Progress of Theoretical Physics, Vol. 128, No. 5, November 2012

Massive Modes in Magnetized Brane Models

Yuta Hamada and Tatsuo Kobayashi

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

(Received August 13, 2012)

We study higher dimensional models with magnetic fluxes, which can be derived from superstring theory. We study mass spectrum and wavefunctions of massless and massive modes for spinor, scalar and vector fields. We compute the 3-point couplings and higher order couplings among massless modes and massive modes in 4D low-energy effective field theory. These couplings have non-trivial behaviors, because wavefunctions of massless and massive modes are non-trivial.

Subject Index: 140, 141, 143

§1. Introduction

Field theory in higher dimensions plays a role in particle physics and cosmology. In particular, extra dimensional field theory derived from superstring theory is important. Their four-dimensional (4D) low-energy effective field theories are determined by geometrical aspects of compact extra dimensions. One of the simplest compact spaces is a torus. However, the simple toroidal compactification does not lead to a chiral theory as a 4D low-energy effective field theory. Hence, it is a key issue to realize a 4D chiral theory when we start with higher dimensional field theory.

Complicated geometrical backgrounds such as Calabi-Yau manifolds would lead to a 4D chiral theory, although it may be difficult to compute explicitly 4D low-energy effective field theories from such geometrical backgrounds. On the other hand, the toroidal compactification can also lead to a 4D chiral theory when we introduce non-vanishing magnetic fluxes in extra dimensions. The numbers of zero-modes are determined by the size of magnetic flux and each zero-mode has a quasi-localized profile. Thus, the toroidal compactification with magnetic fluxes is quite attractive background for higher dimensional field theory[1)–5)] (see also 6), 7)). Its stringy setup corresponds to magnetized D-brane models wrapping cycles on the torus.[8)–11)] Furthermore, magnetized D-brane models are the T-dual of intersecting D-brane models, and many interesting models have been constructed in both types of models.[12),13)]

The Yukawa couplings among massless modes were computed by integrating the overlap of wavefunctions in the extra dimensional space.[1)] If zero-modes are quasi-legalized far away from each other, their couplings are suppressed. On the other hand, if they are localized near each other, their couplings are not suppressed, but would be of $\mathcal{O}(1)$. Thus, these localization behaviors are important from the phenomenological viewpoint, for example, to derive the realistic values of quark and lepton masses and their mixing angles. Furthermore, higher order couplings among massless modes were also computed.[14)] Interestingly, they are written by products of 3-point couplings. These low-energy effective field theories can also lead to Abelian and non-Abelian discrete flavor symmetries, e.g. D_4 and $\Delta(27)$ flavor symmetries.[15),16),*)] These non-Abelian discrete flavor symmetries are important to derive the realistic quark and lepton mass matrices (see e.g. 18) and references therein).

In addition to massless modes, massive modes also have important effects in 4D low-energy effective field theory. For example, they may induce the fast proton decay and flavor changing neutral currents (FCNCs) (see e.g. 19)). Our purpose in this paper is to study massive modes in the extra dimensional models with magnetic fluxes. We study their mass spectrum and wavefunctions explicitly. Then, we study compute 3-point couplings and higher order couplings including these massive modes. These couplings have non-trivial behaviors, because wavefunctions of massless and massive modes are non-trivial.

This paper is organized as follows. In §2, we briefly review the fermion zero-modes on T^2 with the magnetic flux. Then, we study mass spectrum and wavefunctions of higher modes explicitly. These analyses are extended to those for zero-modes and higher modes of scalar and vector fields. Its extension to T^6 is straightforward. In §3, we compute couplings among these modes. In §3.1, we give a brief review on computations of the 3-point couplings and higher order couplings among zero-modes. Then, we extend them to the computations of the 3-point and higher order couplings including higher modes in §3.2. In §3.3, we also consider the couplings including massive modes due to only the Wilson line effect, but not magnetic fluxes. In §4, we give comments on some phenomenological implications of our results. Section 5 is devoted to conclusion and discussion. In Appendix A, we show some useful properties of the Hermite function. In Appendix B, we briefly review the vector field in extra dimensions. In Appendix C, we show useful properties of the products of zero-mode wavefunctions.

§2. Mass spectrum and wavefunctions of massive modes

We consider the $(4+d)$-dimensions, and denote four-dimensional and d-dimensional coordinates by x^μ and y^m with $\mu = 0, \cdots, 3$ and $m = 1, \cdots, d$, respectively. We study the spinor field $\lambda(x^\mu, y^m)$ and the vector field $A_M(x^\mu, y^m)$ with $M = 0, \cdots, (3+d)$. We decompose these fields as follows,

$$\lambda(x^\mu, y^m) = \sum_n \chi_n(x^\mu)\psi_n(y^m), \tag{2·1}$$

$$A_M(x^\mu, y^m) = \sum_n \varphi_{n,M}(x^\mu)\phi_{n,M}(y^m). \tag{2·2}$$

Here we choose the internal wavefunctions $\psi_n(y^m)$ as eigenfunctions of the internal Dirac operator as

$$i\Gamma^m D_m \psi_n = m_n \psi_n, \tag{2·3}$$

*) Similar non-Abelian discrete flavor symmetries are derived in heterotic orbifold models.[17)]

where Γ^m denote the gamma matrices in the internal space. The eigenvalues of m_n become masses of the modes $\chi_n(x^\mu)$ in 4D effective field theory. Similarly, $\phi_{n,M}(y^m)$ correspond to eigenfunctions of the internal Laplace operators, as will be shown explicitly later. The scalar field in the $(4+d)$-dimensions is also decomposed in a similar way.

2.1. T^2 *with magnetic flux*

First, let us consider the 2D torus, T^2. Here, we follow the notation of Ref. 1). Instead of the real coordinates y^1 and y^2, we use the complex coordinate, $z = y^1 + \tau y^2$ with $\tau \in \mathbf{C}$. The metric is given by

$$ds^2 = 2(2\pi R)^2 dz d\bar{z}. \tag{2·4}$$

We identify the complex coordinate as $z \sim z+1$ and $z \sim z+\tau$ on T^2. The area is written by $\mathcal{A} = 4\pi^2 R^2 \mathrm{Im}\tau$.

We introduce the $U(1)$ magnetic flux on T^2 as

$$F_{z\bar{z}} = \frac{\pi i}{\mathrm{Im}\tau} m. \tag{2·5}$$

This magnetic flux is derived, e.g., from the following vector potential,

$$A_{\bar{z}} = \frac{\pi}{2\mathrm{Im}\tau} m z, \qquad A_z = -\frac{\pi}{2\mathrm{Im}\tau} m \bar{z}. \tag{2·6}$$

Their boundary conditions can be written as

$$A_i(z+1) = A_i(z) + \partial_i \chi_1, \qquad A_i(z+\tau) = A_i(z) + \partial_i \chi_2, \tag{2·7}$$

where

$$\chi_1 = \frac{\pi}{\mathrm{Im}\tau} m\ \mathrm{Im} z, \qquad \chi_2 = \frac{\pi}{\mathrm{Im}\tau} m\ \mathrm{Im}\bar{\tau} z. \tag{2·8}$$

Furthermore, we can introduce non-vanishing Wilson lines by using

$$\chi_1 = \frac{\pi}{\mathrm{Im}\tau} \mathrm{Im}(mz+\alpha), \qquad \chi_2 = \frac{\pi}{\mathrm{Im}\tau} \mathrm{Im}\bar{\tau}(mz+\alpha), \tag{2·9}$$

where α is complex and corresponds to the degree of freedom of the Wilson line. It is convenient to use the following notation,

$$\alpha = m\zeta, \tag{2·10}$$

for $m \neq 0$.

2.1.1. Fermion zero-modes

Here, we review the fermion zero-modes, which satisfy Eq. (2·3) with $m_n = 0$.[1)] On T^2, the spinor ψ_n has two components,

$$\psi_n = \begin{pmatrix} \psi_{+,n} \\ \psi_{-,n} \end{pmatrix}. \tag{2·11}$$

We use the gamma matrices on T^2 as

$$\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2·12}$$

Then, the zero-mode equation is written as

$$D\psi_{+,0} = 0, \qquad D^\dagger \psi_{-,0} = 0, \tag{2·13}$$

where

$$D = \frac{1}{\pi R}\left(\bar{\partial} + q\frac{\pi m}{2\mathrm{Im}\tau}(z+\zeta)\right), \tag{2·14}$$

for the spinor with $U(1)$ charge q. The charge q and magnetic flux m should satisfy that $qm =$ integer. They also satisfy the following boundary conditions,

$$\psi_n(z+1) = e^{iq\chi_1(z)}\psi_n(z), \qquad \psi_n(z+\tau) = e^{iq\chi_2(z)}\psi_n(z). \tag{2·15}$$

When $qm > 0$, only the zero-mode $\psi_{+,0}$ has a solution, but $\psi_{-,0}$ has no solution. Then, the chiral spectrum for the zero-modes is realized and the number of zero-modes is equal to qm. Their zero-mode wavefunctions are written explicitly as

$$\psi_+^{j,qm}(z+\zeta) = \left(\frac{2\mathrm{Im}\tau qm}{\mathcal{A}^2}\right)^{1/4} \sum_\ell \Theta_\ell^{j,qm}(z+\zeta,\tau), \tag{2·16}$$

where

$$\begin{aligned}\Theta_\ell^{j,qm}(z+\zeta,\tau) = \exp\Bigg[&-\pi qm\mathrm{Im}\tau\left(\frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau} + \frac{j}{qm} + \ell\right)^2 \\ &+ i\pi qm\mathrm{Re}(z+\zeta)\left(\frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau} + 2\left(\frac{j}{qm}+\ell\right)\right) + i\pi qm\mathrm{Re}\tau\left(\frac{j}{qm}+\ell\right)\Bigg].\end{aligned} \tag{2·17}$$

Note that the effect of the Wilson line ζ is the shift of the wavefunctions $\psi^{j,qm}(z)$ to $\psi^{j,qm}(z+\zeta)$. The zero-mode wavefunction can be written by a product of the Gaussian function and the Jacobi ϑ-function, i.e.,

$$\begin{aligned}\psi_+^{j,qm}(z+\zeta) = \left(\frac{2\mathrm{Im}\tau qm}{\mathcal{A}^2}\right)^{1/4} &\exp\left[i\pi\frac{qm(z+\zeta)\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau}\right] \\ &\times\vartheta\begin{bmatrix} j/qm \\ 0 \end{bmatrix}(qm(z+\zeta), qm\tau),\end{aligned} \tag{2·18}$$

where

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_\ell \exp\left[\pi i(a+\ell)^2\tau + 2\pi i(a+\ell)(\nu+b)\right]. \tag{2·19}$$

$(\psi_+^{j,qm})^*$ represents the anti-particle of $\psi_+^{j,qm}$, and is obtained from Eq. (2·16) by replacing $\Theta_\ell^{j,qm}(z+\zeta,\tau)$ with $\Theta_\ell^{-j,-qm}(\bar{z}+\bar{\zeta},\bar{\tau})$. These zero-mode wavefunctions satisfy the following orthonormal condition,

$$\int_{T^2} dz d\bar{z} \psi_+^{j,qm} (\psi_+^{k,qm})^* = \delta_{jk}. \tag{2·20}$$

When $qm < 0$, there appear the zero-modes for $\psi_{-,0}$, but not for $\psi_{+,0}$. The number of their zero-modes is equal to $|qm|$, and their wavefunctions are obtained similarly. In the following discussion, we assume $qm > 0$.

2.1.2. Fermion massive modes

Here, we study the fermion massive modes with $m_n \neq 0$ in Eq. (2·3). For $m_n \neq 0$, the zero-modes, $\psi_{+,n}$ and $\psi_{-,n}$, mix each other in Eq. (2·3). They satisfy

$$\begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix} \begin{pmatrix} \psi_{+,n} \\ \psi_{-,n} \end{pmatrix} = m_n^2 \begin{pmatrix} \psi_{+,n} \\ \psi_{-,n} \end{pmatrix}. \tag{2·21}$$

The 2D Laplace operator is defined as

$$\Delta = \frac{1}{2}\{D^\dagger, D\}, \tag{2·22}$$

and it satisfies the following algebraic relations,

$$\begin{aligned} \Delta &= D^\dagger D + \frac{2\pi qm}{\mathcal{A}}, \qquad & [D, D^\dagger] &= \frac{4\pi qm}{\mathcal{A}}, \\ [\Delta, D^\dagger] &= \frac{4\pi qm}{\mathcal{A}} D^\dagger, \qquad & [\Delta, D] &= -\frac{4\pi qm}{\mathcal{A}} D. \end{aligned} \tag{2·23}$$

Thus, massive modes are eigenfunctions of the Laplace operator Δ, and their mass spectrum is derived in an analysis similar to that of the quantum harmonic oscillator. It is convenient to use the normalized creation and annihilation operators,

$$a = \sqrt{\frac{\mathcal{A}}{4\pi qm}} D, \qquad a^\dagger = \sqrt{\frac{\mathcal{A}}{4\pi qm}} D^\dagger, \tag{2·24}$$

which satisfy $[a, a^\dagger] = 1$. Then, the eigenvalues of the Laplace operator Δ are given as

$$\lambda_n = 2\pi \frac{qm}{\mathcal{A}}(2n+1), \tag{2·25}$$

and eigenvalues m_n^2 are also written as

$$m_n^2 = 4\pi \frac{qm}{\mathcal{A}} n. \tag{2·26}$$

The corresponding wavefunctions ψ_n are written by

$$\frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_{+,0}^{j.qm}. \tag{2·27}$$

Explicitly, the wavefunctions of massive modes are written as

$$\psi_n^{j,qm} = \frac{(2mq\mathrm{Im}\tau)^{1/4}}{(2^n n!\mathcal{A})^{1/2}} \sum_l \Theta_\ell^{j,qm}(z+\zeta,\tau)$$
$$\times H_n\left(\sqrt{2\pi qm\mathrm{Im}\tau}\left(\frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau}+\frac{j}{qm}+\ell\right)\right), \qquad (2\cdot28)$$

where $H_n(x)$ is the Hermite function. Massive spectra of $\psi_{+,n}$ and $\psi_{-,n}$ are the same and each number of them is equal to qm. Note that $\psi_n^{j,qm}$ satisfy the boundary conditions (2·15). Also, these wavefunctions satisfy the following orthonormal conditions,

$$\int_{T^2} dz d\bar{z} \psi_n^{j,qm} (\psi_\ell^{k,qm})^* = \delta_{jk}\delta_{n\ell}. \qquad (2\cdot29)$$

2.1.3. Scalar and vector modes

Here, we study the scalar and vector modes on T^2. The scalar fields are expanded as eigenfunctions of the Laplace operator,

$$\Delta\phi_n(z) = m_n^2 \phi_n(z). \qquad (2\cdot30)$$

That is, these eigenvalues are obtained as λ_n, i.e.,

$$m_n^2 = \lambda_n = 2\pi\frac{qm}{\mathcal{A}}(2n+1), \qquad (2\cdot31)$$

for the scalar field. All of them including the lightest mode with $n=0$ are massive. Eigenfunctions are the same as those for the fermion, i.e. $\psi_n^{j,qm}$ in Eq. (2·28).

Next, we study the vector field on T^2. We are interested in the charged vector field with the $U(1)$ charge q, where $q \neq 0$.*) For example, they correspond to the $W^\pm$ vector bosons in the $SU(2)$ gauge theory. We decompose the vector fields as Eq. (2·1). From Eq. (B·11) in Appendix B, the mass-squared matrix is written by

$$\mathcal{M}^2 = \begin{pmatrix} \Delta & -i4\pi\frac{qm}{\mathcal{A}} \\ i4\pi\frac{qm}{\mathcal{A}} & \Delta \end{pmatrix}, \qquad (2\cdot32)$$

in the real basis of the 2D vector field $(\phi_{n,1},\phi_{n,2})$. Instead of the real basis, we use the complex basis,

$$\phi_{n,z} = \frac{1}{\sqrt{2}}(\phi_{n,1}+i\phi_{n,2}), \qquad \phi_{n,\bar{z}} = \frac{1}{\sqrt{2}}(\phi_{n,1}-i\phi_{n,2}). \qquad (2\cdot33)$$

The mass spectra of these internal wavefunctions are obtained through solving the following equations,

$$\left(\Delta - \frac{4\pi qm}{\mathcal{A}}\right)\phi_{n,z} = m_n^2\phi_{n,z}, \qquad (2\cdot34)$$
$$\left(\Delta + \frac{4\pi qm}{\mathcal{A}}\right)\phi_{n,\bar{z}} = m_n^2\phi_{n,\bar{z}}. \qquad (2\cdot35)$$

*) Obviously, there is no effect due to magnetic fluxes in the neutral vector fields with $q=0$.

That is, the mass spectrum of $\phi_{n,z}$ is obtained as

$$m_n^2 = \lambda_n - \frac{4\pi qm}{\mathcal{A}} = 2\pi \frac{qm}{\mathcal{A}}(2n-1), \tag{2·36}$$

while the mass spectrum of $\phi_{n,\bar{z}}$ is obtained as

$$m_n^2 = \lambda_n + \frac{4\pi qm}{\mathcal{A}} = 2\pi \frac{qm}{\mathcal{A}}(2(n+1)+1). \tag{2·37}$$

The spectrum of $\phi_{n,z}$ includes the tachyonic mode for $n = 0$, while all modes of $\phi_{n,\bar{z}}$ are massive. Their wavefunctions $\phi_{n,z}$ and $\phi_{n,\bar{z}}$ are the same as those for the fermion, i.e. $\psi_n^{j,qm}$ in Eq. (2·28).

2.1.4. Massive modes only due to Wilson lines

The massive modes also appear only due to non-vanishing Wilson lines α without magnetic fluxes. For completeness, we show their mass spectrum and wavefunctions. The internal wavefunctions for the spinor field as well as the scalar and vector fields satisfy the same boundary condition as Eq. (2·15) with

$$\chi_1 = \frac{\pi}{\mathrm{Im}\tau}\mathrm{Im}\alpha, \qquad \chi_2 = \frac{\pi}{\mathrm{Im}\tau}\mathrm{Im}\bar{\tau}\alpha. \tag{2·38}$$

Then, the wavefunctions satisfying this boundary condition are obtained as

$$\psi^{(W)}_{n_R,n_I}(z) = \frac{1}{\sqrt{\mathcal{A}}}\exp\left[i\pi\left(\frac{\mathrm{Im}\alpha}{\mathrm{Im}\tau}+2n_R\right)\mathrm{Re}z + i\pi\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\left(-\mathrm{Re}\alpha + 2(n_I - \mathrm{Re}\tau n_R)\right)\right], \tag{2·39}$$

where n_R and n_I are integers. Their masses are given as

$$\begin{aligned} m^2_{n_R,n_I} = \frac{4\pi^2\mathrm{Im}\tau}{\mathcal{A}}\Bigg[&\left(\frac{\mathrm{Im}\alpha}{\mathrm{Im}\tau}+n_R\right)^2 \\ &+\left(\frac{1}{\mathrm{Im}\tau}\right)^2\left(-\mathrm{Re}\alpha + (n_I - \mathrm{Re}\tau n_R)\right)^2\Bigg]. \end{aligned} \tag{2·40}$$

2.2. T^6

Here we study the field theory on $(T^2)^3$. It is straightforward to extend the analyses on T^2 and $(T^2)^3$ to one on $(T^2)^2$. We use the complex basis, $z^i = y^{2i-1} + \tau^i y^{2i}$ with $i = 1, 2, 3$ on the i-th T^2, and the metric is written by

$$ds^2 = \sum_i 2(2\pi R^i)^2 dz^i d\bar{z}^i. \tag{2·41}$$

We identify the complex coordinate as $z^i \sim z^i + 1$ and $z^i \sim z^i + \tau^i$, and the area on the i-th T^2 is written by $\mathcal{A}^i = 4\pi^2(R^i)^2\mathrm{Im}\tau^{\mathrm{i}}$.

We introduce the $U(1)$ magnetic flux on the i-th T^2 as

$$F_{z^i\bar{z}^i} = \frac{\pi i}{\mathrm{Im}\tau^{\mathrm{i}}}m^i, \tag{2·42}$$

where qm^i is integer. This magnetic flux is derived from the following vector potential,

$$A_{\bar{z}^i} = \frac{\pi}{2\mathrm{Im}\tau^i} m^i z^i, \qquad A_{z^i} = -\frac{\pi}{2\mathrm{Im}\tau^i} m^i \bar{z}^i. \tag{2·43}$$

We also introduce the Wilson line on the i-th T^2,

$$\alpha^i = m^i \zeta^i. \tag{2·44}$$

Obviously, the mass spectrum and wavefunctions on each T^2 are given as those in §2.1. The full eigenfunctions are the products of the eigenfunctions for the n_i-th modes on the i-th T^2, and the full mass squared is the sum of masses squared for each T^2. The numbers of massless fermions are obtained as $\prod_i qm^i$. The scalar field on T^6 is always massive. The vector field ϕ_{z^r} along the r-th (complex) direction on T^6 has the lowest mass squared with $n^i = 0$ $(i = 1, 2, 3)$ as

$$m^2 = 2\pi q \left(\sum_{i \neq r} \frac{m^i}{\mathcal{A}^i} - \frac{m^r}{\mathcal{A}^r} \right). \tag{2·45}$$

For example, when $m^2/\mathcal{A}^2 + m^3/\mathcal{A}^3 - m^1/\mathcal{A}^1 = 0$, the massless mode appears in ϕ_{z^1}. When $m^2/\mathcal{A}^2 + m^3/\mathcal{A}^3 - m^1/\mathcal{A}^1$ is positive (negative), it becomes massive (tachyonic).

§3. Couplings including massive modes

Here, we study couplings including zero-modes and higher modes in 4D low-energy effective field theory. The 3-point couplings among zero-modes are computed in 1), 20), and higher order couplings among zero-modes are studied in 14). First we briefly review them in §3.1, and extend to 3-point and higher order couplings including higher modes in §3.2. In §3.3, we also consider the couplings including massive modes due to only the Wilson line effect, but not magnetic fluxes.

3.1. *Couplings among zero-modes*

Here we concentrate on the T^2 theory. We consider the coupling among three zero-modes, whose wavefunctions are given as $\psi^{i,q_1m_1}(z+\zeta_1,\tau)$, $\psi^{i,q_2m_2}(z+\zeta_2,\tau)$ and $(\psi^{i,q_3m_3}(z+\zeta_3,\tau))^*$. They have $U(1)$ charges, q_1, q_2 and q_3, respectively, and the magnetic fluxes, m_1, m_2 and m_3 appear in their zero-mode equations. We use the notation, $N_1 = q_1m_1$, $N_2 = q_2m_2$ and $N_3 = q_3m_3$. We assume that $N_1, N_2, N_3 \neq 0$. The gauge invariance requires that $q_1 + q_2 = q_3$, $N_1 + N_2 = N_3$ and $N_1\zeta_1 + N_2\zeta_2 = N_3\zeta_3$. Their 3-point coupling in the 4D low-energy effective field theory is given by the following integral of wavefunctions,

$$y^{ij\bar{k}} = \int d^2z \psi^{i,N_1}\psi^{j,N_2}(\psi^{k,N_3})^*, \tag{3·1}$$

up to the 3-point coupling constant in higher dimensional field theory. Hereafter, we concentrate on the part given as the overlap integral of wavefunctions, omitting the coupling constants in higher dimensions. For the Yukawa coupling, two of these modes correspond to the spinor fields, and the other corresponds to the 4D scalar field. The 4D scalar may be originated from the higher dimensional vector, e.g. on T^6, if the 4D scalar is massless. At any rate, the wavefunctions are the same among the spinor, scalar and vector fields. Thus, we compute the 3-point and higher order couplings without specifying such Lorentz transformation behaviors. However, note that the Lorentz invariance leads to a certain selection rule.

In the computation of the above integral, the important property of zero-mode wavefunctions is that they satisfy the following relation,

$$\psi^{i,N_1}(z_1,\tau)\cdot\psi^{j,N_2}(z_2,\tau) = \frac{1}{\sqrt{N_1+N_2}}\sum_{m=1}^{N_1+N_2}\psi^{i+j+N_1m,\ N_1+N_2}(X,\tau) \times \psi^{N_2i-N_1j+N_1N_2m,\ N_1N_2(N_1+N_2)}(Y,\tau), \tag{3·2}$$

where

$$X = \frac{N_1z_1+N_2z_2}{N_1+N_2}, \qquad Y = \frac{z_1-z_2}{N_1+N_2}, \tag{3·3}$$

as shown in Appendix C (see also 1), 20)).

For example, when all of Wilson lines vanish, i.e. $z_1 = z_2 = z$, the above expansion becomes

$$\psi^{i,N_1}(z,\tau)\cdot\psi^{j,N_2}(z,\tau) = \left(\frac{2\mathrm{Im}\tau N_1N_2}{\mathcal{A}^2(N_1+N_2)}\right)^{1/4}\sum_{m=1}^{N_1+N_2}\psi^{i+j+N_1m,N_1+N_2}(z,\tau) \times\vartheta\begin{bmatrix}\frac{N_2i-N_1j+N_1N_2m}{N_1N_2(N_1+N_2)}\\ 0\end{bmatrix}(0,\tau N_1N_2(N_1+N_2)). \tag{3·4}$$

Then, by using the orthonormal condition (2·20), the 3-point coupling is obtained as

$$y^{ij\bar{k}} = \left(\frac{2\mathrm{Im}\tau N_1N_2}{\mathcal{A}^2(N_1+N_2)}\right)^{1/4}\sum_{m=1}^{N_1+N_2}\delta_{k,i+j+N_1m} \times\vartheta\begin{bmatrix}\frac{N_2i-N_1j+N_1N_2m}{N_1N_2(N_1+N_2)}\\ 0\end{bmatrix}(0,\tau N_1N_2(N_1+N_2)). \tag{3·5}$$

There is the selection rule for allowed couplings as

$$k = i+j. \qquad (\mathrm{mod}\ \ N_1) \tag{3·6}$$

Similarly, we can calculate the 3-point coupling for non-vanishing Wilson lines. Its result leads to the 3-point couplings,

$$y^{ij\bar{k}} = \left(\frac{2\mathrm{Im}\tau N_1N_2}{\mathcal{A}^2(N_1+N_2)}\right)^{1/4}\sum_{m=1}^{N_1+N_2}\delta_{k,i+j+N_1m}e^{i\pi(N_1\zeta_1\mathrm{Im}\zeta_1+N_2\zeta_2\mathrm{Im}\zeta_2-N_3\zeta_3\mathrm{Im}\zeta_3)/\mathrm{Im}\tau} \times\vartheta\begin{bmatrix}\frac{N_2i-N_1j+N_1N_2m}{N_1N_2(N_1+N_2)}\\ 0\end{bmatrix}(N_1N_2(\zeta_1-\zeta_2),\tau N_1N_2(N_1+N_2)). \tag{3·7}$$

Next, we consider the 4-point couplings,

$$y^{ijk\bar{\ell}} = \int d^2z \psi^{i,N_1} \psi^{j,N_2} \psi^{k,N_3} (\psi^{\ell,N_4})^*, \tag{3·8}$$

where the gauge invariance requires $N_1 + N_2 + N_3 = N_4$. For simplicity we consider the case that all of Wilson lines vanish, but it is straightforward to extend to the case with non-vanishing Wilson lines. The direct computation is possible by using the relation (3·4). However, the following calculation is much simpler.[14] We write the above integral

$$y^{ijk\bar{\ell}} = \int d^2z d^2z' \psi^{i,N_1}(z) \psi^{j,N_2}(z) \delta^2(z - z') \psi^{k,N_3}(z') (\psi^{\ell,N_4}(z')))^*. \tag{3·9}$$

We replace the δ function by

$$\delta^2(z - z') = \sum_{s,n} (\psi_n^{s,N_1+N_2}(z))^* \psi_n^{s,N_1+N_2}(z'), \tag{3·10}$$

which is the summation over the complete set corresponding to eigenfunctions for the magnetic flux $N_1 + N_2$. This summation includes higher modes, i.e. $n \neq 0$. Then we can write

$$\begin{aligned} y^{ijk\bar{\ell}} = \sum_{s,n} & \left(\int d^2z \psi^{i,N_1}(z) \psi^{j,N_2}(z) (\psi_n^{s,N_1+N_2}(z))^* \right) \\ & \times \left(\int d^2z' \psi_n^{s,N_1+N_2}(z') \psi^{k,N_3}(z') (\psi^{\ell,N_4}(z')))^* \right). \end{aligned} \tag{3·11}$$

Using the 3-point coupling among zero-modes, the above integral can be obtained as

$$y^{ijk\bar{\ell}} = \sum_s y^{ij\bar{s}} y^{sj\bar{k}}. \tag{3·12}$$

The higher modes $n \neq 0$ do not appear in this summation, because only zero-mode modes $n = 0$ appear on the RHS of Eq. (3·4).

Instead of Eq. (3·9), there is another way to split the integral, e.g.,

$$y^{ijk\bar{\ell}} = \int d^2z d^2z' \psi^{i,N_1}(z) \psi^{j,N_2}(z) \delta^2(z - z') \psi^{k,N_3}(z') (\psi^{\ell,N_4}(z')))^*. \tag{3·13}$$

Then, by replacing the δ function by

$$\delta^2(z - z') = \sum_{t,n} (\psi_n^{t,N_2+N_3}(z))^* \psi_n^{t,N_2+N_3}(z'), \tag{3·14}$$

we obtain

$$y^{ijk\bar{\ell}} = \sum_t y^{jk\bar{t}} y^{si\bar{\ell}}. \tag{3·15}$$

We can show that both of Eqs. (3·12) and (3·15) lead to the same result.[14]

Similarly, we can calculate another type of the 4-point coupling,

$$y^{ij\bar{k}\bar{\ell}} = \int d^2z \psi^{i,N_1}\psi^{j,N_2}(\psi^{k,N_3})^*(\psi^{\ell,N_4})^*, \tag{3·16}$$

where the gauge invariance requires $N_1 + N_2 = N_3 + N_4$. Furthermore, the integrals for 5-point and higher order couplings can be carried out in a similar analysis, and they are written by the proper summations over products of 3-point couplings.

3.2. *Couplings including higher modes*

Here, we study couplings including higher modes. The relation (3·2) among zero-mode wavefunctions plays an important role in the computation of the 3-point couplings for zero-modes. When we operate $(\partial_{z_1} - \frac{\pi N_1}{2\mathrm{Im}\tau}\bar{z}_1)^{n_1}(\partial_{z_2} - \frac{\pi N_2}{2\mathrm{Im}\tau}\bar{z}_2)^{n_2}$ on the LHS of Eq. (3·2), we obtain

$$\begin{aligned}&\left(\partial_{z_1} - \frac{\pi N_1}{2\mathrm{Im}\tau}\bar{z}_1\right)^{n_1}\left(\partial_{z_2} - \frac{\pi N_2}{2\mathrm{Im}\tau}\bar{z}_2\right)^{n_2}\psi_0^{i,N_1}(z_1,\tau)\cdot\psi_0^{j,N_2}(z_2,\tau)\\ &= \sqrt{n_1!n_2!\left(\frac{\pi N_1}{\mathrm{Im}\tau}\right)^{n_1}\left(\frac{\pi N_2}{\mathrm{Im}\tau}\right)^{n_2}}\psi_{n_1}^{i,N_1}(z_1,\tau)\cdot\psi_{n_2}^{j,N_2}(z_2,\tau).\end{aligned} \tag{3·17}$$

On the other hand, when we operate $(\partial_{z_1} - \frac{\pi N_1}{2\mathrm{Im}\tau}\bar{z}_1)^{n_1}(\partial_{z_2} - \frac{\pi N_2}{2\mathrm{Im}\tau}\bar{z}_2)^{n_2}$ on the RHS of Eq. (3·2), we obtain

$$\begin{aligned}&\frac{1}{\sqrt{N_1+N_2}}\sum_{m=1}^{N_1+N_2}\sum_{\ell=0}^{n_1}\sum_{s=0}^{n_2} {}_{n_1}\mathrm{C}_\ell\, {}_{n_2}\mathrm{C}_s \frac{(-1)^{n_2-s}N_1^\ell N_2^s}{(N_1+N_2)^{n_1+n_2}}\\ &\times\left(\partial_X - \frac{\pi}{2\mathrm{Im}\tau}(N_1+N_2)\bar{X}\right)^{\ell+s}\psi_0^{i+j+N_1m,N_1+N_2}(X,\tau)\\ &\times\left(\partial_Y - \frac{\pi}{2\mathrm{Im}\tau}N_1N_2(N_1+N_2)\bar{Y}\right)^{n_1+n_2-\ell-s}\psi_0^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}(Y,\tau)\\ &= \frac{1}{\sqrt{N_1+N_2}}\sum_{m=1}^{N_1+N_2}\sum_{\ell=0}^{n_1}\sum_{s=0}^{n_2} {}_{n_1}\mathrm{C}_\ell\, {}_{n_2}\mathrm{C}_s \frac{(-1)^{n_2-s}N_1^\ell N_2^s}{(N_1+N_2)^{(n_1+n_2)/2}}\left(\frac{\pi}{\mathrm{Im}\tau}\right)^{(n_1+n_2)/2}\\ &\times(N_1N_2)^{(n_1+n_2-\ell-s)/2}\sqrt{(n_1+n_2-\ell-s)!(\ell+s)!}\\ &\times\psi_{l+s}^{i+j+N_1m,N_1+N_2}(X,\tau)\cdot\psi_{n_1+n_2-\ell-s}^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}(Y,\tau),\end{aligned} \tag{3·18}$$

by using the derivatives with respect of X and Y. By identifying Eqs. (3·17) and (3·18), the product of higher modes, $\psi_{n_1}^{i,N_1}(z_1,\tau)$ and $\psi_{n_2}^{j,N_2}(z_2,\tau)$, is expanded as*)

$$\begin{aligned}\psi_{n_1}^{i,N_1}(z_1,\tau)\cdot\psi_{n_2}^{j,N_2}(z_2,\tau) &= \sum_{m=1}^{N_1+N_2}\sum_{\ell=0}^{n_1}\sum_{s=0}^{n_2} {}_{n_1}\mathrm{C}_\ell\, {}_{n_2}\mathrm{C}_s (-1)^{n_2-s}\frac{N_1^{(n_2+\ell-s)/2}N_2^{(n_1-\ell+s)/2}}{(N_1+N_2)^{(n_1+n_2+1)/2}}\\ &\times\sqrt{\frac{(\ell+s)!(n_1+n_2-\ell-s)!}{n_1!n_2!}}\psi_{\ell+s}^{i+j+N_1m,N_1+N_2}(X,\tau)\\ &\times\psi_{n_1+n_2-\ell-s}^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}(Y,\tau).\end{aligned} \tag{3·19}$$

*) A similar relation has been derived in twisted tori.[16)]

When we take $z_1 = z + \zeta_1$ and $z_2 = z + \zeta_2$, it is found that

$$\begin{aligned}
&\psi^{i,N_1}_{n_1}(z+\zeta_1,\tau)\cdot\psi^{j,N_2}_{n_2}(z+\zeta_2,\tau) \\
&= \sum_{m=1}^{N_1+N_2}\sum_{\ell=0}^{n_1}\sum_{s=0}^{n_2} {}_{n_1}\mathrm{C}_\ell\ {}_{n_2}\mathrm{C}_s (-1)^{n_2-s}\frac{N_1^{(n_2+\ell-s)/2}N_2^{(n_1-\ell+s)/2}}{(N_1+N_2)^{(n_1+n_2+1)/2}} \\
&\times\sqrt{\frac{(\ell+s)!(n_1+n_2-\ell-s)!}{n_1!n_2!}}\cdot\psi^{i+j+N_1m,N_1+N_2}_{\ell+s}(z+\zeta_3,\tau) \\
&\times\psi^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}_{n_1+n_2-\ell-s}\left(\frac{\zeta_1-\zeta_2}{N_1+N_2},\tau\right). \qquad (3{\cdot}20)
\end{aligned}$$

Note that the last factor, $\psi^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}_{n_1+n_2-\ell-s}(\frac{\zeta_1-\zeta_2}{N_1+N_2},\tau)$, is constant.

Using the above relation, we can compute the 3-point coupling,

$$y^{ij\bar{k}}_{n_1n_2n_3} = \int dzd\bar{z}\psi^{i,N_1}_{n_1}(z+\zeta_1,\tau)\cdot\psi^{j,N_2}_{n_2}(z+\zeta_2,\tau)\cdot(\psi^{k,N_3}_{n_3}(z+\zeta_3,\tau))^*, \qquad (3{\cdot}21)$$

in a way similar to the 3-point coupling among the zero-modes. The result is obtained as

$$\begin{aligned}
y^{ij\bar{k}}_{n_1n_2n_3} &= \sum_{m=1}^{N_1+N_2}\sum_{\ell=0}^{n_1}\sum_{s=0}^{n_2} {}_{n_1}\mathrm{C}_\ell\ {}_{n_2}\mathrm{C}_s(-1)^{n_2-s} \\
&\times\sqrt{\frac{N_1^{n_2+\ell-s}N_2^{n_1-\ell+s}}{(N_1+N_2)^{n_1+n_2+1}}\frac{(\ell+s)!(n_1+n_2-\ell-s)!}{n_1!n_2!}} \\
&\times\psi^{N_2i-N_1j+N_1N_2m,\ N_1N_2(N_1+N_2)}_{n_1+n_2-\ell-s}\left(\frac{\zeta_1-\zeta_2}{N_1+N_2},\tau\right)\delta_{\ell+s,n_3}\delta_{k,i+j+N_1m} \\
&= \sum_{\ell=max(0,n_3-n_2)}^{min(n_1,n_3)} {}_{n_1}\mathrm{C}_\ell\ {}_{n_2}\mathrm{C}_{n_3-\ell}(-1)^{n_2-n_3-\ell} \\
&\times\sqrt{\frac{N_1^{n_2-n_3+2\ell}N_2^{n_1+n_3-2\ell}}{(N_1+N_2)^{n_1+n_2+1}}\frac{n_3!(n_1+n_2-n_3)!}{n_1!n_2!}} \\
&\times\psi^{N_2k-N_3j,\ N_1N_2(N_1+N_2)}_{n_1+n_2-n_3}\left(\frac{\zeta_1-\zeta_2}{N_1+N_2},\tau\right). \qquad (3{\cdot}22)
\end{aligned}$$

There is the selection rule among i, j and k, which is the same as one for the zero-modes (3·6). Thus, the flavor symmetry appearing only in zero-modes is still exact even when we take into account the effects due to higher modes. In addition, the following relation,

$$n_3 \le n_1 + n_2, \qquad (3{\cdot}23)$$

should be satisfied for the mode numbers, n_1, n_2 and n_3. For example, two zero-modes, $n_1 = n_2 = 0$, can couple with only the zero mode $n_3 = 0$. On the other hand, the two zero-modes, $n_1 = n_3 = 0$, can couple with higher modes, $n_2 \neq 0$, and its coupling is determined by

$$\frac{1}{\sqrt{N_1+N_2}}\left(\frac{N_1}{N_1+N_2}\right)^{n_2/2}\psi_{n_2}^{N_2k-N_3j,\ N_1N_2N_3}\left(\frac{\zeta_1-\zeta_2}{N_1+N_2},\tau\right). \tag{3·24}$$

Similarly, we can compute the 4-point coupling,

$$\begin{aligned} &y^{ijk\bar{\ell}}_{n_1n_2n_3n_4} \\ &= \int dzd\bar{z}\psi_{n_1}^{i,N_1}(z+\zeta_1,\tau)\cdot\psi_{n_2}^{j,N_2}(z+\zeta_2,\tau)\cdot\psi_{n_3}^{k,N_3}(z+\zeta_3,\tau)\cdot(\psi_{n_4}^{\ell,N_4}(z+\zeta_4,\tau))^*. \end{aligned} \tag{3·25}$$

We rewrite it as

$$\begin{aligned} &\int d^2zd^2z'\psi_{n_1}^{i,N_1}(z+\zeta_1,\tau)\cdot\psi_{n_2}^{j,N_2}(z+\zeta_2,\tau)\delta^2(z-z')\psi_{n_3}^{k,N_3}(z'+\zeta_3,\tau) \\ &\times(\psi_{n_4}^{\ell,N_4}(z'+\zeta_4,\tau))^*, \end{aligned} \tag{3·26}$$

and replace the δ function by

$$\delta^2(z-z') = \sum_{n,s}\psi_n^{s,N_1+N_2}(z+\zeta,\tau)(\psi_n^{s,N_1+N_2}(z'+\zeta,\tau))^*. \tag{3·27}$$

Then, the 4-point coupling is given as the summation over products of 3-point couplings,

$$y^{ijk\bar{\ell}}_{n_1n_2n_3n_4} = \sum_{n,s} y^{ij\bar{s}}_{n_1n_2n}\ y^{sk\bar{\ell}}_{nn_3n_4}. \tag{3·28}$$

Similarly, we can compute other higher order couplings by products of the 3-point couplings.

3.3. *Couplings including massive modes only due to Wilson lines*

In the previous section, we have considered the couplings including higher modes under the magnetic flux. Here, we consider the couplings including massive modes only due to the Wilson line. The wavefunctions of such modes are obtained in Eq. (2·39). We compute the following 3-point couplings among two zero-modes $\psi^{j,N_1}(z+\zeta_1,\tau)$ and $(\psi^{k,N_2}(z+\zeta_2,\tau))^*$ and the massive mode $\psi^{(W)}_{n_R,n_I}(z)$, where the two zero-modes have non-vanishing magnetic flux, while the massive mode has no magnetic flux, but the Wilson line. Here, the gauge invariance requires that $N_1 = N_2$ and the Wilson line α_3 of the massive mode $\psi^{(W)}_{n_R,n_I}(z)$ satisfies $N_1\zeta_1 + \alpha_3 = N_1\zeta_2$. Then, the 3-point coupling among these modes is given by the following integral,

$$y^{j,\bar{k}}_{(W)n_Rn_I} = \int dzd\bar{z}\psi^{j,N_1}(z+\zeta_1,\tau)(\psi^{k,N_1}(z+\zeta_2,\tau))^*\psi^{(W)}_{n_R,n_I}(z). \tag{3·29}$$

More explicitly, the integral is written by

$$\begin{aligned}
&y^{j,\bar{k}}_{(W)n_R n_I} \\
&= \frac{\sqrt{2N_1\mathrm{Im}\tau}}{\mathcal{A}^{3/2}}\int dzd\bar{z}\sum_{\ell,n}\exp\left[\frac{i\pi}{N_1\mathrm{Im}\tau}\{(N_1z+\alpha_1)\mathrm{Im}(N_1z+\alpha_1)\right. \\
&-(N_1\bar{z}+\bar{\alpha}_2)\mathrm{Im}(N_1z+\alpha_2)\}+\pi i\left(\frac{j}{N_1}+\ell\right)^2N_1\tau-\pi i\left(\frac{k}{N_1}+n\right)^2N_1\bar{\tau} \\
&+2\pi i\left\{(N_1z+\alpha_1)\left(\frac{j}{N_1}+\ell\right)-(N_1\bar{z}+\bar{\alpha}_2)\left(\frac{k}{N_1}+n\right)\right\} \\
&\left.+\pi i\left\{\mathrm{Re}z\left(\frac{\mathrm{Im}\alpha_3}{\mathrm{Im}\tau}+2n_R\right)+\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\left(-\mathrm{Re}\alpha_3+2(n_I-n_R\mathrm{Re}\tau)\right)\right\}\right]. \qquad (3{\cdot}30)
\end{aligned}$$

The integral over Rez imposes $j=k-n_R$ and $\ell=n$. In addition, the integral over Imz is Gaussian-like. By lengthy computation, it is found that

$$\begin{aligned}
y^{j,\bar{k}}_{(W)n_R n_I} &= \frac{1}{\sqrt{\mathcal{A}}}\exp\left[-\frac{\pi}{2N_1\mathrm{Im}\tau}\{(\mathrm{Im}\alpha_3+n_R\mathrm{Im}\tau)^2+(\mathrm{Re}\alpha_3-n_I+n_R\mathrm{Re}\tau)^2)\}\right. \\
&\left.+i\frac{\pi}{N_1\mathrm{Im}\tau}\left(\mathrm{Im}\bar{\alpha}_2\alpha_1+n_R\mathrm{Im}\bar{\tau}(\alpha_1+\alpha_2)+n_Rn_I\mathrm{Im}\tau-n_I\mathrm{Im}(\alpha_1+\alpha_2)\right)\right]. \qquad (3{\cdot}31)
\end{aligned}$$

This behaves as a Gaussian function for the Wilson line α_3. Thus, this coupling is suppressed depending on the Wilson line as well as n_R and n_I. The mode with the strongest coupling $|y^{j,\bar{k}}_{(W)n_Rn_I}|$ corresponds to the mode with $n_R=n_I=0$, when

$$-\frac{1}{2}\le\frac{\mathrm{Im}\alpha_3}{\mathrm{Im}\tau}\le\frac{1}{2},\qquad -\frac{1}{2}\le\mathrm{Re}\alpha_3\le\frac{1}{2}. \qquad (3{\cdot}32)$$

For other values of α_3, another mode with non-vanishing n_R and/or n_I would have the strongest coupling. For example, for $n_R=n_I=0$, we have

$$|y^{j,\bar{k}}_{(W)n_R=n_I=0}|=\frac{1}{\sqrt{\mathcal{A}}}\exp\left[-\frac{\pi|\alpha_3|^2}{2N_1\mathrm{Im}\tau}\right]. \qquad (3{\cdot}33)$$

This coupling is suppressed depending on $|\alpha_3|^2$. For example, when $|\alpha_3|^2/2N_1\mathrm{Im}\tau=1$, we obtain $|y^{j,\bar{k}}_{(W)n_R=n_I=0}|\approx e^{-\pi}\approx 0.04$. The couplings to other modes with $n_R,n_I\neq 0$ are much more suppressed for the value of α_3, which satisfy Eq. (3·32).

Similarly, we can compute the 3-point coupling among two higher modes, $\psi^{j,N_1}_{n_1}(z+\zeta_1,\tau)$ and $(\psi^{k,N_1}_{n_2}(z+\zeta_2,\tau))^*$ and the massive mode $\psi^{(W)}_{n_R,n_I}(z)$, where the first two modes have non-vanishing magnetic flux, while the last mode has no magnetic flux, but the Wilson line. We assume that $n_1\le n_2$. Such a coupling is obtained as

$$y^{j,\bar{k}}_{n_1n_2(W)n_Rn_I}=\int dzd\bar{z}\psi^{j,N_1}_{n_1}(z+\zeta_1,\tau)(\psi^{k,N_1}_{n_2}(z+\zeta_2,\tau))^*\psi^{(W)}_{n_R,n_I}. \qquad (3{\cdot}34)$$

Again, the integral over Rez imposes $j = k - n_R$ and $\ell = n$. For the integral over Rez, we use Eq. (A·6). Then, the result is written by

$$y^{j,\bar{k}}_{n_1 n_2 (W) n_R n_I} = \frac{y^{j,\bar{k}}_{(W) n_R n_I}}{\sqrt{2^{n_1+n_2} n_2!}} \sum_{k=0}^{n_1} 2^k \frac{(n_1!)^{3/2}}{(k!)^2 (n_1-k)!} \times \left(\sqrt{2\pi N_1 \mathrm{Im}\tau} \left(-\frac{n_R}{N_1} + \frac{\mathrm{Im}\alpha_3}{N_1 \mathrm{Im}\tau} \right) \right)^{n_1+n_2-k}. \qquad (3·35)$$

These couplings include the same suppression factor, $y^{j,\bar{k}}_{(W) n_R n_I}$.

Higher order couplings can be computed similarly. When we consider higher order couplings including more modes such as $\psi^{\ell,N}_n(\tau, z+\zeta)$, we use the technique such as Eqs. (3·26) and (3·27). When we consider higher order coupling including more modes such as $\psi^{(W)}_{n_R,n_I}(z)$, we use the property that the product of two wavefunctions $\psi^{(W)}_{n_R,n_I}(z)$ and $\psi^{(W)}_{m_R,m_I}(z)$ is obtained as $\psi^{(W)}_{n_R+m_R,n_I+m_I}(z)$, and that the Wilson line of $\psi^{(W)}_{n_R+m_R,n_I+m_I}(z)$ is just the sum of two Wilson lines, which $\psi^{(W)}_{n_R,n_I}(z)$ and $\psi^{(W)}_{m_R,m_I}(z)$ have. Using these, we can compute higher order couplings.

§4. Phenomenological comments

We have calculated the couplings among zero-modes and higher modes. They have various important implications from the phenomenological viewpoints. Here, we give some comments.

The first example is about the proton decay. For instance, the proton decay would happen through the heavy X boson in the $SU(5)$ GUT model. It couples with quarks and leptons by the gauge coupling before the gauge symmetry breaking. This coupling does not change in the 4D GUT theory even after the $SU(5)$ group is broken. However, it can change in extra dimensional models, which have been discussed so far. Let us consider the $SU(5) \times U(1)$ GUT model with extra space, T^2 or T^6. We introduce non-vanishing magnetic flux m along the extra $U(1)$ direction. Suppose that the $\bar{\mathbf{5}}$ matter field has a charge q under the extra $U(1)$ symmetry. Before $SU(5)$ breaking, both of the quark and lepton in $\bar{\mathbf{5}}$ are quasi-localized at the same place, and their coupling to the X boson is given by the gauge coupling. Then, we assume the non-vanishing Wilson line α along the $U(1)_Y$ direction in $SU(5)$. It breaks the $SU(5)$ gauge symmetry, the X boson becomes massive and its profile is written by $\psi^{(W)}_{n_R=n_I=0}(z)$ in Eq. (2·39). The quark and lepton in $\bar{\mathbf{5}}$ are still massless, but their profiles split each other, because of Wilson lines. In this case, the coupling among the quark, lepton and the X heavy boson is not equal to the gauge coupling, but it includes the suppression factor, $|y^{j,\bar{k}}_{(W) n_R=n_I=0}|$, as computed in the previous section. That is important to avoid the fast proton decay. For example, when $|\alpha_3|^2/2N_1\mathrm{Im}\tau = 1$, we obtain $|y^{j,\bar{k}}_{(W) n_R=n_I=0}| \approx e^{-\pi} \approx 0.04$. Similarly, the coupling of the X boson with quarks and leptons in the $\mathbf{10}$ matter field can be suppressed. Then, the proton lifetime would drastically change by $\mathcal{O}(10^4 - 10^5)$.

Similarly, we can study the case that $SU(5)$ is broken by the magnetic flux along the $U(1)_Y$ direction, and the X boson becomes massive due to the magnetic flux. In this case, the coupling of the X boson with quark and lepton has the suppression factor given by Eq. (3·24).

Let us comment on another example. The Higgs mode gains its mass by the non-vanishing Wilson line in certain models. That is, the Higgs mode corresponds to the open string stretching between two parallel D-branes (on at least one T^2 of $(T^2)^3$) in the picture of intersecting D-brane models (see e.g. 21)). The Yukawa couplings between this Higgs field and massless matter fields include the factor, $|y^{j,\bar{k}}_{(W)n_R=n_I=0}|$. When the compactification scale is high such as the GUT scale and the Planck scale, this Wilson line α generating the Higgs mass is quite small, $\alpha \ll 1$, and the factor $|y^{j,\bar{k}}_{(W)n_R=n_I=0}|$ is of $\mathcal{O}(1)$. It is also important to see the moduli dependence of these couplings, that is, the dependence of the complex structure τ and the Wilson line α, which is the open string modulus. If F terms of complex structure and/or Wilson line moduli are non-vanishing, the corresponding A terms would appear and they are determined by the moduli dependence. At any rate, it is important to have found that the explicit moduli-dependence of these couplings, even though their values are of $\mathcal{O}(1)$.

We would need heavy right-handed neutrino masses for the seesaw mechanism. These masses may be generated by non-perturbative effects (see e.g. 22), 23)). Alternatively, the right-handed neutrino masses are generated by Wilson lines. If such a mass scale is comparable to the compactification scale, the couplings of the right-handed neutrino with the left-handed neutrino and the Higgs scalar would be suppressed.

Finally, we comment on the Kähler metric. The Kähler metric of the matter fields is diagonal in the flavor basis. However, they couple with massive modes. Such couplings may induce off-diagonal entries in the Kähler metric after integrating out massive modes. Such off-diagonal entries may lead to large FCNCs in the gravity-mediated supersymmetry breaking scenario.[19)] However, when those couplings among massles modes and massive modes are suppressed, off-diagonal entries would be small. We have shown that the selection rule for allowed couplings including higher modes is the same as the one for only zero-modes. Thus, if there is a non-Abelian discrete flavor symmetry in massless modes,[15),16)] that forbids the off-diagonal entries in the Kähler metric. Recall that such a symmetry is not violated by effects due to massive modes.

§5. Conclusion and discussion

We have studied the mass spectrum and wavefunctions of zero-modes and higher modes in extra dimensional models with magnetic fluxes and Wilson lines. Furthermore, we have computed 3-point couplings and higher order couplings included higher modes in the 4D low-energy effective field theory. These couplings have non-trivial behaviors, because wavefunctions of massless and massive modes are quite non-trivial. Using our results, we can write down the 4D low-energy effective field theory with the full modes. Higher modes do not violate the coupling selection rules among only zero-modes. Thus, the flavor symmetry for zero-modes remains exact even when we take into account the effects due to higher modes.

Our results would be important to phenomenological aspects, where couplings between massless and massive modes play a role, for example, the proton decay, the Higgs mass term, right-handed majorana neutrino mass term, FCNCs, etc. We will study in detail phenomenological applications of our results elsewhere. Threshold corrections and their moduli dependence after integrating out the massive modes would be important.

Acknowledgements

The authors would like to thank Hiroyuki Abe, Hiroshi Ohki and Manabu Sakai for useful discussions. The work of T. K. is supported in part by the Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Appendix A
—— Hermite Function ——

Here we show properties of the Hermite function, $H_n(x)$, which is defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{A·1}$$

Its derivative satisfies

$$\frac{d}{dx} H_n(x) = 2x H_n(x) - H_{n+1}(x), \tag{A·2}$$

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x). \tag{A·3}$$

The orthonormal conditions are written as

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \delta_{m,n} 2^n \sqrt{\pi} n!. \tag{A·4}$$

We compute the following integral,

$$I = \int_{-\infty}^{\infty} dx H_n(x+A) H_m(x+B) e^{-(x+A+B)^2}, \tag{A·5}$$

for $n \leq m$. This integral can be calculated as

$$\begin{aligned} I &= \int dx e^{-(x+A)^2} \frac{d^n}{dx^n} (e^{-2B(x+A)-B^2} H_m(x+B)) \\ &= \int dx e^{-(x+A)^2} \sum_{k=0}^{n} H_m^{(k)}(x+B)(-2B)^{n-k} e^{-2B(x+A)-B^2} {}_n\mathrm{C}_k \end{aligned}$$

$$= \sum_{k=0}^{n} 2^k {}_n\mathrm{C}_k \, {}_n\mathrm{P}_k (-2B)^{n-k} \int dx e^{-(x+A+B)^2} H_{m-k}^{(k)}(x+B)$$
$$= \sum_{k=0}^{n} 2^k {}_n\mathrm{C}_k \, {}_n\mathrm{P}_k (-2B)^{n-k} \int dx e^{-(x+B)^2} \frac{d^{m-k}}{dx^{m-k}} e^{-2A(x+B)-A^2}$$
$$= \sum_{k=0}^{n} 2^k {}_n\mathrm{C}_k \, {}_n\mathrm{P}_k (-2B)^{n-k} (-2A)^{m-k} \int dx e^{-(x+A+B)^2}$$
$$= \sum_{k=0}^{n} 2^k {}_n\mathrm{C}_k \, {}_n\mathrm{P}_k (-2B)^{n-k} (-2A)^{m-k} \sqrt{\pi}, \tag{A·6}$$

where

$$H_m^{(k)}(x) = \frac{d^k}{dx^k} H_m(x), \qquad {}_n\mathrm{C}_k = \frac{n!}{k!(n-k)!}, \qquad {}_n\mathrm{P}_k = \frac{n!}{k!}. \tag{A·7}$$

The following integral along the proper path,

$$\int_{-\infty}^{\infty} H_n(x+A) H_m(x+B) e^{-(x+A+B+iC)^2}, \tag{A·8}$$

leads to the same result as the above.

Appendix B
—— *Vector Field* ——

Here, we study the (4+2) dimensional $U(N)$ non-Abelian gauge theory (see also 1), 6)). Its Lagrangian is given as

$$\mathcal{L} = -\frac{1}{4g^2} \mathrm{Tr}\left(F^{MN} F_{MN} \right), \tag{B·1}$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]. \tag{B·2}$$

We compactify the two dimensions on T^2 with magnetic fluxes along $U(1)$ directions. We decompose the $U(1)$ parts B_N and off-diagonal parts W_M,

$$A_M = B_M + W_M = B_M^a U_a + W_M^{ab} e_{ab}, \tag{B·3}$$

with

$$(U_a)^i_j = \delta_{ai}\delta_{aj}, \qquad (e_{ab})_{ij} = \delta_{ai}\delta_{bj}, \quad (a \neq b) \tag{B·4}$$

where $W_M^{ab} = (W_M^{ba})^*$. The quadratic terms of W_M^{ab} in the Lagrangian are relevant to our study, and these appear

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{Tr}\left(D_M W_N D^M W^N - D_M W_N D^N W^M - i G_{MN} [W^M, W^N] \right) + \cdots, \tag{B·5}$$

where

$$G_{MN} = \partial_M B_N - \partial_N B_M, \tag{B·6}$$
$$D_M W_N = \partial_M W_N - i[B_N, W_N]. \tag{B·7}$$

Here, the ellipsis denotes irrelevant terms. Furthermore, these terms are written by

$$\mathcal{L} = \frac{i}{4g^2}\left(G^q_{ij} - G^b_{ij}\right)\left(W^{i,ab}W^{j,ba} - W^{j,ab}W^{i,ba}\right) - \frac{1}{2g^2}\Bigg[\left(D_\mu W^{ba}_i D^\mu W^{i,ab}\right) + \left(\tilde{D}_i W^{ba}_j \tilde{D}^i W^{j,ab}\right) - 2\left(\tilde{D}_i W^{ba}_\mu D^\mu W^{i,ab}\right) - \left(\tilde{D}_i W^{ba}_j \tilde{D}^j W^{i,ab}\right)\Bigg] + \cdots, \tag{B·8}$$

where

$$\tilde{D}_i W^{ab}_j = \partial_i W^{ab}_j - i(B^a_i - B^b_i)W^{ab}_j. \tag{B·9}$$

We expand

$$W^{ab}_i(x,y) = \sum_n \varphi^{ab}_{n,i}(x)\phi^{ab}_{n,i}(y). \tag{B·10}$$

Then, by imposing the gauge-fixing condition $\tilde{D}^i W^{ab}_i = 0$, the equation of motion in the internal space is written by

$$\tilde{D}_i \tilde{D}^i \phi^{ab}_{n,j} + 2i\langle G^{ab,i}_j\rangle \phi_{n,i} = -m^2_n \phi^{ab}_{j,n}. \tag{B·11}$$

Appendix C
—— *Theta Function Identities and Products of Zero-Mode Wavefunctions* ——

Here we study the product of theta functions and product of zero-mode wavefunctions (see also 1), 20)). The theta function satisfies the following identity,

$$\vartheta\begin{bmatrix}\frac{r}{N_1}\\ 0\end{bmatrix}(z_1, \tau N_1)\cdot\vartheta\begin{bmatrix}\frac{s}{N_2}\\ 0\end{bmatrix}(z_2, \tau N_2) = \sum_{m=1}^{N_1+N_2}\vartheta\begin{bmatrix}\frac{r+s+N_1 m}{N_1+N_2}\\ 0\end{bmatrix}(z_1+z_2, \tau(N_1+N_2)) \times\vartheta\begin{bmatrix}\frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2(N_1+N_2)}\\ 0\end{bmatrix}(z_1 N_2 - z_2 N_1, \tau N_1 N_2(N_1+N_2)). \tag{C·1}$$

Using this identity, we can derive the following relations among products of zero-mode wavefunctions,

$$\psi^{i,N_1}(z_1,\tau)\cdot\psi^{j,N_2}(z_2,\tau) = \frac{1}{\sqrt{N_1+N_2}}\sum_{m=1}^{N_1+N_2}\psi^{i+j+N_1 m, N_1+N_2}\left(\frac{N_1 z_1 + N_2 z_2}{N_1+N_2}, \tau\right) \times\psi^{N_2 i - N_1 j + N_1 N_2 m, N_1 N_2(N_1+N_2)}\left(\frac{z_1 - z_2}{N_1+N_2}, \tau\right). \tag{C·2}$$

Its proof is as follows. The LHS is explicitly written as

$$\psi^{i,N_1}(z_1,\tau)\cdot\psi^{j,N_2}(z_2,\tau) = \left(\frac{2\mathrm{Im}\tau\sqrt{N_1N_2}}{\mathcal{A}^2}\right)^{1/2}\exp\left[\frac{i\pi}{\mathrm{Im}\tau}(N_1z_1\mathrm{Im}z_1+N_2z_2\mathrm{Im}z_2)\right]$$
$$\times\,\vartheta\begin{bmatrix}\frac{i}{N_1}\\0\end{bmatrix}(N_1z_1,N_1\tau)\cdot\vartheta\begin{bmatrix}\frac{j}{N_2}\\0\end{bmatrix}(N_2z_2,N_2\tau), \tag{C·3}$$

and it can be rewritten by use of Eq. (C·1) as

$$\psi^{i,N_1}(z_1,\tau)\cdot\psi^{j,N_2}(z_2,\tau) = \left(\frac{2\mathrm{Im}\tau\sqrt{N_1N_2}}{\mathcal{A}^2}\right)^{1/2}\exp\left[\frac{i\pi}{\mathrm{Im}\tau}(N_1z_1\mathrm{Im}z_1+N_2z_2\mathrm{Im}z_2)\right]$$
$$\times\sum_{m=1}^{N_1+N_2}\vartheta\begin{bmatrix}\frac{i+j+N_1m}{N_1+N_2}\\0\end{bmatrix}(N_1z_1+N_2z_2,\tau(N_1+N_2))$$
$$\times\,\vartheta\begin{bmatrix}\frac{N_2i-N_1j+N_1N_2m}{N_1N_2(N_1+N_2)}\\0\end{bmatrix}(N_1N_2(z_1-z_2),\tau N_1N_2(N_1+N_2)). \tag{C·4}$$

Since the exponent part is written as

$$\frac{i\pi}{\mathrm{Im}\tau}(N_1z_1\mathrm{Im}z_1+N_2z_2\mathrm{Im}z_2) = \frac{i\pi}{\mathrm{Im}\tau}\left\{(N_1z_1+N_2z_2)\mathrm{Im}\left(\frac{N_1z_1+N_2z_2}{N_1+N_2}\right)\right.$$
$$\left.+\,N_1N_2(z_1-z_2)\mathrm{Im}\frac{z_1-z_2}{N_1+N_2}\right\}. \tag{C·5}$$

The RHS in Eq. (C·4) is the summation over the products of wavefunctions, $\psi^{i+j+N_1m,N_1+N_2}(\frac{N_1z_1+N_2z_2}{N_1+N_2},\tau)$ and $\psi^{N_2i-N_1j+N_1N_2m,N_1N_2(N_1+N_2)}(\frac{z_1-z_2}{N_1+N_2},\tau)$.

References

1) D. Cremades, L. E. Ibanez and F. Marchesano, J. High Energy Phys. **05** (2004), 079, hep-th/0404229.
2) H. Abe, T. Kobayashi and H. Ohki, J. High Energy Phys. **09** (2008), 043, arXiv:0806.4748.
H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, Nucl. Phys. B **814** (2009), 265, arXiv:0812.3534.
3) P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, J. High Energy Phys. **03** (2009), 029, arXiv:0810.5509.
P. Di Vecchia, R. Marotta, I. Pesando and F. Pezzella, J. of Phys. A **44** (2011), 245401, arXiv:1101.0120.
L. De Angelis, R. Marotta, F. Pezzella and R. Troise, arXiv:1206.3401.
4) K.-S. Choi, T. Kobayashi, R. Maruyama, M. Murata, Y. Nakai, H. Ohki and M. Sakai, Eur. Phys. J. C **67** (2010), 273, arXiv:0908.0395.
T. Kobayashi, R. Maruyama, M. Murata, H. Ohki and M. Sakai, J. High Energy Phys. **05** (2010), 050, arXiv:1002.2828.
5) H. Abe, T. Kobayashi, H. Ohki and K. Sumita, Nucl. Phys. B **863** (2012), 1, arXiv:1204.5327.
6) J. P. Conlon, A. Maharana and F. Quevedo, J. High Energy Phys. **09** (2008), 104, arXiv:0807.0789.
7) F. Marchesano, P. McGuirk and G. Shiu, J. High Energy Phys. **04** (2009), 095, arXiv:0812.2247.
P. G. Camara and F. Marchesano, J. High Energy Phys. **10** (2009), 017, arXiv:0906.3033.

8) C. Bachas, hep-th/9503030.
9) M. Berkooz, M. R. Douglas and R. G. Leigh, Nucl. Phys. B **480** (1996), 265, hep-th/9606139.
10) R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, J. High Energy Phys. **10** (2000), 006, hep-th/0007024.
11) C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, Phys. Lett. B **489** (2000), 223, hep-th/0007090.
12) L. E. Ibanez and A. M. Uranga, *String theory and particle physics: An introduction to string phenomenology* (Cambridge University Press, 2012).
13) R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rep. **445** (2007), 1, hep-th/0610327.
14) H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, J. High Energy Phys. **06** (2009), 080, arXiv:0903.3800.
15) H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, Nucl. Phys. B **820** (2009), 317, arXiv:0904.2631; Phys. Rev. D **80** (2009), 126006, arXiv:0907.5274; Phys. Rev. D **81** (2010), 126003, arXiv:1001.1788.
16) M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, D. Regalado and A. M. Uranga, arXiv:1206.2383.
17) T. Kobayashi, S. Raby and R. J. Zhang, Nucl. Phys. B **704** (2005), 3, hep-ph/0409098.
T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby and M. Ratz, Nucl. Phys. B **768** (2007), 135, hep-ph/0611020.
P. Ko, T. Kobayashi, J. h. Park and S. Raby, Phys. Rev. D **76** (2007), 035005 [Errata; **76** (2007), 059901], arXiv:0704.2807.
18) H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, Prog. Theor. Phys. Suppl. No. 183 (2010), 1, arXiv:1003.3552.
19) P. G. Camara, E. Dudas and E. Palti, J. High Energy Phys. **12** (2011), 112, arXiv:1110.2206.
E. Palti, J. High Energy Phys. **07** (2012), 065, arXiv:1203.4490.
20) I. Antoniadis, A. Kumar and B. Panda, Nucl. Phys. B **823** (2009), 116, arXiv:0904.0910.
21) L. E. Ibanez, F. Marchesano and R. Rabadan, J. High Energy Phys. **11** (2001), 002, hep-th/0105155.
22) R. Blumenhagen, M. Cvetic and T. Weigand, Nucl. Phys. B **771** (2007), 113, hep-th/0609191.
23) L. E. Ibanez and A. M. Uranga, J. High Energy Phys. **03** (2007), 052, hep-th/0609213.