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Title: Intersection in homology through Poincaré Duality

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Abstract

Keywords: Intersection product, multiple intersection, cup product, Poincaré duality, Dehn twist.

The aim of this work is to build an intersection product in homology classes of a smooth compact orientable manifold so that this intersection is Poincaré dual of cup product of its cohomology classes.

To do this, we start from the definition of intersection in complementary grades, and we use the intersection number from differential topology plus stratified transversality theorem to extend this definition to multiple intersection. Then, through a convenient version of Poincaré Duality, we obtain a definition of intersection in any grades. Finally, we calculate the intersection algebra of separating Dehn twist and Johnsons twist.

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1 Introduction

The main goal of this work is to build an intersection product on homology classes of a compact orientable smooth manifolds in a way that this intersection product is the Poincaré dual of cup product.

The reason to do that is that the intersection product is more visual than the cup product. Very often, as in the examples we will see at the end of the document, this visibility simplifies the calculations to a point that they can be solved.

Classic algebraic topologists have been doing calculus with intersections since Poincaré and before (as it can be seen in [8]) but these definitions of intersection cannot be considered strictly correct because they assume transversality between cycles in all steps without justifying that this transversality can be considered.

Precisely due to this problem of lack of proofs, algebraic topologists chose to take the way of cup product (see [6]), which is well defined on topological spaces. However, cup product is not as visual as intersection product and comparison with cycles in complex algebraic geometry and their intersection turns complicated (see [11]).

The aim of this work and its predecessor [10] is to use transversality theorems developed some time after Lefschetz ([5],[3]) in order to build a strict theory of intersection in homology. To do that, we took as a base intersection numbers of submanifolds of complementary dimension, defined in differential topology, to extend this concept to homology classes of complementary grades ([10]).

The proof of Poincaré duality in a convenient way at section 3 allowed us, by using duality, to extend this product to homology classes of any grades in sections 4 and 5, in a Poincaré-dual way to cup product.

In section 6 we apply our results to the study of topology of 3-dimensional mapping tori. We show the universal homology exact sequence which gives us homology groups of these spaces and we compute the intersection algebra on homology classes of these spaces in separating Dehn twist and Johnson twist cases. This homology algebra is Poincaré-dual of cohomology algebra and cup product, which are unpublished on Johnson twist case.

2 Notation and remarks

2.1 Notations

Along the whole document we will refer implicitly to a connected, compact and orientable manifold M, with $n := \dim M$. We will also refer several times to an oriented triangulation of M denoted by T. Then, $\mathrm{sd}(T)$ will denote the barycentric subdivision of T, whose k-simplexes can be thought of as a succession of k simplexes of T, descending in dimension. Any triangulation T induces a complex (T, ∂). Then:

- Let τ^k be a k-simplex of T. It will always have the dimension in the superindex, and if an index is necessary, it will be expressed in the subindex. Furthermore, in coordinates it will always be expressed as

  $$\tau^k = \left(\tau_0^0, \ldots, \tau_k^0\right)$$

  with this orientation.

- In the case of a maximal simplex τ^n, $(-1)^\epsilon$ is the sign which makes its orientation agree with the orientation of M.

- The barycentre of a simplex τ^k will be denoted as $\mathrm{bar}\,\tau^k = \mathrm{bar}\left(\tau_0^0, \ldots, \tau_k^0\right)$.

- A k-simplex of $\mathrm{sd}(T)$ will be expressed by its descending chain of simplexes:

  - For $i_k > \cdots > i_0$, we denote

    $$\tau^{i_k} \succ \ldots \succ \tau^{i_0} := \left(\mathrm{bar}\,\tau^{i_k}, \ldots, \mathrm{bar}\,\tau^{i_0}\right).$$

  - Similarly,

    $$\tau^{i_0} \prec \ldots \prec \tau^{i_k} := \left(\mathrm{bar}\,\tau^{i_0}, \ldots, \mathrm{bar}\,\tau^{i_k}\right) = (-1)^{\frac{k(k+1)}{2}} \tau^{i_k} \succ \ldots \succ \tau^{i_0}.$$

  - We have to make the notation complex in order to make some computations treatable. For this reason, if we put some 0-simplex instead the correspondent index in a chain, it will mean

    $$\tau^{i_0} \prec \ldots \prec \tau^{i_j} \prec \tau_{i_j+1}^0 := \left(\mathrm{bar}\,\tau^{i_0}, \ldots, \mathrm{bar}\,\tau^{i_j}, \mathrm{bar}\left(\tau_0^0, \ldots, \tau_{i_j}^0, \tau_{i_j+1}^0\right)\right),$$

    i.e., "add the 0-simplex to previous i_j-simplex and then compute barycentres". As example:

    $$\tau^0 \prec \ldots \prec \tau^k = \tau_0^0 \prec \ldots \prec \left(\tau_0^0, \ldots, \tau_k^0\right) = \tau_0^0 \prec \ldots \prec \tau_k^0.$$

- Symbols $\prec$ and $\succ$ will denote also a relation of partial ordering whose meaning is "is contained as face" and "contains as a face" respectively. For example, if by context we have a fixed τ^k, $\sum\limits_{\tau^n \succ \tau^k}$ will mean "sum over all maximal simplexes containing τ^k".

Finally, we make some remarks about the symmetric group, since we will use it very often along all document.

Let S_{k+1} be the symmetric group, which will usually act over set $\{0, \ldots, k\}$. Then we define the subset

$$P_{i \to j} := \{l \in S_{k+1} \mid l(i) = j\}.$$

Observe that, if we consider $S_k = P_{k \to k} \subset S_{k+1}$, then $(j\,k) \circ P_{i \to j} \circ (i\,k)$.

2.2 Remarks

In following sections, we will do all the computations taking $\mathbb{Z}$ as group of coefficients. Then, the groups of cochains $\mathcal{C}^{\bullet}(T) = \mathcal{C}^{\bullet}(T; \mathbb{Z}) := \mathrm{Hom}(\mathcal{C}_{\bullet}(T; \mathbb{Z}), \mathbb{Z})$ are isomorphic to the group of chains. Moreover, given a triangulation T, simplexes give a canonical basis of chains, but, in addition, they are also a canonical basis for cochains. Then, when we write a simplex τ^k, we will consider it a chain or cochain depending on context.

All results here can be obtained using a free abelian group G of coefficients, or a ring R when we need multiplication, as in the case of cup product.

Also we do a final observation about triangulations. Polyhedra T and $\mathrm{sd}(T)$ are triangulations of a manifold, which is a very special topological space. This is why these triangulations have special properties that triangulations of generic topological spaces do not have. We list (and we will need) three of them:

(TM1). Every simplex is a face of some n-simplex.

(TM2). Each $(n-1)$-simplex is adjacent to exactly two different n-simplex.

(TM3). If τ^n and σ^n are the two n-simplexes containing a $(n-1)$-simplex μ^{n-1}, then they induce a reverse orientation over μ. That is, if

$$\mu^0 = (\mu_0^0, \ldots, \mu_{n-1}^0)$$
$$\tau^n = (-1)^{\epsilon_\tau} (\tau_0^0, \mu_0^0, \ldots \mu_{n-1}^0)$$

and

$$\sigma^n = (-1)^{\epsilon_\sigma} (\sigma_0^0, \mu_0^0, \ldots \mu_{n-1}^0)$$

then $(-1)^{\epsilon_\tau + \epsilon_\sigma} = -1$.

3 Poincaré Duality

3.1 The dual cell complex $*T$

In the beginning, we only have the manifold M and the triangulation T. Then, we can build the triangulation $\mathrm{sd}(T)$ and we have two triangulations of M, depending on triangulation T, which induces two complexes. The goal of this section is to build a third complex $*T$, also dependent of T, which will be "a kind of" dual triangulation of T. Then, with these T and $*T$ complexes, we will build explicitly the Poincaré Dual isomorphism. The role of $\mathrm{sd}(T)$ is also very important because it acts as a bridge between the other two triangulations.

Since we start only with T and $\mathrm{sd}(T)$, an expected start is to define the inclusion from T to $\mathrm{sd}(T)$. We do it by, giving a simplex τ^k of T,

$$i(\tau^k) := \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\, \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k)}.$$

This map can be extended to group chains by linearity. In fact, we have that

Proposition 3.1. *The inclusion* $i\colon \mathcal{C}_\bullet(T) \longrightarrow \mathcal{C}_\bullet(\mathrm{sd}(T))$ *is a morphism of chain complexes.*

Proof. Equivalently, we see that inclusion commutes with boundary:

$$\begin{aligned} \partial i(\tau^k) &= \partial \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\, \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k)} \\ &= \sum_{j=0}^{k} (-1)^j \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\, \tau^0_{l(0)} \prec \ldots \prec \widehat{\left(\tau^0_{l(0)}, \ldots, \tau^0_{l(j)}\right)} \prec \ldots \prec \left(\tau^0_{l(0)}, \ldots, \tau^0_{l(k)}\right) \end{aligned}$$

but we have that for $j < k$, for all $l \in S_{k+1}$ there exists $l' := l \circ (j\, j+1)$ such that $l \neq l'$, with $\mathrm{sgn}\,(l) \neq \mathrm{sgn}\,(l')$ and such that

$$\tau^0_{l(0)} \prec \ldots \prec \widehat{\left(\tau^0_{l(0)}, \ldots, \tau^0_{l(j)}\right)} \prec \ldots \prec \left(\tau^0_{l(0)}, \ldots, \tau^0_{l(k)}\right)$$

is equal to

$$\tau^0_{l'(0)} \prec \ldots \prec \widehat{\left(\tau^0_{l'(0)}, \ldots, \tau^0_{l'(j)}\right)} \prec \ldots \prec \left(\tau^0_{l'(0)}, \ldots, \tau^0_{l'(k)}\right).$$

Thus terms with $j < k$ cancel by pairs. So we get

$$
\begin{aligned}
\partial i(\tau^k) &= (-1)^k \sum_{l\in S_{k+1}} \operatorname{sgn}(l)\, \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k-1)} \\
&= (-1)^k \sum_{j=0}^{k} \sum_{l\in P_{j\to k}} \operatorname{sgn}(l)\, \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k-1)} \\
&= (-1)^k \sum_{j=0}^{k} \sum_{l\in S_k} \operatorname{sgn}(l\circ (j\ k))\, \tau^0_{l\circ(j\ k)(0)} \prec \ldots \prec \tau^0_{l\circ(j\ k)(k-1)} \\
&= (-1)^{k+1} \sum_{j=0}^{k} \sum_{l\in S_k} \operatorname{sgn}(l)\, \tau^0_{l\circ(j\ k)(0)} \prec \ldots \prec \tau^0_{l\circ(j\ k)(k-1)} \\
&= (-1)^{k+1} \sum_{j=0}^{k} i\left(\left(\tau^0_{(j\ k)(0)}, \ldots, \tau^0_{(j\ k)(k-1)}\right)\right) \\
&= (-1)^{k+1} \sum_{j=0}^{k} i\left(\left(\tau^0_0, \ldots, \tau^0_{j-1}, \tau^0_k, \tau^0 j+1, \ldots, \tau^0_{k-1}\right)\right) \\
&= (-1)^{k+1} \sum_{j=0}^{k} (-1)^{k-1-j}\, i\left(\left(\tau^0_0, \ldots, \widehat{\tau^0_j}, \ldots, \tau^0_k\right)\right) \\
&= \sum_{j=0}^{k} (-1)^{-j}\, i\left(\left(\tau^0_0, \ldots, \widehat{\tau^0_j}, \ldots, \tau^0_k\right)\right) \\
&= i\partial(\tau^k).
\end{aligned}
$$

□

This morphism inclusion becomes an isomorphism (the natural one) when it is extended to homology classes.

Now, the following thing we do is not as expected as before: we define a first version of Poincaré Dual isomorphism. Concretely, we define the *star operator*

$$*\colon \mathcal{C}^k(T) \longrightarrow \mathcal{C}_{n-k}(\mathrm{sd}(T))$$

given by

$$*\tau^k := \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s\in S_{n-k}} \operatorname{sgn}(s)\, \tau^k \prec \tau^0_{s(k+1)} \prec \ldots \prec \tau^0_{s(n)}$$

where this time symmetric group acts over set $\{k+1, \ldots, n\}$.

Observation. The star operator is injective. It can be seen, for example, by observing that $*\tau^k$ is the only simplex whose image contains simplex with the 0-simplex bar τ^k.

Maybe the star operator would seem more natural if we define it in the same way but from chains instead of cochains. Actually, if we do that, we obtain a group morphism so that, in general, it is not a morphism of complexes. We define it in that way because the star operator is almost a morphism between complexes. Concretely

Proposition 3.2. *If δ is the coboundary operator, the equation*

$$\partial * \tau^k = (-1)^{k+1} * \delta\tau^k$$

holds.

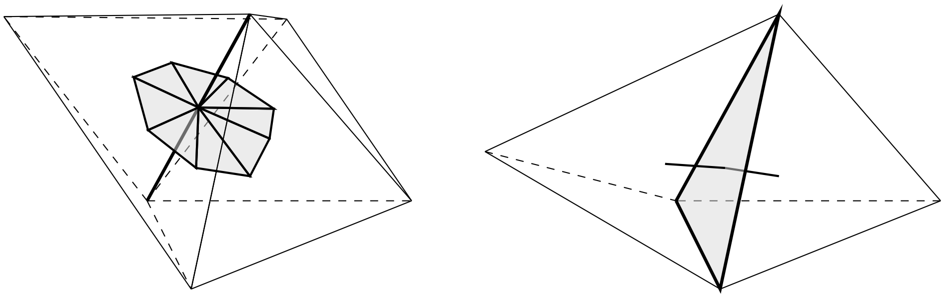

Figure 1: Examples of $*\tau^k$ for $n = 3$. At left, $k = 1$, and at right, $k = 2$.

Proof. To prove that we compute both sides of the equation. By one side, we have

$$\begin{aligned}\partial * \tau^k \quad &= \quad \partial \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \operatorname{sgn}(s)\, \tau^k \prec \tau^0_{s(k+1)} \prec \ldots \prec \tau^0_{s(n)} \\ = \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \operatorname{sgn}(s) \sum_{j=1}^{n-k} (-1)^j \; & \tau^k \prec \ldots \prec \left(\tau^0_0, \widehat{\ldots, \tau^0_{s(k+j)}}\right) \prec \ldots \prec \left(\tau^0_0, \ldots, \tau^0_{s(n)}\right) \\ = \sum_{j=1}^{n-k} (-1)^j \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \operatorname{sgn}(s) \; & \tau^k \prec \ldots \prec \left(\tau^0_0, \widehat{\ldots, \tau^0_{s(k+j)}}\right) \prec \ldots \prec \left(\tau^0_0, \ldots, \tau^0_{s(n)}\right)\end{aligned}$$

as before, for $j \in \{1, \ldots, n-k-1\}$ we have that for every $s \in S_{n-k}$ exists $s' := s \circ (k+j \quad k+j+1)$ such that $s \neq s'$, $\operatorname{sgn}(s) = -\operatorname{sgn}(s')$ and

$$\tau^k \prec \ldots \prec \left(\tau^0_0, \widehat{\ldots, \tau^0_{s(k+j)}}\right) \prec \ldots \prec \left(\tau^0_0, \ldots, \tau^0_{s(n)}\right)$$

is equal to

$$\tau^k \prec \ldots \prec \left(\tau^0_0, \widehat{\ldots, \tau^0_{s'(k+j)}}\right) \prec \ldots \prec \left(\tau^0_0, \ldots, \tau^0_{s'(n)}\right)$$

so terms with these $j \in \{1, \ldots, n-k-1\}$ cancel by pairs. We say that these cancelations are interior because they happen "inside" τ^n. The cancelations where $j = n-k$ will be exterior and a consequence of the fact that T triangulates a manifold.

By (TM2) and (TM3), for every

$$\tau^n = (-1)^{\epsilon} \left(\tau^0_0, \ldots, \tau^0_k, \tau^0_{s(k+1)}, \ldots, \tau^0_{s(n)}\right)$$

always exists a different n-simplex

$$\tilde{\tau}^n = -(-1)^{\epsilon} \left(\tau^0_0, \ldots, \tau^0_k, \tau^0_{s(k+1)}, \ldots, \tau^0_{s(n-1)}, \tilde{\tau}^0_{s(n)}\right)$$

so there are "exterior" cancelations on terms with $j = n - k$. Thus, we obtain:

$$\partial * \tau^k = \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \operatorname{sgn}(s) \; \left(\tau^0_0, \ldots, \tau^0_k, \tau^0_{s(k+1)}\right) \prec \ldots \prec \tau^0_{s(n)}$$

Now, we look at the other side of the equation:

$$(-1)^{k+1} * \delta\left(\tau^{k}\right) = * \sum_{\tau^{k+1} \succ \tau^{k}} \tau^{k+1}$$

where $\tau^{k+1} = \left(\tau_0^0, \ldots, \tau_{k+1}^0\right)$ according to our notation. Thus

$$= \sum_{\tau^{k+1} \succ \tau^{k}} \sum_{\tau^{n} \succ \tau^{k+1}} (-1)^{\epsilon} \sum_{s \in S_{n-k-1}} \operatorname{sgn}(s) \ \tau^{k+1} \prec \tau^0_{s(k+2)} \prec \ldots \prec \tau^0_{s(n)}$$

$$= \sum_{\tau^{k+1} \succ \tau^{k}} \sum_{\tau^{n} \succ \tau^{k+1}} (-1)^{\epsilon} \sum_{s \in S_{n-k-1}} \operatorname{sgn}(s) \ \left(\tau_0^0, \ldots, \tau_{k+1}^0\right) \prec \tau^0_{s(k+2)} \prec \ldots \prec \tau^0_{s(n)}$$

and we arrive at a tedious and delicate point: the time to count. Observe that one may be tempted to replace

$$\sum_{\tau^{k+1} \succ \tau^{k}} \sum_{\tau^{n} \succ \tau^{k+1}} = \sum_{\tau^{n} \succ \tau^{k}}$$

what is *not* true, if one counts the number of times that and the sign with a maximal simplex appears. By basic combinatorial, it is easy to see that in the left expression appears every maximal simplex $n-k$ times. And the sign?

Well, if the maximal simplex τ^n is

$$\left(\tau_0^0, \ldots, \tau_k^0, \tau_{k+1}^0, \ldots, \tau_n^0\right)$$

then, every possible $(k+1)$-simplex τ^{k+1} is of the form

$$\left(\tau_0^0, \ldots, \tau_k^0, \tau_j^0\right) \text{ for } j \in \{k+1, \ldots, n\}$$

so it means that

$$\sum_{\tau^{k+1} \succ \tau^{k}} \sum_{\tau^{n} \succ \tau^{k+1}} (-1)^{\epsilon} = \sum_{j=k+1}^{n} \sum_{\tau^{n} \succ \tau^{k}} (-1)^{\epsilon} \operatorname{sgn}((k+1\, j))$$

and replace every subindex j by a $k+1$ and vice versa. Thus we get

$$= \sum_{j=k+1}^{n} \sum_{\tau^{n} \succ \tau^{k}} (-1)^{\epsilon} \operatorname{sgn}((k+1\, j))$$

$$\sum_{s \in S_{n-k-1}} \operatorname{sgn}(s) \ \left(\tau_0^0, \ldots, \tau_j^0\right) \prec \tau^0_{(k+1 \ \ j) \circ s(k+2)} \prec \ldots \prec \tau^0_{(k+1 \ \ j) \circ s(n)}$$

$$= \sum_{j=k+1}^{n} \sum_{\tau^{n} \succ \tau^{k}} (-1)^{\epsilon}$$

$$\sum_{s \in S_{n-k-1}} \operatorname{sgn}((k+1\, j) \circ s) \ \left(\tau_0^0, \ldots, \tau_j^0\right) \prec \tau^0_{(k+1 \ \ j) \circ s(k+2)} \prec \ldots \prec \tau^0_{(k+1 \ \ j) \circ s(n)}$$

$$= \sum_{j=k+1}^{n} \sum_{\tau^{n} \succ \tau^{k}} (-1)^{\epsilon} \sum_{s \in P_{k+1 \to j}} \operatorname{sgn}(s) \ \left(\tau_0^0, \ldots, \tau^0_{s(k+1)}\right) \prec \tau^0_{s(k+2)} \prec \ldots \prec \tau^0_{s(n)}$$

$$= \sum_{\tau^{n} \succ \tau^{k}} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \operatorname{sgn}(s) \ \left(\tau_0^0, \ldots, \tau^0_{s(k+1)}\right) \prec \tau^0_{s(k+2)} \prec \ldots \prec \tau^0_{s(n)}$$

which is exactly

$$= \partial * \tau^k$$

□

This proposition has several important consequences. The first one:

Definition 3.3. We define the *block complex* $(*T, \partial)$ in the following way:

- The $(n-k)$-chains group of $(*T, \partial)$ is defined as the image of k-chains of T by $*$. That is
$$\mathcal{C}_{n-k}(*T) := *\mathcal{C}^k(T) \subset \mathcal{C}_{n-k}(\mathrm{sd}(T)) .$$
- As a boundary operator, the operator ∂ induced by the boundary operator ∂ of $\mathrm{sd}(T)$.

Corollary 3.4. *The block complex is a chain complex. That is, the boundary operator ∂ send $(k+1)$-chains to k-chains and $\partial^2 = 0$.*

Remark. Note that to show that ∂ is well-defined, we need that terms of $*\mathcal{C}_{k+1}(T)$ can be expressed uniquely, or equivalently, the injectivity of the operator $*$. This is one of the four conditions of the *block complexes.* Block complexes are subcomplexes defined in the natural way to compute the same homology as the whole complexes, so by showing that $*T$ is it, one can prove that $*T$ define the same homology as T, as is done in [8]. However, we will show it in another way. More about block complexes can be found also at [8].

Observation. Operator $*$ viewed as a morphism from $\mathcal{C}^k(T)$ to $\mathcal{C}_{n-k}(*T)$ is an isomorphism, since $*$ is injective and by definition $\mathcal{C}_{n-k}(*T)$ is its image.

A second important corollary:

Corollary 3.5. *The operator $(-1)^{k+1} *$ is a isomorphism between (T, δ) and $(*T, \partial)$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathcal{C}^k(T) \arrow[r, leftrightarrow, "*"] \arrow[d, "\delta"] & \mathcal{C}_{n-k}(*T) \arrow[d, "(-1)^{k+1}\partial"] \\
\mathcal{C}^{k+1}(T) \arrow[r, leftrightarrow, "*"] \arrow[d] & \mathcal{C}_{n-k-1}(*T) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Then, we define

Definition 3.6. The first *Poincaré Dual isomorphism*

$$*^\flat \colon (\mathcal{C}^k(T), \delta) \longrightarrow (\mathcal{C}_{n-k}(*T), \partial)$$

by $*^\flat \tau^k := (-1)^{k+1} * \tau^k$.

Thus, by corollary 3.5, we have the following cohomology-homology isomorphism:

$$\begin{array}{rccc} *^\flat\colon & H^k(T) & \longrightarrow & H_{n-k}(*T) \\ & [a] & \longmapsto & *^\flat[a] = [*^\flat a] = (-1)^{k+1}[*a]. \end{array}$$

Now, once we have built complex of chains $(*T, \partial)$, we can define its dual, the complex of cochains $(*T, \delta)$ in the natural way, that is, cochain groups as dual of chain groups and coboundary operator as dual of boundary operator.

Remark. The boundary operator of $*T$ is induced boundary operator of $\mathrm{sd}(T)$, but the coboundary operator of $*T$ is *not* induced coboundary operator of $\mathrm{sd}(T)$. The adjective "induced" does not pass to dual.

Now, we have built $*T$, so we can redefine the star operator in a way that

$$*\colon \mathcal{C}_k(T) \longrightarrow \mathcal{C}^{n-k}(*T)$$

and so that in coordinates it has the same expression as before.

Then, proposition 3.2 still having important consequences:

Proposition 3.7. *Equation*

$$\delta * \tau^k = (-1)^k * \partial\left(\tau^k\right)$$

holds.

Proof. We know that

$$\partial * \tau^k = (-1)^{k+1} * \delta\tau^k = * \sum_{\tau^{k+1}\succ\tau^k} \tau^{k+1} = \sum_{\tau^{k+1}\succ\tau^k} *\tau^{k+1}$$

since we agreed to the notation $\tau^k := (\tau_0^0, \dots, \tau_k^0)$ and $\tau^{k+1} := (\tau_0^0, \dots, \tau_{k+1}^0)$. It means that

$$\delta * \tau^k = \sum_{\tau^k\succ\tau^{k-1}} *\tau^{k-1}$$

Now,

$$*\partial\tau^k = * \sum_{j=0}^{k} (-1)^j \left(\tau_0^0, \dots, \widehat{\tau_j^0}, \dots, \tau_k^0\right)$$

but for every j we have to sort 0-simplex in order to set τ_j^0 the last element. That is:

$$\begin{array}{rcl} \tau^k & = & \left(\tau_0^0, \dots, \tau_j^0, \dots, \tau_k^0\right) \\ & = & \operatorname{sgn}(k \;\; k-1 \;\; \dots \;\; j)\left(\tau_0^0, \dots, \widehat{\tau_j^0}, \dots, \tau_k^0, \tau_j^0\right) \\ & = & (-1)^{k-j}\left(\tau_0^0, \dots, \widehat{\tau_j^0}, \dots, \tau_k^0, \tau_j^0\right) \end{array}$$

since the term that disappears is the last, we can write

$$*\partial\tau^k = (-1)^k * \sum_{\tau^k \succ \tau^{k-1}} \tau^{k-1} = (-1)^k \sum_{\tau^k \succ \tau^{k-1}} *\tau^{k-1} = (-1)^k \, \delta * \tau^k$$

□

Thus we have again a relation chains - cochains, this time the other way round. Since the expression is the same as before, and cochain groups are isomorphic to chain ones, we have that star operator is still a group isomorphism. Then, as a consequence of the last proposition, we have that

Corollary 3.8. *The operator $(-1)^k *$ is a isomorphism between (T, ∂) and $(*T, \delta)$. Equivalently, following diagram commutes:*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathcal{C}_k(T) \arrow[r, leftrightarrow, "*"] \arrow[d, "\partial"] & \mathcal{C}^{n-k}(*T) \arrow[d, "(-1)^k\delta"] \\
\mathcal{C}_{k-1}(T) \arrow[r, leftrightarrow, "*"] \arrow[d] & \mathcal{C}^{n-k+1}(*T) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

And again, we define

Definition 3.9. The second *Poincaré Dual isomorphism*

$$*^\sharp : \mathcal{C}_k(T) \longrightarrow \mathcal{C}^{n-k}(*T)$$

by $*^\sharp\tau^k := (-1)^k * \tau^k$.

Thus, as before, by corollary 3.8, we have the following homology-cohomology isomorphism:

$$\begin{array}{rccl} *^\sharp : & H_k(T) & \longrightarrow & H^{n-k}(*T) \\ & [a] & \longmapsto & *^\sharp[a] = [*^\sharp a] = (-1)^k [*a]. \end{array}$$

3.2 The homology of $*T$

In the previous section we saw the Poincaré isomorphism between homology/cohomology of T and $*T$. To complete the proof, we should check that homology of $*T$ is the homology of the manifold M. If you let visual intuition run free, it may seem the easy part, but in fact it is the most complicated part of the proof.

3.2.1 Star of a 0-simplex

Definition 3.10. The *open star* of a 0-simplex τ^0 in T is

$$\mathrm{St}\left(\tau^0, T\right) := \bigcup_{\tau \succ \tau^0} \mathrm{int}\left(\tau\right).$$

We also define the *closed star* as

$$\overline{\mathrm{St}}\left(\tau^0, T\right) := \overline{\mathrm{St}\left(\tau^0, T\right)}.$$

Observation. By finiteness of the union, since $\overline{U} \cup \overline{V} = \overline{U \cup V}$, we have

$$\overline{\mathrm{St}}\left(\tau^0, T\right) = \bigcup_{\tau \succ \tau^0} |\tau| = \bigcup_{\tau^n \succ \tau^0} |\tau^n|$$

where the last equality holds because of (TM1). The closed star is a topological subspace with an obvious subtriangulation

$$\sum_{\tau^n \succ \tau^0} (-1)^\epsilon \tau^n$$

where $(-1)^\epsilon$ is the sign that orients τ^n.

Note. Observe that the definition of an open star or closed star can be defined also in triangulation $\mathrm{sd}(T)$. Since all k-simplexes in T correspond to an unique 0-simplex in $*T$, we define the application

$$\overline{\mathrm{St}_\bullet}(\tau^k) \qquad = \qquad \text{Triangulation of } \overline{\mathrm{St}}\left(\mathrm{bar}\,\tau^k, \mathrm{sd}(T)\right)$$

$$= \sum_{\tau^n \succ \tau^0_k} (-1)^\epsilon \sum_{\substack{l \in S_{k+1} \\ s \in S_{k-n}}} \mathrm{sgn}\left(()\, l\right) \mathrm{sgn}\left(()\, s\right) \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k)} \prec \tau^0_{s(k+1)} \prec \ldots \prec \tau^0_{s(n)}.$$

Also as before, S_{k+1} acts over set $\{0, \ldots, k\}$ and S_{n-k} acts over set $\{k+1, \ldots, n\}$.

3.2.2 Join product

Let us just recall the notion of join product.

Definition 3.11. Let $\tau^k = \left(\tau^0_0, \ldots, \tau^0_k\right)$ and $\sigma^l = \left(\sigma^0_0, \ldots, \sigma^0_l\right)$ in some $\mathbb{R}^M$ such that all $k + l + 1$ 0-simplexes are independent points. Then, we define:

$$\left|\tau^k\right| \star \left|\sigma^l\right| := \left|\left(\tau^0_0, \ldots, \tau^0_k, \sigma^0_0, \ldots, \sigma^0_l\right)\right|.$$

This definition results in an operation over (some) topological spaces, sending underlying topological spaces of two simplexes to the underlying topological space of a third simplex.

This fact allows us to extend the definition of $*$ by bilinearity over (some) chains of a triangulation T by saying that, if $\tau^k \in \mathcal{C}_k(T)$ and $\sigma^l \in \mathcal{C}_l(T)$ are simplexes, then

$$a\tau^k \star b\sigma^l = ab(\tau^0_0, \ldots, \tau^0_k, \sigma^0_0, \ldots, \sigma^0_l)$$

whenever $(\tau^0_0, \ldots, \tau^0_k, \sigma^0_0, \ldots, \sigma^0_l) \in \mathcal{C}_{k+l+1}(T)$.

From this simple definition, we can do several observations:

Observation.
- If $\tau^k \star \tau^l$ exists, then $\tau^l \star \tau^k = (-1)^{(k+1)(l+1)}\, \tau^k \star \tau^l$ and also exists.
- If $\tau^k \star \sigma^l$ exists, then $\partial(\tau^k \star \sigma^l) = \partial\tau^k \star \sigma^l + (-1)^{k+1}\, \tau^k \star \partial\sigma^l$ and also exists.
- If $K \subset \mathbb{R}^k$ and $L \subset \mathbb{R}^l$ are simplicial complexes then there exist both $|K| \star |L|$ and $K \star L \subset \mathbb{R}^{k+l+1}$ and moreover $|K| \star |L| = |K \star L|$.

  In particular, the cone of K is $CK = \{c\} \star K$ and the suspension is $SK = \{N, S\} \star K$ where c, N, S are points.

Note. Product $\star$ can be shown to be associative. Further information can be found in [7].

Example 3.12. Let us show several examples that will be useful to us:

- Let $\tau^{i_p} \succ \ldots \succ \tau^{i_0}$ be simplexes of T. Then, in $\mathrm{sd}(T)$ always exists

$$\left(\tau^{i_0} \prec \ldots \prec \tau^{i_l}\right) \star \left(\tau^{i_{l+1}} \prec \ldots \prec \tau^{i_p}\right) = \tau^{i_0} \prec \ldots \prec \tau^{i_p}$$

  and equation holds for all $0 \le l \le p$. This means that we could replace all $\prec$'s by $\star$'s. We will not do it.

- Let $\tau^k \in T$, then:

$$\begin{aligned} i(\tau^k) &= \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\ \tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k)} \\ &= \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\ \left(\tau^0_{l(0)} \prec \ldots \prec \tau^0_{l(k-1)}\right) \star \mathrm{bar}\,\tau^k \\ &= \partial i(\tau^k) \star \mathrm{bar}\,\tau^k = (-1)^k\, \mathrm{bar}\,\tau^k \star \partial i(\tau^k) \end{aligned}$$

  i.e. a simplex of T in $\mathrm{sd}(T)$ is a cone of its boundary with orientation $(-1)^k$.

- Again $\tau^k \in T$, then:

$$\begin{aligned} *\tau^k &= \sum_{\tau^n \succ \tau^k} (-1)^\epsilon \sum_{s \in S_{n-k}} \mathrm{sgn}\,(s)\ \tau^k \prec \tau^0_{l(k+1)} \prec \ldots \prec \tau^0_{l(n)} \\ &= \sum_{\tau^n \succ \tau^k} (-1)^\epsilon \sum_{s \in S_{n-k}} \mathrm{sgn}\,(s)\ \mathrm{bar}\,\tau^k \star \left((\tau^0_0, \ldots, \tau^0_k, \tau^0_{l(k+1)}) \prec \ldots \prec \tau^0_{l(n)}\right) \\ &= \mathrm{bar}\,\tau^k \star \left(\partial * \tau^k\right) \end{aligned}$$

  so blocks $*\tau^k$ are also cones. Moreover, $\left|*\tau^k\right| \cap \left|\tau^k\right| = \left|\mathrm{bar}\,\tau^k\right|$, which is their vertex.

- Now, the key of the main result of this section:

$$\begin{aligned}
&\overline{\mathrm{St}_\bullet}(\tau^k)\\
&=\sum_{\tau^n\succ\tau_k^0}(-1)^\epsilon\sum_{\substack{l\in S_{k+1}\\ s\in S_{k-n}}}\operatorname{sgn}(l)\operatorname{sgn}(s)\,\tau_{l(0)}^0\prec\ldots\prec\tau_{l(j)}^0\prec\tau_{s(j+1)}^0\prec\ldots\prec\tau_{s(n)}^0\\
&=\sum_{\tau^n\succ\tau_k^0}(-1)^\epsilon\sum_{\substack{l\in S_{k+1}\\ s\in S_{k-n}}}\operatorname{sgn}(l)\operatorname{sgn}(s)\left(\tau_{l(0)}^0\prec\ldots\prec\tau_{l(j)}^0\right)\star\left((\tau_0^0,\cdots,\tau_k^0,\tau_{s(j+1)}^0)\prec\ldots\prec\tau_{s(n)}^0\right)\\
&=i(\tau^k)\star\left(\partial*\tau^k\right)\\
&=(-1)^k\operatorname{bar}\tau^k\star\partial i(\tau^k)\star\left(\partial*\tau^k\right)
\end{aligned}$$

  so, again, we have a cone.

- Then, this follows immediately

$$\partial\overline{\mathrm{St}_\bullet}(\tau^k)=(-1)^{k+1}\,\partial i(\tau^k)\star\left(\partial*\tau^k\right)$$

  so

$$\overline{\mathrm{St}_\bullet}(\tau^k)=-\operatorname{bar}\tau^k\star\partial\overline{\mathrm{St}_\bullet}(\tau^k)$$

Observe that natural objects we have been studying are strongly related, in some way, with this join product. Thus, maybe it is not strange at all that to study the homology of these objects, first we study how the homology with join product behaves, at least in easy cases. This is the reason of the following observations:

Observation. The first one is about the homology of the join product in the case of the cone and the suspension. It is easily seen that

$$\widetilde{H}_{i+1}\left(CK\right)\simeq\{0\}$$

and

$$\widetilde{H}_{i+1}\left(SK\right)\simeq\widetilde{H}_i\left(K\right)$$

where $\widetilde{H}_\bullet\left(\cdot\right)$ is the reduced homology.

For example, if S^n is the n-sphere, since

$$\mathrm{S}^n\simeq S\,\mathrm{S}^{n-1}\simeq\{N,S\}\star\ldots\star\{N,S\}$$

we have that

$$\widetilde{H}_{i+k}\left(\mathrm{S}^{k-1}\star K\right)=\widetilde{H}_i\left(K\right).$$

Observation. The second observation is just about the relative homology of the cone with respect its base. Since the cone has the homology of the point:

$$H_{i+1}\left(CX,X\right)\simeq\widetilde{H}_i\left(X\right).$$

After these last observations, we are ready for the main result of this subsection:

Lemma 3.13. *The relative homology of a k-block of $*T$ with respect its boundary is the relative homology of the k-ball to its boundary.*

Proof. We have spent this subsection to justify all the following isomorphisms:

$$\begin{array}{rcll} H_i\left(\left|*\tau^k\right|,\left|\partial*\tau^k\right|\right) & \simeq & H_i\left(\left|\operatorname{bar}\tau^k\star\partial*\tau^k\right|,\left|\partial*\tau^k\right|\right) & \simeq \widetilde{H}_{i-1}\left(\left|\partial*\tau^k\right|\right) \\ & \simeq & \widetilde{H}_{i-1+k}\left(\mathrm{S}^{k-1}\star\left|\partial*\tau^k\right|\right) & \simeq \widetilde{H}_{i-1+k}\left(\left|\partial i\tau^k\star\left(\partial*\tau^k\right)\right|\right) \\ & \simeq & \widetilde{H}_{i-1+k}\left(\left|\partial\overline{\mathrm{St}_\bullet}(\tau^k)\right|\right) & \simeq H_{i+k}\left(\left|\overline{\mathrm{St}_\bullet}(\tau^k)\right|,\left|\partial\overline{\mathrm{St}_\bullet}(\tau^k)\right|\right) \\ & \simeq & H_{i+k}\left(M,M-\operatorname{bar}\tau^k\right) & \simeq H_{i+k}\left(\mathrm{B}^n,\partial\,\mathrm{B}^n\right) \\ & \simeq & H_i\left(\mathrm{B}^k,\partial\,\mathrm{B}^k\right) & \end{array}$$

□

We have not seen that blocks of $*T$ are topologically cells, in which case $*T$ would be a CW complex, but we have just proved in lemma 3.13 that they are homologically cells and therefore suitable for our propose of computing the homology.

Theorem 3.14. *The block complex $(*T,\partial)$ computes the simplicial homology of M.*

Proof. Last lemma gives us the needed condition to repeat the proof of the homology of *CW* complexes getting the same result. This proof can be found with detail in [7], Chapter 4, section 39, *The homology of CW complexes.* □

3.3 Poincaré Duality

All the work done in the previous two sections aims to be the proof of the Poincaré Duality Theorem:

Theorem 3.15 (Poincaré Duality)**.** *Let M be a compact connected oriented manifold. Then there exists the following isomorphism between homology and cohomology:*

$$H_k\left(M\right)\simeq H^{n-k}\left(M\right).$$

In fact, we have found two explicit isomorphisms $*^\flat$ and $*^\sharp$.

4 Cup product - intersection Duality

4.1 Kronecker index

In analogy with the previous section, where we built a relation between T and $*T$, now we have that Kronecker map would be like $\mathrm{sd}(T)$, in the sense that it will play the role of bridge between cup product and intersection, and in the end it will be absolutely hidden.

So we start by some definitions:

Let C be a complex. Then we define the following bilinear pairing

$$\begin{array}{rccl} \langle , \rangle \colon & \mathrm{Hom}\left(\mathcal{C}_k\left(C\right), \mathbb{Z}\right) \otimes \mathcal{C}_k\left(C\right) & \longrightarrow & \mathbb{Z} \\ & \alpha^k \otimes a_k & \longmapsto & \langle \alpha^k, a_k \rangle = \alpha^k(a_k). \end{array}$$

It is just an observation to see that this pairing descend to cohomology - homology. Thus, it induces a map

$$\begin{array}{rccl} \langle , \rangle \colon & H^k\left(C\right) \otimes H_k\left(C\right) & \longrightarrow & \mathbb{Z} \\ & [\alpha^k] \otimes [a_k] & \longmapsto & \langle [\alpha^k], [a_k] \rangle = \langle \alpha^k, a_k \rangle. \end{array}$$

which is called *Kronecker index.* An equivalent version of this map also has a name, in particular

Definition 4.1. We define the *Kronecker map* as

$$\begin{array}{rccl} \kappa \colon & H^k\left(C\right) & \longrightarrow & \mathrm{Hom}\left(H_k\left(C\right), \mathbb{Z}\right) \\ & \alpha^k & \longmapsto & \langle \alpha^k, \cdot \rangle. \end{array}$$

This map relates cohomology with homology by saying that cohomology "is something like" the dual of the homology. Actually, for our purposes, the following theorem says that these spaces are isomorphic:

Theorem 4.2. *Let C be a free chain complex. If $H_k\left(C\right)$ is free for al k, then κ is an isomorphism for all k.*

This theorem is extracted directly from [7], concretely, theorem 45.8, and it is proven there, so we do not prove it. From this theorem also follows that κ is an isomorphism of free parts, by considering coefficients in $\mathbb{Q}$. Thus we obtain an important consequence:

Corollary 4.3. *Under the previous assumptions, the Kronecker index is not degenerated, and with coefficients in $\mathbb{Z}$, κ is an isomorphism of free subgroups.*

And it is important because all complexes we work with are free.

4.2 Cup product

In this subsection we are going to relate cup product with Kronecker index through Poincaré duality.

First of all, we choose to work with simplicial versions of cup product and cap product.

Definition 4.4. Let T be a complex. Choose a partial ordering of the vertices of T that linearly orders the vertices of each simplex of T. Then, we define the *simplicial cup product*

$$\cup \colon \mathcal{C}^{k_1}(T) \times \mathcal{C}^{k_2}(T) \longrightarrow \mathcal{C}^{k_1+k_2}(T)$$

given by

$$\langle \tau^{k_1} \cup \tau^{k_2}, (\sigma_0^0, \ldots, \sigma^{k_1+k_2}) \rangle = \langle \tau^{k_1}, (\sigma_0^0, \ldots, \sigma_{k_1}^0) \rangle \cdot \langle \tau^{k_2}, (\sigma_{k_1}^0, \ldots, \sigma_{k_1+k_2}^0) \rangle$$

where $\sigma_0^0 < \cdots < \sigma_{k_1+k_2}^0$ in the given ordering.

Definition 4.5. Let T be a complex, and again choose a partial ordering. We define the *simplicial cap product*

$$\cap \colon \mathcal{C}^{k_1}(T) \times \mathcal{C}_{k_1+k_2}(T) \longrightarrow \mathcal{C}_{k_2}(T)$$

given by

$$\tau^{k_1} \cap (\sigma_0^0, \ldots, \sigma^{k_1+k_2}) = (\sigma_0^0, \ldots, \sigma_{k_2}^0) \otimes \langle \tau^{k_1}, (\sigma_{k_2}^0, \ldots, \sigma_{k_1+k_2}^0) \rangle$$

where $\sigma_0^0 < \cdots < \sigma_{k_1+k_2}^0$ in the given ordering.

Both definitions are given in [7], and we will suppose basic results. Then, we start by some results:

Proposition 4.6. *Since M is orientable, it has a fundamental class. Let Γ be a generator, then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C}^k(T) \arrow[r, "(-1)^{nk+1} *^\flat"] & \mathcal{C}_{n-k}(*T) \arrow[d, "i_*"] \\ \mathcal{C}^k(\mathrm{sd}(T)) \arrow[u, "i^*"] \arrow[r, "\cap\,\mathrm{sd}\,\Gamma"] & \mathcal{C}_{n-k}(\mathrm{sd}(T)) \end{tikzcd}$$

Proof. First, to have a well defined expression of simplicial cap product, we have to choose a partial ordering of the vertices of $\mathrm{sd}(T)$, in such a way that to compute the expression of cap product we must express simplexes in the given ordering. Thus we choose the following partial ordering: $\mathrm{bar}\,\tau^k < \mathrm{bar}\,\tau^l$ if and only if $\tau^k \succ \tau^l$.

Now, by definition, expressed in coordinates

$$\begin{aligned} \mathrm{sd}(\Gamma) \quad &= \sum_{\tau^n} (-1)^\epsilon \sum_{s \in S_n} \mathrm{sgn}(s)\, \tau_{s(0)}^0 \prec \ldots \prec \tau_{s(n)}^0 \\ &= (-1)^{\frac{n(n+1)}{2}} \sum_{\tau^n} (-1)^\epsilon \sum_{s \in S_n} \mathrm{sgn}(s)\, \tau_{s(n)}^0 \succ \ldots \succ \tau_{s(0)}^0 \end{aligned}$$

where this is expressed in the given ordering. Thus, let $\tau^{i_k} \succ \ldots \succ \tau^{i_0}$ a generic k-simplex of $\mathrm{sd}(T)$. Then

$$\left(\tau^{i_k} \succ \ldots \succ \tau^{i_0}\right) \cap \left(\tau_{s(n)}^0 \succ \ldots \succ \tau_{s(0)}^0\right)$$

$$= \tau^0_{s(n)} \succ \ldots \succ \left(\tau^0_{s(0)}, \ldots, \tau^0_{s(k)}\right) \cdot \langle \tau^{i_k} \succ \ldots \succ \tau^{i_0}, \tau^0_{s(k)} \succ \ldots \succ \tau^0_{s(0)} \rangle$$

so the pairing vanishes when $\tau^{i_j} = \left(\tau^0_{s(0)}, \ldots, \tau^0_{s(j)}\right)$ for all $j \leq k$. Otherwise, the pairing becomes 1. Thus $\cap\, \mathrm{sd}(\Gamma)$ is so that vanishes in the simplexes that are not of the form $\tau^0_{r(0)} \prec \ldots \prec \tau^0_{r(k)}$ for some simplex τ^k of T and for some $r \in S_{k+1}$, and

$$\begin{aligned}
&\tau^0_{r(0)} \prec \ldots \prec \tau^0_{r(k)} \cap \mathrm{sd}(\Gamma) \\
&= (-1)^{\frac{n(n+1)}{2} + \frac{k(k+1)}{2}} \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{\substack{s \in S_n \\ s(j)=r(j) \\ \forall j \leq k}} \mathrm{sgn}\,(s)\, \tau^0_{s(n)} \succ \ldots \succ \left(\tau^0_{s(0)}, \ldots, \tau^0_{s(k)}\right) \\
&= (-1)^{\frac{n(n+1)}{2} + \frac{k(k+1)}{2} + \frac{(n-k)(n-k+1)}{2}} \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{\substack{s \in S_n \\ s(j)=r(j) \\ \forall j \leq k}} \mathrm{sgn}\,(s) \left(\tau^0_{s(0)}, \ldots, \tau^0_{s(k)}\right) \prec \ldots \prec \tau^0_{s(n)}.
\end{aligned}$$

It is just an exercise to see that

$$(-1)^{\frac{n(n+1)}{2} + \frac{k(k+1)}{2} + \frac{(n-k)(n-k+1)}{2}} = (-1)^{k(k-n)}$$

so

$$\begin{aligned}
&= (-1)^{k(k-n)} \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{\substack{s \in S_n \\ s(j)=r(j) \\ \forall j \leq k}} \mathrm{sgn}\,(s) \left(\tau^0_{s(0)}, \ldots, \tau^0_{s(k)}\right) \prec \ldots \prec \tau^0_{s(n)} \\
&= (-1)^{k(k-n)} \sum_{\tau^n \succ \tau^k} (-1)^{\epsilon} \sum_{s \in S_{n-k}} \mathrm{sgn}\,(s \circ r) \left(\tau^0_{r(0)}, \ldots, \tau^0_{r(k)}\right) \prec \ldots \prec \tau^0_{s(n)}
\end{aligned}$$

if we let S_{n-k} act over set $\{k+1, \ldots, n\}$ and $r \in S_{k+1}$ act over set $\{0, \ldots, k\}$. Observe that it is exactly

$$= (-1)^{k(k-n)} \mathrm{sgn}\,(r) * \tau^k = (-1)^{k(k-n)+k+1} \mathrm{sgn}\,(r) *^{\flat} \tau^k = (-1)^{nk+1} \mathrm{sgn}\,(r) *^{\flat} \tau^k.$$

On the other hand we have to check the $i_* *^{\flat} i^*$. To do that, we are going to express the morphism i^* in coordinates. We have that

$$\langle i^* \left(\tau^{i_0} \prec \ldots \prec \tau^{i_k}\right), \sigma^k \rangle = \langle \tau^{i_0} \prec \ldots \prec \tau^{i_k}, i_* \sigma^k \rangle$$

$$= \sum_{l \in S_{k+1}} \mathrm{sgn}\,(l)\, \langle \tau^{i_0} \prec \ldots \prec \tau^{i_k}, \sigma^0_{l(0)} \prec \ldots \prec \sigma^0_{l(k)} \rangle,$$

so, as before, i^* vanishes unless the simplex is of the form $\tau^0_{r(0)} \prec \ldots \prec \tau^0_{r(k)}$, and in that case

$$i^*(\tau^0_{r(0)} \prec \ldots \prec \tau^0_{r(k)}) = \mathrm{sgn}\,(r)\, \tau^k.$$

So it follows that

$$i_* *^{\flat} i^* = (-1)^{nk+1} \cap \mathrm{sd}(\Gamma)$$

as we desired.

□

All morphisms involved in the last proposition become morphisms in homology/cohomology, and inclusions become isomorphisms. So we obtain

Corollary 4.7. *Diagram*

$$\begin{tikzcd} H^k(T) \arrow[r, "(-1)^{nk+1}*^\flat"] & H_{n-k}(*T) \arrow[d, "i_*"] \\ H^k(\mathrm{sd}(T)) \arrow[u, "i^*"] \arrow[r, "\cap\,\mathrm{sd}[\Gamma]"] & H_{n-k}(\mathrm{sd}(T)) \end{tikzcd}$$

commutes, and all morphisms are isomorphisms.

However, the important result is the following, in which we hide $\mathrm{sd}(T)$. Observe that this behaviour is the general philosophy: we used $\mathrm{sd}(T)$ because it gave us a good partial ordering of their vertices, and now we forget it.

Proposition 4.8. *Diagram*

$$\begin{tikzcd} H^k(T) \arrow[r, "(-1)^{nk+1}*^\flat"] \arrow[dr, "\cap[\Gamma]"'] & H_{n-k}(*T) \arrow[d, leftrightarrow, "\simeq"] \\ & H_{n-k}(T) \end{tikzcd}$$

commutes. Furthermore the isomorphism is the expected one: if i_T and i_{*T} are the inclusions to $\mathrm{sd}(T)$ *from T and $*T$, the isomorphism is $i_T \circ (i_{*T})^{-1}$.*

Proof. It follows directly from the previous result and from naturality of cap product. In this case, naturality implies that the diagram

$$\begin{tikzcd} H^k(T) \arrow[r, "\cap[\Gamma]"] & H_{n-k}(*T) \arrow[d, "i_*"] \\ H^k(\mathrm{sd}(T)) \arrow[u, "i^*"] \arrow[r, "\cap\,\mathrm{sd}[\Gamma]"] & H_{n-k}(\mathrm{sd}(T)) \end{tikzcd}$$

commutes. More about naturality of cap product is explained in [7] page 390. □

Remark. This is a second version of Poincaré Duality, in which $\cap[\Gamma]$ is the isomorphism.

We recall now the following relation between cup and cap products, that is: if we let $\alpha^k \in H^k(T)$, $\beta^l \in H^l(T)$ and $c_{k+l+m} \in H_{k+l+m}(T)$ then

$$(\alpha^k \cup \beta^l) \cap c_{k+l+m} = \alpha^k \cap (\beta^l \cap c_{k+l+m}) \in H_m(T)$$

which is the key, together with the last proposition, of the consequence:

Corollary 4.9. *Let $\alpha^k \in H^k(T)$, $\beta^{n-k} \in H^{n-k}(T)$, then*

$$*^\flat(\alpha^k \cup \beta^{n-k}) = (-1)^{nk}\, \alpha^k \cap *^\flat \beta^{n-k}.$$

But we know that in the case of complementary grades, if we identify $H_0\left(T\right)$ with $\mathbb{Z}$, cap product is the Kronecker index ([7], theorem 66.3).

That is the last piece we need to relate cup product with Kronecker index, and we do it with the following result

Proposition 4.10. *All squares in the next diagram commute.*

$$\begin{tikzcd}
H^k\left(T\right) \otimes H^{n-k}\left(T\right) \arrow[r, "(-1)^{nk}\cup"] \arrow[dd, "id\otimes(\simeq\circ *^\flat)"] & H^n\left(T\right) \arrow[dr, "*^\flat"] \arrow[dd, "(-1)^{n+1}\langle\cdot,\Gamma\rangle"] & \\
 & & H_0\left(*T\right) \\
H^k\left(T\right) \otimes H_k\left(T\right) \arrow[r, "\langle,\rangle"] & \mathbb{Z} \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

Keep in mind this diagram, because we will complete it when we talk about intersection.

Corollary 4.11. *For cohomology groups with coefficients over $\mathbb{Z}$, cup product in cohomology classes of complementary grades is a non-degenerated pairing of cohomology groups modulo torsion.*

4.3 Intersection

This section aims to be similar to the previous one: we are going to relate intersection product with Kronecker index. Thus we will have a relation between intersection and cup product. To do that, we will continue the work done in [10].

In this section, we will suppose the triangulation T' is a *smooth* triangulation. That is, if

$$\phi\colon T' \longrightarrow M$$

is the homeomorphism between the triangulation and the manifold, and if $\mathcal{T}^k$ is k-simplex of T', then morphism $\operatorname{int}\left(\phi|_{\mathcal{T}^k}\right) := \phi|_{\operatorname{int}(\mathcal{T}^k)}$ is a smooth map between manifolds.

Then, let us consider $T = \operatorname{sd}(T')$. Thus T is also a smooth triangulation. If we let $\mathcal{S}_\bullet\left(T\right)$ denote the smooth singular chains, then we can see T', T, $\operatorname{sd}(T)$ and $*T$ as subcomplexes of the complex of smooth singular chains. Concretely, we have an inclusion defined as

$$\begin{array}{rccc} i\colon & \mathcal{C}_\bullet\left(T\right) & \longrightarrow & \mathcal{S}_\bullet\left(M\right) \\ & \tau^k & \longmapsto & \phi|_{\tau^k} \end{array}$$

and similarly with other triangulations. These inclusions became isomorphisms when we pass to homology.

Then, we recall that we have an *intersection pairing*

$$I\colon \mathcal{S}_k\left(M\right) \otimes \mathcal{S}_{n-k}\left(M\right) \longrightarrow \mathcal{S}_0\left(M\right)$$

which is only defined over transversal pairs. However, we can extend this pairing to homology

$$I\colon H_k\left(M\right) \otimes H_{n-k}\left(M\right) \longrightarrow H_0\left(M\right) \simeq \mathbb{Z}$$

because we saw in section 4.2 of [10], that

- In every pair of homology classes there are always transversal representatives, thanks to stratified transversality theorem.
- Intersection between a cycle and a boundary is a boundary.

and so intersection is well defined in homology classes.

Now,

Definition 4.12. We define *simplicial intersection* as the only map I that makes the following diagram commute

$$\begin{tikzcd}
H_k\left(T\right) \otimes H_{n-k}\left(T\right) \arrow[dr, "I"] \arrow[dd, "\simeq"'] & \\
& H_0\left(*T\right) \\
H_k\left(M\right) \otimes H_{n-k}\left(M\right) \arrow[ur, "I"'] &
\end{tikzcd}$$

or what is the same, the composition with the inclusion:

$$I([a^i\tau_i^k],[b^j\sigma_j^{n-k}]) := I([\phi|_{a^i\tau_i^k}],[\phi|_{b^j\sigma_j^{n-k}}]) = I([a^i\ \phi|_{\tau_i^k}],[b^j\ \phi|_{\sigma_j^{n-k}}]).$$

Note. We will also denote intersection in the following way:

$$[a^i\tau_i^k]\cdot[b^j\sigma_j^{n-k}] := I([a^i\tau_i^k],[b^j\sigma_j^{n-k}]).$$

Definition 4.13. Let τ^k be a simplex of T and $\mathcal{T}^l$ be a simplex of T'. We say that τ^k is *interior* to $\mathcal{T}^l$ iff $\operatorname{int}\left(\tau^k\right) \subset \operatorname{int}\left(\mathcal{T}^l\right)$.

Observation. τ^k is interior to $\mathcal{T}^l$ iff $k \leq l$ and $\operatorname{bar}\mathcal{T}^l$ is one of the 0-simplexes of τ^k.

Proposition 4.14. *Every k-chain of T admits an homologous representative so that every k-simplex τ^k of the chain is interior to some maximal simplex $\mathcal{T}^n$ of T'.*

Proof. For every simplex τ^k of the chain, if it is already interior, we do nothing. If not, it is on some face of some maximal simplex $\mathcal{T}^n$. Then consider

$$\tau^{k+1} := \tau^k \prec \operatorname{bar}\mathcal{T}^n$$

and by adding $(-1)^{k+1}\,\partial\tau^{k+1}$ to the chain, we are removing this τ^k simplex and adding only interior simplexes. $\square$

As a consequence we get

Corollary 4.15. *If $a \in H_\bullet\left(T\right)$, then it admits representatives such that all their simplexes are interior to maximal simplexes of T'.*

Proposition 4.16. *If τ^k, k-simplex of T, is interior to a maximal simplex $\mathcal{T}^n$ of T', then $\left|*\tau^k\right| \subset \operatorname{int}\left(\mathcal{T}^n\right)$.*

Proof. All simplexes of $\left|*\tau^k\right|$ are of the form $\sigma_s^{n-k} := \tau^k \prec \tau^0_{s(k+1)} \prec \ldots \prec \tau^0_{s(n)}$ for some maximal simplex $\tau^n \succ \tau^k$ of T. Then, all 0-simplexes of σ_s^{n-k} are the barycentres of simplexes interiors to $\mathcal{T}^n$, thus $\left|\sigma_s^{n-k}\right| \subset \mathcal{T}^n$, and the result follows. $\square$

And now, the main result of this section:

Proposition 4.17. *Let τ^k be a k-simplex of T interior to some maximal simplex of T'. Then, there exists a deformation $\widetilde{\phi|_{\tau^k}}$ as small as we want of $\phi|_{\tau^k}$ so that it is transversal to the whole $i\left(\mathcal{C}_{n-k}\left(*T\right)\right) \subset \mathcal{S}_{n-k}\left(M\right)$. Furthermore,*

$$I(\widetilde{\phi|_{\tau^k}}, \phi|_{*\sigma^k}) = \begin{cases} 1 & \textit{if } \tau^k = \sigma^k \textit{ as simplexes} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. To prove that, we are going to build explicitly this deformation. First, let us see $*\tau^k$ as simplexes of $\mathrm{sd}(T)$. Now, choose one of these simplexes by choosing a maximal simplex $\tau^n \succ \tau^k$ and an element $s \in S_{n-k}$, so that we choose the simplex $\sigma_s^{n-k} := \tau^k \prec \tau^0_{s(k+1)} \prec \ldots \prec \tau^0_{s(n)}$. Choose a point $a \in \operatorname{int}\left(\sigma_s^{n-k}\right)$. Then, we are going to build a deformation of τ^k such that:

- Deformation happens in τ^n and only deforms a neighbourhood of $\operatorname{bar}\tau^k$, so around its boundary $\partial\tau^k$ deformation does not take place.
- Deformation intersects $*\tau^k$ transversality in point a with positive sign, if we consider the orientations induced by $\mathbb{R}^n$, H_1 and H_2.

To do that, we put τ^n inside $\mathbb{R}^n$, with canonical basis $\{e_i\}_{i\leq n}$, in a way that we send $\operatorname{bar}\tau^k = 0$, $\tau_i^0 = e_{i+1}$ for $i < k$ and $\tau_i^0 = e_i$ for $i > k$. Then, let call

$$H_1 := \langle e_i \rangle_{i\leq k}$$

so that $\tau^k \subset H_1$ and

$$H_2 := \langle e_i \rangle_{i>k}.$$

Observe that $\sigma_s^{n-k} \subset H_2$, and thus its intersection with τ^k corresponds to the origin, $\operatorname{bar}\tau^k$. Since $H_1 \oplus H_2 = \mathbb{R}^n$, we see that if $\operatorname{bar}\tau^k$ were an interior point of σ_s^{n-k}, the intersection would be transversal.

The idea is to "push up" τ^k around the origin direction to point a so that deformation carries $\operatorname{bar}\tau^k$ to a. Now, let

$$b_\epsilon \colon H_1 \longrightarrow \mathbb{R}$$

be a bump function such that

- $b_\epsilon(x^1) = 0$ for $|x^1| > \epsilon$.
- $b_\epsilon(x^1) = 1$ for $|x^1| < \frac{\epsilon}{2}$.
- $0 \leq b_\epsilon \leq 1$.

Now, we propose the following deformation:

$$\begin{array}{rccl} g\colon & \mathbb{R}^n \times I = (H_1 \oplus H_2) \times I & \longrightarrow & \mathbb{R}^n = H_1 \oplus H_2 \\ & (x,t) = (x^1, x^2, t) & \longmapsto & g_t(x) = \left(g_t^1(x^1,x^2), g_t^2(x^1,x^2)\right) \end{array}$$

defined in the following way:

- $g_t(x^1, x^2) = g_t(x^1, 0) + Ax^2$, where A is a $(n-k) \times (n-k)$ matrix. For our purposes, we can suppose $A = Id$.
- $g_t^2(x^1, 0) = tb_\epsilon(x^1) \cdot a$
- $g_t^1(x^1, 0) = x^1$.

Thus, for a fixed t

$$Dg_t = \begin{pmatrix} Id & 0 \\ tb'_\epsilon \cdot a & A \end{pmatrix}.$$

So, if we have chosen $A = Id$, we have that determinant of Dg_t is positive. We can also check that

- g_t is a diffeomorphism.
- $g_0 = id$, $g_1(0) = a$ and $g_t(x) = x$ for $x^1 > \epsilon$.
- $g_t(H_1) \cap H_2 = \{ta\}$, since g_t only "moves" a point by direction a. In particular, $g_1(H1) \cap H_2 = \{a\}$, and the intersection continues being transversal and positive.
- For a small enough $\epsilon > 0$, $g_t(\tau^k) \subset \tau^n$.

Secondly, we are going to carry this deformation to simplicial chains. We have that σ_s^{n-k} and $\operatorname{int}\left(\tau^k\right)$ are in the interior of maximal simplex $\mathcal{T}^n$ of T', so homeomorphism ϕ restricted to its interior is smooth and injective. Now, choose a point a of σ_s^{n-k} so that $D\phi$ has non-vanishing determinant (by injectivity, they form a dense subset). Since ϕ conserves orientations, this determinant would be positive. Then, we define

$$\widetilde{\phi|_{\tau^k}} = \phi|_{g_1(\tau^k)}.$$

Now, is it clear that $\widetilde{\phi|_{\tau^k}}$ and $\phi|_{*\tau^k}$ intersect transversaly in $\{a\}$. The sign is positive for the following reason:

Orientation of τ^k and τ^n with respect orientation induced by H_1 and $\mathbb{R}^n$ is $(-1)^k$. Orientation of τ^n respect orientation of T is, by definition, $(-1)^\epsilon$. Orientation of σ_s^{n-k} with respect H_2 is positive, while with respect $*\tau^k$ is $(-1)^\epsilon \operatorname{sgn}(s)$. So if we do all these changes, we see that intersection in $\mathbb{R}^n$ has the same sign as intersections in T. Since $D\phi$ in a is positive, the sign of intersection is positive.

Finally, we have to see that $\widetilde{\phi|_{\tau^k}}$ does not intersect any other $*\sigma^k$. That is because $g_1(\tau^k)$ lies in τ^n, so at most, it can intersect $\frac{(n+1)!}{(k+1)!}$ of them. But we know τ^k only intersects $*\tau^k$ and by choosing a and ϵ, we can do the deformation as small as we want.

$\square$

This proposition gives us the relation with Kronecker index:

Corollary 4.18. *The following diagram commutes:*

$$\begin{tikzcd}
H^k(T) \otimes H_k(T) \arrow[r, "\langle , \rangle"] \arrow[d, "*^\flat \otimes id"] & \mathbb{Z} \\
H_{n-k}(*T) \otimes H_k(T) \arrow[r, "(-1)^{nk+1} I"] & H_0(T) \arrow[u, "\simeq"]
\end{tikzcd}$$

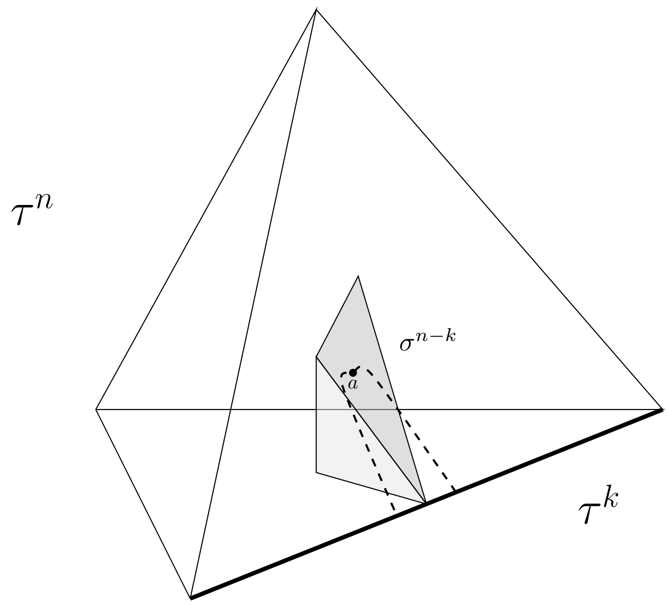

Figure 2: Transversal deformation $g_1(\tau^k)$ of τ^k

Proof. Let $[b^j \sigma_j^k] \in H^k(T)$ and $[a^i \tau_i^k] \in H_k(T)$. By previous results, we can suppose that $a^i \tau_i^k$ is an interior representative. Then

$$\begin{aligned}
I(*^\flat [b^j \sigma_j^k], [a^i \tau_i^k]) \qquad &= (-1)^{k(n-k)} \, I([a^i \tau_i^k], [b^j *^\flat \sigma_j^k]) \\
&= (-1)^{k(n-k)} \, I([a^i \ \phi|_{\tau_i^k}], [b^j \ \phi|_{*^\flat \sigma_j^k}]) \\
&= (-1)^{k(n-k)} \, I([a^i \ \widetilde{\phi|_{\tau_i^k}}], [b^j \ \phi|_{*^\flat \sigma_j^k}]) \\
&= (-1)^{k(n-k)} \, I(a^i \ \widetilde{\phi|_{\tau_i^k}}, b^j \ \phi|_{*^\flat \sigma_j^k}) \\
&= (-1)^{k(n-k)+k+1} \, a^i b^i I(\widetilde{\phi|_{\tau_i^k}}, \phi|_{*\sigma_j^k}) \\
&= (-1)^{nk+1} \, a^i b^j \langle \sigma_j^k, \tau_i^k \rangle \cdot a_{i,j}
\end{aligned}$$

where $a_{i,j}$ are intersection points. Now, if we sum coefficients we get

$$(-1)^{nk+1} \, a^i b^j \langle \sigma_j^k, \tau_i^k \rangle = (-1)^{nk+1} \, \langle b^j \sigma_j^k, a^i \tau_i^k \rangle = (-1)^{nk+1} \, \langle [b^j \sigma_j^k], [a^i \tau_i^k] \rangle$$

as we desired. $\square$

This is the relation between intersection and the Kronecker index. So, we can now complete the diagram of proposition 4.10:

Corollary 4.19. *All squares in the following diagram commute.*

$$
\begin{tikzcd}
H^k(T) \otimes H^{n-k}(T) \arrow[r, "(-1)^{nk}\cup"] \arrow[d, "id\otimes(\simeq\circ *^\flat)"] & H^n(T) \arrow[d, "(-1)^{n+1}\langle\cdot,\Gamma\rangle"] \arrow[dr, bend left, "*^\flat"] & \\
H^k(T) \otimes H_k(T) \arrow[r, "\langle,\rangle"] \arrow[d, "*^\flat\otimes i"] & \mathbb{Z} \arrow[r, leftrightarrow, "\simeq"'] & H_0(*T) \arrow[dl, bend left, "\simeq"] \\
H_{n-k}(*T) \otimes H_k(T) \arrow[r, "(-1)^{nk+1}I"] & H_0(T) \arrow[u, "\simeq"] &
\end{tikzcd}
$$

So, finally, we have found the relation between cup product and intersection in homology classes.

Corollary 4.20. *If homology groups $H_\bullet(T)$ are free, intersection product in homology classes of complementary grades is a not-degenerated pairing.*

Remark. If we take coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$ we get the same result, and the torsion part of homology groups disappears. So we get that the intersection product is a non-degenerated pairing.

Corollary 4.21. *With coefficients in $\mathbb{Z}$, the intersection product in homology classes module torsion of complementary grades is a non-degenerated pairing.*

5 Intersection in any grades

5.1 Multiple intersection

The goal of this section is to generalize intersection product to multiple homology classes. As in the case of the intersection pairing, we have that they must satisfy a generalization of "complementary grades" condition. Specifically, if we have r homology classes to intersect, and k_i are their grades, the condition is that

$$\sum k_i = (r-1)n$$

and it is equivalent to

$$\sum (n-k_i) = n$$

where n is the dimension of the manifold M.

To define multiple intersection in homology classes, we will proceed as in the case of the pairing, and we start by defining intersection in smooth singular chains:

$$I\colon \mathcal{S}_{k_1}(M) \otimes \cdots \otimes \mathcal{S}_{k_r}(M) \longrightarrow \mathcal{S}_0(M)$$

only in transversal tuples.

We start by recalling the notion of transversality:

Note. We denote de *diagonal* as $\Delta_r(M) := \left\{(x^1, \ldots, x^r) \in M^r \mid x^1 = \cdots = x^r\right\}$. We also will denote $(f^i) := f^1 \times \cdots \times f^r$, $(X_i) := X_1 \times \cdots \times X_r$ and $(x^i) := (x^1, \ldots, x^r)$.

Definition 5.1. Let X_i be manifolds, and $f_i\colon X_i \to M$ smooth maps. Then, we say f_i are transversal iff

$$T_{(f_i)(x^i)}\Delta_r(M) + T_{(x^i)}(f_i)T_{(x^i)}(X_i) = T_{(f_i)(x^i)}M^r$$

for all $(x^i) \in (X_i)$ so that $(f_i)(x^i) \in \Delta_r(M)$.

Observe that, by dimensionality, to be transversal they must satisfy

$$\sum \dim(X_i) \geq (r-1)n$$

or not to intersect with the diagonal. Observe also, that in case of "complementary grades" condition, the sum of spaces becomes a direct sum.

There happens that if (f_i) are transversal, $(f_i)^{-1}(\Delta_r(M))$ is a submanifold of (X_i) of codimension the sum of codiemnsions, or what is the same, of dimension $\sum \dim(X_i) - (r-1)n$.

Note. Since we are going to work in simplicial chains, let

$$\tau^k\colon \Delta^k \longrightarrow M$$

denote a smooth simplex, where Δ^k is the standard k-simplex. We may or may not indicate its dimension.

Observe that smooth singular simplexes are (stratified) smooth functions from (stratified) manifolds to a manifold. So we can define over them (a stratified version of) transversality notion.

Definition 5.2. Let τ_i be smooth simplexes. Then, we say they are *transversal* as simplexes if the restriction to their interiors are transversal as smooth function between manifolds and if the restriction to any face of any of the τ_i are transversal as a simplexes.

This definition proceed by induction over the sum of dimensions. We extend this definition to chains by saying that they are transversal if simplexes involved are.

Definition 5.3. Let τ_i be transversal smooth simplexes satisfying the complementary grades condition $\sum k_i = (r-1)n$. Then we define its *multiple intersection*

$$I^r : \mathcal{S}_{k_1}(M) \otimes \cdots \otimes \mathcal{S}_{k_r}(M) \longrightarrow \mathcal{S}_0(M)$$

by

$$I^r(\tau_i) = I^r(\tau_1, \ldots, \tau_r) := \sum_{(p) \in (\tau_i)^{-1}((\tau_i)(\Delta_i) \cap \Delta_r(M))} (-1)^\theta \, p$$

where the sign $(-1)^\theta$ is either positive or negative if both sides of the equation

$$T_{(\tau_i)(x^i)} \Delta_r(M) \oplus T_{(x^i)}(\tau_i) T_{(x^i)}(\Delta_i) = T_{(\tau_i)(x^i)} M^r$$

have or not the same orientation.

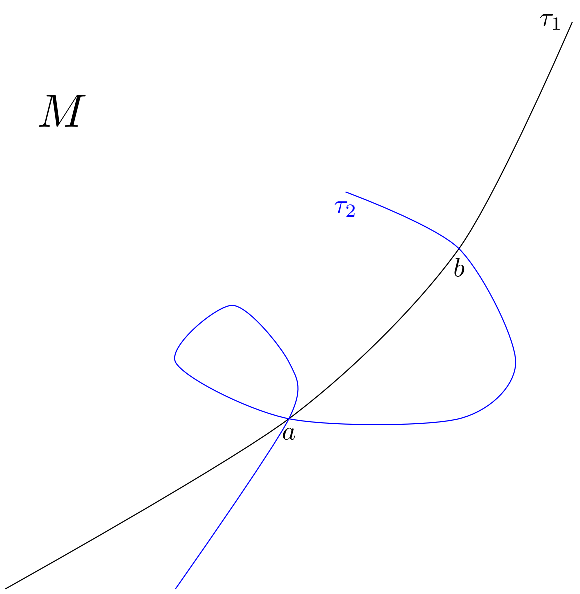

Figure 3: In the example, τ_1 and τ_2 are transversal. Moreover, with appropriate orientations of M, τ_1 and τ_2, $I^2(\tau_1, \tau_2) = +a - a + b = b$.

Observation. The previous definition is well defined over transversal smooth simplexes, but we have to check several things. First, tangent space $T_{(x^i)}(\Delta_i)$ is well defined, because by transversality plus dimensionality, all points (x^i) belong to $(\mathrm{int}\,(\Delta_i))$, which is a manifold.

Lemma 5.4. *The resultant* 0*-chain of* I^r *is finite.*

Proof. Since M is Hausdorff, and standard simplexes are compact, τ_i are closed as maps. Thus so is (τ_i). Since diagonal is closed, $(\tau_i)(\Delta_i) \cap \Delta_r(M)$ is closed and then all $(x_i) \in (\tau_i)^{-1}((\tau_i)(\Delta_i) \cap \Delta_r(M))$ form a closed set.

But, by transversality, these points are a 0-manifold, which means, they are isolated points. Since all of them belong to a compact set, there must be a finite number of them. $\square$

This definition extends to transversal chains.

Observation. In the case of $r = 1$, I^r is the identity.

Remark. Let us consider a positive basis of $T_{\tau_i(x^i)}M$ and a positive basis of $T_{x_i}\Delta_i$. Then, sign $(-1)^\theta$ is the sign of the determinant of matrix

$$\begin{pmatrix} \boxed{Id} & \boxed{D\tau_1} & & \\ \vdots & & \ddots & \\ \boxed{Id} & & & \boxed{D\tau_r} \end{pmatrix}$$

at points p, where $(p, \ldots, p) \in (\tau_i)^{-1}((\tau_i)(\Delta_i) \cap \Delta_r(M))$.

In the case of $r = 2$, this definition of transversality is not exactly the definition we had in [10]: they differ in a sign. Concretely, in intersection defined in [10] we considered the sign of the determinant

$$\begin{pmatrix} \boxed{D\tau_1} & \boxed{D\tau_2} \end{pmatrix}$$

that differs in a sign $(-1)^k$ from determinant of

$$\begin{pmatrix} \boxed{Id} & \boxed{D\tau_1} & \\ \boxed{Id} & & \boxed{D\tau_2} \end{pmatrix}$$

where $k_1 := k$ and then $k_2 := n - k$.

Proposition 5.5. *Multiple intersection is commutative in the following way:*

$$I^r(\tau_1^{k_1}, \ldots, \tau_i^{k_i}, \tau_{i+1}^{k_{i+1}}, \ldots, \tau_r^{k_r}) = (-1)^{n^2 + k_i k_{i+1}} I^r(\tau_1^{k_1}, \ldots, \tau_{i+1}^{k_{i+1}}, \tau_i^{k_i}, \ldots, \tau_r^{k_r}).$$

Proof. Of course, being transversal does not depends of the sorting of τ_i. To check the sign we must see how the sign changes when we permute boxes $D\tau_i$ and $D\tau_{i+1}$ in the determinant of matrix

$$\begin{pmatrix} \boxed{Id} & \boxed{D\tau_r} & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & \boxed{D\tau_i} & & \\ \vdots & & & & \boxed{D\tau_{i+1}} & \\ \vdots & & & & & \ddots & \\ \boxed{Id} & & & & & & \boxed{D\tau_r} \end{pmatrix}$$

$\square$

Now, our goal is to see that we can extend it homology to classes. To do that, we will proceed as in [10], that is, first show that for a tuple of cycles there are homologous ones that are transversal, and secondly, see that the intersection of transversal cycles, with one of them a boundary, gives a boundary.

5.1.1 Transversality of stratifications

Now we are going to do a little introduction to stratifications simply to justify that we can always think that "almost all" simplexes are transversal. All definitions and results we are going to introduce are in [3], pages 36-38.

Definition 5.6. Let Z be a topological space and $\mathscr{S}$ a partially ordered set. A *$\mathscr{S}$-decomposition* of Z is a locally finite family of disjoint locally closed subsets called *pieces* $S_i \subset Z$, with $i \in \mathscr{S}$, so that

(1). $Z = \bigcup_{i\in\mathscr{S}} S_i$.

(2). $i \leq j \Leftrightarrow S_i \cap \overline{S_j} \neq \emptyset \Leftrightarrow S_i \subset \overline{S_j}$.

In the case of $i < j$ we will denote $S_i < S_j$.

As examples of decompositions we have:

- A manifold Z, with $\mathscr{S} = \{0\}$ and $S_0 = Z$.
- More generally, a manifold Z with boundary, with $\mathscr{S} = \{0, 1\}$ and $S_0 = \partial Z$, $S_1 = \operatorname{int}(Z)$.
- Even more generally, a manifold Z with corners, with $\mathscr{S} = \{0, \cdots, \dim Z\}$ and S_i =interior of corner of codimension $\dim Z - i$. As a particular case we have cartesian product of simplexes.
- A triangulation, with $\mathscr{S} = \{0, \cdots, \dim Z\}$ and and S_i =interior of simplexes of codimension $\dim Z - i$

Definition 5.7. Now let A and Z be two $\mathscr{S}$-decomposed spaces with

$$Z = \bigcup_{i\in\mathscr{S}} S_i \quad \text{and} \quad A = \bigcup_{i\in\mathscr{S}} R_i.$$

A *decomposed map* $f: A \to Z$ is a continuous map so that $f(R_i) \subset S_i$ for all $i \in \mathscr{S}$.

Definition 5.8. Now, let Z be a closed subset of a manifold M and $Z = \bigcup_{i\in\mathscr{S}} S_i$ a $\mathscr{S}$-decomposition. Then, we say that this decomposition is a *Whitney stratification* if

(1). Every piece S_i is a locally closed smooth submanifold of M.

(2). Every pair $S_\alpha < S_\beta$ satisfies the *Whitney conditions A and B*: let $x_i \in S_\beta$ and $y_j \in S_\alpha$ be sequences of points converging to a point $y \in S_\alpha$ and suppose that (in some coordinate system) the sequence of lines $l_i = \overline{x_i y_i}$ converges to a line l, and tangent spaces $T_{x_i} S_\beta$ converge to a tangent space π. Then:

(WC A) $T_y S_\alpha \subset \pi$.

(WC B) $l \subset \pi$.

All examples seen before are Whitney stratifications. Observe that all these objects we have introduced are a generalization of objects we work with: standard simplexes are in particular Whitney stratifications. So in some way it is natural to define a generalization of definition 5.2.

Definition 5.9. Let $Z_1 \subset M_1$ and $Z_2 \subset M_2$ be Whitney subsets of smooth manifolds and $f\colon M_1 \to M_2$ a smooth map. Then we say that the restriction $f|_{Z_1} : Z_1 \to M_2$ is *transverse* to Z_2 if for each stratum A of Z_1 and every stratum B of Z_2 the map $f|_A : A \to M_2$ is transverse to B, or what is the same,

$$T_x f\, T_x A + T_{f(x)} B = T_{f(x)} M_2.$$

And we have just arrived to the point that enables us to show the main result of the section:

Proposition 5.10 (Proposition in section 1.3.2 of [3])**.** *If $Z_1 \subset M_1$ and $Z_2 \subset M_2$ are closed subsets with Whitney stratifications, then*

$$T = \left\{ f \in \mathcal{C}^\infty(M_1, M_2) \mid f|_{Z_1} \text{ is transverse to } Z_2 \right\}$$

is open and dense (in the Whitney $\mathcal{C}^\infty$ topology) in $\mathcal{C}^\infty(M_1, M_2)$.

To understand with more precision what this theorem is exactly saying, we remit to *goresky*.

Corollary 5.11. *If we have a non-transversal map, we can make a as-small-as-we-want deformation to get a transversal one.*

5.1.2 Homology invariance

Now we are going to see the homology invariance.

Proposition 5.12. *Let τ_i be transversal smooth simplexes so that their dimensions sum $(r-1)n+1$. Then $(\tau_i)^{-1}\left((\tau_i)(\Delta_i) \cap \Delta_r(M)\right) \in (\Delta_i)$ is a compact 1-manifold with boundary. Moreover, its boundary is $(\tau_i)^{-1}\left((\tau_i)(\delta(\Delta_i)) \cap \Delta_r(M)\right)$.*

Proof. (*Ref. theorem on page 60 of [4] for a comparison*)

By using the same arguments as before, we know that preimage is closed and compact. By transversality, if we restrict to the interior, we are in the case of smooth manifolds and then the preimage is a family of disjoint 1-manifolds. Also by transversality, we know that intersection also can happen in the interior of a face of codimension 1 of one simplex, and in the interior of all others simplexes. Furthermore, in this case, intersection is a finite set of isolated points. Because of this, the intersection on the boundary is a finite set of isolated points.

Thus we have to check that two interior 1-manifolds do not join in one of these isolated points, and that the isolated points are end points of a 1-manifold. To do that, we use again transversality.

Let (x^i) be one of these isolated points. Then, locally, in some coordinate system, (τ_i) is $f\colon U \subset H^{(r-1)n+1} \to \mathbb{R}^{(r-1)n} \times \mathbb{R}^n$ with U a neighbourhood of 0, and where $H^l := \left\{ (x^i) \in \mathbb{R}^l \mid x^1 \geq 0 \right\}$ is the upper semispace. By smoothness of f, it can be extended to neighbourhood of 0 in $\mathbb{R}^{(r-1)n+1}$.

Let us call δf the restriction of f to $\delta H^{(r-1)n+1}$. Then, by transversality we have

$$T_0\Delta_r(M) \oplus T_0\delta f \;\; T_0\delta H^{(r-1)n+1} = T_0\mathbb{R}^{rn}$$

and thus

$$T_0\delta f \;\; T_0\delta H^{(r-1)n+1} = T_0\mathbb{R}^{(r-1)n}.$$

This means that the extended function f is also transversal at point 0, so preimage is a 1-manifold. Thus (x^i) is the end point of exactly one 1-manifold. □

Corollary 5.13. *Under the above assumptions the sum of coefficients of* $I^r(\partial\tau_1, \ldots, \tau_r) + \cdots + I^r(\tau_1, \ldots, \partial\tau_r)$ *vanishes.*

Proposition 5.14. *Let a_i be transversal cycles, with a_r a boundary, such that they satisfy the complementary grades condition $\sum k_i = (r-1)n$. Let*

$$\epsilon\colon \mathcal{S}_0(M) \longrightarrow \mathbb{Z}$$

the map which sums the coefficients. Then,

$$\epsilon \circ I^r(a_i) = 0.$$

Proof. We can suppose, without loss of generality, that $a_r = \partial\sigma$ where σ is a smooth simplex. Since σ is transversal on its boundary, it is in a neighbourhood too. So we can make a deformation of σ small enough so that it does not change its boundary and is transversal.

By linearity of ϵ and I, and by Corollary 5.13 we have that

$$\epsilon \circ I^r(\partial a_1, a_2, \ldots, \sigma) + \cdots + \epsilon \circ I^r(a_1, \ldots, a_{r-1}, \partial\sigma) = 0$$

but a_i are cycles, so

$$\epsilon \circ I^r(a_1, \ldots, \partial\sigma) = \epsilon \circ I^r(a_1, \ldots, a_r) = 0$$

as we wanted to see. □

But the sum of coefficients equal to 0 is exactly the expression of boundaries if 0-simplexes belong to the same connected component, so

Corollary 5.15. *Let a_i be transversal cycles, with some of them a boundary, such that they satisfy the complementary grades condition $\sum k_i = (r-1)n$. Then $I(a_i)$ is a boundary.*

And finally we reach the result we were looking for:

Corollary 5.16. *Multiple intersection is well defined over homology classes. That is, we have well defined intersection mapping*

$$I^r\colon H_{k_1}(M) \otimes \cdots \otimes H_{k_r}(M) \longrightarrow H_0(M)$$

where $\sum k_i = (r-1)n$.

5.2 Intersection in any grades

We have defined multiple intersection of degree 0 in homology classes. The reason to do that is to be able to define intersection in any grades, which is the aim of this subsection. We will work throughout this section with $\mathbb{Z}$ coefficients as group of coefficients, and when we refer to homology classes as their free part, i.e. $H_\bullet(M) := \frac{H_\bullet(M)}{\operatorname{Tor} H_\bullet(M)}$.

Definition 5.17. We define *intersection*

$$\begin{array}{rccl} I\colon & H_{k_1}(M)\otimes\cdots\otimes H_{k_r}(M) & \longrightarrow & H_{k_1+\cdots+k_r-(r-1)n}(M) \\ & a_1\otimes\cdots\otimes a_r & \longmapsto & I(a_i)=I(a_1,\ldots,a_r) \end{array}$$

where $k_1+\cdots+k_r\geq(r-1)n$ as the only homology class such that

$$I^2(I(a_i),a_{r+1})=I^{r+1}(a_1,\ldots,a_{r+1})$$

for all $a_{r+1}\in H_{rn-k_1-\cdots-k_r}(M)$.

Observation. We have that

$$H_k(M)\overset{\kappa}{\simeq}H^k(M)\overset{*^\flat}{\simeq}H_{n-k}(M).$$

Moreover, we have seen that under these assumptions, intersection is a non-degenerated pairing. So there exists a unique class $I(a_i)$.

Observation. Intersection in any grades I commutes in the same way as I^r, that is

$$I(\tau_1^{k_1},\ldots,\tau_i^{k_i},\tau_{i+1}^{k_{i+1}},\ldots,\tau_r^{k_r})=(-1)^{n^2+k_ik_{i+1}}\,I(\tau_1^{k_1},\ldots,\tau_{i+1}^{k_{i+1}},\tau_i^{k_i},\ldots,\tau_r^{k_r}).$$

.

Proposition 5.18. *Let*

$$I^r\colon H_{k_1}(M)\otimes\cdots\otimes H_{k_r}(M)\longrightarrow H_0(M)$$

and

$$I^{r+1}\colon H_{k_1}(M)\otimes\cdots\otimes H_{k_r}(M)\otimes H_n(M)\longrightarrow H_0(M)$$

denote de multiple intersection of degree 0 *of* r *and* $r+1$ *homology classes. Let* $[\Gamma]$ *be the positive generator of* $H_n(M)\simeq\mathbb{Z}$. *Then*

$$I^r(\cdot,\ldots,\cdot)=I^{r+1}(\cdot,\ldots,\cdot,[\Gamma]).$$

Proof. In fact, the proposition is true in smooth singular transversal chains: first recall that for every transversal r-tuple of simplexes in (τ_i) in $\mathcal{S}_{k_1}(M)\otimes\cdots\otimes\mathcal{S}_{k_r}(M)$ we have that the set of points (x^i) such that $(\tau_i)(x^i)\in\Delta_r(M)$ is finite. Thus there exits a representative Γ so that is transversal. Then the set of points (x^i) still being the same, with a last component x^{r+1} of the interior of the standard simplex Δ^n added.

Then we just have to look for the sign. Fix a point (x^i), and let $\tau_{r+1}^n\colon\Delta_{r+1}^n\to M$ be the simplex of Γ which contains x^{r+1} in its domain. Then we recall for the definition that the sign $(-1)^\theta$ depends on the conservation or not conservation of the orientation on both sides of the equation

$$T_{(\tau_i)(x^i)}\Delta_r(M)\oplus T_{(x^i)}(\tau_i)T_{(x^i)}(\Delta_i)=T_{(\tau_i)(x^i)}M^r.$$

Since Γ is a positive representative, the equation,

$$T_{x^{r+1}}\tau_{r+1}^{n}T_{x^{r+1}}\Delta_{r+1}^{n} = T_{\tau_{r+1}^{n}(x^{r+1})}M$$

conserves the signs. So sign $(-1)^{\theta}$ still depends on the equation

$$\left(T_{(\tau_i)(x^i)}\Delta_r\left(M\right)\times\{0\}^n\right)\oplus T_{(x^i)}(\tau_i)T_{(x^i)}(\Delta_i)\oplus T_{x^{r+1}}\tau_{r+1}^{n}T_{x^{r+1}}\Delta_{r+1}^{n} = T_{(\tau_i)(x^i)}M^{r+1}$$

and, since $T_aA\oplus T_bB\simeq T_{(a,b)}A\times B$, it is equivalent to

$$T_{(\tau_i)(x^i)}\Delta_r\left(M\right)\times\{0\}^n\oplus T_{(x^i)}(\tau_i)T_{(x^i)}(\Delta_i) = T_{(\tau_i)(x^i)}M^{r+1}.$$

Now, proving that we can replace $\Delta_r\left(M\right)\times\{0\}^n$ by $\Delta_{r+1}\left(M\right)$ is equivalent to checking that matrixes

$$\begin{pmatrix} \boxed{Id} & \boxed{D\tau_1} & & & \\ \vdots & & \ddots & & \\ \boxed{Id} & & & \boxed{D\tau_r} & \\ & & & & \boxed{D\tau_{r+1}} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{Id} & \boxed{D\tau_1} & & & \\ \vdots & & \ddots & & \\ \boxed{Id} & & & \boxed{D\tau_r} & \\ \boxed{Id} & & & & \boxed{D\tau_{r+1}} \end{pmatrix}$$

have the same determinant. $\square$

What this proposition tells us is that, if one of the homology groups we intersect is $H_n\left(M\right)$, then this component does not really give us information and we can omit it. It has several consequences:

Corollary 5.19. *The intersection in any grades I coincide with intersection I^r of degree 0. That is, if $k_1+\cdots+k_r=(r-1)n$, then*

$$I(a_i) = I^r(a_i).$$

Corollary 5.20. *If we consider intersection in any grades*

$$I\colon H_{k_1}\left(M\right)\otimes\cdots\otimes H_{k_{r-1}}\left(M\right)\otimes H_{k_r}\left(M\right)\longrightarrow H_{k_1+\cdots+k_r-(r-1)n}\left(M\right)$$

with $k_r=n$, then

$$I(a_1,\ldots,a_r) = (-1)^{n(k_1+\cdots+k_r-(r-1)n)}\,I(a_1,\ldots,a_{r-1}).$$

Proof. Let us call

$$k_{r+1} := nr-k_1-\cdots-k_r = n(r-1)-k_1-\cdots-k_{r-1}.$$

For all $b\in H_{k_{r+1}}\left(M\right)$ we have

$$I^2(I(a_1,\ldots,a_{r-1},[\Gamma]),b) = I^{r+1}(a_1,\ldots,a_{r-1},[\Gamma],b)$$

$$= (-1)^{n(n+k_{r+1})}\,I^r(a_1,\ldots,a_{r-1},b) = (-1)^{n(n+k_{r+1})}\,I^2(I(a_1,\ldots,a_{r-1}),b).$$

$\square$

So we have translated proposition 5.18 to intersection in any grades.

Observation. Let $0 \leq k_i \leq n$ for all i. If there exists $l \leq r$ so that $k_1 + \cdots + k_l = (l-1)n$ then $a_i = \alpha_i[\Gamma]$ for all $i > l$ and

$$I^r(a_1, \ldots, a_r) = \alpha_{l+1} \cdots \alpha_r I^l(a_1, \ldots, a_l)$$

by proposition 5.18. This is also true in the case of intersection in any grades, with some sign. When we have the particular case $l = 1$, we have that intersection with $[\Gamma]$ becomes the identity.

From this follows that "interesting cases" of multiple intersection are when $k_1 + \cdots + k_l > (l-1)n$ for all $l \leq r$. In particular it implies $0 < k_i < n$. Since we have the requirement $k_1 + \cdots + k_r \geq (r-1)n$ and $k_1 + \cdots + k_r \leq (n-1)r$, we have that $r \leq n$. The limit case $r = n$ is the case of intersection of n hypersurfaces.

In conclusion, we observe that if we consider only "interesting intersections", which means intersection without points or total space, their multiplicities have to be at least 2 and at most n.

6 Examples

6.1 Mapping tori

Definition 6.1. Let X some manifold, and $\phi\colon X \longrightarrow X$ a diffeomorphism. Then, the *mapping torus* is the topological space defined as

$$M_\phi := \frac{X \times I}{(x,0) \sim (\phi(x),1)}$$

together with the natural projection $\pi\colon M_\phi \to \mathrm{S}^1$.

It has the following properties (see [9]):

- M_ϕ is a manifold of dimension $\dim X + 1$.
- If X is orientable and ϕ preserves the orientation, M_ϕ is orientable.
- M_ϕ is a smooth fibration over S^1 locally trivial, where fibres are isomorphic to X.
- If $\widetilde{\phi}$ is diffeotopic to ϕ, then $M_{\widetilde{\phi}}$ is diffeomorphic to M_ϕ.

These facts allow us to compute homology groups of M_ϕ as a function of ϕ and homology groups of X, by Mayer-Vietoris exact sequence.

Let us use $\mathrm{S}^1 = \frac{\mathbb{R}}{x\sim x+1}$ with orientation given by $\mathbb{R}$. Then we highlight the cardinal points $E := 0$, $N := \frac{1}{4}$, $W := \frac{1}{2}$, $S := \frac{3}{4}$. Now, for some $\epsilon > 0$, let

$$\widetilde{U} := (E-\epsilon, W+\epsilon) \text{ and } \widetilde{V} := (W-\epsilon, E+\epsilon)$$

and then we define

$$U := \pi^{-1}(\widetilde{U}) \text{ and } V := \pi^{-1}(\widetilde{V})$$

where $\pi\colon M_\phi \to \mathrm{S}^1$ is the projection. Since the base of U and V are contractible, they are the trivial bundle. In fact, they are contractible to the fibres $X_N := \pi^{-1}(N)$ and $X_S := \pi^{-1}(S)$ respectively. The same happens with both connected components of the intersection

$$U \cap V = \pi^{-1}(W-\epsilon, W+\epsilon) \sqcup \pi^{-1}(E-\epsilon, E+\epsilon)$$

which are contractible respectively to $X_W := \pi^{-1}(W)$ and $X_N := \pi^{-1}(E)$. We may regard M_ϕ as the gluing of $X \times \widetilde{U}$ and $x \times \widetilde{V}$ by identity around W and by ϕ around E. Since $M_\phi = U \cup V$, we have the Mayer-Vietoris sequence

$$0 \longrightarrow \mathcal{C}_\bullet(U \cap V) \longrightarrow \mathcal{C}_\bullet(U) \otimes \mathcal{C}_\bullet(V) \longrightarrow \mathcal{C}_\bullet(M_\phi) \longrightarrow 0$$

which, by contractions, can be reduced to

$$0 \longrightarrow \mathcal{C}_\bullet(X_W \sqcup X_E) \xrightarrow{i} \mathcal{C}_\bullet(X_N) \otimes \mathcal{C}_\bullet(X_S) \xrightarrow{\pi} \mathcal{C}_\bullet(M_\phi) \longrightarrow 0$$

with

$$i(a_W, b_E) = (a_N + b_N, a_N + \phi_\bullet b_N)$$

and

$$\pi(a_N, b_S) = a_W - b_W.$$

Then, the exact long homology sequence determines the homology of M_ϕ.

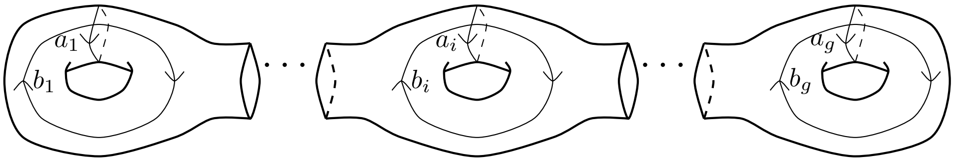

Figure 4: Surface Σ.

We are going to compute homology of M_ϕ with $X = \Sigma$ a compact connected orientable surface of genus g and ϕ a oriented diffeomorphism. Observe that due to diffeotopy invariance, we regard ϕ in

$$M(g,0) := \frac{\mathrm{Diff}^+(\Sigma_g)}{\mathrm{Diff}_0(\Sigma_g)},$$

which is called the *mapping class group* of genus g (see [2], chapter 4).

We have the long exact sequence

$$0 \longrightarrow H_3(M_\phi) \xrightarrow{\delta} H_2(\Sigma_W \sqcup \Sigma_E) \xrightarrow{i_2} H_2(\Sigma_N) \oplus H_2(\Sigma_S) \xrightarrow{\pi_2} H_2(M_\phi)$$

$$\xrightarrow{\delta} H_1(\Sigma_W \sqcup \Sigma_E) \xrightarrow{i_1} H_1(\Sigma_N) \oplus H_1(\Sigma_S) \xrightarrow{\pi_1} H_1(M_\phi) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H_0(\Sigma_W \sqcup \Sigma_E) \xrightarrow{i_0} H_0(\Sigma_N) \oplus H_0(\Sigma_S) \xrightarrow{\pi_0} H_0(M_\phi) \xrightarrow{\delta} 0$$

which splits in three exact sequences

$$0 \longrightarrow H_3(M_\phi) \longrightarrow \ker i_2 \longrightarrow 0,$$

$$0 \longrightarrow \frac{H_2(\Sigma_N) \oplus H_2(\Sigma_S)}{\ker \pi_2} \xrightarrow{\tilde{\pi}_2} H_2(M_\phi) \xrightarrow{\delta_2} H_1(\Sigma_W \sqcup \Sigma_E)$$

$$\xrightarrow{i_1} H_1(\Sigma_N) \oplus H_1(\Sigma_S) \xrightarrow{\pi_1} H_1(M_\phi) \xrightarrow{\delta} \ker i_0 \longrightarrow 0$$

and

$$0 \longrightarrow \frac{H_0(\Sigma_N) \oplus H_0(\Sigma_S)}{\ker \pi_0} \longrightarrow H_0(M_\phi) \longrightarrow 0$$

where the middle one is the important. First, observe that

$$\ker \pi_k = \mathrm{im}\, i_k = \langle ([a_N^k], [a_S^k]) \rangle \simeq H_k(\Sigma)$$

and thus

$$\frac{H_k(\Sigma_N) \oplus H_k(\Sigma_S)}{\ker \pi_k} \simeq H_k(\Sigma).$$

Secondly, we have that

$$i_k([a_W], [b_E]) = \begin{pmatrix} 1 & 1 \\ 1 & \phi_\bullet \end{pmatrix} \begin{pmatrix} [a] \\ [b] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \phi_\bullet - 1 \end{pmatrix} \begin{pmatrix} [a] \\ [b] \end{pmatrix}$$

so doing this change of basis, we see that in the exact sequence we can replace $H_1(\Sigma_W \sqcup \Sigma_E)$, i_1 and $H_1(\Sigma_N) \oplus H_1(\Sigma_S)$ by $H_1(\Sigma_E)$, $\phi_\bullet - Id$ and $\langle([a_N],[a_S])\rangle \simeq H_1(\Sigma)$ respectively. Finally, since we have

$$\begin{tikzcd} 0 \arrow[r] & \ker i_0 \arrow[r] & H_0(\Sigma_W \sqcup \Sigma_E) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & H_0(\Sigma_N) \oplus H_0(\Sigma_S) \arrow[d, leftrightarrow, "\simeq"] \\ 0 \arrow[r] & H_1(\mathrm{S}^1) \arrow[r] & H_0\left(\widetilde{U} \cap \widetilde{V}\right) \arrow[r] & H_0\left(\widetilde{U}\right) \oplus H_0\left(\widetilde{U}\right) \end{tikzcd}$$

then $\ker i_0 \simeq H_1(\mathrm{S}^1)$. Thus, we obtain

Theorem 6.2. *Let Σ be a closed, oriented and compact surface. Let $\phi\colon \Sigma \to \Sigma$ be an oriented diffeomorphism and M_ϕ its induced mapping torus. Then, the homology groups of H_ϕ with integer coefficients satisfy the exact sequence:*

$$0 \longrightarrow H_2(\Sigma) \longrightarrow H_2(M_\phi) \xrightarrow{\delta_2} H_1(\Sigma)$$

$$\xrightarrow{\phi_\bullet - Id} H_1(\Sigma) \longrightarrow H_1(M_\phi) \longrightarrow H_1(\mathrm{S}^1) \longrightarrow 0$$

which determines homology groups in terms of $\phi_\bullet - Id$.

Now we are going to see the case with ϕ equal to the identity in homology. We have that

$$\ker i_k = \langle([a_W^k], -[a_E^k])\rangle,\ \operatorname{im} i_k = \langle([a_N^k],[a_S^k])\rangle = \ker \pi_k \text{ and } \operatorname{im} \pi_k = H_k(M_\phi)$$

and so we can break the long sequence to sequences

$$0 \longrightarrow H_3(M_\phi) \xrightarrow{\delta} \langle([a_W^2], -[a_E^2])\rangle \longrightarrow 0$$

$$0 \longrightarrow \frac{H_2(\Sigma_N)\oplus H_2(\Sigma_S)}{\langle([a_N^2],[a_S^2])\rangle} \xrightarrow{\pi_2} H_2(M_\phi) \xrightarrow{\delta} \langle([a_W^1], -[a_E^1])\rangle \longrightarrow 0$$

$$0 \longrightarrow \frac{H_1(\Sigma_N)\oplus H_1(\Sigma_S)}{\langle([a_N^1],[a_S^1])\rangle} \xrightarrow{\pi_1} H_1(M_\phi) \xrightarrow{\delta} \langle([a_W^0], -[a_E^0])\rangle \longrightarrow 0$$

$$0 \longrightarrow \frac{H_1(\Sigma_N)\oplus H_0(\Sigma_S)}{\langle([a_N^0],[a_S^0])\rangle} \xrightarrow{\pi_0} H_0(M_\phi) \longrightarrow 0$$

Since

$$\langle([a_W^2], -[a_E^2])\rangle \simeq H_2(\Sigma) \simeq \mathbb{Z}, \qquad \langle([a_W^1], -[a_E^1])\rangle \simeq H_1(\Sigma) \simeq \mathbb{Z}^{2g}$$

$$\langle([a_W^0], -[a_E^0])\rangle \simeq H_0(\Sigma) \simeq \mathbb{Z}, \qquad \frac{H_2(\Sigma_N)\oplus H_2(\Sigma_S)}{\langle([a_N^2],[a_S^2])\rangle} \simeq H_2(\Sigma) \simeq \mathbb{Z}$$

$$\frac{H_1(\Sigma_N)\oplus H_1(\Sigma_S)}{\langle([a_N^1],[a_S^1])\rangle} \simeq H_1(\Sigma) \simeq \mathbb{Z}^{2g}, \qquad \frac{H_0(\Sigma_N)\oplus H_0(\Sigma_S)}{\langle([a_N^0],[a_S^0])\rangle} \simeq H_0(\Sigma) \simeq \mathbb{Z}$$

we deduce

$$H_3(M_\phi) \simeq \mathbb{Z} \qquad H_2(M_\phi) \simeq \mathbb{Z} \oplus \mathbb{Z}^{2g}$$

$$H_1(M_\phi) \simeq \mathbb{Z}^{2g} \oplus \mathbb{Z} \qquad H_0(M_\phi) \simeq \mathbb{Z}$$

Let us denote homology classes of Σ as in figure 4, and let $[p]$ and $[\Sigma]$ be the positive generators of $H_0(\Sigma)$ and $H_2(\Sigma)$ respectively. As it is observed in the figure 4, we have $[a_i] \cdot [b_j] = \delta_{i,j}[p]$.

Suppose ϕ is the identity. Then $M_\phi = \Sigma \times \mathrm{S}^1$, and exact sequences tell us that its homology classes are:

- For $H_3(M_{id})$ the group generated by the fundamental class, since it is a compact orientable manifold. Call it $[M]$.
- Group $H_2(M_{id})$ splits into two. First, the fundamental class of Σ along a fibre that we will denote as $[\Sigma] = [\Sigma \times \{W\}]$ again. Secondly, all the 1-homology classes of Σ along S^1. Call them $[a_i^2] := [a_i \times \mathrm{S}^1]$ and $[b_i^2] := [b_i \times \mathrm{S}^1]$.
- Similarly, group $H_1(M_{id})$ also splits into two. This time, $[a_i^1] := [a_i \times \{W\}]$ and $[b_i^1] := [b_i \times \{W\}]$ are those which generate classes "parallel" to fibres, and $[p^1] := [p \times \mathrm{S}^1]$ the one which generates the class "parallel" to S^1.
- Finally, $H_0(M_{id})$ is generated by $[p^0] := [p \times \{W\}]$.

With these provided representatives of homology classes is it easy to get intersection algebra. It is not difficult to check that complementary grade intersection is

	$H_0(M_{id})$	$H_1(M_{id})$			$H_2(M_{id})$			$H_3(M_{id})$
I^2	$[p^0]$	$[p^1]$	$[a_i^1]$	$[b_i^1]$	$[a_i^2]$	$[b_i^2]$	$[\Sigma]$	$[M]$
$[p^0]$								$[p^0]$
$[p^1]$					0	0	$-[p^0]$	
$[a_i^1]$					0	$-[p^0]$	0	
$[b_i^1]$					$[p^0]$	0	0	
$[a_i^2]$		0	0	$-[p^0]$				
$[b_i^2]$		0	$[p^0]$	0				
$[\Sigma]$		$[p^0]$	0	0				
$[M]$	$-[p^0]$							

and pairs of a_i's with b_j's with $i \neq j$ have intersection zero. Now, by computing triple intersections we can fill the table:

	$H_0(M_{id})$	$H_1(M_{id})$			$H_2(M_{id})$			$H_3(M_{id})$
I	$[p^0]$	$[p^1]$	$[a_i^1]$	$[b_i^1]$	$[a_i^2]$	$[b_i^2]$	$[\Sigma]$	$[M]$
$[p^0]$								$[p^0]$
$[p^1]$					0	0	$-[p^0]$	$-[p^1]$
$[a_i^1]$					0	$-[p^0]$	0	$-[a_i^1]$
$[b_i^1]$					$[p^0]$	0	0	$-[b_i^1]$
$[a_i^2]$		0	0	$-[p^0]$	0	$-[p^1]$	$[a_i^1]$	$[a_i^2]$
$[b_i^2]$		0	$[p^0]$	0	$[p^1]$	0	$[b_i^1]$	$[b_i^2]$
$[\Sigma]$		$[p^0]$	0	0	$-[a_i^1]$	$-[b_i^1]$	0	$[\Sigma]$
$[M]$	$-[p^0]$	$-[p^1]$	$-[a_i^1]$	$-[b_i^1]$	$-[a_i^2]$	$-[b_i^2]$	$-[\Sigma]$	$-[M]$

6.1.1 Dehn twist

Definition 6.3. Let γ be a smooth closed curve in Σ. Then choose an open neighbourhood of γ diffeomorphic to a cylinder $\mathrm{S}^1 \times I$, where $\mathrm{S}^1 \times \left\{\frac{1}{2}\right\}$ is γ. Now consider the operation "twist smoothly the cylinder exactly a turn", as for example, if we choose a bump function b_ϵ so that $b_\epsilon(x) = 0$ for $x < \frac{1}{2} - \epsilon$ and $b_\epsilon(x) = 1$ for $x > \frac{1}{2} + \epsilon$, then the map

$$\begin{array}{rccc} T\colon & \mathrm{S}^1 \times I & \longrightarrow & \mathrm{S}^1 \times I \\ & (\theta, t) & \longmapsto & (\theta + b_\epsilon(t), t) \end{array}$$

is an example. Observe that T is a diffeomorphism such that it is the identity outside some compact set in the cylinder, so it can be extended to the whole surface Σ. Then, this extension, say T_γ, is called a *Dehn twist along* γ.

Of course Dehn twists along a curve are not unique, but they have the following properties (see [2]):

- If we choose another neighbourhood of γ, or we choose another diffeomorphism to $\mathrm{S}^1 \times I$, or we twist cylinder in a different way, we obtain a diffeotopic Dehn twist.
- Dehn twist of diffeotopic curves give us diffeotopic Dehn twists.

So Dehn twist is well defined in the mapping class group.

Observe that if we have a 1-cycle a in Σ, if a does not intersect γ, T_γ has any effect over a. Otherwise, if a intersects γ, T_γ has the effect of adding the path γ to a. This is, in fact, the well-known result (see [2], [1]):

Proposition 6.4 (Picard - Lefschetz formula)**.** *If T_γ is a Dehn twist on Σ, then, in homology,*

$$T_\gamma([a]) = [a] + ([a] \cdot [\gamma])[\gamma].$$

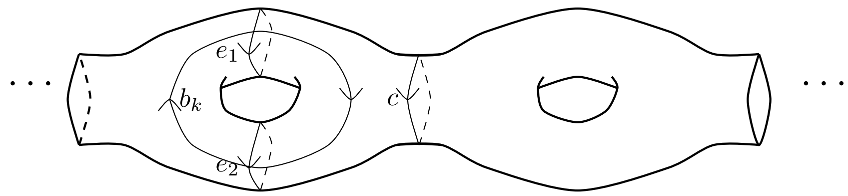

Figure 5: Paths c and $e = e_1 + e_2$.

If we consider the surface Σ with $g \geq 2$ and this time we choose $\phi = T_c$, with c a path so that disconnect Σ, as in figure 5, then M_ϕ is called *separating Dehn twist.*

Since c is it a boundary, by proposition 6.4, Dehn twist along c is the identity as homology map, so homology classes of M_ϕ are isomorphic to homology classes in the case of the identity. For seeing intersection, first note that c disconnect $\Sigma - c$ in two components Σ_L and Σ_R, and every homology class has support on one of this two components, except $[\Sigma]$. Then, intersection, in each component, is like in the case of identity. So both sides have the same intersection tables, with intersection in classes $[p_L]$ and $[p_R]$, where $p_L \in \Sigma_L$ and $p_R \in \Sigma_R$. Let h be a chain. Then $\mathrm{Tr}(h)$ denote the transport, that is the chain of dimension one more

$$\mathrm{Tr}(h) = \frac{h \times I}{(x,0) \sim (\phi(x), 1)}.$$

Observe that if chain h has support disjoint from c, $\mathrm{Tr}(h) = h \times \mathrm{S}^1$.

Now choose any 1-cycle γ in Σ so that $\partial\gamma = p_L - p_R$. Then γ intersects c, we can suppose transversaly. Now, we transport γ along S^1 obtaining the 2-chain $\mathrm{Tr}(\gamma) \subset M_\phi$, since $\phi_\bullet(\gamma) = \gamma + c$, we obtain that

$$\partial(\mathrm{Tr}(\gamma)) = (p_L \times \mathrm{S}^1) - (p_R \times \mathrm{S}^1) - c = p_L^1 - p_R^1 - c.$$

Now, since c is a boundary, we have that $[p_L^1] = [p_R^1]$.

Finally, intersections with $[\Sigma]$ are like in trivial case, and thus we deduce that intersection algebra is exactly as in trivial case.

As a no trivial example, we are going to compute intersection algebra of mapping torus with $g \geq 3$ and $\phi = T_e$, with e as is shown in figure 5. This twist is called *Johnson twist.* As before, we see that curves e split Σ into two components, Σ_L at left and Σ_R at right, in figure 5. Since $\partial\Sigma_L = -e$, e is a boundary and again we have $\phi_\bullet = Id$ in homology, we obtain the same homology groups. Moreover, as before, all homology classes except $[b_k]$ have representatives far away of e, and thus they induce exactly the same representatives as in above cases. However $[b_k]$ have not. If we consider the transport $\mathrm{Tr}(b_k)$ as before, we see that its boundary is $-e$, because when we transport b_k along S^1, when we complete a turn it becomes $b_k + e$, so if we do the boundary of the torus, we are forgetting this e. But if we consider $\mathrm{Tr}(b_k) + (\Sigma_L \times \{E\})$, it becomes a cycle. What we are saying is that, in chains

$$\begin{tikzcd} & (b_k \times (E,W) + \Sigma_L \times \{E\}, b_k \times (W,E)) \arrow[r] \arrow[d, "\partial"] & \mathrm{Tr}(b_k) + \Sigma_L \times \{E\} \\ (b_k, \phi_\bullet b_k) \arrow[r, "i"] & (-e, 0) & \end{tikzcd}$$

So, by exact sequences we found before, it is a representative of the homology class of $H_2(M_\phi)$ generated by $[b_k]$. Also by exact sequences we see that if we sum any multiple of $\Sigma \times \{E\}$, it continues being a representative. Thus, in particular, $\mathrm{Tr}(b_k) - (\Sigma_R \times \{E\})$ is a representative too.

With respect to the first homology group, as before, it is generated by $[a_i \times \{E\}]$, $[b_i \times \{E\}]$ and $[p \times \mathrm{S}^1]$. But observe the following fact: if we choose p_L and p_R points in Σ_L and Σ_R, and we choose a 1-cycle γ in Σ such that $\partial\gamma = p_L - p_R$, then γ intersects e_1 or e_2, which are homologic to a_k and $-a_k$ respectively. Concretely, let γ_1 and γ_2 be paths which intersect $e - 1$ and e_2 respectively.

So when we consider $\mathrm{Tr}(\gamma_1)$ we will obtain, for the same reasons as before, that its boundary is $(p_L \times \mathrm{S}^1) - (p_R \times \mathrm{S}^1) - a_k$. So we obtain that $[p_L \times \mathrm{S}^1]$ and $[p_R \times \mathrm{S}^1]$ are not the same homology class, but they are related. So we have to choose one of these classes to generate de first homology group. Let us denote $p_R^1 = p_R \times \mathrm{S}^1$ and $p_L^1 = p_L \times \mathrm{S}^1$.

If we choose $\mathrm{Tr}(\gamma_2)$ instead $\mathrm{Tr}(\gamma_1)$, since $b_k = \gamma_2 - \gamma_1$, we have that $-e = \mathrm{Tr}(b_k) = \mathrm{Tr}(\gamma_2) - \mathrm{Tr}(\gamma - 1)$ and so $[\mathrm{Tr}(\gamma_1)] = [\mathrm{Tr}(\gamma_2)]$, so we arrive to same relation.

Now, we are going to compute intersection algebra and, as before, we start computing intersection in complementary grades. Observe that all intersections which do not involve b_k are still like in cases before. Also as before, let us call

$$b_k^2 := \mathrm{Tr}(b_k) + \Sigma_L \times \{E\},$$

then

- If $L < k$, $[a_L^1] \cdot [b_k^2] = 0$ since we can choose a representative of $[a_L^1]$ in a different fibre than $\Sigma \times \{E\}$.
- Similarly $[b_L^1] \cdot [b_k^2] = 0$.
- If $R > k$, $[a_R^1] \cdot [b_k^2] = 0$ and $[b_R^1] \cdot [b_k^2] = 0$.
- $[a_k^1] \cdot [b_k^2] = -[p_R^0]$, if we choose a representative of $[a_k^1]$ contained in Σ_R.
- $[p_R^1] \cdot [b_k^2] = 0$ and thus $[p_L^1] \cdot [b_k^2] = ([p_R^1] + [a_k^1]) \cdot [b_k^2] = -[p_R^0]$

So we get the table

I^2	$[a_L^2]$	$[b_L^2]$	$[a_k^2]$	$[b_k^2]$	$[a_R^2]$	$[b_R^2]$	$[\Sigma]$	$[M]$
$[p^0]$								$[p^0]$
$[a_L^1]$	0	$-[p^0]$	0	0	0	0	0	
$[b_L^1]$	$[p^0]$	0	0	0	0	0	0	
$[a_k^1]$	0	0	0	$-[p^0]$	0	0	0	
$[b_k^1]$	0	0	$[p^0]$	0	0	0	0	
$[a_R^1]$	0	0	0	0	0	$-[p^0]$	0	
$[b_R^1]$	0	0	0	0	$[p^0]$	0	0	
$[p_R^1]$	0	0	0	0	0	0	$-[p^0]$	

and we can swap rows with columns by multiplying by sign -1. Observe that, as we knew, if we identify $H_0(M_\phi) \simeq \mathbb{Z}$, this table becomes a squared invertible matrix. Now, by doing triple intersection and applying this matrix we obtain intersection in complementary grades. As examples, we have

I^3	$[a_L^2]$	$[b_L^2]$	$[a_k^2]$	$[b_k^2]$	$[a_R^2]$	$[b_R^2]$	$[\Sigma]$
$[b_L^2],[\Sigma]$	$[p^0]$	0	0	0	0	0	0
$[b_k^2],[\Sigma]$	0	0	$[p^0]$	0	0	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

and so on. So we can deduce the table

I	$[a_L^2]$	$[b_L^2]$	$[a_k^2]$	$[b_k^2]$	$[a_R^2]$	$[b_R^2]$	$[\Sigma]$
$[a_L^2]$	0	$-[p_L^1]$	0	0	0	0	$[a_L^1]$
$[b_L^2]$	$[p_L^1]$	0	0	0	0	0	$[b_L^1]$
$[a_k^2]$	0	0	0	$-[p_R^1]$	0	0	$[a_k^1]$
$[b_k^2]$	0	0	$[p_R^1]$	0	0	0	$[b_k^1]$
$[a_R^2]$	0	0	0	0	0	$-[p_R^1]$	$[a_R^1]$
$[b_R^2]$	0	0	0	0	$[p_R^1]$	0	$[b_R^1]$
$[\Sigma]$	$-[a_L^1]$	$-[b_L^1]$	$-[a_k^1]$	$-[b_k^1]$	$-[a_R^1]$	$-[b_R^1]$	0

and intersections with $[M]$ by the formula

$$I([c],[M]) = (-1)^{|c|+1}\, I([M],[c]) = (-1)^{|c|}\,[c]$$

since, for all homology class $[e]$ of grade $3-|c|$, we have that

$$I^2(I([c],[M]),[e]) = I^3([c],[M],[e]) = (-1)^{9+3\cdot(3-|c|)}\, I^2([c],[e]) = (-1)^{|c|}\, I^2([c],[e]).$$

Thus, we have obtained intersection algebra of Johnson twist. If we choose $[p_R^1]$ as a generator in $H_1(M_\phi)$, then we have that

$$I([a_L^2],[b_L^2]) = -[p_L^1] = -[p_R^1] - [a_k^1],$$

so it has a different intersection algebra than trivial bundle and separating Dehn twist cases.

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