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Masses and Thermodynamic Properties of a Quarkonium System

E. P. Inyang[1, 2], E. P. Inyang[2], I. O. Akpan[2], J. E. Ntibi[2] and E. S. William[2]

[1]*Department of Pure and Applied Sciences, National Open University of Nigeria, Abuja*

[2]*Theoretical Physics Group, Department of Physics, University of Calabar, P.M.B 1115 Calabar Nigeria*

Corresponding author email: etidophysics@gmail.com

Abstract

Hulthen plus Hellmann potentials are adopted as the quark-antiquark interaction potential for studying the thermodynamic properties and the mass spectra of heavy mesons. The potential was made to be temperature dependent by replacing the screening parameter with Debye mass. We solved the radial Schrödinger equation analytically using the Nikiforov-Uvarov method. The energy eigenvalues and corresponding wave function in terms of Laguerre polynomials were obtained. The present results are applied for calculating the mass of heavy mesons such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, and thermodynamic properties such as the mean energy, the specific heat, the free energy, and the entropy. Four special cases were considered when some of the potential parameters were set to zero, resulting in Hellmann potential, Yukawa potential, Coulomb potential, and Hulthen potential, respectively. The present potential provides satisfying results in comparison with experimental data and the work of other researchers.

Keywords: Hulthen potential; Hellmann potential; Thermodynamic properties; Schrödinger equation; Nikiforov-Uvarov method **;** heavy mesons

1.0 Introduction

Thermodynamics is the branch of physics that is concerned with temperature and its relation to energy. This area of physics plays an essential role in high energy physics [1]. The study of thermodynamic properties is significant in various areas of physical and chemical sciences. This is made possible using the solutions of the quantum mechanical problems, which contain all the necessary information to describe the quantum system under study [2-4]. Thermodynamic properties play an essential role in describing quark-gluon plasma. Its properties also play an essential role in calculating heavy quarks in comparison to the strange quark matter [5]. The quarkonia with heavy quark and antiquark and their interaction are well described by the Schrödinger equation (SE). The solution of the spectral problem for the SE with spherically symmetric potentials is of significant concern in describing the spectra of quarkonia. Potential models offer a rather good description of the mass spectra of systems such as bottomonium $\left(b\bar{b}\right)$, charmonium $\left(c\bar{c}\right)$, etc [6]. In simulating the interaction potentials for these systems, confining-type potentials are generally used. The holding potential is the so-called

Cornell potential or Killingbeck potential with two terms, one of which is responsible for the Coulomb interaction of the quarks, and the other corresponds to a confining term [7].

The Hulthen potential takes the form [8]

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1-e^{-\alpha r}}, \tag{1}$$

where α is the screening parameter, and A_0 is the potential strength constant which is sometimes identified with the atomic number when the potential is used for atomic phenomena [9]. It is a short-range potential that behaves like a Coulomb potential for small values and decreases exponentially for large values. It has been used in many branches of physics, such as nuclear and particle physics, atomic physics, solid-state physics, and chemical physics [10,11]. The Hellmann potential which is a superposition of an attraction Coulomb potential and a Yukawa potential can be expressed as [12]

$$V(r) = -\frac{A_1}{r} + \frac{A_2 e^{-\alpha r}}{r}, \tag{2}$$

where the parameters A_1 and A_2 denote the strength of Coulomb and Yukawa potentials respectively, α denotes the screening parameter, and r is the distance between two particles. Over the past years, the potential model has received much concern from many authors [13-16]. Many authors have provided both exact and approximate solutions to SE using different methods with Cornell potential.

Abu-Shady *et al*. [17] studied the thermodynamic properties of heavy mesons in the non- relativistic quark model using the Nikiforov-Uvarov method. Okorie et al.[18] obtained the eigenvalues and eigenfunction, vibrational partition function, and other relevant thermodynamic properties of the SE with modified Mobius squared potential using a modified factorization method. Prasanth et al.[19] used the equation of state (EoS) of Quark-Gluon Plasma(QGP) with Mayer's theory of plasma to derived a semi-classical EoS of QGP including the contribution of quantum effects near the transition temperature.

Carrington et al.[20] studied the transport of heavy quarks across a system, which is called plasma, using a Fokker-Plank equation where the quarks interact with long-wavelength chromodynamic fields. Abu-Shady and Ikot [21] solved N-radial SE analytically using the supersymmetric quantum mechanics method (SUSYQM), in which the heavy quarkonia potential is introduced at finite temperature and baryon chemical potential.

Vega and Flores [22] solved approximately, the Schrödinger equation with the Cornell potential using the variational method and supersymmetric quantum mechanics method (SUSYQM). Furthermore, Ciftci and Kisoglu [23] solved non-relativistic arbitrary l-states of quarkonium through the asymptotic iteration method (AIM). The energy eigenvalues with any $l \neq 0$ states and mass of the heavy quark-antiquark system (quarkonium) were obtained, in which the quarks are considered as spinless for easiness and are bounded by Cornell potential. An analytic solution of the N-dimensional radial Schrödinger equation with the mixture of vector and scalar potentials via the Laplace transformation method (LTM) was studied by [24]. The results were employed to study the different properties of the heavy-light mesons. Al-Jamel and Widyan [25] studied the spin- averaged mass spectra of heavy quarkonia in a Coulomb plus quadratic potential within the framework of the non-relativistic Schrödinger equation using the Nikiforov-Uvarov method. Al-Oun *et al*. [26] examined heavy quarkonia ($c\bar{c}$ and $b\bar{b}$) characteristics properties in the general framework of non-relativistic potential models consisting of a Coulomb plus quadratic potential. Kumar and Chand [27] carried out an asymptotic study to the N-dimensional radial Schrödinger equation for the quark-antiquark interaction potential employing the asymptotic iteration method (AIM). The complete energy spectra of the consigned system are obtained by computing and adding energy eigenvalues for the ground state of large r and small r. Mansour and Gamal [28] also studied the bound state of heavy quarks using a general polynomial potential with the NU method. Abu-Shady *et al.* [29] studied the N-dimensional radial Schrödinger equation using the analytical exact iteration method (AEIM). The Cornell potential is generalized to chemical potential and finite temperature. Ibekwe et al.[30] solved the radial SE with an exponential, generalized, harmonic Cornell potential using the series expansion method. They applied the bound state eigenvalues to study the energy spectra for CO, NO, CH, and N_2 diatomic molecules and the mass spectra of heavy quarkonium systems. Inyang et al.[31] obtained the Klein-Gordon equation solutions for the Yukawa potential using the Nikiforov-Uvarov method. The energy eigenvalues were obtained both in a relativistic and non-relativistic regime. They applied the results to calculate heavy-meson masses of charmonium $c\bar{c}$ and bottomonium $b\bar{b}$.

Recently, an effort has been made with great interest in a combination of two or more potentials in both the relativistic and non-relativistic approach. The essence of combining two or more physical potential models is to have a wider range of applications. For example, Edet et al. [32] obtained an approximate solution of the SE for the modified Kratzer potential plus screened Coulomb potential model using the Nikiforov–Uvarov method. Also,William et al.[33] obtained bound state solutions of the radial Schrödinger equation by the combination of Hulthén and Hellmann potential within the framework of Nikiforov-Uvarov (NU) method. With the above studies in mind, we seek to study the SE by adopting a potential obtained from the combination of Hulthen potential [Eq.(1)] and Hellmann potential [Eq.(2)] analytically by using the NU method and the results applied to calculate the properties of quarkonium particles such as masses and thermodynamic properties that have not been considered before using this potential to the best of our knowledge. The potential takes the form:

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1-e^{-\alpha r}} - \frac{A_1}{r} + \frac{A_2 e^{-\alpha r}}{r}, \tag{3}$$

where A_0, A_1 and A_2 are potential strength parameters and α is the screening parameter. In order to make Eq.(3) temperature-dependent, the screening parameter is replaced with Debye mass $\left(m_D(T)\right)$ which is temperature-dependent and vanishes at $T \to 0$, this gives

$$V(r,T) = -\frac{A_0 e^{-m_D(T)r}}{1-e^{-m_D(T)r}} - \frac{A_1}{r} - \frac{A_2 e^{-m_D(T)r}}{r} \tag{4}$$

We carry out Taylor series expansion of the exponential terms in Eq.(4) up to order three, in order to make the potential to interact in the quark-antiquark system and this yields,

$$\frac{e^{-m_D(T)r}}{r}=\frac{1}{r}-m_D(T)+\frac{m_D^2(\mathrm{T})\mathrm{r}}{2}-\frac{m_D^3(T)r^2}{6}+\ldots \tag{5}$$

$$\frac{e^{-m_D(T)r}}{1-e^{-m_D(T)\mathrm{r}}}=\frac{1}{m_D(T)r}-\frac{1}{2}+\frac{m_D(T)r}{12}+\ldots \tag{6}$$

We substitute Eqs.(5) and (6) into Eq.(4) and obtain

$$V(r,T)=-\frac{\beta_0}{r}+\beta_1 r-\beta_2 r^2+\beta_3, \tag{7}$$

where

$$-\beta_0=A_2-A_1-\frac{A_0}{m_D(T)},\quad \beta_1=\frac{A_2 m_D^2(T)}{2}-\frac{A_0 m_D(T)}{12},\quad \beta_2=\frac{A_2 m_D^3(T)}{6},\quad \beta_3=\frac{A_0}{2}-A_2 m_D(T) \tag{8}$$

The paper is structured as follows: The NU method solution of the SE for the Hulthen plus Hellmann potentials is composed in section 2. In section 3, the thermodynamic properties of the Schrödinger equation with Hulthen plus Hellmann potentials are obtained. In Section 4, the results of heavy quarkonia, such as charmonium and bottomonium, are presented. Finally, the concluding remarks are presented in section 5.

2. Approximate solutions of the Schrödinger equation with Hulthen plus Hellmann potentials

The Schrödinger equation (SE) for two particles interacting via potential $V(r)$ is given by [34]

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]R(r)=0, \tag{9}$$

where l,μ,r and $\hbar$ are the angular momentum quantum number, the reduced mass for the quarkonium particle, inter-particle distance and reduced plank constant, respectively.

We substitute Eq.(7) into Eq.(9) and obtain

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu E_{nl}}{\hbar^2}+\frac{2\mu\beta_0}{\hbar^2 r}-\frac{2\mu\beta_1 r}{\hbar^2}+\frac{2\mu\beta_2 r^2}{\hbar^2}-\frac{2\mu\beta_3}{\hbar^2}-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{10}$$

Let,

$$\left.\begin{aligned}&\zeta=\frac{2\mu}{\hbar^2}\left(E-\beta_3\right),\quad \alpha_0=\frac{2\mu\beta_0}{\hbar^2},\quad \alpha_1=\frac{2\mu\beta_1}{\hbar^2}\\&\alpha_2=\frac{2\mu\beta_2}{\hbar^2},\quad \gamma=l(l+1)\end{aligned}\right\} \tag{11}$$

Substituting Eq.(11) into Eq.(10), we have

$$\frac{d^2R(r)}{dr^2}+\left[\zeta+\frac{\alpha_0}{r}-\alpha_1 r+\alpha_2 r^2-\frac{\gamma}{r^2}\right]R(r)=0 \tag{12}$$

We transform the coordinate of Eq.(12) from r to x by setting

$$x = \frac{1}{r} \tag{13}$$

Upon differentiating Eq.(13) and simplifying we have

$$\frac{d^2R}{dr^2} = \frac{2}{r^3}\frac{dR}{dx} + \frac{1}{r^4}\frac{d^2R}{dx^2} \tag{14}$$

Substituting Eqs.(13) and (14) into Eq.(12) we have

$$\frac{d^2R(\mathrm{x})}{dx^2} + \frac{2}{x}\frac{dR}{dx} + \frac{1}{x^4}\left[\zeta + \alpha_0 x - \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} - \gamma x^2\right]R(x) = 0 \tag{15}$$

Next, we suggest the subsequent approximation scheme on $\frac{\alpha_1}{x}$ and $\frac{\alpha_2}{x^2}$ terms.

Suppose that there is a distinguishing radius r_0 of the meson. Then the system is based on the expansion of $\frac{\alpha_1}{x}$ and $\frac{\alpha_2}{x^2}$ in a power series about r_0; i.e. $\delta \equiv \frac{1}{r_0}$, in the x -space up to the second order. This is analogous to Pekeris approximation, which helps to distort the centrifugal term such that the modified potential can be solved by NU method [35].

Setting $y = x - \delta$ and around $y = 0$ it can be expanded into a series of powers as;

$$\frac{\alpha_1}{x} = \frac{\alpha_1}{y+\delta} = \frac{\alpha_1}{\delta\left(1+\frac{y}{\delta}\right)} = \frac{\alpha_1}{\delta}\left(1+\frac{y}{\delta}\right)^{-1} \tag{16}$$

which yields

$$\frac{\alpha_1}{x} = \alpha_1\left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2}{\delta^3}\right) \tag{17}$$

Similarly,

$$\frac{\alpha_2}{x^2} = \alpha_2\left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4}\right) \tag{18}$$

By substituting Eqs.(17) and (18) into Eq.(15) , we obtain

$$\frac{d^2R(x)}{dx^2} + \frac{2x}{x^2}\frac{dR(x)}{dx} + \frac{1}{x^4}\left[-\varepsilon + \alpha x - \beta x^2\right]R(x) = 0 \tag{19}$$

where

$$-\varepsilon = \left(\zeta + \frac{6\alpha_2}{\delta^2} - \frac{3\alpha_1}{\delta}\right),\ \alpha = \left(\frac{3\alpha_1}{\delta^2} + \alpha_0 - \frac{8\alpha_2}{\delta^3}\right),\ \beta = \left(\gamma + \frac{\alpha_1}{\delta^3} - \frac{3\alpha_2}{\delta^4}\right)\Bigg\} \tag{20}$$

Comparing Eq.(19) and Eq.(A1), we obtain

$$\left.\begin{array}{l}\tilde{\tau}(x)=2x,\ \sigma(x)=x^{2},\ \tilde{\sigma}(x)=-\varepsilon+\alpha x-\beta x^{2}\\ \sigma'(x)=2x,\ \sigma''(x)=2\end{array}\right\} \tag{21}$$

We substitute Eq.(21) into Eq.(A9) and obtain

$$\pi(x)=\pm\sqrt{\varepsilon-\alpha x+\left(\beta+k\right)x^{2}} \tag{22}$$

To determine k, we take the discriminant of the function under the square root, which yields

$$k=\frac{\alpha^{2}-4\beta\varepsilon}{4\varepsilon} \tag{23}$$

We substitute Eq.(23) into Eq.(22) and have

$$\pi(x)=\pm\left(\frac{\alpha x}{2\sqrt{\varepsilon}}-\frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{24}$$

Differentiating the negative part of Eq.(24) yields

$$\pi_{-}'(x)=-\frac{\alpha}{2\sqrt{\varepsilon}} \tag{25}$$

Substituting Eqs. (21) and (25) into Eq.(A7) we have

$$\tau(x)=2x-\frac{\alpha x}{\sqrt{\varepsilon}}+\frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{26}$$

Differentiating Eq. (26) we have

$$\tau'(x)=2-\frac{\alpha}{\sqrt{\varepsilon}} \tag{27}$$

By using Eq.(A10), we obtain

$$\lambda=\frac{\alpha^{2}-4\beta\varepsilon}{4\varepsilon}-\frac{\alpha}{2\sqrt{\varepsilon}} \tag{28}$$

And using Eq.(A11), we obtain

$$\lambda_{n}=\frac{n\alpha}{\sqrt{\varepsilon}}-n^{2}-n \tag{29}$$

Equating Eqs.(28) and (29) , the energy eigenvalues of Eq.(10) is as given

$$E_{nl}=A_{0}\left(\frac{1}{2}-\frac{m_{D}(T)}{4\delta}\right)+A_{2}m_{D}(T)\left(\frac{3m_{D}(T)}{2\delta}-m_{D}^{2}(T)-1\right)-\frac{\hbar^{2}}{8\mu}\left[\frac{\frac{2\mu}{\hbar^{2}}\left(A_{2}-A_{1}+\frac{A_{0}}{m_{D}(T)}\right)+\frac{\mu m_{D}(T)}{\hbar^{2}\delta^{2}}\left(3A_{2}m_{D}(T)-\frac{A_{0}}{2}\right)-\frac{8\mu A_{2}m_{D}^{3}(T)}{3\hbar^{2}\delta^{3}}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^{2}+\frac{\mu A_{2}m_{D}^{2}(T)}{\hbar^{2}\delta^{3}}\left(1-\frac{m_{D}(T)}{\delta}\right)-\frac{\mu A_{0}m_{D}(T)}{6\hbar^{2}\delta^{3}}}}\right]^{2} \tag{30}$$

Special cases

1. When we set $A_0 = A_1 = 0$, we obtain the energy eigenvalues for Yukawa potential

$$E_{nl} = A_2 m_D(T)\left(\frac{3m_D(T)}{2\delta} - m_D^2(T) - 1\right) - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu A_2}{\hbar^2} + \frac{3\mu A_2 m_D^2(T)}{\hbar^2\delta^2} - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2\delta^3}\left(1 - \frac{m_D(T)}{\delta}\right)}}\right]^2 \quad (31)$$

2. When we set $A_1 = A_2 = 0$, we obtain the energy eigenvalues for Hulthen potential

$$E_{nl} = A_0\left(\frac{1}{2} - \frac{m_D(T)}{4\delta}\right) - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu A_0}{\hbar^2 m_D(T)} - \frac{A_0 \mu m_D(T)}{2\hbar^2\delta^2}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\mu A_0 m_D(T)}{6\hbar^2\delta^3}}}\right]^2 \quad (32)$$

3. When we set $A_0 = 0$ we obtain the energy eigenvalues for Hellmann potential

$$E_{nl} = A_2 m_D(T)\left(\frac{3m_D(T)}{2\delta} - m_D^2(T) - 1\right) - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}(A_2 - A_1) + \frac{3A_2\mu m_D^2(T)}{\hbar^2\delta^2} - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2\delta^3}\left(1 - \frac{m_D(T)}{\delta}\right)}}\right]^2 \quad (33)$$

4. When we set $A_2 = A_3 = m_D(T) = 0$, we obtain the energy eigenvalues for the Coulomb potential

$$E_{nl} = -\frac{\mu A_1^2}{2\hbar^2 (n + l + 1)^2} \quad (34)$$

The result of Eq.(34) is very consistent with the result obtained in Eq.(36) of Ref.[32].

To determine the wavefunction, we substitute Eqs. (21) and (24) into Eq.(A4) and obtain

$$\frac{d\phi}{\phi} = \left(\frac{\varepsilon}{x^2\sqrt{\varepsilon}} - \frac{\alpha}{2x\sqrt{\varepsilon}}\right)dx \quad (35)$$

Integrating Eq.(35) ,we obtain

$$\phi(x) = x^{-\frac{\alpha}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{x\sqrt{\varepsilon}}} \quad (36)$$

By substituting Eqs.(21) and (24) into Eq.(A6) and integrating, thereafter simplify we obtain

$$\rho(x) = x^{-\frac{\alpha}{\sqrt{\varepsilon}}} e^{-\frac{2\varepsilon}{x\sqrt{\varepsilon}}} \quad (37)$$

Substituting Eqs.(21) and(37) into Eq.(A5) we have

$$y_n(x) = N_{nl} e^{\frac{2\varepsilon}{x\sqrt{\varepsilon}}} x^{\frac{\alpha}{\sqrt{\varepsilon}}} \frac{d^n}{dx^n}\left[e^{-\frac{2\varepsilon}{x\sqrt{\varepsilon}}} x^{2n - \frac{\alpha}{\sqrt{\varepsilon}}}\right] \quad (38)$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{x\sqrt{\varepsilon}}\right)=\frac{1}{n!}e^{\frac{2\varepsilon}{x\sqrt{\varepsilon}}}x^{\frac{\alpha}{\sqrt{\varepsilon}}}\frac{d^n}{dx^n}\left(e^{-\frac{2\varepsilon}{x\sqrt{x}}}x^{2n-\frac{\alpha}{\sqrt{\varepsilon}}}\right) \tag{39}$$

where

$$\frac{1}{n!}=B_n \tag{40}$$

Hence,

$$y_n(x)\equiv L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{x\sqrt{\varepsilon}}\right) \tag{41}$$

Substituting Eqs.(36) and (41) into Eq.(A2) we obtain the wavefunction of Eq.(10) in terms of Laguerre polynomial as

$$\psi(x)=N_{nl}x^{-\frac{\alpha}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{x\sqrt{\varepsilon}}}L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{x\sqrt{\varepsilon}}\right) \tag{42}$$

where N_{nl} is normalization constant, which can be obtained from

$$\int_0^{\infty}|N_{nl}(r)|^2dr=1 \tag{43}$$

3. Thermodynamic properties of the Schrödinger equation with Hulthen plus Hellmann potential

Thermodynamic properties of Hulthen plus Hellmann potential can be obtained from the partition function by setting Temperature $T\rightarrow 0$, which vanishes Debye mass and reduces Eq.(30) to

$$E_{nl}=P_1-\frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2, \tag{44}$$

where,

$$\sigma=\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2} \tag{45}$$

$$P_1=\frac{A_0}{2} \tag{46}$$

$$P_2=\frac{2\mu}{\hbar^2}(A_1-A_2) \tag{47}$$

3.1 Partition function $Z(\beta)$

The partition function takes the form,

$$Z(\beta)=\sum_{n=0}^{\lambda}e^{-\beta E_{nl}} \tag{48}$$

where,

$$\beta = \frac{1}{KT} \tag{49}$$

K is the Boltzman constant, T is the absolute temperature, n is the principal quantum number, $n = 0,1,2,3...$ and λ is the maximum or upper bound quantum number.

Substituting Eq.(44) into Eq.(48) we obtain

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta\left(P_1 - \frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\sigma)}\right]^2\right)} \tag{50}$$

In the classical limit, at high temperature T , the sum is replaced by an integral,

$$Z(\beta) = \int_0^{\lambda} e^{M_1\beta + \frac{N\beta}{\rho^2}} d\rho \tag{51}$$

where,

$$n + \sigma = \rho \tag{52}$$

$$M_1 = -P_1 \tag{53}$$

$$N = \frac{\hbar^2 P_2^2}{8\mu} \tag{54}$$

Integrating Eq.(51) we obtain the partition function as,

$$Z(\beta) = \frac{1}{2} e^{M_1\beta} \sqrt{N\beta} \left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}} - 2\sqrt{N\beta}\sqrt{\pi} erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}} - 2\sqrt{\pi} \right) \tag{55}$$

and the imaginary error function $erfi(x)$ is defined as follows [16],

$$erfi(x) = \frac{erf(ix)}{i} = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt. \tag{56}$$

3.2 Mean energy $U(\beta)$

$$U(\beta) = -\frac{\partial}{\partial\beta} InZ(\beta), \tag{57}$$

Substituting Eq.(65) into Eq.(67) we obtain,

$$U(\beta) = -\frac{\left[M_1 e^{M_1\beta} \sqrt{N\beta}\Delta_1 + \frac{\frac{1}{4} e^{M_1\beta}\Delta_1 N}{\sqrt{N\beta}} + \frac{1}{2} e^{M_1\beta}\sqrt{N\beta}\Delta_2 \right]}{e^{M_1\beta}\sqrt{N\beta}\Delta_1} \tag{58}$$

where,

$$\Delta_1 = \frac{2\lambda e^{\frac{N\beta}{\lambda^2}} - 2\sqrt{N\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}} - 2\sqrt{\pi} \tag{59}$$

$$\Delta_2 = -\frac{\lambda e^{\frac{N\beta}{\lambda^2}} N}{(N\beta)^{\frac{3}{2}}} - \frac{\sqrt{N}\sqrt{\pi}\, erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta^2}} + \frac{N^{\frac{3}{2}}\sqrt{\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{(N\beta)^{\frac{3}{2}}} \tag{60}$$

3.3 Free energy $F(\beta)$

$$F(\beta) = -KT \ln Z(\beta) \tag{61}$$

We substitute Eqs.(49) and (55) into Eq.(61) and obtain

$$F(\beta) = -\frac{1}{\beta}\ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}} - 2\sqrt{N\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}} - 2\sqrt{\pi}\right)\right] \tag{62}$$

3.4 Entropy $S(\beta)$

$$S(\beta) = K \ln Z(\beta) - K\beta \frac{\partial}{\partial\beta}\ln Z(\beta) \tag{63}$$

We substitute Eqs.(55) and (58) into Eq.(63) and obtain

$$S(\beta) = K \ln\left[\frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}} - 2\sqrt{N\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)}{\sqrt{N\beta}} - 2\sqrt{\pi}\right)\right]$$
$$-K\beta \frac{\left(M_1 e^{M_1\beta}\sqrt{N\beta}\Delta_1 + \frac{\frac{1}{4}e^{M_1\beta}\Delta_1 N}{\sqrt{N\beta}} + \frac{1}{2}e^{M_1\beta}\sqrt{N\beta}\Delta_2\right)}{e^{M_1\beta}\sqrt{N\beta}\Delta_1} \tag{64}$$

3.5 Specific heat $C(\beta)$

$$C(\beta) = \frac{\partial U}{\partial T} = -K\beta^2 \frac{\partial U}{\partial \beta} \tag{65}$$

Substituting Eqs.(59) and (60) into Eq.(58) and then substitute into Eq.(65) we obtain

$$C(\beta)=-K\beta^{2}\left\{\begin{aligned}&\frac{1}{\gamma_1}\left[\begin{aligned}&\left(M_1^2\right)\gamma_1+\frac{M_1e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N}{\sqrt{N\beta}}-\frac{2M_1e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\\&-\frac{1}{4}\frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N^2}{(N\beta)^{\frac{3}{2}}}+\frac{1}{2}\frac{e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta^2}-\frac{e^{M_1\beta}Ne^{\frac{N\beta}{\lambda^2}}}{3\lambda}\end{aligned}\right]\\&-\frac{1}{\gamma_1}\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\right]\\&-\frac{1}{\gamma_3}\left[\begin{aligned}&\left[\frac{1}{2}\left(M_1e^{M_1\beta}\right)\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)+\frac{\gamma_2}{4}-\frac{1}{2}\frac{e^{M_1\beta}}{\beta}\right]N\\&+\left[M_1\gamma_1+\frac{\gamma_2}{2}-\frac{e^{M_1\beta}+\lambda e^{\frac{N\beta}{\lambda^2}}}{\beta}\right]\lambda e^{\frac{N\beta}{\lambda^2}}\end{aligned}\right]\end{aligned}\right\} \tag{66}$$

where

$$\gamma_1=e^{M_1\beta}\sqrt{N\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right) \tag{67}$$

$$\gamma_2=\frac{e^{M_1\beta}\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)N}{\sqrt{N\beta}} \tag{68}$$

$$\gamma_3=e^{M_1\beta}N\beta^2\left(\frac{2\lambda e^{\frac{N\beta}{\lambda^2}}}{\sqrt{N\beta}}-2\sqrt{\pi}erfi\left(\frac{\sqrt{N\beta}}{\lambda}\right)-2\sqrt{\pi}\right)^2 \tag{69}$$

4. Results

Using the relation in Refs. [36, 37], we calculate the mass spectra of the heavy quarkonia such as charmonium and bottomonium.

$$M=2m+E_{nl} \tag{70}$$

where m is quarkonium bare mass and E_{nl} is energy eigenvalues.

By substituting Eq.(30) into Eq.(70) we obtain the mass spectra for Hulthen plus Hellmann potential as,

$$M = 2m + A_0\left(\frac{1}{2} - \frac{m_D(T)}{4\delta}\right) + A_2 m_D(T)\left(\frac{3m_D(T)}{2\delta} - m_D^2(T) - 1\right)$$
$$-\frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(A_2 - A_1 + \frac{A_0}{m_D(T)}\right) + \frac{\mu m_D(T)}{\hbar^2\delta^2}\left(3A_2 m_D(T) - \frac{A_0}{2}\right) - \frac{8\mu A_2 m_D^3(T)}{3\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{\mu A_2 m_D^2(T)}{\hbar^2\delta^3}\left(1 - \frac{m_D(T)}{\delta}\right) - \frac{\mu A_0 m_D(T)}{6\hbar^2\delta^3}}}\right]^2 \quad (71)$$

Table 1

Mass spectra of charmonium in (GeV) for Hulthen plus Hellmann potential (m_c =1.209 GeV, μ = 0.6045 GeV, A_0 = -1.693 GeV, A_1 = 20.654 GeV, A_2 = 0.018 GeV, δ = 0.2 GeV, $m_D(T)$ = 1.52 GeV, $\hbar$ = 1)

State	Present work	[35]	[25]	Experiment[38,39]
1S	3.096	3.078	3.096	3.097
2S	3.686	4.187	3.686	3.686
1P	3.295	3.415	3.433	3.525
2P	3.802	4.143	3.910	3.773
3S	4.040	5.297	3.984	4.040
4S	4.269	6.407	4.150	4.263
1D	3.583	3.752	3.767	3.770
2D	3.976	-	-	4.159
1F	3.862	-	-	-

Table 2

Mass spectra of bottomonium in (GeV) for Hulthen plus Hellmann potential (m_b = 4.823 GeV, μ = 2.4115 GeV, A_0 = -1.591 GeV, A_1 = 9.649 GeV, A_2 = 0.028 GeV, δ = 0.25 GeV, $m_D(T)$ = 1.52 GeV, $\hbar$ = 1)

State	Present work	[35]	[25]	Experiment[38,39]
1S	9.460	9.510	9.460	9.460
2S	10.023	10.627	10.023	10.023
1P	9.661	9.862	9.840	9.899
2P	10.138	10.944	10.160	10.260
3S	10.355	11.726	10.280	10.355
4S	10.567	12.834	10.420	10.580
1D	9.943	10.214	10.140	10.164
2D	10.306	-	-	-
1F	10.209	-	-	-

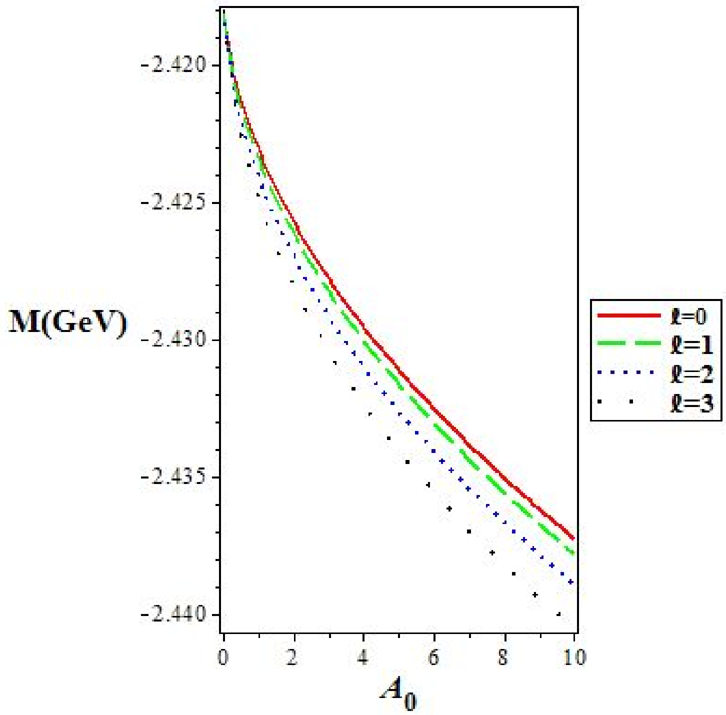

Fig. 1: Variation of mass spectra with potential strength (A_0) for different quantum numbers

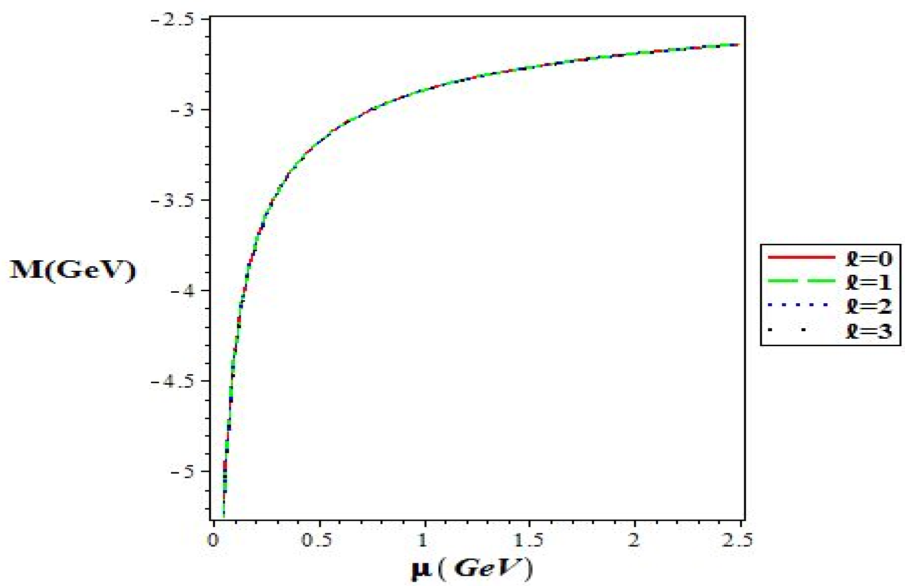

Fig. 2: Variation of mass spectra with reduced mass μ for different quantum numbers

4.1 Thermodynamic properties plots

In this subsection, we present the plots of thermodynamic properties.

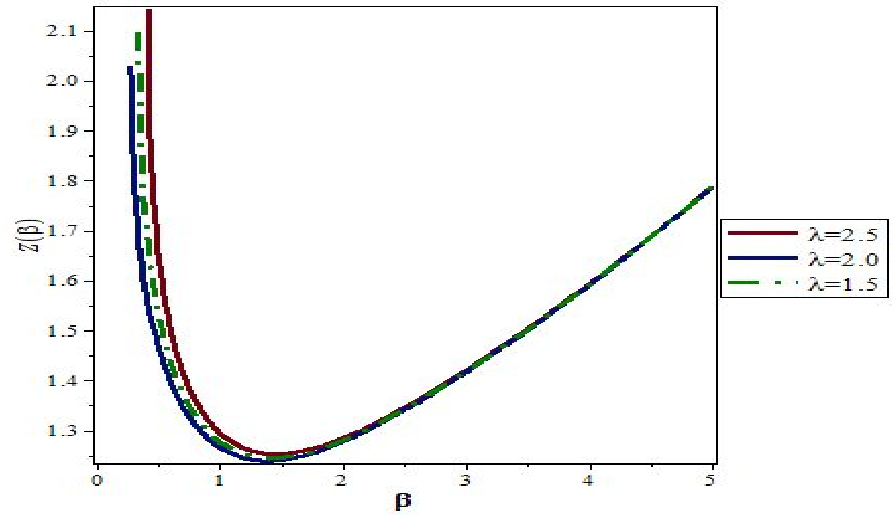

Fig.3: Plots of the partition function $Z(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

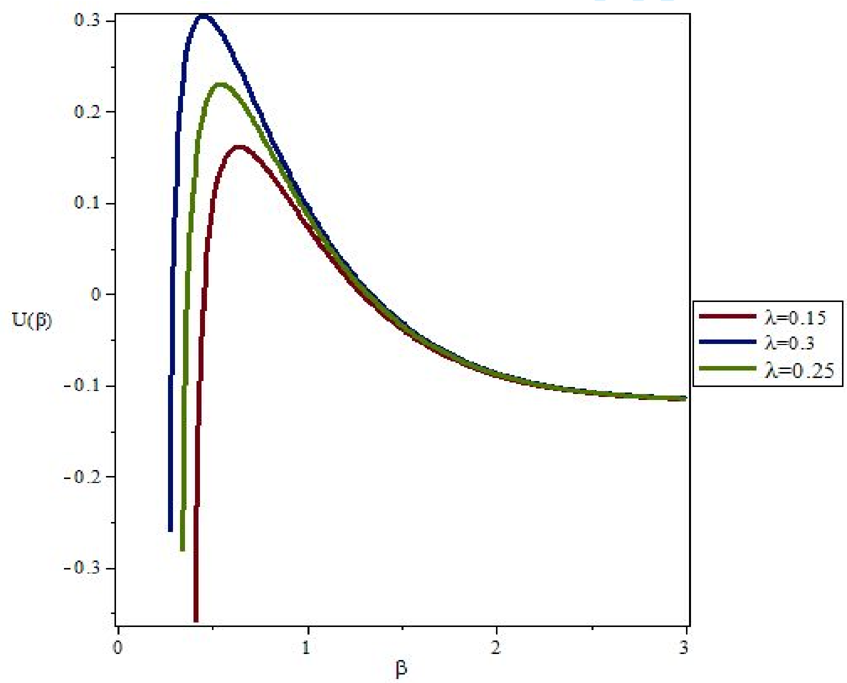

Fig. 4: Plots of the mean energy $U(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

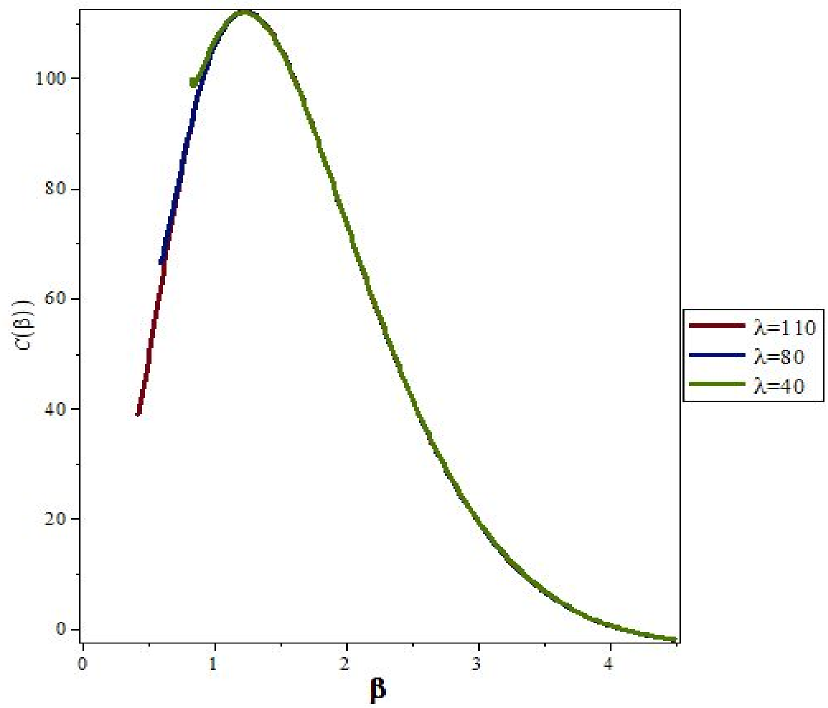

Fig. 5: Plots of the specific heat $C(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

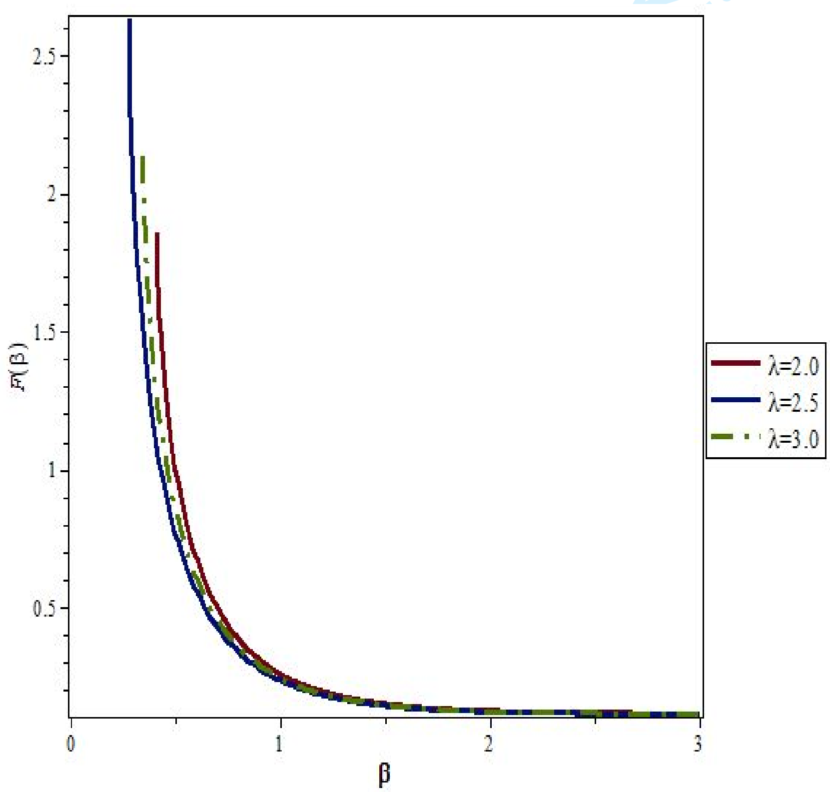

Fig. 6: Plots of the free energy $F(\beta)$ versus temperature (β) for different values of maximum quantum number (λ)

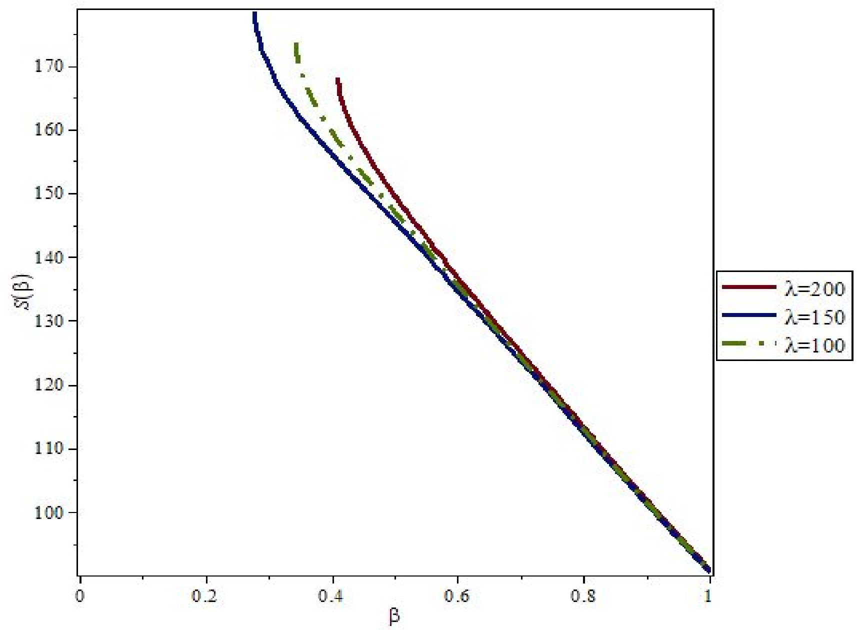

Fig.7: Plots of the entropy $S(\beta)$ versus temperature (β) for different values of maximum quantum number(λ

4.2 Discussion of results

We calculate mass spectra of charmonium and bottomonium for states from 1S to 1F, as presented in Tables 1 and 2. The free parameters of Eq.(71) were obtained by solving two algebraic equations of mass 2S,2P in Eq.(71) in the case of charmonium.

We followed the same procedure for bottomonium and obtained the free parameters of mass spectra for $1S, 2S$ in Eq.(71). For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems we adopt the numerical values of $m_b = 4.823$ GeV and $m_c = 1.209\, GeV$ [40], and the corresponding reduced mass are $\mu_b = 2.4115\, GeV$ and $\mu_c = 0.6045\, GeV$, respectively. The Debye mass $m_D(T)$ is 1.52 GeV by fitting with experimental data. We note that the calculation of mass spectra of charmonium and bottomonium are in good agreement with experimental data and the work of other researchers like; Ref.[35] as shown in Tables 1 and 2 in which the author investigated the N-radial SE analytically when the Cornell potential was extended to finite temperature.

We plotted mass spectra energy as a function of potential strength (A_0) and reduced mass (μ), respectively, as shown in Figs. 1 and 2. In Fig. 1, the mass spectra energy converges at the beginning but spread out. Also, there is a monotonic decrease with an increase in potential strength (A_0). Figure 2 shows the convergence of the mass spectra energy as the reduced mass (μ) increases for various angular quantum numbers. The thermodynamic properties were obtained by first obtaining the partition function. The variation of partition function $Z(\beta)$ as a function of temperature is presented in Fig. 3. Here the partition function decreases exponentially with increase in temperature β for different values of maximum quantum number λ, then later increases with increasing temperature which is the same as reported in Ref.[17].

The plot of mean energy $U(\beta)$ with different values of β and λ is as presented in Fig 4. The mean energy increases monotonically and then decreases with increasing β and λ. Figure 5 shows the variation of specific heat $C(\beta)$ as a function of temperature. There is a monotonic increase in specific heat as β increases and then decreases as β and λ increases with each plot converging. The convergent of each plot is an indication of the range of temperature that charmonium melts to its constituents as charm quark. The free energy $F(\beta)$ is plotted as a function of temperature as shown in Fig 6. The free energy decreases exponentially as β and λ increases and converges at a point close to zero. The plot of entropy $S(\beta)$ as function of temperature β and maximum quantum number λ is shown in Fig 7. We note that the entropy decreases with increasing β. This finding is in agreement with Ref.[18] in which the entropy increases with increasing temperature for the diatomic molecules.

5. Conclusion

In this study, we adopted a combination of Hulthen and Hellmann potentials for quark-antiquark interactions. The potential model was made to be temperature-dependent by replacing the screening parameter with Debye mass. We obtained the approximate solutions of the Schrödinger equation for energy eigenvalues and the corresponding eigenfunction in terms of Laguerre polynomials using the NU method. Four special cases were considered which result in Yukawa, Hulthen, Hellmann, and Coulomb potentials. We apply the present results to compute heavy-meson masses of charmonium and bottomonium for different quantum states. We also obtained thermodynamic properties such as free energy, mean energy, entropy, and specific heat by setting the temperature $T = 0$ which vanishes the Debye mass, and their plots were in good agreement with the works of Ref.[17] and Ref.[18]. The prediction of heavy mesons could be used to predict the newly identified particle made of four quarks of the same flavor. Particles made of four quarks are exotic. These exotic heavy particles provide extreme and theoretically fairly simple cases with which to test models that can then be used to explain the nature of ordinary matter particles, like protons or neutrons.

DECLARATIONS:

AVAILABILITY OF DATA AND MATERIALS

All data generated during this study are included in the references in the paper.

COMPETING INTERESTS

The authors declare that they have no competing interests.

FUNDING

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AUTHORS CONTRIBUTIONS

EPI suggested the point research and follows up with writing the literature. EPI carried out the calculations and wrote it. IOA carried out the results and reviewed it. JEN and ESW carried out the writing of the full manuscript. All authors read and approved the final manuscript.

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Not applicable

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Appendix A: Review of Nikiforov-Uvarov (NU) method

The NU method according to Nikiforov and Uvarov is used to transform Schrödinger-like equations into a second-order differential equation through a coordinate transformation $x = x(r)$, of the form [41-45]

$$\psi''(x)+\frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x)+\frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x)=0 \tag{A1}$$

where $\tilde{\sigma}(x)$, and $\sigma(x)$ are polynomials, at most second degree and $\tilde{\tau}(x)$ is a first-degree polynomial.

The exact solution of Eq. (A1) can be obtained by using the transformation

$$\psi(x)=\phi(x)y(x) \tag{A2}$$

This transformation reduces Eq.(A1) into a hypergeometric-type equation of the form

$$\sigma(x)y''(x)+\tau(x)y'(x)+\lambda y(x)=0 \tag{A3}$$

The function $\phi(x)$ can be defined as the logarithm derivative

$$\frac{\phi'(x)}{\phi(x)}=\frac{\pi(x)}{\sigma(x)} \tag{A4}$$

With $\pi(x)$ being at most a first-degree polynomial. The second part of the wave functions in Eq. (A2) is a hypergeometric-type function obtained by Rodrigues relation:

$$y(x)=y(x)=\frac{N_{nl}}{\rho(x)}\frac{d^n}{dx^n}\left[\sigma^n(x)\rho(x)\right] \tag{A5}$$

where N_{nl} is the normalization constant and $\rho(x)$ the weight function which satisfies the condition below;

$$\left(\sigma(x)\rho(x)\right)'=\tau(x)\rho(x) \tag{A6}$$

where also

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{A7}$$

For bound solutions, it is required that

$$\frac{d\tau(x)}{dx} < 0 \tag{A8}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(x)$ and parameter λ, respectively:

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \tag{A9}$$

and

$$\lambda = k_- + \pi'_-(x) \tag{A10}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda_n + n\tau'(x) + \frac{n(n-1)}{2}\sigma''(x) = 0, (n = 0, 1, 2, \ldots) \tag{A11}$$