Mass versus relativistic and rest masses

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The concept of relativistic mass, which increases with velocity, is not compatible with the standard language of relativity theory and impedes the understanding and learning of the theory by beginners. The same difficulty occurs with the term rest mass. To get rid of relativistic mass and rest mass it is appropriate to replace the equation $E=mc^2$ by the true Einstein's equation $E_0=mc^2$, where E_0 is the rest energy and m is the mass.
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I. MASS IN RELATIVITY

In 1946, while explaining the equivalence of mass and energy Albert Einstein called the relation $E=mc^2$ "the most urgent problem of our time."[1] Only two sentences in this paper were devoted to nuclear weapons, the rest to the physical meaning of $E=mc^2$. Today the subject of the relation between mass and energy remains as urgent as it was 60 years ago, not only because energy is so vital for our civilization, but because relativity theory, for which $E=mc^2$ has become an icon, is pivotal for science and education.

According to the standard model all processes in nature are in principle reducible to the interactions of elementary particles either massive such as electrons and protons, or very light such as neutrinos, or massless such as photons. In all these interactions the energy E and momentum $\mathbf{p}$ of an isolated system of particles are conserved. The mass of the system is defined in terms of energy and momentum by the most fundamental equation of relativity theory

$$m^2 = (E/c^2)^2 - (\mathbf{p}/c)^2. \tag{1}$$

The second basic equation connects momentum and the velocity $\mathbf{v}$,

$$\mathbf{p} = E\mathbf{v}/c^2. \tag{2}$$

These equations are consistent with the central idea of relativity theory—the idea of four dimensional space-time introduced by Minkowski, according to which the positions of a particle in space $\mathbf{r}$ and in time t form a four-vector. More precisely, the components of a four-vector have equal dimensions: $\{ct,\mathbf{r}\}$ or $\{t,\mathbf{r}/c\}$. Analogously, E and $\mathbf{p}c$ are components of a four-vector. The mass m is a four-scalar, which means that m has the same value in all reference frames, and hence, does not depend on velocity.

It follows from the definition of mass that the energy of a particle at rest (with $\mathbf{p}=0$) is given by

$$E_0 = mc^2, \tag{3}$$

where E_0 is rest energy. The history of the concept of rest energy started in 1905 with the famous article by Einstein.[2] He elaborated on it in 1921 (Ref. 3) and in 1934 (Ref. 4) stressing that the relation between energy and mass is given by $E_0=mc^2$.

II. THE UNIT OF VELOCITY

The speed c enters the basic equations not as the speed of light, but as the maximal speed in nature. Were photons—particles of light—slightly massive, their velocities would be smaller than c, but the basic equations would remain the same. It is appropriate for relativistic processes to use c as the unit of velocity and hence to use the system of units in which c is unity ($c=1$). It is convenient to denote the energy of a particle by e and the energy of a system of particles by E. In units in which $c=1$ the energy of a particle e and its momentum $\mathbf{p}$ have the same units as its mass m. Equations (1) and (2) take the form

$$m^2 = e^2 - p^2 \tag{4}$$

and

$$\mathbf{p} = e\mathbf{v}, \tag{5}$$

where $p=|\mathbf{p}|$.

By using the fundamental laws of conservation of energy and momentum and by applying the Pythagorean theorem ($a^2+b^2=c^2$) it is easy to describe the relativistic properties of individual particles as well as of composite systems of particles.[5]

III. RIGHT TRIANGLE

It is convenient to represent Eq. (4) by a right triangle with sides m (horizontal), p (vertical), and e (hypotenuse). A simple transfer of p^2 onto the left-hand side of Eq. (4) lets us display this pseudo-Euclidean equation on a Euclidean plane without the "handles" used for that purpose by Taylor and Wheeler in Ref. 7. This simple transfer implies replacement of the axis e by the axis m.

For a particle at rest $p=0$ and the triangle "collapses" to a horizontal line segment (biangle). In accordance with Eq. (3) we obtain $e_0=m$.

When $p \ll m$ we rewrite Eq. (4) as $(e-m)(e+m)=p^2$, realize that $e-m=e_k$ and $e+m \approx 2m$, and obtain the nonrelativistic expression for the kinetic energy, $e_k=p^2/2m$. Similarly, when $m \ll p$ we obtain from $(e-p)(e+p)=m^2$ the expression for ultrarelativistic particles $e-p=m^2/2e$. For massless particle the triangle collapses to a vertical biangle with $p=e$ and, hence, $v=1$.

IV. AN ASIDE ON CLOCKS AND RODS

A similar replacement of the temporal axis t by the axis corresponding to the interval s between two events immediately allows us to obtain the contraction of measuring rods and the time dilation of clocks in motion. These phenomena can be directly read off the definition of the interval s between two events: $s^2 = t^2 - \mathbf{r}^2$, where the space-time coordinate of one event is $(t, \mathbf{r})$ and that of the other is (0,0), with $\mathbf{r}^2 = x^2 + y^2 + z^2$.

The Lorentz transformations for motion in direction $+x$,

$$t = t'(e/m) + x'(p/m), \tag{6}$$

$$x = t'(p/m) + x'(e/m). \tag{7}$$

Here, the primed coordinates t', x' refer to the reference frame of the object, and the unprimed ones t, x to that of the observer.

For time dilation take the first event to be the creation of a particle with mass m and energy e, and as the second event take its decay; both events occur at $x'=0$. It follows from Eq. (6) that the dilation factor is $t/t' = e/m$.

For length contraction consider not a rod, but two particles of mass m and energy e situated at a distance $x' = \ell$ where ℓ is the length of a rod at rest which simultaneously fall down through a slot of length less than ℓ. The word "simultaneously" means here that $t'=0$. It follows from Eq. (7) that the contraction factor here is $x'/x = m/e$.

V. TWO PARTICLES

The mass of a system of two free particles is defined by a relation analogous to the equation for one particle

$$M^2 = E^2 - \mathbf{P}^2, \tag{8}$$

where

$$E = e_1 + e_2, \tag{9}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2. \tag{10}$$

It follows from these equations that masses are additive only for particles at rest. For particles in motion they are nonadditive, slightly in the Newtonian limit, and drastically in the ultrarelativistic limit. The mass of a system of free particles is not equal to the sum of the masses of its constituents. For instance the mass of a system of two massless photons in positronium decay is equal to the mass of positronium (see Ref. 5, Secs. 8–10). Equations (8)–(10) are fundamental for particle colliders such as the Large Hadron Collider at CERN.

VI. TEXTBOOKS AND MASS MEDIA

Unfortunately, sometimes and especially in his popular writings Einstein was careless about the subscript 0 and spoke about the equivalence of mass and energy and omitted the attribute "rest" for the energy. As a result Einstein's equation $E_0 = mc^2$ became known in its famous but misleading form $E = mc^2$. One of the most unfortunate consequences is the concept that the mass of a relativistic body increases with its velocity. This velocity dependent mass is known as "relativistic mass." Another consequence is the term "rest mass" and the corresponding symbol m_0. These confusing concepts and notations prevail in such classic texts as the ones by Born[8] and Feynman.[9] Moreover, in these texts the dependence of mass on velocity is presented as an experimental fact predicted by relativity theory and proving its correctness.

To substantiate the formula $m = E/c^2$ some authors use the connection between momentum and velocity in Newtonian mechanics, $\mathbf{p} = m\mathbf{v}$, forgetting that this relation is valid only when $v \ll c$ and that it contradicts the basic equation $m^2 = (E/c^2)^2 - (\mathbf{p}/c)^2$. Einstein's tolerance of $E = mc^2$ is related to the fact that he never used in his writings the basic equation of relativity theory, Eq. (1) (see Ref. 6). However, in 1948 he forcefully warned against the concept of mass increasing with velocity (see his letter quoted in Ref. 10). Unfortunately this warning was ignored. The formula $E = mc^2$, the concept relativistic mass, and the term rest mass are widely used even in the recent popular science literature (such as Refs. 11–13), and thus create serious stumbling blocks for beginners in relativity.

VII. CONCLUSION

The present round of discussions of the concept of mass started on the pages of the *American Journal of Physics* in the witty article by Adler[14] and has been continued by a number of authors.[15–18] The time is ripe today to ascend from $E = mc^2$ to $E_0 = mc^2$.

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[1] A. Einstein, "$E = mc^2$: The most urgent problem of our time," Sci Illustrated **I**, 16–17 (1946), reprinted in A. Einstein, *Out of My Later Years* (Gramercy, New York, 1993), pp. 49–53.
[2] A. Einstein, "Does the inertia of a body depend on its energy content?," Ann. Phys. **18**, 639–641 (1905).
[3] A. Einstein, *The Meaning of Relativity* (Princeton U. P., Princeton, 1921). See Eq. (44).
[4] A. Einstein, "Elementary derivation of the equivalence of mass and energy," Bull. Am. Math. Soc. **41**, 223–230 (1935).
[5] L. B. Okun, "The theory of relativity and the Pythagorean theorem," Phys. Usp. **51**, 622–631 (2008); arXiv:0809.2379.
[6] L. B. Okun, "The Einstein formula $E_0 = mc^2$ 'Isn't the Lord laughing?'," Phys. Usp. **51**, 513–527 (2008); arXiv:0808.0437.
[7] E. F. Taylor and J. A. Wheeler, *Spacetime Physics: Introduction to Special Relativity*, 2nd ed. (W. H. Freeman, New York, 1992), pp. 246–252.
[8] M. Born, *Einstein's Relativity Theory* (Dover, New York, 1962).
[9] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963–1965).
[10] L. B. Okun, "The concept of mass," Phys. Today **42**(6), 31–36 (1989).
[11] S. Hawking, *The Universe in a Nutshell* (Bantam, 2001).
[12] M. Jammer, *Concepts of Mass in Contemporary Physics and Philosophy* (Princeton U. P., Princeton, 2000).
[13] R. Stannard, *Relativity: A Very Short Introduction* (Oxford U. P., Oxford, 2008).
[14] C. G. Adler, "Does mass really depend on velocity, dad?," Am. J. Phys. **55**, 739–743 (1987).
[15] L. B. Okun, "The concept of mass (mass, energy, relativity)," Sov. Phys. Usp. **32**, 629–638 (1989).
[16] Wolfgang Rindler, Michael A. Vandyck, Poovan Murugesan, Siegfried Ruschin, Catherine Sauter, and Lev B. Okun, "Putting to rest mass misconceptions," Phys. Today **43**(5), 13–15 (1990); **43**(5), 115–117 (1990).
[17] T. R. Sandin, "In defense of relativistic mass," Am. J. Phys. **59**, 1032–1036 (1991).
[18] "Mass in special relativity," Wikipedia, 2008.