

MASTER-SLAVE GLOBAL STOCHASTIC SYNCHRONIZATION OF CHAOTIC OSCILLATORS

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Abstract. We study synchronization of two chaotic oscillators in a master-slave configuration. The two dynamic systems are coupled via a directed feedback that randomly switches among a finite set of given constant functions at a prescribed time rate. We use stochastic Lyapunov stability theory and partial averaging techniques to show that global synchronization is possible if the switching period is sufficiently small and if the two systems globally exponentially synchronize under an average feedback coupling. The approach is applied to the synchronization of two Chua's circuits.

Key words. master-slave synchronization, global synchronization, stochastic synchronization, chaos, chua's circuit, exponential stability

AMS subject classifications. 34C15, 34C29, 34D23, 74H65, 93C10, 93E15

1. Introduction. Chaos synchronization is a topic of great interest, due to its observation in a huge variety of phenomena of different nature. In many biological systems, synchronization plays an important role in self-organization of organisms' groups [10]. Examples of synchronization include communication among fireflies [9, 36], locomotion of animals [13], molecular and cellular activity [24] and cardiac stimulation [21, 27, 42]. The study of neural activity [44, 54, 63] and brain disorders [3, 53] is a correlated issue as well. Other examples and applications can also be found in ecological systems [5], meteorology [16], chemistry [24, 37], gas-liquid bubbling dynamics [57, 61] and optics [55, 62]. Many reviews on chaos synchronization are currently available (see for example [2, 6, 12, 23, 43, 48, 50]).

In the literature, different paradigms have been studied to describe synchronization of two or more chaotic oscillators. We mention, among the others, peer-to-peer coupling [22, 52, 56, 59], back-stepping [7], generalized synchronization [64, 65], phase synchronization [6] and master-slave synchronization [11, 33, 40, 60]. In this work, we focus on master-slave synchronization. In this case, one system acts as a "master" by driving the other system that behave consequently as a "slave".

Most of the research efforts on master-slave synchronization focus on time invariant coupling (see for example [11, 26, 33, 40, 45, 46, 47]). Nevertheless, experimental and numerical evidences show that synchronization can also be achieved using intermittent, time-varying master-slave coupling [20, 32, 65, 66]. In [20], experimental results on synchronization of two periodically coupled chaotic circuits are presented. In [32, 65], the slave system is driven by a sequence of samples of the master's state (*impulsive synchronization*). In [32, 66], the signal transmission from the master to the slave system is adaptively controlled. That is, the driving signal is transmitted only when it is expected to reduce the synchronization error (*selective synchronization*).

The main goal of this work is to establish sufficient conditions for global synchronization of master-slave coupled chaotic systems with time-varying coupling. We focus on the general case where the intermittent coupling changes randomly over time. Intermittent coupling is made possible through a switching function, that changes randomly over time, assuming values among a finite set of constant functions. The

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synchronization problem is transformed into a nonlinear stochastic stability problem and it is studied using partial averaging techniques (see for example [25, 29, 49]), nonlinear system theory (see for example [35]) and stochastic stability theory (see for example [38, 39]). We associate to the stochastic system a partially averaged system characterized by a constant coupling. This auxiliary system can be studied using well-established and manageable Lyapunov based techniques as those presented in [33]. Under suitable regularity conditions, we show that if the partially averaged system is globally exponentially stable and the switching period is sufficiently small, the stochastic system is globally asymptotically synchronized.

The type of intermittent coupling considered in this paper has been also analyzed in the framework of consensus theory [30, 51]. We note that, in consensus theory, the individual systems' dynamics is linear while in the present case the coupled systems are strongly nonlinear. Partial averaging techniques have been used in the synchronization literature by [52, 59]. Both these efforts deal with peer-to-peer coupling and only local asymptotic synchronization results based on linearized dynamics are presented. In this paper the inherent non linear nature of the coupled systems is retained and global exponential results are presented.

The system studied in this paper finds many practical applications. For example, in communication and signal processing, chaotic behavior can be used for message encryption and secure communication [15, 17, 18, 31, 34, 41]. Higher communication efficiency can potentially be achieved through sporadic transmission of the driving signal. This is particularly useful when the available resources are shared and the amount of information that can be transmitted is limited (for example this is the case of Internet communication). Furthermore, in many biological systems, interactions happen only sporadically and randomly [4] (see for example the neurons' activity in the brain or the synchronous flashing behavior of the fireflies).

We organize the paper as follows. In Section 2, we define the master-slave synchronization problem in the case of a stochastic linear feedback. In Section 3, we present a few general results on stability of nonlinear stochastic systems. In Section 4, we apply these results to the synchronization problem. As a sample case, in Section 5 we consider the case of two stochastically coupled Chua's circuits. Section 6 is left for the conclusions.

2. Problem Statement.

We consider the master system

$$\dot{x}(t) = Ax(t) + g(x(t)) + u(t) \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^n$ is the input vector, $A \in \mathbb{R}^{n \times n}$ is a constant matrix, g is a non linear function, n is a positive integer and $t \in \mathbb{R}^+$ indicates the time variable. We construct a slave system for (2.1)

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + g(\tilde{x}(t)) + u(t) + K(t)(x(t) - \tilde{x}(t)) \quad (2.2)$$

System (2.2) is unidirectionally coupled to the master system (2.1) through the feedback matrix function $K(t)$. We consider the case where $K(t)$ is a piecewise constant signal that in every time interval $[\sigma_k, \sigma_{k+1}]$, with $k \in \mathbb{Z}^+$ and $\sigma_0 = 0$, equals the random variable \mathfrak{K}_k . We assume that the random variables \mathfrak{K}_k are finite-state, independent and identically distributed in the state space $\{K_1, K_2, \dots, K_N\}$, with $N \in \mathbb{Z}^+$. We further assume that switching occurs at equally spaced time instants σ_k , with $|\sigma_{k+1} - \sigma_k| = \varepsilon$, where $\varepsilon > 0$ is a fixed time duration.

Following [33], we assume that

$$g(x) - g(\tilde{x}) = M_{x,\tilde{x}}(x - \tilde{x})$$

for some bounded matrix $M_{x,\tilde{x}}$, whose elements depend on x and \tilde{x} . As discussed in [33], this condition applies to a large variety of chaotic systems.

We express the system of equations (2.1) and (2.2) in terms of the error function $e = x - \tilde{x}$

$$\begin{aligned} \dot{e}(t) &= Ae(t) + g(x(t)) - g(\tilde{x}(t)) - K(t)e(t) \\ &= (A - K(t))e(t) + M_{x(t),x(t)-e(t)}e(t) \end{aligned} \quad (2.3)$$

Equation (2.3) can be compactly rewritten as

$$\dot{e}(t) = y(e(t), t) - K^\diamond(t/\varepsilon)e(t) \quad (2.4)$$

where $y(e(t), t) = (A - M_{x(t),x(t)-e(t)})e(t)$ and $K^\diamond(t/\varepsilon) = K(t)$. We note that the matrix function K^\diamond switches at a unit rate. Equation (2.4) shows that two different time scales are involved in the problem: a fast time scale t/ε representing the switching process and a slow time scale t describing the chaotic dynamics. By considering the error function e , the synchronization problem reduces to the stability analysis of the stochastic and nonautonomous nonlinear system in equation (2.3).

We associate to the stochastic system (2.4) a partially averaged deterministic system whose synchronizability can be assessed through well-established Lyapunov stability techniques [33]. We show that if the switching rate is sufficiently fast and the Lyapunov function of the deterministic system is sufficiently regular, the stability properties of the partially averaged system are inherited by the stochastic system.

3. Preliminary results on global stability through fast switching. We consider the integral equation in \mathbb{R}^n

$$x(t) = x(\sigma_k) + \int_{\sigma_k}^t f(x(\xi), \xi, \Omega) d\xi \quad (3.1)$$

where $t \in [\sigma_k, \sigma_{k+1}]$, $\sigma_0 = 0$, $|\sigma_{k+1} - \sigma_k| = \varepsilon$, and $k \in \mathbb{Z}^+$. The function f is defined in $\mathbb{R}^n \times \mathbb{R}^+ \times \Theta$. Here, Ω is a finite-state random variable taking values in $\Theta = \{\omega_1, \dots, \omega_N\}$, with $N \in \mathbb{Z}^+$. We assume that $f(0, t, \omega_j) = 0$, $\forall t \in \mathbb{R}^+$, $j = 1, \dots, N$ and that for every $\omega \in \Theta$ the function $f_\omega(x, t) = f(x, t, \omega)$ is globally Lipschitz in \mathbb{R}^+ , with Lipschitz constant $L_{\omega,\varepsilon}$. We further require that $L_{\omega,\varepsilon} < L$, where L is a constant independent of ω and ε . We note that (3.1) describes a Markovian nonlinear, nonhomogeneous jump system (see for example [14]). We look for a solution of (3.1) for $t \geq t_0$ and initial conditions $x(t_0) = x_0$. In the sequel, we use $\mathbb{E}[\bullet]$ to indicate expectation and we denote probability with $P\{\bullet\}$.

In this section, we establish sufficient conditions for global stability of the stochastic system (3.1). First we recall the definitions of global mean square exponential stability (see for example [19]), global almost surely stability and global asymptotic stability (see for example [39]).

DEFINITION 3.1. *The system (3.1) is globally mean square exponentially stable if there exist $\alpha \geq 0$ and $\beta > 0$ such that for any $t_0 \in \mathbb{R}^+$, and any $x_0 \in \mathbb{R}^n$*

$$\mathbb{E}[\|x(t)\|^2] \leq \alpha \|x_0\|^2 e^{-\beta(t-t_0)}$$

DEFINITION 3.2. *The system (3.1) is globally almost surely stable if for any $\rho > 0$ and $\mu > 0$, there is $\delta(\rho, \mu) > 0$ such that if $\|x_0\| < \delta(\rho, \mu)$, then*

$$P \left\{ \sup_{\infty > t \geq 0} \|x(t)\| \geq \mu \right\} \leq \rho \quad (3.2)$$

DEFINITION 3.3. *The system (3.1) is globally almost surely asymptotically stable if it is globally almost surely stable and $x(t) \xrightarrow{a.s.} 0$ for all $x_0 \in \mathbb{R}^n$.*

Here, the notation $x(t) \xrightarrow{a.s.} 0$ indicates almost sure convergence (see for example [28]).

From classical Lyapunov stability theory, it is well known that a deterministic dynamical system is asymptotically stable if there exists a Lyapunov function whose time derivative along the solutions of the system is negative definite (see for example [35]). In [1], this condition is relaxed and it is shown that if the Lyapunov function decreases when evaluated at a discrete sequence of time instants, the system is asymptotically stable. In this case, the time derivative of the Lyapunov function can assume positive and negative values. The following Theorem extends the results of [1] from the deterministic to the stochastic case and it is used in what follows to establish our main claim.

THEOREM 3.4. *Consider the system (3.1) and suppose that there exists a function $V : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+$*

$$\lambda_{min} \|x\|^2 \leq V(x, t) \leq \lambda_{max} \|x\|^2 \quad (3.3)$$

with λ_{min} and λ_{max} positive nonzero real constants. Assume also that there exists ν , with $0 < \nu \leq 1$, such that

$$E[V(x(\sigma_{k+1}), \sigma_{k+1}) | x(\sigma_k)] - V(x(\sigma_k), \sigma_k) \leq -\nu V(x(\sigma_k), \sigma_k) \quad (3.4)$$

for every $k \in \mathbb{Z}^+$. Then (3.1) is globally mean square exponentially stable and globally almost surely asymptotically stable.

Proof. See Appendix A \square

REMARK 1. *The arguments given in the proof of Theorem (3.4) not only establish the global mean square exponential stability and the global almost sure asymptotic stability, but provide further insights on the asymptotic behavior of the solution of (3.1). In fact, from (A.16) and (A.17) we have*

$$P \left\{ \sup_{\infty > (t-t_0) \geq T} \|x(t)\| \geq \psi\gamma \right\} \leq \bar{c}e^{-\varphi T} \quad (3.5)$$

where $\bar{c} = c(1 - \nu)\|x(t_0)\|^2$ and $\varphi = -\ln(1 - \nu)/\varepsilon$. Equation (3.5) implies that the probability that $\|x(t)\|$ is greater or equal than a certain quantity decreases with an exponential decay. This corresponds to the definition of global exponential stability presented in [39].

We associate to (3.1) the partially averaged system

$$\dot{x}(t) = \bar{f}(x(t), t) = E[f(x(t), t, \Omega)] \quad (3.6)$$

Equation (3.6) represents a deterministic, time-varying, non linear system. We notice that $\bar{f}(0, t) = 0, \forall t \in \mathbb{R}^+$. If (3.6) is globally exponentially stable, by the Converse Theorem of Lyapunov (see [35], Theorem 3.12) we know that there exists a Lyapunov function $V(x, t)$, whose time derivative is negative definite along its trajectories. In the following theorem, we show that if $V(x, t)$ satisfies further regularity conditions and the switching period is sufficiently small, the original system (3.1) is globally mean square exponentially stable and almost surely asymptotically stable.

THEOREM 3.5. *Consider the system (3.1) and the associated partially averaged system (3.6) and suppose that there exists a Lyapunov function $V(x, t)$ which satisfies the following conditions:*

1. $V(0, t) = 0, \forall t \in \mathbb{R}^+$ and there exist positive numbers $\lambda_{min}, \lambda_{max}$ such that for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$\lambda_{min}\|x\|^2 \leq V(x, t) \leq \lambda_{max}\|x\|^2 \quad (3.7)$$

2. there exists a positive number w such that for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t)\bar{f}(x, t) \leq -w\|x\|^2 \quad (3.8)$$

3. $\forall t \in \mathbb{R}^+, \frac{\partial V}{\partial x}(0, t) = 0$ and $\frac{\partial V}{\partial x}(x, t)$ is globally Lipschitz, with Lipschitz constant K_v . Moreover, $\forall t \in \mathbb{R}^+, \frac{\partial^2 V}{\partial x \partial t}(0, t) = 0$, and $\frac{\partial^2 V}{\partial x \partial t}(x, t)$ is globally Lipschitz, with Lipschitz constant K_{vt} .

There exists an $\varepsilon^* > 0$ such that $\forall \varepsilon < \varepsilon^*$ system (3.1) is globally mean square exponentially stable and globally almost surely asymptotically stable.

Proof. Consider the Lyapunov function $V(x, t)$. Its derivative along the solution of (3.1) is

$$\dot{V}(x(t), t) = \frac{\partial V}{\partial t}(x(t), t) + \frac{\partial V}{\partial x}(x(t), t)f(x(t), t, \Omega) \quad (3.9)$$

For every nonnegative integer k , we define

$$\Delta V(\sigma_{k+1}, \sigma_k) = \mathbb{E}[V(x(\sigma_{k+1}), \sigma_{k+1})|x(\sigma_k)] - V(x(\sigma_k), \sigma_k). \quad (3.10)$$

From (3.1), (3.9) and (3.10) we have

$$\begin{aligned} \Delta V(\sigma_{k+1}, \sigma_k) &= \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \dot{V}(x(t), t) dt \right] = \\ &= \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(t), t) + \frac{\partial V}{\partial x}(x(t), t)f(x(t), t, \Omega) dt \right] = \\ &= \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial x}(x(t), t)f(x(t), t, \Omega) - \frac{\partial V}{\partial x}(x(\sigma_k), t)f(x(\sigma_k), t, \Omega) dt \right] \\ &\quad + \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(t), t) - \frac{\partial V}{\partial t}(x(\sigma_k), t) dt \right] \\ &\quad + \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(\sigma_k), t) + \frac{\partial V}{\partial x}(x(\sigma_k), t)f(x(\sigma_k), t, \Omega) dt \right] \end{aligned} \quad (3.11)$$

We seek an upper bound for the absolute values of the three terms in the summation above.

We start our analysis by considering the first and the second terms. Using the Lipschitz conditions on f_ω and on the first and second derivatives of V and following the argument of [49] (see proof of Theorem 2, part II and III), for each realization ω of Ω , we have

$$\left| \int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial x}(x(t), t) f(x(t), t, \omega) - \frac{\partial V}{\partial x}(x(\sigma_k), t) f(x(\sigma_k), t, \omega) dt \right| \leq 2L_{\omega, \varepsilon}^2 K_v e^{2\varepsilon L_{\omega, \varepsilon}} \varepsilon^2 \|x(\sigma_k)\|^2$$

$$\left| \int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(t), t) - \frac{\partial V}{\partial t}(x(\sigma_k), t) dt \right| \leq L_{\omega, \varepsilon} K_{vt} e^{2\varepsilon L_{\omega, \varepsilon}} \varepsilon^2 \|x(\sigma_k)\|^2$$

Since $\sum_{i=1}^N P\{\Omega = \omega_i\} = 1$ and $L_{\omega, \varepsilon} < L$ for each ω , the absolute value of the first term of the summation (3.11) is less than or equal to

$$2L^2 K_v e^{2\varepsilon L} \varepsilon^2 \|x(\sigma_k)\|^2$$

In addition, the absolute value of second term is less than or equal to

$$L K_{vt} e^{2\varepsilon L} \varepsilon^2 \|x(\sigma_k)\|^2$$

Now, we consider the third term in the right side of (3.11)

$$\begin{aligned} \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(\sigma_k), t) + \frac{\partial V}{\partial x}(x(\sigma_k), t) f(x(\sigma_k), t, \Omega) dt \right] &= \\ &= \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(\sigma_k), t) + \frac{\partial V}{\partial x}(x(\sigma_k), t) \bar{f}(x(\sigma_k), t) dt \right] \\ &+ \mathbb{E} \left[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial x}(x(\sigma_k), t) \{f(x(\sigma_k), t, \Omega) - \bar{f}(x(\sigma_k), t)\} dt \right] \end{aligned}$$

By hypothesis, (3.8) provides a bound for the first term in the summation above, while the second term is equal to zero. Using the bounds above in equation (3.11), we find

$$\Delta V(\sigma_{k+1}, \sigma_k) \leq [g(\varepsilon) - w] \|x(\sigma_k)\|^2 \quad (3.12)$$

where the function $g(\varepsilon)$ is defined by

$$g(\varepsilon) = (2L^2 K_v e^{2\varepsilon L} + L K_{vt} e^{2\varepsilon L}) \varepsilon^2 \quad (3.13)$$

Noticing that $g(0) = 0$ and $g'(0) = 0$, we have that there exists $\varepsilon^* > 0$ such that

$$\Delta V(\sigma_{k+1}, \sigma_k) \leq -\bar{w} \|x(\sigma_k)\|^2 \quad (3.14)$$

where $\bar{w} = [w - g(\varepsilon)] > 0$, for every $\varepsilon < \varepsilon^*$.

In conclusion, if the switching period ε is sufficiently small, (3.14) and (3.10) imply

that the hypotheses of Theorems 3.4 are satisfied. Thus the claim follows. \square

REMARK 2. *We assumed in Section 2 that ε is a fixed period of time. This hypothesis can be relaxed. In fact, Theorem 3.5 can be generalized considering not equally spaced switching instants. If the switching interval is bounded in the interval $[0, \varepsilon_{max}]$, equation (A.9) in Theorem 3.4 still holds with $\gamma = e^{L\varepsilon_{max}}$, while Theorem 3.5 holds with $\varepsilon_{max} \leq \varepsilon^*$.*

4. Stochastic chaos synchronization. In this section we combine the general findings of Section 3 on stochastic stability of non linear systems with available results on synchronizability of deterministic systems, to provide sufficient conditions for the global synchronization of system (3.1). In particular, we make use of the results of [33], where a criterion for global exponential synchronization of (2.1) and (2.2) is given in the case of constant feedback gain. The error system equation (2.3), when $K(t)$ equals the constant K^* , becomes

$$\dot{e}(t) = (A - K^*)e(t) + M_{x(t),x(t)-e(t)}e(t) \quad (4.1)$$

For clarity we restate here the main theorem of [33].

THEOREM 4.1. *The system (4.1) is globally exponentially stable, if the feedback gain matrix K^* is chosen such that*

$$l_i(t) \leq -w < 0, \quad i = 1, 2, \dots, n$$

where $l_i(t)$'s are the eigenvalues of the matrix

$$Q(t) = (A - K^* + M_{x(t),x(t)-e(t)})^T P + P(A - K^* + M_{x(t),x(t)-e(t)})$$

and P is a positive definite symmetric constant matrix. A Lyapunov function for (4.1) can be constructed as

$$V(e(t), t) = e(t)^T P e(t) \quad (4.2)$$

with

$$\dot{V}(e(t), t) = e(t)^T Q(t) e(t) \leq -w \|e(t)\|^2 < 0 \quad (4.3)$$

We note that, for $i = 1, \dots, N$, the function $f(e(t), t, K_i) = (A + M_{x,x-e} - K_i)e(t)$ is globally Lipschitz in \mathbb{R}^+ with Lipschitz constant $L_i = \|A\| + m + \|K_i\|$, where $\|M\| \leq m$. The Lipschitz constants L_i are bounded by $L = \|A\| + m + \max_{1 \leq i \leq N} \{\|K_i\|\}$. We further notice that $f(0, t, K_i) = 0, \forall t \in \mathbb{R}^+$. We associate to the system (2.3) the partially averaged system

$$\dot{e}(t) = (A + M_{x(t),x(t)-e(t)})e(t) - \bar{K}e(t) \quad (4.4)$$

where $\bar{K} = \mathbb{E}[K(t)] = \sum_{i=1}^N p_i K_i$. Here, p_i indicates the probability of $K(t)$ assuming value K_i , that is $p_i = P\{K(t) = K_i\}$.

Since \bar{K} is constant, we can apply Theorem 4.1 to (4.4). The Lyapunov function (4.2) constructed for the partially averaged system, can be used to assess the stability of the stochastic system. In fact $V(0, t) = 0$ and (3.7) holds for $\lambda_{min} = \min\{\lambda(P)\}$ and

$\lambda_{max} = \max\{\lambda(P)\}$, since P is a constant matrix (here $\lambda(\bullet)$ indicates the spectrum of the matrix). Furthermore, (4.3) is equivalent to (3.8) and Condition 3 of Theorem 3.5 is satisfied with $K_v = 2\|P\|$ and $K_{vt} = 0$. Thus equation (3.14), specified for the case at hand, reads

$$2L^2K_v e^{2L\varepsilon}\varepsilon^2 - w = 0 \quad (4.5)$$

and it yields the sought value of ε^* . By applying Theorem 3.5, we claim that the system (2.3) is globally mean square exponentially stable and globally almost surely asymptotically stable $\forall \varepsilon < \varepsilon^*$. We summarize the above arguments in the following Corollary.

COROLLARY 4.2. *Consider the system (2.3) and the corresponding partially averaged system (4.4). If the feedback gain matrix $K(t)$ is chosen such that*

$$\bar{l}_i(t) \leq -w < 0, \quad i = 1, 2, \dots, n$$

where $\bar{l}_i(t)$'s are the eigenvalues of the matrix

$$Q(t) = (A - \bar{K} + M_{x(t),x(t)-e(t)})^T P + P(A - \bar{K} + M_{x(t),x(t)-e(t)})$$

and P is a positive definite symmetric constant matrix, then there exists an $\varepsilon^* > 0$ such that such that $\forall \varepsilon < \varepsilon^*$ the system (2.3) is mean square globally exponentially stable and globally almost surely asymptotically stable.

5. Case study: synchronization of two chaotic Chua's circuit. As an example, we apply our results to synchronization of Chua's circuits (see for example [58]). A Chua's circuit system is described by

$$\begin{cases} \dot{x} &= a(y - x - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -by \end{cases} \quad (5.1)$$

where $a > 0$, $b > 0$ and the nonlinear function h has the form

$$h(x) = m_1x + \frac{1}{2}(m_0 - m_1)\{|x + 1| - |x - 1|\} \quad (5.2)$$

with $m_0 < 0$ and $m_1 < 0$. We define

$$h(x) - h(\tilde{x}) = w_{x,\tilde{x}}(x - \tilde{x}) \quad (5.3)$$

where $w_{x,\tilde{x}}$ depends on x and \tilde{x} and is bounded by $m_0 \leq w_{x,\tilde{x}} \leq m_1$ (see for example [33]).

We consider the case where $K(t)$ is a diagonal matrix. Following (2.2), the slave system of (5.1) is constructed

$$\begin{cases} \dot{\tilde{x}} &= a(\tilde{y} - \tilde{x} - h(\tilde{x})) + k_1(t)(x - \tilde{x}) \\ \dot{\tilde{y}} &= \tilde{x} - \tilde{y} + \tilde{z} + k_2(t)(x - \tilde{x}) \\ \dot{\tilde{z}} &= -b\tilde{y} + k_3(t)(x - \tilde{x}) \end{cases} \quad (5.4)$$

Combining (5.1) and (5.4), we obtain equation (2.3), with

$$A = \begin{bmatrix} -a & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad K(t) = \begin{bmatrix} k_1(t) & 0 & 0 \\ 0 & k_2(t) & 0 \\ 0 & 0 & k_3(t) \end{bmatrix}, \quad g(x) = \begin{bmatrix} -ah(x) \\ 0 \\ 0 \end{bmatrix}.$$

We observe that $g(x) - g(\tilde{x}) = M_{x,x-e}e$, where

$$M_{x,x-e} = \begin{bmatrix} -aw_{x,x-e} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.5)$$

and $\|M\| \leq m_0m_1$.

We associate to the system (2.3) the partially averaged system

$$\dot{e} = Ae + M_{x,x-e}e - \bar{K}e \quad (5.6)$$

where

$$\bar{K} = \begin{bmatrix} \bar{k}_1 & 0 & 0 \\ 0 & \bar{k}_2 & 0 \\ 0 & 0 & \bar{k}_3 \end{bmatrix} \quad (5.7)$$

By choosing $P = I$, and by setting

$$\begin{aligned} \bar{k}_1 &\geq \frac{1}{2}(1 - a - 2m_0 - w) \\ \bar{k}_2 &\geq \frac{1}{2}(a - 1 + |1 - b| - w) \\ \bar{k}_3 &\geq \frac{1}{2}(|1 - b| - w) \end{aligned} \quad (5.8)$$

The partially averaged system is globally exponentially stable [33]. We also obtain $K_v = 2$ and $L = \|A\| + m_0m_1 + \max_{1 \leq i \leq N} \{\|K_i\|\}$. Equation (4.5) gives the value of ε^* that assures the global mean square exponential stability and the global almost sure asymptotic stability of the stochastic system $\forall \varepsilon < \varepsilon^*$.

Here, we present a few numerical results that illustrate how two stochastically coupled Chua's circuits synchronize for a sufficiently fast switching rate. We select $a = 9.78$, $b = 14.97$, $m_0 = -1.31$ and $b = -0.75$ in order to have chaotic behavior of the system [33]. We let $K(t)$ switching randomly between the two constant matrices K_1 and K_2 , where K_1 is the zero matrix and

$$K_2 = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 27.5 & 0 \\ 0 & 0 & 20 \end{bmatrix} \quad (5.9)$$

For these parameters we have $\|A\| = 18.8859$, $\max_{1 \leq i \leq N} \{\|K_i\|\} = \|K_2\| = 27.5$ and $L = 47.3684$. Selecting $p_1 = 0.6$ and $p_2 = 0.4$, w can be chosen from (5.8) to be equal to 0.5. From equation (4.5) we have that for $\varepsilon = 10^{-3}$ the system synchronizes globally mean square exponentially and globally almost surely asymptotically. Figure 5.1 depicts the trajectories of the master and slave systems on the xy and xz planes. This figure shows that the two systems synchronize even if the initial conditions are significantly different.

6. Conclusions. In this paper we presented a general criterion for global synchronization of two chaotic oscillators in a master-slave configuration. The two systems are coupled through a stochastic unidirectional feedback, realized through a

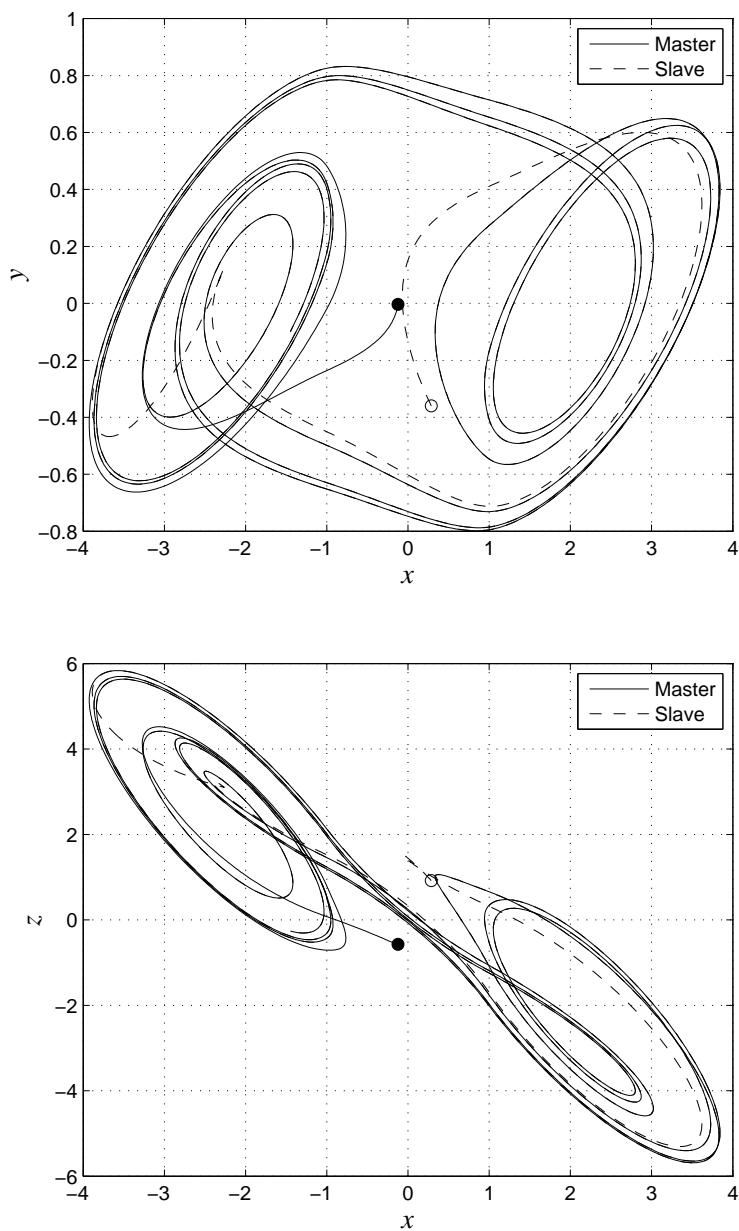


FIG. 5.1. Trajectories of the master and slave systems in the xy and xz plane

switching function, that switches randomly among a finite set of constant values. Using tools based on Lyapunov stability and partially averaging we showed that, under suitable regularity conditions, the synchronization characteristics of the partially averaged system are inherited by the stochastic system. Our findings are illustrated through numerical simulations on Chua's circuits.

Appendix A. Proof of Theorem 3.4.

Proof. We start by proving the global mean square exponential stability. Consider arbitrary initial time $t_0 \in \mathbb{R}^+$ and initial condition $x_0 \in \mathbb{R}^n$. We define the index \hat{k} so that $t_0 \in [\sigma_{\hat{k}-1}, \sigma_{\hat{k}}]$. Specifying equation (3.4) at the \hat{k} -th and $(\hat{k} + 1)$ -th switching instants, we have

$$\mathbb{E}[V(x(\sigma_{\hat{k}+1}), \sigma_{\hat{k}+1}) | x(\sigma_{\hat{k}})] \leq (1 - \nu)V(x(\sigma_{\hat{k}}), \sigma_{\hat{k}}) \quad (\text{A.1})$$

$$\mathbb{E}[V(x(\sigma_{\hat{k}+2}), \sigma_{\hat{k}+2}) | x(\sigma_{\hat{k}+1})] \leq (1 - \nu)V(x(\sigma_{\hat{k}+1}), \sigma_{\hat{k}+1}) \quad (\text{A.2})$$

By taking the conditional expected value of (A.2) we obtain

$$\mathbb{E}[\mathbb{E}[V(x(\sigma_{\hat{k}+2}), \sigma_{\hat{k}+2}) | x(\sigma_{\hat{k}+1})] | x(\sigma_{\hat{k}})] \leq (1 - \nu)\mathbb{E}[V(x(\sigma_{\hat{k}+1}), \sigma_{\hat{k}+1}) | x(\sigma_{\hat{k}})] \quad (\text{A.3})$$

Using the smoothing lemma for martingales (see e.g. [28] Lemma 1.1 p. 474) and (A.1) in (A.3), we have

$$\mathbb{E}[V(x(\sigma_{\hat{k}+2}), \sigma_{\hat{k}+2}) | x(\sigma_{\hat{k}})] \leq (1 - \nu)^2 V(x(\sigma_{\hat{k}}), \sigma_{\hat{k}})$$

Iterating the argument above for any $n > \hat{k}$, we obtain

$$\mathbb{E}[V(x(\sigma_n), \sigma_n) | x(\sigma_{\hat{k}})] \leq (1 - \nu)^{n-\hat{k}} V(x(\sigma_{\hat{k}}), \sigma_{\hat{k}}) \quad (\text{A.4})$$

By using the bounds in (3.3), equation (A.4) gives

$$\mathbb{E}[\|x(\sigma_n)\|^2 | x(\sigma_{\hat{k}})] \leq \lambda_{max}/\lambda_{min} (1 - \nu)^{n-\hat{k}} \|x(\sigma_{\hat{k}})\|^2 \quad (\text{A.5})$$

Equation (A.5) can be used to derive an upper bound for the unconditioned expected value, that is needed to assess the global mean square exponential stability according to Definition 3.1. Since \hat{k} is a given instant of time and x_0 is a prescribed initial condition, $x(\sigma_{\hat{k}})$ is a finite-state random variable taking values in $\{x_1(\sigma_{\hat{k}}), \dots, x_N(\sigma_{\hat{k}})\}$. From the definition of conditional expectation (see for example [8]) we have

$$\mathbb{E}[\|x(\sigma_n)\|^2] = \sum_{i=1}^N \mathbb{E}[\|x(\sigma_n)\|^2 | x_i(\sigma_{\hat{k}})] P\{x_i(\sigma_{\hat{k}})\} \quad (\text{A.6})$$

where $P\{x_i(\sigma_{\hat{k}})\}$ is the probability that $x_i(\sigma_{\hat{k}})$ is the realization of the random variable $x(\sigma_{\hat{k}})$. Hence, using inequality (A.5), equation (A.6) yields

$$\mathbb{E}[\|x(\sigma_n)\|^2] \leq \sum_{i=1}^N \frac{\lambda_{max}}{\lambda_{min}} (1 - \nu)^{n-\hat{k}} \|x_i(\sigma_{\hat{k}})\|^2 P\{x_i(\sigma_{\hat{k}})\} \quad (\text{A.7})$$

In order to assess the global mean square exponential stability we need to analyze the system dynamics inside every switching interval. Given a generic switching interval $[\sigma_k, \sigma_{k+1}]$ and an instant $\bar{t} \in [\sigma_k, \sigma_{k+1}]$, using the triangle inequality, $\forall t \geq \bar{t}$ in $[\sigma_k, \sigma_{k+1}]$, equation (3.1) yields

$$\|x(t)\| \leq \|x(\bar{t})\| + \int_{\bar{t}}^t \|f(x(\xi), \xi, \Omega)\| d\xi \quad (\text{A.8})$$

Since the functions f_ω are globally Lipschitz in \mathbb{R}^+ and all the corresponding Lipschitz constants are bounded by a constant L , equation (A.8) yields

$$\|x(t)\| \leq \|x(\bar{t})\| + \int_{\bar{t}}^t L \|x(\xi)\| d\xi$$

Using the Gronwall-Bellman inequality (see [35], Lemma 2.1) we have

$$\|x(t)\| \leq \gamma \|x(\bar{t})\| \quad (\text{A.9})$$

with $\gamma = e^{L\varepsilon}$. Therefore, using (A.9) in (A.7), we find that $\forall t \in [\sigma_n, \sigma_{n+1}]$

$$\mathbb{E}[\|x(t)\|^2] \leq \gamma \mathbb{E}[\|x(\sigma_n)\|^2] \leq \gamma \sum_{i=1}^N \frac{\lambda_{max}}{\lambda_{min}} (1-\nu)^{n-\hat{k}} \|x_i(\sigma_{\hat{k}})\|^2 P\{x_i(\sigma_{\hat{k}})\} \quad (\text{A.10})$$

Inequality (A.9) can also be used to find an upper bound for $\|x(\sigma_{\hat{k}})\|$ in terms of the initial conditions. In fact, since $t_0 \in [\sigma_{\hat{k}-1}, \sigma_{\hat{k}}]$ according to the definition of \hat{k} , (A.9) holds for $\bar{t} = t_0$ and $t \geq t_0$. Since $\sigma_{\hat{k}} \geq t_0$, we obtain

$$\|x(\sigma_{\hat{k}})\| \leq \gamma \|x(t_0)\| \quad (\text{A.11})$$

Finally, using (A.11) to bound the right side of (A.10), we obtain

$$\begin{aligned} \mathbb{E}[\|x(t)\|^2] &\leq \gamma^2 \sum_{i=1}^N \frac{\lambda_{max}}{\lambda_{min}} (1-\nu)^{n-\hat{k}} \|x(t_0)\|^2 P\{x_i(\sigma_{\hat{k}})\} \\ &\leq \gamma^2 \frac{\lambda_{max}}{\lambda_{min}} (1-\nu)^{n-\hat{k}} \|x(t_0)\|^2 \\ &\leq \alpha \|x(t_0)\|^2 e^{-\beta(t-t_0)} \end{aligned} \quad (\text{A.12})$$

where $\alpha = \gamma^2(1-\nu)\lambda_{max}/\lambda_{min}$ and $\beta = -\ln(1-\nu)/\varepsilon$. Therefore, according to Definition 3.1, the system (3.1) is globally mean square exponentially stable.

In the second part of the proof, we establish the global almost surely asymptotic stability. We notice that, since (3.4) holds and $V(x(\sigma_k), \sigma_k)$ is a positive quantity, the sequence of $V(x(\sigma_k), \sigma_k)$ is a supermartingale (see for example Definition 2.4 in [28]). Therefore, we can apply the supermartingale inequality (see [38], Proposition 1 p. 31) and obtain that for every $\eta > 0$

$$P \left\{ \sup_{\infty > k \geq n} V(x(\sigma_k), \sigma_k) \geq \eta \right\} \leq \frac{\mathbb{E}[V(x(\sigma_n), \sigma_n) | x(\sigma_{\hat{k}})]}{\eta} \quad (\text{A.13})$$

Substituting (A.4) into (A.13) we obtain

$$P \left\{ \sup_{\infty > k \geq n} V(x(\sigma_k), \sigma_k) \geq \eta \right\} \leq \frac{(1-\nu)^{n-\hat{k}}}{\eta} V(x(\sigma_{\hat{k}}), \sigma_{\hat{k}}) \quad (\text{A.14})$$

By hypothesis $V(x, t) \leq \lambda_{max} \|x\|^2$. Thus, equation (A.14) yields

$$P \left\{ \sup_{\infty > k \geq n} \|x(\sigma_k)\|^2 \geq \frac{\eta}{\lambda_{max}} \right\} \leq \frac{\lambda_{max}}{\eta} (1-\nu)^{n-\hat{k}} \|x(\sigma_{\hat{k}})\|^2 \quad (\text{A.15})$$

Using (A.11) in (A.15) we have

$$P \left\{ \sup_{\infty > k \geq n} \|x(\sigma_k)\| \geq \psi \right\} \leq c(1-\nu)^{n-\hat{k}} \|x(t_0)\|^2 \quad (\text{A.16})$$

where $\psi = \sqrt{\eta/\lambda_{max}}$ and $c = \gamma^2/\psi^2$.

Defining the events $A = \{\|x(\sigma_k)\| \geq \varrho\}$ and $B = \{\|x(t)\|/\gamma \geq \varrho\}$ we notice, since (A.9) holds for $\bar{t} = \sigma_k$, that B is included in A and therefore

$$P \left\{ \sup_{\infty > (t-t_0) \geq T} \|x(t)\| \geq \varrho\gamma \right\} \leq P \left\{ \sup_{\infty > k \geq n} \|x(\sigma_k)\| \geq \varrho \right\} \quad (\text{A.17})$$

with $T = n\varepsilon$. Since $0 \leq (1 - \nu) < 1$ and (A.9) holds with $\bar{t} = \sigma_k$ and $t \in [\sigma_k, \sigma_{k+1}]$, from (A.16) and (A.17) we derive

$$P \left\{ \sup_{\infty > (t-t_0) \geq T} \|x(t)\| \geq \psi\gamma \right\} \leq c\|x(t_0)\|^2 \quad (\text{A.18})$$

which provides the global almost surely stability of system (3.1).

From (A.16) we also obtain that for every $\psi > 0$

$$\sum_{n=\hat{k}}^{\infty} P \left\{ \sup_{\infty > k \geq n} \|x(\sigma_k)\| \geq \psi \right\} < \infty \quad (\text{A.19})$$

Thus, by directly applying the Borel-Cantelli lemma (see [28] Corollary 18.1), we find that

$$\sup_{\infty > k \geq n} \|x(\sigma_k)\| \xrightarrow{a.s.} 0 \quad (\text{A.20})$$

Since $\|x(\sigma_k)\|$ is a positive sequence, from (A.20) we have that $\|x(\sigma_k)\| \xrightarrow{a.s.} 0$ as $k \rightarrow \infty$. As (A.9) holds for $\bar{t} = \sigma_k$, it follows that $x(t) \xrightarrow{a.s.} 0$ as $t \rightarrow \infty$. Therefore, according to Definition 3.3, system (3.1) is globally almost surely asymptotically stable. \square

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