instance [114,

§§14.5, 15.1, 15.2] and [64]. In particular, let $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$ be the category of sheaves over $\mathcal{D}_{\mathrm{T}}$ with the topology of upward closed sets. In light of Theorem 4.1.2 we refer to $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$ as *the Muchnik topos*. Regrettably, the Muchnik topos has not been studied extensively. Like $\mathrm{Sh}(X)$ for any topological space X, the Muchnik topos is a model of intuitionistic higher-order logic, intuitionistic arithmetic, and intuitionistic analysis. However, the Muchnik topos has the advantage of carrying with it the original intuitionistic motivation in terms of Kolmogorov's Aufgabenrechnung. We may therefore expect the Muchnik topos to yield new foundational insights. This is a topic of ongoing investigation.

4.2. Non-interpretability of intuitionism in $\mathcal{E}_{\mathrm{w}}$. In view of the Kolmogorov/Muchnik interpretation of intuitionism in $\mathcal{D}_{\mathrm{w}}$, one may ask whether intuitionism can be similarly interpreted in various sublattices of $\mathcal{D}_{\mathrm{w}}$. Following Birkhoff [12, 13] (first and second editions) we define a *Brouwerian lattice* to be a distributive lattice L with a top element and a bottom element such that for all $a, b \in L$ there is a unique smallest $c \in L$ such that $\sup(a, c) \geq b$. Just as classical propositional calculus may be viewed as the theory of Boolean lattices, so intuitionistic propositional calculus may be viewed as the theory of Brouwerian lattices. Theorem 4.1.1 says that $\mathcal{D}_{\mathrm{w}}$ is Brouwerian, and Sorbi and Terwijn [108] have investigated Brouwerian sublattices of $\mathcal{D}_{\mathrm{w}}$. In particular, each initial segment of $\mathcal{D}_{\mathrm{w}}$ is Brouwerian.

Recall from Subsection 1.3 that $\mathcal{E}_{\mathrm{w}}$ is a distributive sublattice of $\mathcal{D}_{\mathrm{w}}$ with top and bottom elements $\mathbf{1}$ and $\mathbf{0}$. In view of the great interest of $\mathcal{E}_{\mathrm{w}}$ as documented in Sections 2 and 3 above, it is natural to ask whether $\mathcal{E}_{\mathrm{w}}$ is Brouwerian. The answer is negative, as shown by the following theorem from [101].

THEOREM 4.2.1. *Given $\mathbf{p} \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{0} < \mathbf{p}$, we can find $\mathbf{p}_1 < \mathbf{p}$ such that $\mathbf{p}_1 \in \mathcal{E}_{\mathrm{w}}$ and there is no smallest $\mathbf{z} \in \mathcal{E}_{\mathrm{w}}$ such that* $\sup(\mathbf{p}_1, \mathbf{z}) \geq \mathbf{p}$.

PROOF. By the first part of Theorem 3.4.5, let $\mathbf{a}$ be a Turing degree such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ and $\mathbf{p} \not\leq \mathbf{a}$. Let $\mathbf{p}_1 = \inf(\mathbf{p}, \mathbf{a})$. Clearly $\mathbf{p}_1 < \mathbf{p}$, and by Theorem 3.3.1 we have $\mathbf{p}_1 \in \mathcal{E}_{\mathrm{w}}$. Now given $\mathbf{z} \in \mathcal{E}_{\mathrm{w}}$ such that $\sup(\mathbf{p}_1, \mathbf{z}) \geq \mathbf{p}$, we clearly have $\mathbf{0} < \mathbf{z}$ and $\mathbf{z} \not\leq \mathbf{a}$, so by the second part of Theorem 3.4.5 let $\mathbf{b}$ be a Turing degree such that $\mathbf{0} < \mathbf{b} < \mathbf{0}'$ and $\mathbf{z} \not\leq \mathbf{b}$ and $\sup(\mathbf{a}, \mathbf{b}) = \mathbf{0}'$. Let $\mathbf{z}_1 = \inf(\mathbf{z}, \mathbf{b})$. Clearly $\mathbf{z}_1 < \mathbf{z}$, and by Theorem 3.3.1 we have $\mathbf{z}_1 \in \mathcal{E}_{\mathrm{w}}$. Since $\mathbf{p} \leq \mathbf{1} < \mathbf{0}'$ we have $\sup(\mathbf{a}, \mathbf{b}) > \mathbf{p}$, hence $\sup(\mathbf{p}_1, \mathbf{z}_1) \geq \mathbf{p}$. This completes the proof. □

Theorem 4.2.1 implies that there are many pairs of degrees $\mathbf{p}_1, \mathbf{p} \in \mathcal{E}_{\mathrm{w}}$ such that $\mathrm{imp}(\mathbf{p}_1, \mathbf{p}) \notin \mathcal{E}_{\mathrm{w}}$ and moreover $\mathcal{E}_{\mathrm{w}}$ is not Brouwerian. Comparing Remark 4.1.3 with Theorem 4.2.1, we may say that $\mathcal{D}_{\mathrm{w}}$ provides an interesting model of intuitionistic higher-order arithmetic and analysis, while $\mathcal{E}_{\mathrm{w}}$ does not even provide a model of intuitionistic propositional calculus.

5. $\mathcal{E}_{\mathrm{w}}$ and symbolic dynamics. In this section we present an application of $\mathcal{E}_{\mathrm{w}}$ to symbolic dynamics. As explained below, symbolic dynamics is the study of subshifts. We are interested specifically in $\mathbf{Z}^d$-subshifts. A standard reference for symbolic dynamics is Lind/Marcus [63] which also includes an appendix on $\mathbf{Z}^d$-subshifts [63, §13.2].

5.1. Tiling problems. Historically, the subject of $\boldsymbol{Z}^2$-subshifts began with *tiling problems* in the sense of Wang [117]. A *Wang tile* is a unit square with colored edges. Given a finite set A of Wang tiles, let P_A be the problem of tiling the plane with copies of tiles from A. More formally, P_A is the set of mappings $x : \boldsymbol{Z} \times \boldsymbol{Z} \to A$ such that for all $(i, j) \in \boldsymbol{Z} \times \boldsymbol{Z}$ the right edge of $x(i, j)$ matches the left edge of $x(i+1, j)$ and the top edge of $x(i, j)$ matches the bottom edge of $x(i, j+1)$. Clearly P_A is an effectively closed set in the effectively compact space $A^{\boldsymbol{Z}\times\boldsymbol{Z}}$. From this it follows that $\deg_{\mathrm{w}}(P_A) \in \mathcal{E}_{\mathrm{w}}$ provided $P_A \neq \emptyset$.

In 1966 Berger [5] proved that, given A, it is algorithmically undecidable whether $P_A = \emptyset$. From this it follows that there exists an A such that $P_A \neq \emptyset$ but no $x \in P_A$ is periodic. In 1971 Robinson [82] gave an elegant simplified treatment of Berger's results. In 1974 Myers [72] used Robinson's method to construct an A such that $P_A \neq \emptyset$ but no $x \in P_A$ is computable. Thus $\mathbf{0} < \deg_{\mathrm{w}}(P_A) \leq \mathbf{1}$, and indeed, for the A constructed by Myers one has $\deg_{\mathrm{w}}(P_A) = \mathbf{1}$. My contribution in 2007 [105] was to show that for each $\mathbf{p} \in \mathcal{E}_{\mathrm{w}}$ one can find an A such that $\deg_{\mathrm{w}}(P_A) = \mathbf{p}$. Thus the Muchnik degrees of tiling problems are precisely characterized in terms of $\mathcal{E}_{\mathrm{w}}$. A new treatment of these results and many others is in Durand/Romashchenko/Shen [30].

5.2. Symbolic dynamics. Given a dynamical system, one may partition the state space into a finite number of regions. Then, each orbit of the system has a symbolic representation obtained by ignoring the actual states and considering only the regions. In this way one obtains a symbolic representation of the given system. The existence of these symbolic representations is part of the reason for the importance of the symbolic case in dynamical systems theory.

Some key definitions for symbolic dynamics are as follows. Fix a countable semigroup G. Specifically, let G be the additive group $(\boldsymbol{Z}^d, +)$ or the additive semigroup $(\boldsymbol{N}^d, +)$ where d is a positive integer. Let A be a finite set of symbols. The *shift action* of G on A^G is defined by $(S^g x)(h) = x(g + h)$ for all $g, h \in G$ and $x \in A^G$. We endow A with the discrete topology and A^G with the product topology. A *G-subshift* is a nonempty set $X \subseteq A^G$ which is topologically closed and *shift-invariant*, i.e., $x \in X$ implies $S^g x \in X$ for all $g \in G$. The study of subshifts is called *symbolic dynamics*.

Let $X \subseteq A^G$ and $Y \subseteq B^G$ be G-subshifts. A *shift morphism* is a continuous mapping $\Phi : X \to Y$ such that $\Phi(S^g x) = S^g \Phi(x)$ for all $g \in G$ and $x \in X$. Two G-subshifts are said to be *conjugate* if they are topologically isomorphic, i.e., there is a shift isomorphism between them. A compactness argument shows that any shift morphism is given by a *block code*, i.e., a finite mapping $\phi : A^F \to B$, where F is a fixed finite subset of G, such that $\Phi(x)(g) = \phi(S^g x \upharpoonright F)$ for all $x \in X$ and all $g \in G$. Since block codes are Turing functionals (in fact, bounded truth-table functionals), the existence of a shift morphism $\Phi : X \to Y$ implies that Y is weakly reducible to X. In particular, the Muchnik degree of X is a conjugacy invariant of X.

Since $\deg_{\mathrm{w}}(X)$ is a conjugacy invariant, it is appropriate to compare $\deg_{\mathrm{w}}(X)$ with other conjugacy invariants which have arisen previously in dynamical systems theory. One of the most important conjugacy invariants is the *topological entropy*,

$$\mathrm{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x \upharpoonright F_n \, ; \; x \in X\}|}{|F_n|}$$

where $F_n = \{-n, \dots, n\}^d$ if $G = \mathbf{Z}^d$, or $\{0, 1, \dots, n\}^d$ if $G = \mathbf{N}^d$. Here the cardinality of a finite set F is denoted $|F|$. As a guiding principle, one may say that $\deg_{\mathrm{w}}(X)$ represents a lower bound on the complexity of the orbits of X, while $\mathrm{ent}(X)$ represents an upper bound. In [106] we provide a precise characterization of $\mathrm{ent}(X)$ in terms of the initial segment complexity of the orbits of X. Relationships of this kind are a subject of ongoing investigation.

A G-subshift X is said to be *of finite type* if it is defined by a finite set E of excluded finite configurations. More precisely, X is of finite type if

$$X = \{x \in A^G \, ; \; (\forall g \in G)\, (S^g x \upharpoonright F \notin E)\}$$

where F and E are finite. Many of the subshifts which arise in practice (see for instance [15, 63]) are of finite type. Moreover, this property of subshifts is again a conjugacy invariant.

In 1989 [70] it was realized that $\mathbf{Z}^2$-subshifts of finite type are essentially the same thing as tiling problems. Clearly each tiling problem $P_A \neq \emptyset$ is a $\mathbf{Z}^2$-subshift of finite type. Conversely, it is easy to see that each $\mathbf{Z}^2$-subshift of finite type is conjugate to a tiling problem. Thus, all of the results and methods which were originally developed for tiling problems [5, 82, 72, 105] apply equally well to the study of $\mathbf{Z}^2$-subshifts of finite type. Hochman and Meyerovitch [42] have used these methods to show that a nonnegative real number is the entropy of a $\mathbf{Z}^2$-subshift of finite type if and only if it is right recursively enumerable. In addition, my result from [105] (see Subsection 5.1 above) provides the following characterization of the Muchnik degrees of such subshifts.

THEOREM 5.2.1. *Let* $\mathbf{p}$ *be a Muchnik degree. For each* $d \geq 2$ *the following statements are pairwise equivalent.*

1. $\mathbf{p} = \deg_{\mathrm{w}}(X)$ *where* X *is a* $\mathbf{Z}^d$*-subshift of finite type.*
2. $\mathbf{p} = \deg_{\mathrm{w}}(X)$ *where* X *is an* $\mathbf{N}^d$*-subshift of finite type.*
3. $\mathbf{p}$ *belongs to* $\mathcal{E}_{\mathrm{w}}$.

PROOF. See [105]. Another proof is implicit in [30]. □

5.3. An application. We shall now present an application of Theorem 5.2.1 which is stated purely in terms of subshifts, with no reference to Muchnik degrees. Namely, we shall construct an infinite collection of $\mathbf{Z}^2$-subshifts of finite type which are, in a certain sense, mutually incompatible. This application is intended to suggest that the Muchnik degrees may provide a potentially significant method for the classification of subshifts. In particular, each of the Muchnik degrees in Figure 1 represents a possibly interesting class of subshifts of finite type.

If X and Y are G-subshifts on k and l symbols respectively, let $X + Y$ and $X \times Y$ be the disjoint union and Cartesian product of X and Y. These are G-subshifts on $k + l$ and kl

symbols respectively. If $\mathcal{U}$ is a collection of G-subshifts, let $\mathrm{cl}(\mathcal{U})$ be the closure of $\mathcal{U}$ under $+$ and $\times$.

THEOREM 5.3.1. *We can find an infinite collection of $\mathbf{Z}^2$-subshifts of finite type, $\mathcal{W}$, such that for all partitions of $\mathcal{W}$ into two subcollections, $\mathcal{U}$ and $\mathcal{V}$, there is no shift morphism of X into Y for any $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$.*

PROOF. By Theorem 3.1.1.2 let $\mathbf{p}_i$ for $i = 1, 2, \ldots$ be an infinite family of Muchnik degrees in $\mathcal{E}_{\mathrm{w}}$ which are *independent*, i.e.,

$$\inf(\mathbf{p}_{i_i}, \ldots, \mathbf{p}_{i_m}) \not\leq \sup(\mathbf{p}_{j_1}, \ldots, \mathbf{p}_{j_n})$$

whenever $\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_n\} = \emptyset$. By Theorem 5.2.1, for each $i = 1, 2, \ldots$ let X_i be a $\mathbf{Z}^2$-subshift of finite type such that $\deg_{\mathrm{w}}(X_i) = \mathbf{p}_i$. Let $\mathcal{W}$ be the collection X_i, $i = 1, 2, \ldots$, and let $\mathcal{U}, \mathcal{V}$ be a partition of $\mathcal{W}$. By induction on $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$ we can easily show that neither of X and Y is Muchnik reducible to the other. Since each shift morphism is given by a block code, it follows that there is no shift morphism of X into Y or vice versa. □

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