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Massive gauge particles versus Goldstone bosons in non-Hermitian non-Abelian gauge theory

Andreas Fring[a], **Takanobu Taira**[b]

Department of Mathematics, City, University of London, Northampton Square, London EC1V 0HB, UK

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Abstract We investigate the Englert–Brout–Higgs–Guralnik–Hagen–Kibble mechanism for non-Hermitian field theories with local non-Abelian gauge symmetry in different regions of their parameter spaces. We demonstrate that the two aspects of the mechanism, that is giving mass to gauge vector bosons and at the same time preventing the existence of massless Goldstone bosons, remain to be synchronized in all regimes characterized by a modified CPT symmetry. In the domain of parameter space where the "would be Goldstone bosons" can be identified the gauge vector bosons become massive and the Goldstone bosons cease to exist. The mechanism is also in tact at the standard exceptional points. However, at the zero exceptional points, that is when the eigenvalues of the mass squared matrix vanish irrespective of the symmetry breaking, the mechanism breaks down as the Goldstone bosons cannot be identified and the gauge vector bosons remain massless. This breakdown coincides with the vanishing of the CPT inner product of symmetry breaking vacua defined on the eigenvector space of mass squared matrix. We verify this behaviour for a theory with SU(2) symmetry in which the complex scalar fields are taken in the fundamental as well as in the adjoint representation.

1 Introduction

Our main objective is to extend the Englert–Brout–Higgs–Guralnik–Hagen–Kibble mechanism [1–4], hereafter simply referred to as Higgs mechanism, to non-Hermitian field theories with a local non-Abelian gauge symmetry using a pseudo-Hermitian approach. We focus on the two key aspects for which the mechanism was originally developed, that is to give mass to gauge vector bosons and at the same time prevent the existence of massless Goldstone bosons. When keeping the symmetry global one may adopt different starting points for the study of Goldstone phases, such as the field content of local operators, a scattering matrix based on a particle picture or an explicit Lagrangian.

For instance, two-dimensional conformal quantum field theories are well-understood in terms of their operator content characterized by infinite-dimensional algebras of local conformal transformations [5]. A large class of such theories, minimal models [6], are known to possess a finite operator content, and the treatment of unitary and non-unitary theories is formally identical. The simplest massive non-unitary field theory consisting of only one real scalar field describing in its ultraviolet limit the critical point of the Ising model in a purely imaginary magnetic field, the Yang-Lee edge singularity [7, 8], is known for a long time to correspond to the non-Hermitian Lagrangian [9, 10]

$$\mathcal{L}=\int \mathrm{d}^d x\left[\frac{1}{2}(\nabla\phi)^2+i(h-h_c)\phi+\frac{1}{3}ig\phi^3\right]. \tag{1}$$

Exact scattering theories for two-dimensional models have also been identified [11] that can be used to probe the ultraviolet limit most easily by employing the thermodynamic Bethe Ansatz [12]. These techniques have also been employed for hypothetical scattering matrices for massless Goldstone fermions (Goldstinos) [13] and scattering matrices that reduce to them in certain limits [14]. Despite the fact that the Mermin–Wagner theorem prevents the validity of the Goldstone theorem in dimensions $d \leq 2$, it was argued in [15] that for certain symmetry groups, e.g. $SO(N)$ with formally taking $N < 2$, this restriction can be circumvented so that Goldstone phases maybe be identified in such type of non-Hermitian systems.

Rather than taking an operator content or a scattering matrix as a starting point, one may of course also commence directly with a non-Hermitian Lagrangian. From that perspective, it is natural to try to extend techniques and methods developed for the treatment of non-Hermitian quantum mechanics [16–19] to a quantum field theory setting. Such considerations have been carried out for a scalar field theory with imaginary cubic self-interaction terms [20, 21], with a Lagrangian identical to (1) but for $h = h_c$ without a linear term, deformed harmonic oscillators [22], non-Hermitian versions with a field theoretic Yukawa interaction [23–26], free

[a] e-mail: a.fring@city.ac.uk (corresponding author)
[b] e-mail: takanobu.taira@city.ac.uk

fermion theories with a γ_5-mass term and the massive Thirring model [27], $\mathcal{PT}$-symmetric versions of quantum electrodynamics [28–30] and other types of $\mathcal{PT}$-symmetric quantum field theories in higher dimensions [22] than (1).

Non-Hermitian quantum field theories are also viewed as possible options to overcome the limitations of the Standard Model of particle physics, which is well-known to fail in describing gravity and neutrino oscillations in a consistent manner. In [31], a study was initiated to potentially resolving the latter issue by extending the ordinary two-flavour neutrino oscillation to a non-Hermitian PT-symmetric/pseudo Hermitian setting. A well-studied simple Hermitian extension of the Standard Model referred to as the two-Higgs doublet-model [32] involves a second Higgs doublet or possibly more [33]. Here, we investigate non-Hermitian variants of these models.

Here, we are especially interested in complex non-Hermitian scalar field theories and the question of how the aforementioned Goldstone phases manifest in these theories, together with the subsequent extension to the Higgs mechanism [1–4] in Abelian and non-Abelian gauge theories. These issues have been studied recently by various groups in different approaches, which differ from their very onset: Given a generic action for a complex scalar field theory of the form $\mathcal{I} = \int d^4x \mathcal{L}(\phi, \phi^*)$, one has two options in a Hermitian theory to derive the equations of motion by means functional variation, either to calculate $\delta\mathcal{I}/\delta\phi = 0$ or $\delta\mathcal{I}/\delta\phi^* = 0$. Since the standard $\mathcal{CPT}$-theorem [34] applies, the two resulting equations are the same. In contrast, in a non-Hermitian theory, one no longer has $\mathcal{I} = \mathcal{I}^*$, so that the two equations are not only not the same, but in addition one also has the new options $\delta\mathcal{I}^*/\delta\phi = 0$ and $\delta\mathcal{I}^*/\delta\phi^* = 0$. In the first approach, we refer to as the "*surface term approach*", it was suggested [35, 36] to take of only the two equations resulting from $\delta\mathcal{I}/\delta\phi = 0$, $\delta\mathcal{I}^*/\delta\phi^* = 0$ and neglect the remaining two. As the resulting equations are in general not compatible, the authors propose to use some non-vanishing surface terms to compensate for the discrepancy. The second approach [37] consists of taking $\delta\mathcal{I}/\delta\phi = 0$ or $\delta\mathcal{I}/\delta\phi^* = 0$, and when determining the vacuum allowing the real and imaginary parts of the complex scalar field to be also complex. Thus, in this approach the field content is doubled or re-defined.

Here, we follow an approach, we refer to as the "*pseudo-Hermitian approach*" [38, 39], more aligned to the procedure pursued in non-Hermitian versions of quantum mechanics, in which one employs so-called Dyson maps [40] to transform a non-Hermitian Hamiltonian to a Hermitian Hamiltonian. Since the action $\mathcal{I}$ contains a Lagrangian, rather than a Hamiltonian, we need to first Legendre transform the complex Lagrangian $\mathcal{L}$ to a non-Hermitian Hamiltonian $\mathcal{H}$, carry out the similarity transformation by means of a Dyson map, while preserving equal time commutation relations, to obtain a Hermitian Hamiltonian $\mathfrak{h}$, which we then inverse Legendre transform to a real Lagrangian $\mathfrak{l}$

$$\mathcal{L} \overset{\text{Legendre}}{\rightarrow} \mathcal{H} \overset{\text{Dyson}}{\rightarrow} \eta\mathcal{H}\eta^{-1} = \mathfrak{h} \overset{\text{Legendre}^{-1}}{\rightarrow} \mathfrak{l}. \tag{2}$$

A consistent set of equations of motion is then obtained by functionally varying the action $\mathfrak{s} = \int d^4x \mathfrak{l}(\varphi, \chi)$ involving this real Lagrangian $\mathfrak{l}$ with respect to the real field components $\varphi, \chi \in \mathbb{R}$ of the complex scalar field $\phi = 1/\sqrt{2}(\varphi + i\chi)$, i.e. $\delta\mathfrak{s}/\delta\varphi = 0$ and $\delta\mathfrak{s}/\delta\chi = 0$. In order to perform the similarity transformation, one needs to canonically quantize the theory first, which is what we will do below.

In fact, this will be the only quantum aspect in our manuscript. Our discussion here is kept classical and we will not carry out a second quantization of the scalar fields, nor BRST quantization for the gauge fields, etc. Supported by our analysis below and in our previous papers [38, 39], our view here is that much insight can be gained by a classical treatment, apart from the appeal to the quantum field theory for the canonical commutation relations. One may of course fully analyse the theory in its quantum aspects, but our view is that this should be performed on the transformed, i.e. Hermitian, theory rather than directly on the non-Hermitian theory. In the former scenario one would not expect any issues to arise as one is simply dealing with a Hermitian system, whereas in the latter case one may encounter a number of what we would refer to as "*pseudo-problems*".

The above-mentioned incompatibility of the variational principle is an example for such a problem that already occurs on the level of the classical theory. One may try to fix this problem directly for the non-Hermitian system or simply consider the equivalent Hermitian system in which the problem is entirely absent. Another well-known example for a pseudo-problem in the quantum mechanical context is for instance to take the variables x and p, that might appear in the definition of a non-Hermitian Hamiltonian, to be physical observables. However, only the pseudo-Hermitian counterparts or the Dyson mapped quantities $\eta x\eta^{-1}$ and $\eta p\eta^{-1}$ can be observed and should be interpreted as being physical. Taking instead the original operators x and p from the non-Hermitian theory one may derive violations of the uncertainly relations and other properties that contradict standard quantum mechanical principles. Again these problems entirely disappear in the equivalent Hermitian system when considering the correct variables.

Previously, we have analysed the Goldstone theorem for a non-Hermitian scalar field theory with an Abelian [38] and a non-Abelian symmetry [39]. Here, the main purpose is to investigate the extension to the Higgs mechanism. As it is well-known, in the standard Higgs mechanism the Goldstone bosons acquires a mass so that our previous findings will inevitably have a bearing on the investigation to be carried out here. Let us therefore recall the key finding from [38, 39]: The main object of study has been the eigenvalue spectra of the non-Hermitian squared mass matrix M^2, obtained by expanding around the symmetry breaking or preserving vacua. The reality of these eigenvalues has been guaranteed by a modified $\mathcal{CPT}$ -symmetry of the original action $\mathcal{I}$. Hence, we distinguished in the usual fashion between *$\mathcal{CPT}$-symmetry regime* characterized by M^2 commuting with this symmetry operator and its eigenstates being simultaneous eigenstates of the symmetry operator. When the latter is not the case, one refers to that regime as the *$\mathcal{CPT}$-spontaneously broken regime* in which some eigenvalues become complex conjugate pairs. The points in parameter space at which this occurs are commonly referred to as *exceptional point*. As physical masses are positive and real, we also require the eigenvalues of M^2 to be non-negative. We encountered a special behaviour at the transition points when the eigenvalues become zero, which we referred to [38, 39] as *zero exceptional points of type I and type II*.[1] At the type I points, the mass matrix is non-diagonalizable and the continuous symmetry is broken, whereas at the type II points, the mass matrix can be diagonalized and the vacuum with broken continuous symmetry re-acquires the symmetry at this point.

Using the above-mentioned approaches, the Higgs mechanism was previously studied for Abelian [37, 41] as well as non-Abelian gauge theories [36] leading to slightly different findings. In [37], the interesting observation made, that the mass of the gauge vector boson vanishes at the zero exceptional point, was not confirmed in [41]. In addition, for the non-Abelian gauge theories, it was found in [36] that the Higgs mechanism even applies in the spontaneously broken $\mathcal{CPT}$-regime. Our aim is here to compare the various observations made using these alternative approaches with a pseudo-Hermitian approach, extend the studies to other models, symmetries and representations within this framework.

Our manuscript is organized as follows: In Sect. 2.1, we introduce first our non-Hermitian model with scalar field taken in the fundamental representation possessing a local $SU(2) \times U(1)$-symmetry, discuss the symmetry breaking vacuum of this theory and subsequently the Higgs mechanism. We indicate how to extend the model from $SU(2)$ to $SU(N)$. We repeat the same discussion in Sect. 2.2 for a non-Hermitian model with scalar field taken in the adjoint representation. Our conclusions are stated in Sect. 3. The distinction between the two types of exceptional points is crucial and for that reason, we include in our appendix a discussion for part of the squared mass matrix that explains the difference.

2 Pseudo-Hermitian approach to the Higgs mechanism

In this section, we commence by investigating the same model considered in [36] using, however, a pseudo-Hermitian method to compare our results with the findings in [36]. We will observe that the mass spectrum of the fields in the $SU(2)$ fundamental representation coincides with the one found in [36], but the masses for the gauge vector bosons differ and in particular vanish at the zero-exceptional points. We will extend this model to incorporate a $SU(N)$-symmetry and continue to observe this phenomena also for these more general systems. Finally, we will consider a new model for which the fields are taken in a different representation, the adjoint representation of $SU(2)$, making similar observations.

2.1 A $SU(2)$-model in the fundamental representation

We start by applying the pseudo-Hermitian approach to a model with local $SU(2) \times U(1)$-symmetry previously studied using the surface term approach in [36]. The model corresponds to the gauged version of the one for which the Goldstone mechanism was studied in [38]

$$\mathcal{L}_2 = \sum_{i=1}^{2} \left| D_\mu \phi_i \right|^2 + m_i^2 |\phi_i|^2 - \mu^2(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1) - \frac{g}{4}(|\phi_1|^2)^2 - \frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{3}$$

Here, $g, \mu \in \mathbb{R}$, $m_i \in \mathbb{R}$ or $m_i \in i\mathbb{R}$ are constants. When compared to [38], we have replaced here as usual the standard derivatives ∂_μ by covariant derivatives $D_\mu := \partial_\mu - ieA_\mu$, involving a charge $e \in \mathbb{R}$ and the Lie algebra-valued gauge fields $A_\mu := \tau^a A_\mu^a$. Here, the τ^a, $a = 1, 2, 3$, are taken to be Pauli matrices, which when re-defined as $i(-1)^{a+1}\tau^a$ are the generators of $SU(2)$. We have also added the standard Yang–Mills term comprised of the Lie algebra valued field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$. The two complex scalar fields ϕ_i are taken to be in the representation space of fundamental representation of $SU(2)$. The model described by $\mathcal{L}_2$ admits a global continuous $U(1)$-symmetry, a local continuous $SU(2)$-symmetry and two discrete antilinear $\mathcal{CPT}$-symmetries as described in more detail in [38]. Crucially, $\mathcal{L}_2$ is not Hermitian, which at this point is simply to be understood as not being invariant under complex conjugation. The Abelian version of $\mathcal{L}_2$ was discussed in [37, 39].

As argued in [38], it is useful to decompose the complex fields into their real components $\phi_j^k = 1/\sqrt{2}(\varphi_j^k + i\chi_j^k)$ with $\varphi_j^k, \chi_j^k \in \mathbb{R}$. Thus simply rewriting the complex scalar fields in Eq. (3) in terms of their real and imaginary components, we obtain the following Lagrangian:

$$\begin{aligned}
\mathcal{L}_2 = \frac{1}{2}\sum_{k=1}^{2}\sum_{j=1}^{2} \Bigg\{ & \Big[\partial_\mu \varphi_j^k + e(A_\mu \chi_j)^k\Big]\Big[\partial^\mu \varphi_j^k + e(A^\mu \chi_j)^k\Big]^* \\
& + \Big[\partial_\mu \chi_j^k - e(A_\mu \varphi_j)^k\Big]\Big[\partial^\mu \chi_j^k - e(A^\mu \varphi_j)^k\Big]^* - 2\mathrm{Im}\Big[\Big[\partial_\mu \varphi_j^k + e(A_\mu \chi_j)^k\Big]^*\Big[\partial^\mu \chi_j^k - e(A^\mu \varphi_j)^k\Big]\Big] \\
& + m_j^2\Big[(\varphi_j^k)^2 + (\chi_j^k)^2\Big] - 2i\mu^2(\varphi_1^k \chi_2^k - \chi_1^k \varphi_2^k) - \frac{1}{4}F_{\mu\nu}^k\Big(F^k\Big)^{\mu\nu}\Bigg\} - \frac{g}{16}\left[\sum_{k=1}^{2}(\varphi_1^k)^2 + (\chi_1^k)^2\right]^2.
\end{aligned} \tag{4}$$

[1] The difference between these two types of points is discussed in the appendix in form of a general discussion for key blocks of the squared mass matrix.

We use here the standard notation * for complex conjugation and † for the simultaneous conjugation with transposition. See also [38] for further details.

The $SU(2)$-symmetry manifests itself as follows: A change in the complex scalar fields due to this symmetry is $\delta\phi_j^k = i\alpha_a T_a^{kl}\phi_j^l$, where the generators T_a of the symmetry transformation are the standard Pauli matrices σ_a, $a = 1, 2, 3$. The infinitessimal changes for the real component fields are then identified as

$$\delta\varphi_j^1 = -\alpha_1\chi_j^2 + \alpha_2\varphi_j^2 - \alpha_3\chi_j^1, \quad \delta\chi_j^1 = \alpha_1\varphi_j^2 + \alpha_2\chi_j^2 + \alpha_3\varphi_j^1, \tag{5}$$

$$\delta\varphi_j^2 = -\alpha_1\chi_j^1 - \alpha_2\varphi_j^1 + \alpha_3\chi_j^2, \quad \delta\chi_j^2 = \alpha_1\varphi_j^1 - \alpha_2\chi_j^1 - \alpha_3\varphi_j^2, \tag{6}$$

which leave the above Lagrangian invariant. The discrete antilinear $\mathcal{CPT}_\pm$-symmetries, see [38] for more details, manifest themselves as

$$\mathcal{CPT}_\pm: \ \varphi_j^k(x_\mu) \to \mp(-1)^j\varphi_j^k(-x_\mu), \quad \chi_j^k(x_\mu) \to \pm(-1)^j\chi_j^k(-x_\mu). \tag{7}$$

A noteworthy remark is that it is straightforward to generalize the model from a locally $SU(2)$ -invariant one to a locally $SU(N)$-invariant one, by extending the sum over k from 2 to N, while keeping the $U(1)$-symmetry global. In what follows, we will focus on $N = 2$.

A crucial feature of $\mathcal{L}_2$ is that its $\mathcal{CPT}$ -invariance translates into pseudo-Hermiticity [18, 42], meaning that it can be mapped to a Hermitian Lagrangian $\mathfrak{l}_2$ by means of the adjoint action of a Dyson map η as $\mathfrak{l}_2 = \eta\mathcal{L}_2\eta^{-1}$. This may be achieved by the slightly modified version of the Dyson map used in [37, 38]

$$\eta_2^\pm = \exp\left(\pm\sum_{i=1}^{2}\int \mathrm{d}^3x\,\Pi^{\varphi_2^i}(t',\vec{x})\varphi_2^i(t',\vec{x}) + \Pi^{\chi_2^i}(t',\vec{x})\chi_2^i(t',\vec{x})\right). \tag{8}$$

We denote here the time dependence by t' to indicate that commutators are understood as equal time commutators for the canonical momenta $\Pi^{\varphi_2^i} = \partial_t\varphi_2^i$ and $\Pi^{\chi_2^i} = \partial_t\chi_2^i$, $i = 1, 2$. satisfying $\left[\psi_j^k(\mathbf{x}, t), \Pi^{\psi_l^m}(\mathbf{y}, t)\right] = i\delta_{jl}\delta_{km}\delta(\mathbf{x} - \mathbf{y})$, $j, k, l, m = 1, 2$, for $\psi = \varphi, \chi$.

Hence, $\eta_2^\pm$ is not to be viewed as explicitly time-dependent as discussed in much detail for instance in [43]. The adjoint action of η_2^+ on the individual fields maps as

$$\varphi_1^k \to \varphi_1^k, \quad \varphi_2^k \to -i\varphi_2^k, \quad \chi_1^k \to \chi_1^k, \quad \chi_2^k \to -i\chi_2^k, \quad A_\mu \to A_\mu, \quad k = 1, 2. \tag{9}$$

Thus, we convert the complex Lagrangian into the real Lagrangian

$$\begin{aligned}\mathfrak{l}_2 = {} & \frac{1}{2}\sum_{j=1}^{2}(-1)^{j+1}\Big\{\left|\partial_\mu\varphi_j + e(A_\mu\chi_j)\right|^2 + \left|\partial_\mu\chi_j - e(A_\mu\varphi_j)\right|^2 + m_j^2\left[\varphi_j\cdot\varphi_j + \chi_j\cdot\chi_j\right] \\ & - 2\mathrm{Im}\left[\left[\partial_\mu\varphi_j + e(A_\mu\chi_j)\right]^*\cdot\left[\partial^\mu\chi_j - e(A^\mu\varphi_j)\right]\right] + (-1)^j 2\mu^2(\varphi_1\cdot\chi_2 - \chi_1\cdot\varphi_2)\Big\} \\ & - \frac{g}{16}\left[\varphi_1\cdot\varphi_1 + \chi_1\cdot\chi_1\right]^2 - \frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right).\end{aligned} \tag{10}$$

We may transform here directly the Lagrangian rather than the Hamiltonian, as suggested in (2), since the kinetic energy term is real and the complexity only results from the potential term. Introducing the two two-component fields of the form

$$\Phi^k := \begin{pmatrix}\varphi_1^k \\ \chi_2^k\end{pmatrix}, \quad \Psi^k := \begin{pmatrix}\chi_1^k \\ \varphi_2^k\end{pmatrix}, \qquad k = 1, 2, \tag{11}$$

we can re-write the Lagrangians $\mathcal{L}_2$ and $\mathfrak{l}_2$ more compactly. Defining the 2×2 matrices

$$H_\pm := \begin{pmatrix}-m_1^2 & \pm\mu^2 \\ \pm\mu^2 & m_2^2\end{pmatrix}, \quad \mathcal{I} := \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \quad E := \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}, \tag{12}$$

the real Lagrangian $\mathfrak{l}_2$ acquires the form

$$\begin{aligned}\mathfrak{l}_2 = {} & \frac{1}{2}\Big\{\left[\partial_\mu\Phi + e\mathcal{I}A_\mu\Psi\right]^*\mathcal{I}\left[\partial^\mu\Phi + e\mathcal{I}A^\mu\Psi\right] + \left[\partial_\mu\Psi - e\mathcal{I}A_\mu\Phi\right]^*\mathcal{I}\left[\partial^\mu\Psi - e\mathcal{I}A^\mu\Phi\right] \\ & - 2\mathrm{Im}\left[\left(\partial_\mu\Phi + e\mathcal{I}A_\mu\Psi\right)^*\left(\partial^\mu\Psi - e\mathcal{I}A^\mu\Phi\right)\right] - \Phi^T H_+\Phi - \Psi^T H_-\Psi\Big\} \\ & - \frac{g}{16}\left(\Phi^T E\Phi + \Psi^T E\Psi\right)^2 - \frac{1}{4}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right).\end{aligned} \tag{13}$$

We have simplified here the index notation by implicitly contracting, keeping in mind that we are summing over two separate index sets $k \in \{1,2\}$ and $j \in \{1,2\}$. For instance, we set

$$(I A_\mu \Phi)^k_\alpha \rightarrow \mathcal{I}_{\alpha\beta} A^{kj}_\mu \Phi^j_\beta, \tag{14}$$

$$\left[\partial_\mu \Phi^k_j + e\big(\mathcal{I} A_\mu \Psi\big)^k_j\right]^* \mathcal{I}_{j\ell}\left[\partial^\mu \Phi^k_\ell + e\big(\mathcal{I} A^\mu \Psi\big)^k_\ell\right] \rightarrow \big[\partial_\mu \Phi + e\mathcal{I} A_\mu \Psi\big]^* \mathcal{I}\big[\partial^\mu \Phi + e\mathcal{I} A^\mu \Psi\big], \tag{15}$$

$$\Phi^{k^T} H_+ \Phi^k \rightarrow \Phi^T H_+ \Phi. \tag{16}$$

In this formulation, we may think of the real and complex Lagrangians, $\mathfrak{l}_2$ and $\mathcal{L}_2$, as being simply related by a kind of Wick rotation in the field-configuration space

$$\Phi^k \rightarrow T\Phi^k, \quad \Psi^k \rightarrow T\Psi^k, \quad \text{with } T := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{17}$$

One may of course wonder about the negative sign in the kinetic term of $\mathfrak{l}_2$ and whether these lead to ghost fields. Thus, we finish this subsection with a short discussion that establishes that these signs disappear when the Lagrangian is properly diagonalized. As this argument is the same in the global and local theory, we set the gauge fields to zero without loss of generality for this purpose and consider the corresponding action

$$S = \frac{1}{2}\int \mathrm{d}^4x \Big(\partial_\mu \Phi^\top \mathcal{I} \partial^\mu \Phi - \Phi^\top H_\pm \Phi\Big) + S_{\text{int}}, \tag{18}$$

$$= -\frac{1}{2}\int \mathrm{d}^4x\, \Phi^\top \mathcal{I}\big(\partial_\mu\partial^\mu + M^2\big)\Phi + S_{\text{int}}, \tag{19}$$

where S_{int} contains all terms of higher order than Φ^2. We also assumed that surface terms vanish at infinity when integrating by parts and used $\mathcal{I}^2 = \mathbb{I}$, $M^2 = \mathcal{I}H_\pm$. Next, we diagonalize the squared mass matrix as $M^2 = T^{-1}DT$ and consider next only the integrant of the first term in (2)

$$\Phi^\top \mathcal{I}\big(\partial_\mu\partial^\mu + M^2\big)\Phi = \Phi^\top \mathcal{I}\big(\partial_\mu\partial^\mu + TDT^{-1}\big)\Phi = \Psi^\top\big(\partial_\mu\partial^\mu + D\big)\Psi, \tag{20}$$

where we introduced the new field $\Psi := T^\top \mathcal{I}\Phi$ and used $T^{-1} = T^\top \mathcal{I}$.

The latter relation is derived as follows: We start by defining the right and left eigenvectors of M^2 as

$$\mathcal{I}H_\pm v_i = \lambda_i v_i, \qquad \text{and} \qquad (\mathcal{I}H_\pm)^\dagger u_i = \lambda_i u_i, \tag{21}$$

respectively. Noting that $\mathcal{I}^\dagger = \mathcal{I}$ and $H_\pm^\dagger = H_\pm$, the last relation implies that $\mathcal{I}H_\pm\mathcal{I}u_i = \lambda_i \mathcal{I}u_i$. Therefore, we can express the right eigenvectors in terms of the left eigenvectors as $v_i = \mathcal{I}u_i$. The matrix T is made out of the column vectors of v_i, i.e. $T = (v_1, \ldots)$ so that $\mathcal{I}T = (u_1, \ldots)$. Since the left and right eigenvector form a biorthonormal basis, $v_i \cdot u_j = \delta_{ij}$, it follows that $T^\top \mathcal{I} T = \mathbb{I}$ and hence $T^{-1} = T^\top \mathcal{I}$.

2.1.1 The symmetry breaking vacuum

The vacuum solutions Φ^k_0, Ψ^k_0 by solving $\delta V = 0$, with V denoting the potential

$$V = \Phi^\top H_+ \Phi + \Psi^\top H_- \Psi + \frac{g}{16}\Big(\Phi^\top E \Phi + \Psi^\top E \Psi\Big)^2 \tag{22}$$

amounts to solving the two equations

$$\Big(H_- + \frac{g}{4}R^2 E\Big)\Psi^k_0 = 0, \quad \Big(H_+ + \frac{g}{4}R^2 E\Big)\Phi^k_0 = 0, \qquad k = 1, 2, \tag{23}$$

with $R^2 := \left|\left(\phi^0_1\right)^1\right|^2 + \left|\left(\phi^0_1\right)^2\right|^2 = \frac{1}{2}\sum_{k=1}^2 \Phi_0^{k^T} E \Phi^k_0 + \Psi_0^{k^T} E \Psi^k_0$=const. Hence, in the real component field configuration space, the vacuum manifold is a S^3-sphere with radius R. Consequently, we may consider equations (23) as two eigenvalue equations. Thus, besides the trivial $SU(2)$-invariant vacuum $\Phi^k_0 = \Psi^k_0 = 0$, $k = 1, 2$, we must have zero eigenvalues in both equations, which is equivalent to requiring

$$R^2 = \frac{4}{g m_2^2}(\mu^4 + m_1^2 m_2^2). \tag{24}$$

Since R^2 is positive, this equality imposes restrictions on the parameters g, μ and the possible choices for $m_1 \in \mathbb{R}$, $m_2 \in i\mathbb{R}$ or $m_1 \in i\mathbb{R}$, $m_2 \in \mathbb{R}$. The corresponding vectors that satisfy equation (23), suitably normalized with regard to the standard inner product, are

$$\Psi^2_0 = N_\Psi \begin{pmatrix} m_2^2 \\ \mu^2 \end{pmatrix}, \quad \Phi^2_0 = N_\Phi \begin{pmatrix} -m_2^2 \\ \mu^2 \end{pmatrix}. \tag{25}$$

Imposing now the constraint on R^2 as stated after equation (23), a possible solution is $\Phi_0^1 = \Psi_0^1 = \Phi_0^2 = 0$ and Ψ_0^2 as defined in (25) with normalization constant $N_\Psi = \pm\sqrt{2}R/m_2^2$. Hence, we recover the symmetry breaking vacuum used in [39].

2.1.2 *The Higgs mechanism*

Let us now demonstrate how the gauge vector boson acquires a finite mass and how at the same time the emergence of a Goldstone boson is prevented by the Higgs mechanism [1–4] in the $\mathcal{CPT}$-symmetric regime, at the exceptional points and even in the spontaneously broken $\mathcal{CPT}$-symmetric regime. The mechanism breaks down at the two types of zero exceptional points.

Expanding the potential V in (22) around the vacuum specified at the end of Sect. 2.1.1 leads to

$$V(\Phi_0+\Phi,\Psi_0+\Psi) = V(\Phi_0,\Psi_0) + \frac{1}{2}\Phi^i \frac{\partial^2 V(\Phi_0,\Psi_0)}{\partial\Phi^i\partial\Phi^j}\Phi^j + \frac{1}{2}\Psi^i \frac{\partial^2 V(\Phi_0,\Psi_0)}{\partial\Psi^i\partial\Psi^j}\Psi^j + \Phi^i \frac{\partial^2 V(\Phi_0,\Psi_0)}{\partial\Phi^i\partial\Psi^j}\Psi^j + \cdots \tag{26}$$

$$= \frac{1}{2}\sum_{i=1}^{2} -\Phi^{i\top}\Big(H_+ + \frac{g}{4}R^2 E\Big)\Phi^i - \Psi^{1\top}\Big(H_- + \frac{g}{4}R^2E\Big)\Psi^1 - \Psi^{2\top}\Big[H_- + \frac{g}{4}R^2E + -\frac{g}{2}(E\Psi_0^2)^2E\Big]\Psi^2 + \cdots \tag{27}$$

As expected, multiplying the Hessians in (27) by $\mathcal{I}$ gives back the squared mass matrix we found in [39]. The kinetic term is almost unchanged except for the term involving Ψ^2

$$T = \frac{1}{2}\big[\partial_\mu\Phi + e\mathcal{I}A_\mu\Psi\big]^\dagger \mathcal{I}\big[\partial^\mu\Phi + e\mathcal{I}A^\mu\Psi\big] + \mathrm{Re}\Big\{\big(\partial_\mu\Phi + e\mathcal{I}A_\mu\Psi\big)^\dagger\mathcal{I}\big(e\mathcal{I}A^\mu\Psi_0\big)\Big\} - \mathrm{Im}\Big\{\big(\partial_\mu\Phi + e\mathcal{I}A_\mu\Psi + e\mathcal{I}A_\mu\Psi_0\big)^\dagger\big(\partial^\mu\Psi - e\mathcal{I}A^\mu\Phi\big)\Big\} + \frac{1}{2}e^2\big(A_\mu\Psi_0\big)^\dagger\mathcal{I}\big(A^\mu\Psi_0\big) \tag{28}$$

The last term corresponds to the mass term of the gauge vector boson that we evaluate to

$$\frac{1}{2}e^2\big(A_\mu\Psi_0\big)^*\mathcal{I}\big(A^\mu\Psi_0\big) = \frac{1}{2}e^2\big(A_\mu\Psi_0\big)^{*k}_{\alpha}\mathcal{I}_{\alpha\beta}\big(A^\mu\Psi_0\big)^k_\beta \tag{29}$$

$$= \frac{1}{2}e^2\big(A^\dagger_\mu A^\mu\big)^{kj}(\Psi_0)^k_\alpha\mathcal{I}_{\alpha\beta}(\Psi_0)^j_\beta$$

$$= \frac{1}{2}e^2\big(A^\dagger_\mu A^\mu\big)^{22}(\Psi_0)^2_\alpha\mathcal{I}_{\alpha\beta}(\Psi_0)^2_\beta$$

$$= \frac{1}{2}e^2A^a_\mu A^{b\mu}(\tau^{a\dagger}\tau^b)^{22}\frac{2R^2}{m_2^4}\big(m_2^4-\mu^4\big)$$

$$= \frac{1}{2}m_g^2A^a_\mu A^{a\mu}, \tag{30}$$

where we used the standard relation $\tau^{a\dagger}\tau^b = \tau^a\tau^b = \delta_{ab}\mathbb{I} + i\varepsilon_{abc}\tau^c$. Therefore, we read off the mass of each of the three components of the gauge vector boson as

$$m_g := \frac{\sqrt{2}eR}{m_2^2}\sqrt{m_2^4-\mu^4}. \tag{31}$$

In [39], we identified the physical regions in the parameter space in which the squared mass matrix has non-negative eigenvalues and in which the Goldstone bosons can be identified. Let us now compare those regions with the values for which the gauge vector boson becomes massive. We immediately see from the expression in (31) that the gauge vector boson remains massless when $\mu^4 = m_2^4$ or when $R = 0$, i.e. $\mu^4 = -m_1^2m_2^2$. The two sets of values correspond precisely to the two types of zero exceptional points, type I and II, respectively, at which the squared mass matrix develops zero eigenvalues $\lambda = 0$. These points are distinct from standard exceptional points where two eigenvalues coalesce and become complex thereafter, here at $\lambda = \frac{\mu^4}{m_2^2} - m_2^2$. See the appendix for a more detailed explanation about the distinction between these types of exceptional points.

Thus, the two aspects of the Higgs mechanism, i.e. giving mass to the gauge vector boson and at the same time preventing the existence of the Goldstone bosons, remain to go hand in hand. In the $\mathcal{CPT}$-symmetric regime, the mechanism applies, but at the zero exceptional points, the Higgs mechanism breaks down as the Goldstone bosons are not identifiable [39], and at the same time, the gauge vector boson remains massless. In contrast, at the exceptional point, the Goldstone bosons are identifiable [39], although in a different manner, and the gauge vector bosons become massive.

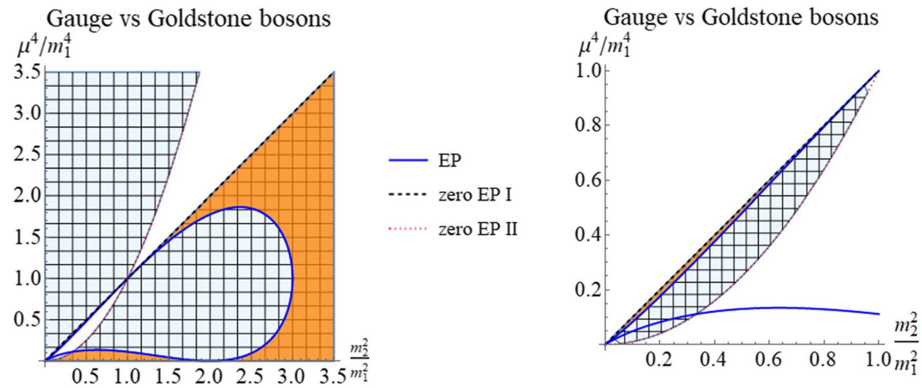

Fig. 1 Regions, for which the gauge vector boson is massive (blue with mesh) versus physical regions (orange) in which the would be Goldstone boson can be identified, bounded by exceptional and zero exceptional points as function of $(\mu^4/m_1^4, m_2^2/m_1^2)$ for the theory expanded around the SU(2)-symmetry breaking vacuum. Left panel for $c_1 = -c_2 = 1$ and right panel for $c_1 = -c_2 = -1$. The coupling constant g must be positive

Let us see this in detail by following [39] and replacing $m_i^2 \to c_i m_i^2$, with $c_i = \pm 1$ to account for all possibilities in signs. We found that physical regions only exist for the two cases $c_1 = -c_2 = 1$ and $c_1 = -c_2 = -1$. For the two cases, we may then write

$$\frac{m_g^2}{m_1^2} = c_2 \frac{4e^2}{g} \frac{m_1^6}{m_2^6} \left(\frac{m_2^4}{m_1^4} - \frac{\mu^4}{m_1^4} \right) \left(\frac{\mu^4}{m_1^4} - \frac{m_2^2}{m_1^2} \right), \tag{32}$$

noting that m_g^2/m_1^2 only depends on the two parameters m_2^2/m_1^2 and m_2^4/m_1^4 similarly as the eigenspectrum of the squared mass matrix [36, 39]. We require the right-hand side of (32) to be positive as shown in Fig. 1.

We observe in Fig. 1 that while the region in which the Goldstone boson can be identified is bounded by exceptional as well as zero exceptional points, the exceptional points lie well inside the region for which the gauge vector boson is massive, i.e. they acquire a mass in the $\mathcal{CPT}$-symmetric regime as well as in the spontaneously broken $\mathcal{CPT}$-symmetric regime. In the $\mathcal{CPT}$-symmetric regime, this agrees well with the findings that at these points the "would be Goldstone boson" is prevented from existing as a massless particle. We use here "would be Goldstone bosons" rather than "Goldstone boson", as they do not exist in the gauged theory, but could be constructed by setting the gauge fields to zero. We may think of the sign change in front of the mass terms, $c_i \to -c_i$, that relates the left to the right panel as a phase transition [44].

Let us now demonstrate this behaviour in detail and expand for this purpose the Lagrangian around the symmetry broken vacuum up to second order in the fields

$$\begin{aligned} \mathfrak{l}_2 = & \sum_{k=1}^{2} \frac{1}{2} \partial_\mu \Phi^{kT} \mathcal{I} \partial^\mu \Phi^k + \frac{1}{2} \partial_\mu \Psi^{kT} \mathcal{I} \partial^\mu \Psi^k - \frac{1}{2} \Phi^{kT} \left(H_+ + \frac{g}{4} R^2 E \right) \Phi^k \\ & - \frac{1}{2} \Psi^{1T} \left(H_- + \frac{g}{4} R^2 E \right) \Psi^1 - \frac{1}{2} \Psi^{2T} \left(H_+ + \frac{g}{4} R^2 E + \frac{g}{2} (E \Psi_{(0)}^2)^2 E \right) \Psi^2 \\ & + e \mathrm{Re} \left[\partial_\mu \Phi^\dagger (A^\mu \Psi_0) \right] + e \mathrm{Im} \left[\left(\mathcal{I} A_\mu \Psi_0 \right)^\dagger \partial^\mu \Psi \right] + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu} + \cdots \end{aligned} \tag{33}$$

We recall now from [39] that the first two lines of the Lagrangian $\mathfrak{l}_2$ can be diagonalized and the Goldstone bosons can be identified in terms of the field content of the model. Furthermore, the Goldstone modes are eigenvectors in the null space of squared mass matrices

$$M_\pm^2 := \mathcal{I} \left(H_\pm + \frac{g}{4} R^2 E \right), \tag{34}$$

computed above as Ψ_0^2 and $\mathcal{I}\Psi_0^2$, so that the Goldstone modes are proportional to these two vectors. The explicit forms of the Goldstone fields were found in [38] (equation (3.40) therein), denoted as ψ_5^{Gb}, $\psi_3^{\mathrm{Gb}}$and ψ_1^{Gb}, therein. We express them here as

$$G^1 := \frac{e}{m_g} (\Psi_0^2)^T \Phi^1, \quad G^3 := \frac{e}{m_g} (\Psi_0^2)^T \Phi^2, \quad G^2 := -\frac{e}{m_g} (\Psi_0^2)^T \mathcal{I} \Psi^1, \tag{35}$$

respectively. As expected for the Higgs mechanism, the number of "would be Goldstone bosons" equals the amount of massive vector gauge bosons. The fact that the Goldstone modes are inverse proportional to the mass of the gauge bosons explains that they cannot be identified for massless gauge bosons. Keeping now only the Goldstone kinetic term from the first two lines of the Lagrangian $\mathfrak{l}_2$ and the one involving the gauge fields in Eq. (33), we obtain

$$\mathfrak{l}_2 = \sum_{a=1}^{3} \frac{1}{2} \partial_\mu G^a \partial^\mu G^a + e \mathrm{Re} \left[\partial_\mu \Phi^\dagger (A^\mu \Psi_0) \right] + e \mathrm{Im} \left[\left(\mathcal{I} A_\mu \Psi_0 \right)^\dagger \partial^\mu \Psi \right] + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu} + \cdots \tag{36}$$

Using the explicit representations of the Pauli matrices, the real and imaginary parts are determined as

$$\mathrm{Re} \left[\partial_\mu \Phi^T A^\mu \Psi_0 \right] = A_\mu^a \mathrm{Re} \left[\partial_\mu \Phi^T \tau^a \Psi_0 \right] = A_\mu^1 \partial_\mu \Phi^T \tau^1 \Psi_0 + A_\mu^3 \partial_\mu \Phi^T \tau^3 \Psi_0$$

$$
\begin{aligned}
&= A_\mu^1 \partial_\mu \left(\Phi^1\right)^T \Psi_0^2 - A_\mu^3 \partial_\mu \left(\Phi^2\right)^T \Psi_0^2 \\
&= A_\mu^1 \frac{m_g}{e} \partial^\mu G^1 - \frac{m_g}{e} A_\mu^3 \partial^\mu G^3 \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Im}\Big[\left(\mathcal{I} A_\mu \Psi_0\right)^\dagger \partial^\mu \Psi\Big] &= A_\mu^a \mathrm{Im}\Big[\Psi_0^T \tau^a \mathcal{I} \partial^\mu \Psi\Big] = -i\left(A_\mu^2 \Psi_0^T \tau^2 \mathcal{I} \partial^\mu \Psi\right) \\
&= A_\mu^2 \left(\Psi_0^2\right)^T \mathcal{I} \partial^\mu \Psi^1 = -A_\mu^2 \frac{m_g}{e} \partial^\mu G^2. \qquad (38)
\end{aligned}
$$

Finally, the Lagrangian in (36) can be simplified to

$$
\begin{aligned}
\mathfrak{l}_2 &= \sum_{a=1}^{3} \frac{1}{2} \partial_\mu G^a \partial^\mu G^a - m_g A_\mu^1 \partial^\mu G^1 + m_g A_\mu^2 \partial^\mu G^2 - m_g A_\mu^3 \partial^\mu G^3 + \frac{1}{2} m_g^2 A_\mu^a A^{a\mu} + \cdots \\
&= \frac{1}{2} m_g^2 \left(A_\mu^1 - \frac{1}{m_g} \partial_\mu G^1\right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^2 + \frac{1}{m_g} \partial_\mu G^2\right)^2 + \frac{1}{2} m_g^2 \left(A_\mu^3 + \frac{1}{m_g} \partial_\mu G^3\right)^2 + \cdots \\
&= \frac{1}{2} \sum_{a=1}^{3} m_g^2 B_\mu^a B^{a\mu} + \cdots, \qquad (39)
\end{aligned}
$$

where we defined the new vector gauge particle with component fields $B_\mu^a := A_\mu^a - \frac{1}{m_g} \partial_\mu G^a$. We may also replace A_μ^a by B_μ^a in the field strength $F_{\mu\nu}$ so that A_μ can be eliminated entirely from the Lagrangian. We see that the Higgs mechanism applies as long as $m_g \neq 0$. However, at the zero exceptional points, not only the gauge boson mass vanishes, but the Higgs mechanism no longer applies, in the sense that we cannot remove the degrees of freedom of Goldstone bosons.

We summarize the behaviour we found in the different types of regimes in the following Table.

	$\mathcal{CPT}$	Sp. broken $\mathcal{CPT}$	EP	Zero EP I	Zero EP II
Gauge bosons	Massive	Massive	Massive	Massless	Massless
Goldstone bosons	$\exists$	$\exists$	$\exists$	$\nexists$	$\nexists$

Thus, we encounter three different types of behaviour: in the $\mathcal{CPT}$-symmetric regime, at the standard exceptional points as well as in the spontaneously broken $\mathcal{CPT}$-symmetric regime, the Higgs mechanism applies in the usual way. However, in the latter regime, other particles in the theory become non-physical. At the zero exceptional points, the vector gauge bosons remain massless, and no Goldstone bosons can be identified in the global theory.

2.1.3 From SU(2) to SU(N)

We will now follow the same line of reasoning as in the previous subsection and generalize our model from possessing a $SU(2)$-symmetry to one with a $SU(N)$-symmetry. For this purpose, we simply replace the Pauli matrices in all our expressions by the traceless and skew-Hermitian $N \times N$-matrices corresponding to the $SU(N)$-generators T^a with $a = 1, \ldots, (N^2 - 1)$. The vacua are still determined by the solutions of the eigenvalue problem (23) with zero eigenvalue condition

$$
R^2 = \frac{1}{2} \sum_{i=1}^{N} \Phi_0^{i\,T} E \Phi_0^i + \Psi_0^{i\,T} E \Psi_0^i = \text{constant} = \frac{4}{g m_2^2} (\mu^4 + m_1^2 m_2^2). \qquad (40)
$$

The zero eigenvalue condition implies that the vacuum manifold is a S^{2N-1}-sphere with radius R. This follows from the fact that $SU(N)$ acts on the $2N$-dimensional space spanned by $(\varphi_1^0)^i$, $(\chi_1^0)^i$, $i = 1, \ldots, N$, with norm equal to R^2. On this space $SU(N-1)$ simply permutes the fields among themselves, hence acting as a stabilizer or isotropy subgroup. Thus, the vacuum manifold corresponds to the coset $SU(N)/SU(N-1) \cong S^{2N-1}$.

As we discussed in detail in [39], we may utilize the symmetry of the Lagrangian to transform the vacua into convenient forms without changing the eigenvalue spectrum of the mass matrix. Thus, using the elements $T \in SU(N)/SU(N-1) \subset SU(N)$, we may transform the vacuum into the form

$$
\Phi_0^i = 0, \quad \Psi_0^i = \frac{\sqrt{2} R}{\sqrt{N} m_2^2} \begin{pmatrix} m_2^2 \\ \mu^2 \end{pmatrix}, \quad \text{for } i = 1, \ldots, N, \qquad (41)
$$

satisfying the constraint (40). Let us now use this $SU(N)$-symmetry breaking vacuum to calculate the mass of the gauge vector boson. Taking the proper $SU(N)$-algebra rather than the physicist's version, as in the last subsection for $SU(2)$, we also change $e \to ie$. Dropping here the kinetic term reported in (28) and considering only the relevant term in the Lagrangian we obtain

$$
\mathfrak{l}_A := \frac{1}{2} e^2 A_\mu^a A^{b\mu} \left(T^{a\dagger} T^b\right)_{ij} (\Psi_0)_\alpha^i \mathcal{I}_{\alpha\beta} (\Psi_0)_\beta^j
$$

$$= \frac{1}{N} e^2 A^a_\mu A^{b\mu} R^2 \left(1 - \frac{\mu^4}{m_2^4}\right) \sum\nolimits_{i,j=1}^{N} \left(T^a T^b\right)_{ij}. \tag{42}$$

We evaluate the last factor using the identity $T^a T^b = \frac{1}{2N}\delta_{ab}\mathbb{I}_N + \frac{1}{2}\sum_{c=1}^{N^2-1}(f_{abc} + i g_{abc})T^c$, where the g_{abc} and f_{abc} are completely symmetric and anti-symmetric tensors, respectively. We note that $\sum_{i,j=1}^{N}(T^c)_{ij} = \mathrm{Tr}T^c = 0$ due to the skew-Hermitian nature of T^c and $\sum_{i,j=1}^{N}(\mathbb{I}_N)_{ij} = \mathrm{Tr}\mathbb{I}_N = N$. Thus, we can diagonalize $\mathfrak{l}_A$, computing

$$\mathfrak{l}_A = \frac{R^2}{2N} e^2 \left(1 - \frac{\mu^4}{m_2^4}\right) A^a_\mu A^{a\mu} = \frac{1}{2} m_g^2 A^a_\mu A^{a\mu}, \tag{43}$$

from which we read off the masses $m_g^{(a)}$ of the $N^2 - 1$ gauge vector bosons. We note that once again they vanish at the zero exceptional points, but now for all $SU(N)$-models.

2.2 A $SU(2)$-symmetric model in adjoint representation

As we have demonstrated, the gauge vector boson becomes massive for the $SU(N)$-symmetric model in the $\mathcal{CPT}$-symmetric regime and at the exceptional point when the fields are taken to be in the representation space of the fundamental representation. On the other hand, the Higgs mechanism breaks down at the zero exceptional points. Remarkably, it still applies when the $\mathcal{CPT}$-symmetry is broken, although in that regime other particles acquire complex masses so that the region is non-physical. Recall that the regions in which the $\mathcal{CPT}$-symmetry holds are identified in Fig. 1.

Let us now see whether we encounter a similar behaviour when the fields are taken in adjoint representation. We consider here a slightly different non-Hermitian $SU(2)$-invariant Lagrangian

$$\begin{aligned}\mathcal{L}_2^{\mathrm{ad}} = &\frac{1}{2}\mathrm{Tr}(D\phi_1)^2 + \frac{1}{2}\mathrm{Tr}(D\phi_2)^2 - \frac{m_1^2}{2}\mathrm{Tr}(\phi_1^2) + \frac{m_2^2}{2}\mathrm{Tr}(\phi_2^2) - i\mu^2\mathrm{Tr}(\phi_1\phi_2)\\ &- \frac{g}{4}\left[\mathrm{Tr}(\phi_1^2)\right]^2 - \frac{1}{4}\mathrm{Tr}\left(F^2\right),\end{aligned} \tag{44}$$

where as in Eq. (4), we take $g, \mu \in \mathbb{R}$, $m_i \in \mathbb{R}$ or $m_i \in i\mathbb{R}$, to be constants. The two complex scalar fields are expressed as $\phi_i = \phi_i^a T^a$, $i = 1, 2$ and $a = 1, 2, 3$, where the T^a are the three $SU(2)$-generators in the adjoint representation that, up to a factor of 2, satisfy the same algebra as the Pauli spin matrices, that is $[T^a, T^b] = i\varepsilon_{abc}T^c$. Hence, the adjoint representation is $(T^a)_{bc} = -i\varepsilon_{abc}$, i.e. to be explicit

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{45}$$

such that $\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$ and therefore $\mathrm{Tr}(\phi^2) = 2\sum_{a=1}^{3}\phi^a\phi^a$. The $SU(2)$-symmetry in the adjoint representation for each generator T^a is therefore

$$\phi_j \to e^{i\alpha T^a}\phi_j e^{-i\alpha T^a} \approx \phi_j - \alpha\varepsilon_{abc}\phi_j^b T^c, \tag{46}$$

so that the infinitesimal changes to the fields ϕ_i^a result to

$$\delta\phi_i^a = -\alpha\varepsilon_{abc}\phi_i^b. \tag{47}$$

We will utilize this expression below.

In more a compact form, the Lagrangian in (44) can be expressed equivalently as

$$\mathcal{L}_2^{\mathrm{ad}} = D_\mu\phi_i^a D^\mu\phi_i^a - \phi_i^a M_{ij}^2\phi_j^a - g\left(\phi_i^a E_{ij}\phi_j^a\right)^2 - \frac{1}{4}F_{\mu\nu}^a\left(F^{\mu\nu}\right)^a, \tag{48}$$

where repeated indices are summed over the appropriate index sets $i, j, \mu, \nu \in \{1, 2\}$ and $a, b \in \{1, 2, 3\}$. The matrix M^2 is defined as

$$M^2 = \begin{pmatrix} m_1^2 & i\mu^2 \\ i\mu^2 & -m_2^2 \end{pmatrix}, \tag{49}$$

and E as in (12). The covariant derivative in the adjoint representation acting on a complex field takes on the form

$$(D_\mu\phi_i)^a := \partial_\mu\phi_i^a + e\varepsilon_{abc}A_\mu^b\phi_i^c \tag{50}$$

Pursuing here a pseudo-Hermitian approach we perform a similarity transformation on the Lagrangian in (48) with Dyson map

$$\eta = \prod_{a=1}^{3} e^{\frac{\pi}{2}\int d^3x \Pi_2^a \phi_2^a}, \tag{51}$$

that maps the complex Lagrangian $\mathcal{L}_2^{\text{ad}}$ to a real Lagrangian

$$\mathfrak{l}_2^{\text{ad}} = (D_\mu \phi_i)^a \mathcal{I}_{ij} (D^\mu \phi_j)^a - \phi_i^a H_{ij} \phi_j^a - g\left(\phi_i^a E_{ij} \phi_j^a\right)^2, \tag{52}$$

where the matrix H is defined as

$$H := \begin{pmatrix} m_1^2 & \mu^2 \\ \mu^2 & m_2^2 \end{pmatrix}, \tag{53}$$

and $\mathcal{I}$ as in (12).

2.2.1 The SU(2)-symmetry preserving and breaking vacua

To find the different types of vacua ϕ^0, we need to solve again $\delta V = 0$. The corresponding functional variation of the Lagrangian in (52) leads to the three sets of equations

$$\left(H + 2gR^2 E\right)\left(\phi^0\right)^a = 0, \quad a = 1, 2, 3, \tag{54}$$

with $R^2 := \left(\phi_i^0\right)^a E_{ij} \left(\phi_j^0\right)^a$. Next to the trivial $SU(2)$-symmetry preserving solution $\left(\phi^0\right)^a = 0$, a $SU(2)$-symmetry breaking solution is obtained by requiring $\left(\phi^0\right)^a$ to become a vector in the null space of the matrix $H + 2gR^2 E$, which is the case when

$$\left(\phi^0\right)^a = \frac{N_a}{m_2^2}\begin{pmatrix} m_2^2 \\ -\mu^2 \end{pmatrix}, \quad \text{and} \quad R^2 = \frac{\mu^4 - m_1^2 m_2^2}{2g m_2^2}, \tag{55}$$

where the N_a are normalization constants. Given the solution in (55), the relation for R^2 imposes the additional constraint $R^2 = N_1^2 + N_2^2 + N_3^2$ on these constants. Expressing the Lie algebra valued vacuum field $\phi_i^0 = \left(\phi_i^0\right)^a T^a$ in the matrix form of the adjoint representation (45), we obtain

$$\phi_1^0 = i\begin{pmatrix} 0 & -N_3 & N_2 \\ N_3 & 0 & -N_1 \\ -N_2 & N_1 & 0 \end{pmatrix}, \quad \text{and} \quad \phi_2^0 = -\frac{\mu^2}{m_2^2}\phi_1^0. \tag{56}$$

We can now apply the $SU(2)$-symmetry to the vacuum state in the form

$$\phi^{\text{vac}} = \left[\left(\phi_1^0\right)^1, \left(\phi_2^0\right)^1, \left(\phi_1^0\right)^2, \left(\phi_2^0\right)^2, \left(\phi_1^0\right)^3, \left(\phi_2^0\right)^3\right], \tag{57}$$

so that the infinitesimal changes $\delta\phi_i(\phi^{\text{vac}})$ with (47) and (55) yield the following states for each generator

$$v_1^0 = \frac{\alpha_1}{m_2^2}\left(0, 0, N_3 m_2^2, -N_3\mu^2, -N_2 m_2^2, N_2\mu^2\right), \tag{58}$$

$$v_2^0 = \frac{\alpha_2}{m_2^2}\left(-N_3 m_2^2, N_3\mu^2, 0, 0, N_1 m_2^2, -N_1\mu^2\right), \tag{59}$$

$$v_3^0 = \frac{\alpha_3}{m_2^2}\left(N_2 m_2^2, -N_2\mu^2, -N_1 m_2^2, N_1\mu^2, 0, 0\right), \tag{60}$$

as solutions for ϕ^{vac}. Evidently, these states are linearly dependent as

$$\sum\nolimits_{i=1}^{3} \frac{N_i v_i^0}{\alpha_i} = 0. \tag{61}$$

According to Goldstone's theorem, the states v_i^0 should be eigenvectors of the squared mass matrix with eigenvalue 0. As only two of them are linearly independent we expect to find two massless Goldstone bosons, which in our gauged model correspond to "would be Goldstone bosons". Hence, the $SU(2)$-symmetry has been broken down to a $U(1)$-symmetry, so that the group theoretical argument predicts two Goldstone bosons equal to the dimension of the coset $SU(2)/U(1)$.

2.2.2 The squared mass matrix

Expanding the Lagrangian in Eq. (48) about the vacuum solution gives

$$\mathcal{L}_2^{\mathrm{ad}} = (D_\mu \phi_i)^a \mathcal{I}_{ij} (D_\mu \phi_j)^a - \frac{1}{2} \phi_i^a H_{ij}^{(a)} \phi_j^a + 2(D_\mu \phi_i^0)^a \mathcal{I}_{ij} (D^\mu \phi_j)^a + (D_\mu \phi_i^0)^a \mathcal{I}_{ij} (D_\mu \phi_j^0)^a + \mathcal{O}(\phi^3), \tag{62}$$

where the last two terms originate from expanding the covariant kinetic term. The Hessian matrix is then computed by differentiating (54) once more

$$\hat{H}_{ij}^{ab} := \frac{\partial^2 \mathcal{V}}{\partial \phi_i^a \partial \phi_j^b} = 2H_{ij} + 4gR^2 E_{ij} \delta^{ab} + 8g \left(E\phi^a\right)_i \left(E\phi^b\right)_j, \tag{63}$$

from which we obtain the non-Hermitian squared mass matrix as

$$\begin{aligned} M^2 &= \frac{1}{2} \mathcal{I} \hat{H} \bigg|_{\phi^{\mathrm{vac}}} \\ &= \begin{pmatrix} m_1^2 + 2gR^2 + 4gN_1^2 & \mu^2 & 4gN_1N_2 & 0 & 4gN_1N_3 & 0 \\ -\mu^2 & -m_2^2 & 0 & 0 & 0 & 0 \\ 4gN_1N_2 & 0 & m_1^2 + 2gR^2 + 4gN_2^2 & \mu^2 & 4gN_2N_3 & 0 \\ 0 & 0 & -\mu^2 & m_2^2 & 0 & 0 \\ 4gN_1N_3 & 0 & 4gN_2N_3 & 0 & m_1^2 + 2gR^2 + 4gN_3^2 & \mu^2 \\ 0 & 0 & 0 & 0 & -\mu^2 & -m_2^2 \end{pmatrix}. \end{aligned} \tag{64}$$

The entries in the rows and columns of M^2 are labelled as $(\phi_1^1, \phi_2^1, \phi_1^2, \phi_2^2, \phi_1^3, \phi_2^3) =: \Psi$. The six eigenvalues λ of M^2 are then computed to

$$\lambda_{1,2} = 0; \qquad \lambda_{3,4} = \frac{\mu^4 - m_2^4}{m_2^2}, \qquad \lambda_\pm = \kappa \pm \sqrt{2(\mu^4 - m_1^2 m_2^2) + \kappa^2}, \tag{65}$$

with $\kappa := 3\mu^4/2m_2^2 - m_2^2/2 - m_1^2$. We can now verify that the three vectors v_i^0 in (58)–(60), corresponding to the infinitesimal changes of the vacuum (55) under the action of the $SU(2)$-symmetry, are indeed vectors in the null space of M^2. Due to their linear dependence, we may choose two of them to be associated with the two massless "would be Goldstone bosons".

We note that there are zero exceptional points at $\mu^4 = m_2^4$ when $\lambda_{3,4} = 0$, and at $\mu^4 = m_1^2 m_2^2$ when either $\lambda_- = 0$ or $\lambda_+ = 0$. The standard exceptional point for which the two eigenvalues λ_- and λ_+ coalesce occurs when $m_1^2 = 3\mu^4/2m_2^2 + m_2^2/2 \pm \mu^2$. The Jordan normal form for the mass squared matrix becomes

$$\mathrm{diag} D_e = (0, \lambda_e^b, 0, \lambda_e^b, 0, \lambda_e^b, \Lambda), \quad \lambda_e^b = \frac{\mu^4}{m_2^2} - m_2^2, \quad \Lambda = \begin{pmatrix} \pm\mu^2 - m_2^2 & \pm(\alpha - \beta)\mu^2 \\ 0 & \pm\mu^2 - m_2^2 \end{pmatrix}, \tag{66}$$

for some arbitrary constants α and β.

We notice that the eigenvalues in (65) do not depend on the choice of the three normalization constants N_a, since all of these vacua are equivalent as they are related by $SU(2)$-symmetry transformations. The physical regions of the model are determined by the requirement that the eigenvalues are real and positive. Taking now account of the possibility that $m_i \in \mathbb{R}$ or $m_i \in i\mathbb{R}$, by allowing for different signs in front of the m_i^2 terms in setting $m_i^2 \to c_i m_i^2$, we find that the model does not possess any physical region when $c_1 = c_2 = \pm 1$ and physical regions when $c_1 = -c_2 = \pm 1$ as argued also in the previous section.

2.2.3 The would be Goldstone bosons

Let us now identify the two massless Goldstone bosons $\psi_{1,2}^{\mathrm{Gb}}$ in the different $\mathcal{PT}$-regimes by the same procedure as previously explained in [38, 39], with the difference that they will be made to vanish due to the presence of the gauge bosons. In terms of the original scalar fields in the model we identify the Goldstone bosons by evaluating

$$\psi_{1,2}^{\mathrm{Gb}} := \sqrt{(\Psi^T \hat{I} U)_{1,2} (U^{-1} \Psi)_{1,2}}, \tag{67}$$

where the matrix U diagonalizes the squared mass matrix by $U^{-1} M^2 U = D$ with $\mathrm{diag} D = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_-, \lambda_+)$ and $\mathrm{diag} \hat{I} = \{\mathcal{I}, \mathcal{I}, \mathcal{I}\}$. In the $\mathcal{PT}$-symmetric regime the similarity transformation U is well-defined by

$$U := (v_1, v_2, v_3, v_4, v_-, v_+), \tag{68}$$

where the v_i are the eigenvectors of M^2. Up to normalizations constants for each eigenvector, we obtain in our example the concrete expressions

$$v_i = \left[(m_2^2 + \lambda_i) \tau_{i1}, -\mu^2 \tau_{i1}, (m_2^2 + \lambda_i) \tau_{i2}, -\mu^2 \tau_{i2}, (m_2^2 + \lambda_i) \tau_{i3}, -\mu^2 \tau_{i3} \right], \tag{69}$$

with $\tau_{12} = \tau_{23} = \tau_{32} = \tau_{43} = 0$, $\tau_{33} = \tau_{42} = \tau_{\pm 1} = -\tau_{13} = -\tau_{22} = N_1$, $\tau_{21} = \tau_{41} = \tau_{\pm 2} = N_2$ and $\tau_{11} = \tau_{31} = \tau_{\pm 3} = N_3$. Defining a $\mathcal{PT}$-inner product as $\langle a|b\rangle_{\mathcal{PT}} := a\hat{I}b$, these vectors can be orthonormalized $\langle v_i|v_j\rangle_{\mathcal{PT}} = \delta_{ij}$. Recall from the argument after (21) that indeed $v_i = \mathcal{I}u_i$, so that $\langle v_i|v_j\rangle_{\mathcal{PT}} = v_i\mathcal{I}^2 u_j = \delta_{ij}$. Here, we should stress that M^2 is only the analogue to a Hamiltonian, and we are not attempting to construct an associated Hilbert space.

For convenience, we take now $N_1 = N_2 = 0$, $N_3 = R$ and compute

$$\psi_1^{\mathrm{Gb}} := \frac{m_2^2\phi_1^3 + \mu^2\phi_2^3}{\sqrt{m_2^4 - \mu^4}}, \quad \text{and} \quad \psi_2^{\mathrm{Gb}} := \frac{m_2^2\phi_1^2 + \mu^2\phi_2^2}{\sqrt{m_2^4 - \mu^4}}. \tag{70}$$

We note that $\det U = \lambda_3\lambda_4(\lambda_- - \lambda_+)\mu^6 R^4$, indicating the breakdown of these expressions at the exceptional points when $\lambda_- = \lambda_+$, the zero exceptional point when $\lambda_3 = \lambda_4 = 0$ and at the trivial vacuum when $R = 0$, as previously observed in [38, 39]. However, at the exceptional point we may still calculate the expressions for the Goldstone boson when taking into account that in this case, the two eigenvectors v_- and v_+ become identical. In order to obtain two linearly independent eigenvectors when the squared mass matrix is converted into its Jordan normal form, we multiply two entries of the vector v_+ by some arbitrary constants $\alpha \neq \beta$ as $(v_+)_1 \to \alpha(v_+)_1$ and $(v_+)_2 \to \beta(v_+)_2$. With this change the matrix U becomes invertible as $\det U = \lambda_3\lambda_4(\beta - \alpha)(m_2^2 + \kappa)N_1^2\mu^6 R^2$. We may now evaluate the expression in (67) obtaining the same formulae for the Goldstone bosons as in (70). At the zero exceptional point, it is not possible to identify the Goldstone in terms of the original fields in the model.

2.2.4 The mass of the vector gauge boson

Finally, we calculate the mass of the gauge vector bosons by expanding the minimal coupling term in equation (52) around the symmetry breaking vacuum (57)

$$\begin{aligned}
\left[D_\mu(\phi + \phi^0)\right]^T \mathcal{I}\left[D^\mu(\phi + \phi^0)\right] &= (D_\mu\phi^0)^T \mathcal{I}(D^\mu\phi^0) + \cdots \\
&= e^2\left[\varepsilon_{abc}A_\mu^b\left(\phi_i^0\right)^c\right]\mathcal{I}_{ij}\left(\varepsilon_{ade}A^{d\mu}\left(\phi_j^0\right)^e\right) + \cdots \\
&= e^2\left(A_\mu^a A^{a\mu}\left(\phi_i^0\right)^b\mathcal{I}_{ij}\left(\phi_j^0\right)^b - A_\mu^a A^{b\mu}\left(\phi_i^0\right)^b\mathcal{I}_{ij}\left(\phi_j^0\right)^a\right) + \cdots,
\end{aligned} \tag{71}$$

where we used the standard identity $\varepsilon_{abc}\varepsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}$. A convenient choice for the normalization constants N_i that is compatible with (55) and diagonalizes (71) is to set two constants to zero and the remaining one to R. For instance, taking $N_1 = N_2 = 0$, $N_3 = R$, the only nonvanishing terms in (71) are

$$= e^2\left(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu}\right)\left(\phi_i^0\right)^3\mathcal{I}_{ij}\left(\phi_j^0\right)^3, \tag{72}$$

$$= e^2 R^2\left(1 - \frac{\mu^4}{m_2^4}\right)\left(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu}\right). \tag{73}$$

Thus for $\mu^4 \neq m_2^4$ and $R \neq 0$, we obtain two massive vector gauge bosons $m_g^{(1)}$ and $m_g^{(2)}$, that is one for each "would be Goldstone boson". When $\mu^4 = m_2^4$, that is then model is at the zero exceptional point of type I, the gauge mass vector bosons remain massless. This feature is compatible with our previous observations in [38, 39] and above that at these points, the Goldstone bosons cannot be identified.

We notice here that the two massive vector gauge bosons are proportional to the $\mathcal{CPT}$-inner product of the symmetry-broken vacuum solution

$$m_{\mathrm{gauge}}^2 \propto \langle 0|0\rangle_{\mathcal{CPT}} \propto \phi^{\mathrm{vac}}\hat{I}\phi^{\mathrm{vac}} \propto R^2\left(1 - \frac{\mu^4}{m_2^4}\right). \tag{74}$$

Hence, the vanishing of the mass for the vector gauge bosons at the two types of zero exceptional points can be associated to the vanishing of the $\mathcal{CPT}$-inner product at these points. This is reminiscent of the vanishing of the $\mathcal{CPT}$-inner product at the standard exceptional points, which is responsible for interesting phenomena such as the stopping of light at these locations in the parameter space [45, 46]. We note, however, a key difference between the two scenarios: While the $\mathcal{CPT}$ -inner product in (74) is devised on the eigenvector space of squared mass matrix, the latter is a $\mathcal{CPT}$-inner product on the Hilbert space.

3 Conclusions

Employing a pseudo-Hermitian approach, we found that the Higgs mechanism applies in the usual way in the $\mathcal{CPT}$-symmetric regime by giving a mass to the vector gauge bosons and preventing Goldstone bosons to exist, which is also found in [37, 41] using different approaches. As in [41], we also observed that in the spontaneously broken $\mathcal{CPT}$-symmetric regime, the vector gauge bosons become massive and the Higgs mechanism is intact. However, as in this regime, other particles acquire complex masses, it has to be discarded as non-physical for that reason. Even though technically one needs to treat the standard exceptional point differently from the other regimes, the main principle of the Higgs mechanism still holds up. In contrast to the finding in [41], we observed that the Higgs mechanism breaks down at the zero exceptional points, which was also observed in [37]. We find the same characteristic behaviour, i.e. the matching of the amounts of massive vector gauge bosons and "would be Goldstone bosons", for the complex scalar fields taken in the fundamental as well as in the adjoint representation. The vanishing of the mass for the vector gauge bosons coincides with the vanishing of the $\mathcal{CPT}$-inner product on the eigenvector space of squared mass matrix.

Obviously, there are many interesting extensions to these investigations, such as for instance the treatment of models with a more involved field content or different types of continuous symmetries.

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Appendix: Type I (standard) versus type II (zero) exceptional points

In this appendix, we present a discussion that illustrates the difference between the two types of exceptional points. The main distinction in their behaviour is that a one-dimensional parameter space the $\mathcal{PT}$-symmetry is spontaneously broken on one side of the type I (standard) exceptional point, whereas for type II (zero) exceptional point the $\mathcal{PT}$-symmetry is preserved on both sides. The zero exceptional point occurs when two eigenvalues coalesce at zero, hence the name.

We consider here a (3×3)-matrix of a very generic form that occurs for instance as a building block of the squared mass matrix in the model discussed in [39], see equation (3.48), therein,

$$H = \begin{pmatrix} A & W & 0 \\ -W & B & -V \\ 0 & V & -C \end{pmatrix} \tag{75}$$

Here, we carry out the discussion for a Hamiltonian H, having in mind the analogy to the squared mass matrix. The determinant is easily computed to $\det H = A\kappa - CW^2$, $\kappa := V^2 - BC$. In order to obtain a zero eigenvalue, $\lambda_0 = 0$, we enforce now the determinant to vanish by setting $A = W^2 C/\kappa$. The other two eigenvalues then become

$$\lambda_\pm = \frac{\kappa(B-C) + CW^2 \pm \tau}{2\kappa} \tag{76}$$

with $\tau = \sqrt{\kappa^2\big((B+C)^2 - 4V^2\big) + 2\kappa W^2\big(C(B+C) - 2V^2\big) + C^2W^4}$.

The exceptional points are identified by simultaneously solving the two equations:

$$\det(H - \lambda\mathbb{I}) = 0, \quad \text{and} \quad \frac{d}{d\lambda}\det(H - \lambda\mathbb{I}) = 0, \tag{77}$$

for W and λ, obtaining the two sets of eigenvalues

$$\lambda_\pm^e = \frac{\hat{\kappa}^2 - \sqrt{\kappa}V}{C}, \lambda_0 = 0 \quad \text{and} \quad \lambda_-^0 = \lambda_0 = 0, \lambda_+^0 = \frac{C^3 + BV^2 - 2CV^2}{\hat{\kappa}^2} \tag{78}$$

for the critical parameters

$$W^e = \frac{\tilde{\kappa}}{C}, \quad \text{and} \quad W^0 = i\frac{\kappa}{\hat{\kappa}}, \tag{79}$$

respectively. We abbreviated $\tilde{\kappa} := \pm\sqrt{\kappa\left(\kappa + V^2 - C^2\right) \pm 2\kappa^{3/2}V}$ and $\hat{\kappa} := \sqrt{V^2 - C^2}$. The first set of eigenvalues in (78) correspond to the standard exceptional point and the second set to the zero exceptional point.

Next, we calculate the bi-orthonormal basis from the normalized left and right eigenvectors u_i, v_i, $i = 0, \pm$, respectively, for H

$$v_0 = \frac{1}{\sqrt{N_0}}(-\kappa, CW, VW), \quad v_\pm = \frac{1}{\sqrt{N_\pm}}(W(\hat{\kappa}^2 - C\lambda_\pm), \kappa(C+\lambda_\pm), V\kappa), \quad u_i = Uv_i, \tag{80}$$

with $U = \mathrm{diag}(1, -1, 1)$ and normalization constants $N_0 = \kappa^2 + W^2\hat{\kappa}^2$, $N_\pm = V^2\kappa^2 + W^2[V^2 - C(C+\lambda_\pm)]^2 - (C+\lambda_\pm)^2\kappa^2$. By construction, these vectors satisfy the orthonormality relation $u_i \cdot v_j = \delta_{ij}$.

We observe now that at the standard exceptional point the two eigenvectors for the non-normalized ($N_\pm$ become zero at the exceptional points) eigenvalues $\lambda_\pm^e$ coalesce, which distinguishes exceptional points from standard degeneracy. The left and right eigenvectors become in this case

$$v_\pm^{e,r} = \left(\tilde{\kappa}, V\sqrt{\kappa} - \kappa, C\sqrt{\kappa}\right), \qquad v_0^{e,r} = (-C\kappa, C\tilde{\kappa}, V\tilde{\kappa}), \tag{81}$$

$$v_\pm^{e,l} = \left(\tilde{\kappa}, \kappa - V\sqrt{\kappa}, C\sqrt{\kappa}\right), \qquad v_0^{e,l} = (-C\kappa, -C\tilde{\kappa}, V\tilde{\kappa}), \tag{82}$$

with

$$v_\pm^{e,l} \cdot v_\pm^{e,r} = 0, \quad \text{and} \quad v_0^{e,l} \cdot v_0^{e,r} = C^2\left(\kappa^2 - \tilde{\kappa}^2\right) + V^2\tilde{\kappa}^2. \tag{83}$$

Similarly, at the zero exceptional point, the eigenvectors for the eigenvalues λ_0 and λ_-^0 coalesce, which qualifies this point also to be called "exceptional" in the standard terminology. In this case, the left and right eigenvectors become

$$v_+^{0,r} = \left(V\kappa, iV\hat{\kappa}(C - B), i\hat{\kappa}^3\right), \qquad v^{0,r} = v_-^{0,r} = \left(i\hat{\kappa}, C, V\right), \tag{84}$$

$$v_+^{0,l} = \left(V\kappa, iV\hat{\kappa}(B - C), i\hat{\kappa}^3\right), \qquad v^{0,l} = v_-^{0,r} = \left(i\hat{\kappa}, -C, V\right), \tag{85}$$

with

$$v_+^{e,l} \cdot v_+^{e,r} = \left(C^3 + BV^2 - 2CV^2\right)^2, \quad \text{and} \quad v^{0,l} \cdot v^{0,r} = 0. \tag{86}$$

In order to understand the key difference between these two types of exceptional points, we consider at first the eigenvalues (76) near the critical values in (79). Concerning the standard exceptional points, we note that the two eigenvalues become identical when $\tau \to 0$. Thus, writing $\tau/C^2 = (W^2 - (W^e)^2)(W^2 - \tilde{W})$, with $\tilde{W}$ being the second root of the polynomial in W^2 under the square root, it is now clear that if we consider the eigenvalues as functions of W^2 the argument of the square root has different signs for $W^2 = (W^e)^2 + \epsilon$ and $W^2 = (W^e)^2 - \epsilon$. Hence, the eigenvalues are real on one side of the exceptional point in the W^2-parameter space and complex on the other. In contrast, none of the eigenvalues becomes complex in the neighbourhood of the critical value W^0.

For completion, we also report the Dyson map and hence the metric operator for which the same behaviour may be observed. Using the operator that diagonalizes the non-Hermitian Hamiltonian H

$$\eta = (v_0, v_+, v_-), \qquad \rho = \eta\eta^\dagger, \tag{87}$$

with determinant

$$\det\eta = \frac{V\kappa}{\sqrt{N_0 N_+ N_-}}(\lambda_- - \lambda_+)(\kappa^2 + W^2\hat{\kappa}^2), \tag{88}$$

we verify the pseudo- and quasi-Hermiticity relations

$$\eta^{-1}H\eta = h = h^\dagger, \qquad \rho H = H^\dagger\rho. \tag{89}$$

We observe that the map breaks down at both exceptional points, i.e. $\det\eta = 0$ for the critical values W^e and W^0, and on one side of the standard exceptional point. In all other regions of the parameter space, it holds. Thus, we find the same behaviour as already observed for the analysis of the eigenvalues.

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