Progress of Theoretical Physics, Vol. 81, No. 2, February 1989

Mastersymmetries and Multi-Hamiltonian Formulations for Some Integrable Lattice Systems

W. OEVEL, H. ZHANG and B. FUCHSSTEINER

Universität Paderborn, D4790 Paderborn

(Received September 6, 1988)

Conserved quantities, bi-hamiltonian formulation, recursive structure and hereditary symmetries are obtained for a number of lattice systems with physical significance. Furthermore, for the multisoliton solutions the gradients of the angle variables are given. Apart from the well investigated Toda lattice these systems include: Volterra lattice, lumped Network system, Kac-Moerbeke-Langmuir lattice and a class of Network equations. No use is made of the Lax representation or any other additional information about the equations under consideration. All quantities are found in a purely algorithmic way by use of mastersymmetries.

§ 1. Introduction

Ever since the pioneering work on the Fermi-Pasta-Ulam problem[1] discrete nonlinear systems have been in the focus on nonlinear studies since many interesting physical phenomena can be modeled by them. Of special interest are those systems admitting large symmetry groups since contrary to early expectations usually no equipartition of energy between the different modes of the system takes place. This shows that systems with nontrivial nonlinear interactions can behave like linear systems. Apart from such phenomena of physical relevance these systems are interesting from the mathematical viewpoint since they usually provide excellent discrete approximations for completely integrable nonlinear partial differential equations.

Nowadays, since the introduction of a system with exponential interaction by Toda[2] describing vibrations of particles in a one-dimensional lattice, many such systems with infinite dimensional symmetry groups are known. These systems play a major role in solid state physics.

Usually, the complete integrability of such a system is shown via the exhibition of a Lax pair which mostly is found by ingenuity, and no algorithm seems to be known telling whether or not a given system is completely integrable. Even then the explicit treatment of a completely integrable nonlinear lattice system is difficult because the computation of the relevant quantities (symmetry generators, conservation laws, angle variables) out of a given Lax pair is a nontrivial task.

We have developed an algorithm which tells whether a system (of a certain class) is completely integrable. Furthermore, if the answer is affirmative, then the recursive structure of the system under consideration is computed automatically. This algorithm is briefly sketched and will be reported in full detail elsewhere. Since we have implemented a prerelease version of this algorithm by means of computer algebra we are able to exhibit easily the recursive structure and the explicit form of relevant quantities for many nonlinear systems. The results of these studies are given in this paper, they include: recursion formulas for symmetry group generators, bi-hamiltonian formulations, hereditary symmetries (recursion operators) yielding Lax pairs, mastersymmetries and gradients of angle variables for the multisoliton solutions. These structures are found for the following equations: the Toda lattice,[2] a Volterra-type lattice,[3] lumped Network systems,[4] Kac-Moerbeke-Langmuir lattice,[5] and a class of Network equations.[3),6] Many new explicit formulas for these systems can be found in this paper. It should be remarked that similar studies have been performed[7] for the relativistic Toda lattice of Schneider and Ruijsenaars and nonlinear quantum mechanical spin-1/2 systems.[8]

§ 2. Basic notation

We will consider evolution equations of the form

$$u_t = K(u)\,, \tag{2·1}$$

where u is a point in a space $S=\{u=(u_1, \cdots, u_m)|u: \boldsymbol{Z}\to\boldsymbol{R}^m\}$ of (vector valued) sequences and K is a vectorfield over this space. We will assume certain boundary conditions for these sequences, which are to ensure that the infinite summations turning up in the following make sense. The dual space S^* is again a space of (vector valued) sequences acting on S via the pairing

$$\langle u^*, u\rangle = \sum_{n\in \boldsymbol{Z}} (u^*(n), u(n))\,; \quad u^*=(u^*(n))\in S^*\,, \quad u=(u(n))\in S\,, \tag{2·2}$$

where $(.,.)$ is the usual euclidean scalar product on $\boldsymbol{R}^m$. If A is some function of $u\in S$, then its directional derivative at the point u into the direction of a vector $v\in S$ is given by

$$A'(u)[v] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} A(u+\varepsilon v)\,. \tag{2·3}$$

For example, if A takes values in S, then this derivative has the form

$$(A'(u)[v])(n) = \sum_{k\in \boldsymbol{Z}} v(n+k)\frac{\partial A(u)}{\partial u(n+k)}\,. \tag{2·4}$$

Regarding S as a manifold the vectorfields, i.e., the maps $K: S\to S$, carry the usual Lie algebra structure given by the commutator

$$[K_1, K_2] = K_2'[K_1] - K_1'[K_2]\,. \tag{2·5}$$

Vector fields commuting with a given vectorfield K are considered as generators of 1-parameter groups of symmetry transformations for the dynamical system (2·1), we will call them **symmetries** for short. In the following we want to construct a set of such symmetries for some given equations, a helpful tool will be so-called **recursion operators** (Refs. 9) and 10)). These are operators $\Phi(u): S\to S$ satisfying

$$[K, \Phi G] - \Phi[K, G] \equiv \Phi'[K]G - K'[\Phi G] + \Phi K'[G] = 0 \tag{2·6}$$

for all vectorfields G. Obviously such an operator maps symmetries to symmetries.

Another powerful tool for the construction of symmetries had been introduced in

Ref. 11) with the notion of **mastersymmetries.** A mastersymmetry τ for a given vectorfield K is a vectorfield satisfying $[K, [\tau, K]]=0$ and $[\tau, K]\neq 0$, i.e., it sends K to a (non-vanishing) symmetry $[\tau, K]$ of K via the commutator. From the Jacobi identity one immediately concludes that $[\tau, [\tau, K]]$ is again a further symmetry of K. Without additional assumptions no further algebraic relations can be derived, hence a mastersymmetry essentially has the property of generating 2 symmetries out of K by applying the map (**Lie derivative**) $L_\tau: A\rightarrow[\tau, A]$. But assuming the group of (time-independent) symmetries of K to be abelian, i.e., all the symmetry generators of K have to commute, one trivially derives, that the Lie derivative into the direction of such a mastersymmetry always maps a symmetry of K to another symmetry. Of course, for a given K it is hard —or impossible—to show that all its symmetries will commute. But for integrable equations it is known that a large set of commuting symmetries (corresponding to action variables in involution) exists, i.e., the Lie derivative into the direction of a mastersymmetry will be a selfmap on this set of vectorfields. Hence, for integrable K, a mastersymmetry as defined above will be an important **heuristic** tool: find a mastersymmetry τ for K, construct further vectorfields by iteratively applying L_τ to known symmetries of K (e.g., K itself) and then try to verify **a posteriori** that these vectorfields form an abelian set of symmetries for K. A typical way of such an a posteriori proof is given for hamiltonian systems, as very often the mastersymmetries lead to the construction of Hamiltonian pairs[10] and hereditary recursion operators,[9),10)] from which the commutativity of the constructed vectorfields can be derived easily.

We briefly review the necessary notation: The **gradient** of a scalar valued function $f: S\rightarrow \boldsymbol{R}$ is the element of S^* given by

$$\langle \nabla f(u), v\rangle=f'(u)[v], \text{ i.e., } (\nabla f(u))(n):=\frac{\partial f(u)}{\partial u(n)}. \tag{2·7}$$

A vectorfield of the form $K=P\nabla f$ is called **hamiltonian,** where P is a **Poisson** (hamiltonian, implectic[10]) operator, i.e., $P(u): S^*\rightarrow S$ is a skewsymmetric linear operator satisfying the "Jacobi-identity"

$$\langle a^*, P'[Pb^*]c^*\rangle+\text{cyclic permutations}=0 \tag{2·8}$$

for arbitrary elements $a^*, b^*, c^*\in S^*$. As a consequence the **Poisson bracket** $\{f_1, f_2\}=\langle\nabla f_2, P\nabla f_1\rangle$ defines a Lie algebra structure on the space of scalar fields over S. A **conservation law** for (2·1) is a scalar valued function f such that $f(u(t))$ is constant for all solutions of (2·1), i.e., $\langle\nabla f, K\rangle=0$. For a hamiltonian vectorfield $K=P\nabla f$ the function f automatically is a conservation law and P will map the gradient of any conservation law to a symmetry for K. The **Lie-derivative** of such an operator P into the direction of a vectorfield τ is given by

$$L_\tau P=P'[\tau]-\tau' P-P\tau'^*, \tag{2·9}$$

where τ'^* denotes the adjoint (w.r.t. the duality (2·2)) of the linearization τ'. If this resulting operator turns out to be Poisson again, then P and $L_\tau P$ automatically form a **compatible Hamiltonian pair** (Ref. 10)), i.e., their sum is again a Poisson operator.

In the examples of the next section the relevant vectorfields and operators will be formulated in terms of the following basic operations $S\to S$: An element $a\in S$ (or S^*) gives rise to a multiplication operator

$$a: u\to(au)(n)=a(n)u(n)\,, \tag{2·10}$$

and by $[n]$ we will denote the multiplication

$$[n]: u\to([n]u)(n)=nu(n)\,. \tag{2·11}$$

Let T_+ and T_- be the shift operators given by $(T_\pm u)(n):=u(n\pm1)$, we abbreviate

$$\Delta=T_+-T_-\,. \tag{2·12}$$

Apart from these local operations we will need the linear operator Δ^{-1} given by

$$(\Delta^{-1}u)(n):=\frac{1}{2}\Big(\sum_{k=-\infty}^{-1}u(n+1+2k)-\sum_{k=1}^{\infty}u(n-1+2k)\Big) \tag{2·13}$$

(assuming suitable boundary conditions for u). Note that $\Delta\Delta^{-1}=\Delta^{-1}\Delta=1$, i.e., (2·13) defines the inverse of the difference operator Δ when acting on elements of S. For a constant sequence $c(n)=1$ we define $(\Delta^{-1}c)(n)=n/2$. We remark that the inverses of the difference operators $1-T_-$ and T_+-1 can be expressed in terms of the above operator, one finds

$$(1-T_-)^{-1}=\Delta^{-1}(1+T_+)\,,\qquad (T_+-1)^{-1}=\Delta^{-1}(1+T_-)\,. \tag{2·14}$$

All these operations can also be applied to elements of S^*. For the transposed operators (w.r.t. (2·2)) one finds $T_+^*=T_-$, $T_-^*=T_+$. Hence Δ and Δ^{-1} are skew symmetric operators, whereas all multiplication operators are symmetric.

§ 3. Results for some integrable lattice equations

We will construct infinite sets of commuting symmetries and conservation laws for some lattice equations. For those of the examples admitting a bi-hamiltonian formulation the results can be summarized by the following:

THEOREM[10),12),13)]

Let $K_1=P_0\nabla f_1=P_1\nabla f_0$ be a vectorfield admitting two different hamiltonian formulations w.r.t. two compatible Poisson operators P_0 and P_1. Then (for invertible P_0) the operator $\Phi=P_1P_0^{-1}$ is a hereditary[10)] recursion operator for all the vectorfields K_i defined by $K_{i+1}:=\Phi K_i$, and there exists a set of functions $\{f_i, i=0, 1, 2, \cdots\}$ such that $K_{i+1}=P_0\nabla f_{i+1}=P_1\nabla f_i$. The operators defined by $P_{i+1}=\Phi P_i$ are all Poisson and the vectorfields K_i admit further hamiltonian formulations w.r.t. these "higher" Poisson operators: $K_{i+j}=P_i\nabla f_j$. The functions f_i are in involution w.r.t all the Poisson brackets given by the P_j's, the vectorfields K_i commute.

Assume that the pair P_0, P_1 as well as the function f_0 admit a conformal symmetry generated by a vectorfield τ_0, i.e.,

$$L_{\tau_0}P_0=\alpha P_0,\quad L_{\tau_0}P_1=\beta P_1,\quad \nabla(\langle\nabla f_0, \tau_0\rangle)=\gamma\nabla f_0 \tag{3·1}$$

for some scalar factors α, β and γ. Defining a set of vectorfields $\tau_{i+1}=\Phi\tau_i$ one finds

$$[\tau_i, K_j]=(\beta+\gamma+(j-1)(\beta-\alpha))K_{i+j}\,,$$

$$[\tau_i, \tau_j]=(\beta-\alpha)(j-i)\tau_{i+j}\,,$$

$$L_{\tau_i}P_j=(\beta+(j-i-1)(\beta-\alpha))P_{i+j} \qquad (3\cdot2)$$

for all admissible indices i and j. The set of functions in involution can be constructed using the recursion relations

$$\langle \nabla f_j, \tau_i\rangle=(\gamma+(i+j)(\beta-\alpha))f_{i+j}\,. \qquad (3\cdot3)$$

So obviously the above vectorfields K_i form an infinite set of commuting symmetries, the functions f_i are conservation laws in involution for all the hamiltonian equations $u_t=K_j(u)$. In this sense all these dynamical systems are integrable. The set of vectorfields τ_i shall be called the **mastersymmetries** for this integrable system (cf. Refs. 11) and 12)), they obviously provide an iterative scheme to obtain the higher symmetries and conservation laws for the K's using (3·2) and (3·3). These τ's are related to time-dependent symmetries of the K's (cf. Refs. 11) and 12)).

Having constructed a set of functions f_i in involution, i.e., the action variables of the integrable system, the natural question for the angle variables arises. It turns out that for many systems the mastersymmetries are linked to these quantities:

If the mastersymmetries are hamiltonian, one can apply the Lie algebra homomorphism induced by the hamiltonian operator to map the time-dependent symmetries onto the scalar fields yielding conservation laws linear in time (i.e., angle variables). This happens for integro-differential equations such as the Benjamin-Ono, the Kadomtsev-Petviashvili and all the other completely integrable systems in (2+1) dimensions. In these cases no recursion operators (in the usual sense) have been found although recently recursion operators in an extended sense as well as (generalized) non-hamiltonian mastersymmetries have been discovered.[14] The situation changes drastically when the mastersymmetries are not hamiltonian, i.e., the Lie derivative of the hamiltonian operator P_0 into the direction of the mastersymmetries does not vanish. This Lie derivative then yields a second invariant operator P_1, which usually gives rise to a second hamiltonian formulation for the system under consideration. Hence, in this case each nontrivial mastersymmetry immediately leads to a bi-hamiltonian formulation. This happens for equations like the Korteweg de Vries, the modified Korteweg de Vries, the sine-Gordon, the Nonlinear Schrödinger and so on. Now, taking

$$\Phi=P_1P_0^{-1}\,, \qquad (3\cdot4)$$

one finds a recursion operator for the system, which is automatically hereditary, if the second operator P_1 was found to be hamiltonian.

So in any case finding a nontrivial mastersymmetry yields the recursive structure of the hierarchy under consideration. But in addition the angle-variables, at least for the multisoliton manifolds, can be obtained by the mastersymmetries even if those are not hamiltonian. In the first case, where the mastersymmetries are hamiltonian, the construction of the angle-variables is obvious: They are given by the hamiltonian functions of the mastersymmetries. In the other case additional arguments have to be applied, this will be reported in detail elsewhere,[15] here we just review the results:

Let $u_t=K_1(u)$ be a dynamical system admitting a hereditary recursion operator Φ and a mastersymmetry τ_1. Application of Φ to the first mastersymmetry τ_1 yields a sequence of mastersymmetries

$$\tau_j=\Phi^{j-1}\tau_1\,, \qquad j=1, 2, 3, \cdots \tag{3·5}$$

satisfying the relations

$$[\tau_j, \tau_k]=(k-j)\tau_{k+j} \tag{3·6}$$

(after an appropriate rescaling of τ_1). The vectorfields defined via

$$K_{j+1}:=\Phi^j K_1 \tag{3·7}$$

are the symmetry generators of the system (or the members of the hierarchy, if one likes). Now, if one considers the following invariant submanifold:

$$\{u|\sum_{i=1}^{N+1} a_i K_i=0,\ a_i\in \boldsymbol{R}\}\,, \tag{3·8}$$

then this turns out to be the N-soliton manifold[9),16)] being of dimension $2N$. The parametrization of this manifold is given by time, the $N-1$ phases and the N different asymptotic speeds (represented by the variables a_i). A detailed study reveals that, although the τ_n's are not hamiltonian, the following linear combinations

$$A_r=B_r\sum_{i=1}^{N} a_i(r)\tau_{i+r}\,, \quad r=1, \cdots, N \tag{3·9}$$

are hamiltonian on the reduced manifold given by (3·8). Here the $a_i(r)$ are the coefficients of the polynomials

$$\sum_{i=1}^{N} a_i(r)\lambda^i=\frac{1}{\lambda-\lambda_r}\sum_{i=1}^{N+1} a_i\lambda^i\,, \tag{3·10}$$

the B_r are suitable integrating factors and the λ_r are the zeros of the polynomial

$$P(\lambda)=\sum_{i=1}^{N+1} a_i\lambda^i\,. \tag{3·11}$$

The hamiltonian functions of the vectorfields A_r given by (3·9) correspond to conservation laws with a linear time dependence, i.e., we thus have found the angle variables on the above multisoliton manifolds.

Example 1: The Toda lattice[2)]

The bi-hamiltonian formulation of the Toda lattice

$$\frac{d}{dt}\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix}=K_1(u)=\begin{bmatrix} u_1(n)(u_2(n+1)-u_2(n)) \\ 2(u_1^2(n)-u_1^2(n-1)) \end{bmatrix} \tag{3·12}$$

is given by $K_1=P_0\nabla f_1=P_1\nabla f_0$, with (cf. Ref. 2))

$$P_0(u)=\begin{bmatrix} 0 & ;\, u_1(T_+-1) \\ (1-T_-)u_1\,; & 0 \end{bmatrix},\quad f_1=\sum_{n\in Z}\left(u_1^2(n)+\frac{1}{2}u_2^2(n)\right),$$

$$P_1(u)=\begin{bmatrix} \frac{1}{2}u_1\Delta u_1 & ;\quad u_1(T_+-1)u_2 \\ u_2(1-T_-)u_1 & ;\, 2(u_1^2T_+-T_-u_1^2) \end{bmatrix},\quad f_0=\sum_{n\in Z}u_2(n)\,. \tag{3·13}$$

As all these quantities are homogeneous expressions in the field variable u, we have a conformal symmetry transformation $u\to\exp(\varepsilon)u$ generated by $\tau_0(u)=u$, one finds:

$$L_{\tau_0}P_0=-P_0\,,\quad L_{\tau_0}P_1=0\,,\quad \langle\nabla f_0,\,\tau_0\rangle=f_0\,. \tag{3·14}$$

With

$$P_0^{-1}(u)=\begin{bmatrix} 0 & ;\,\frac{1}{u_1}\Delta^{-1}(1+T_+) \\ \Delta^{-1}(1+T_-)\frac{1}{u_1}\,; & 0 \end{bmatrix}, \tag{3·15}$$

one finds the recursion operator $\Phi=P_1P_0^{-1}$ and the first nontrivial mastersymmetry

$$\tau_1(u)=[u]K_1(u)+\begin{bmatrix} \frac{3}{2}u_1u_2+\frac{1}{2}u_1T_+u_2 \\ u_2^2+4T_-u_1^2 \end{bmatrix}. \tag{3·16}$$

Verifying $L_{\tau_1}P_0=-2P_1$, one checks the compatibility of the hamiltonian pair P_0 and P_1. Hence all the assumptions of the theorem are satisfied and we can construct all the higher invariants of the Toda lattice using Φ or τ_1.

We remark that a further conservation law $C(u)=\sum\ln(u_1(n))$ exists for the integrable system. This function C turns out to be a Casimir (see e.g., Ref. 17)) for the operators P_0 and P_1, i.e., $P_0\nabla C=P_1\nabla C=0$. From $L_{\tau_0}P_1=0$ one concludes that τ_0 should be hamiltonian w.r.t. P_1. Indeed, one finds $\tau_0=P_1\nabla\sum n\ln(u_1(n))$. We mention that τ_0 and τ_1 are the only local mastersymmetries, i.e., all higher τ's involve the nonlocal operator Δ^{-1}. Among the hamiltonian operators the first three, i.e., P_0, P_1 and

$$P_2(u)=\begin{bmatrix} u_1(T_+u_2-u_2T_-)u_1 & ;\quad u_1(u_1^2(T_+-1)+(T_+-1)u_2^2-T_-u_1^2 \\ & \qquad +T_+u_1^2T_+) \\ ((1-T_-)u_1^2+u_2^2(1-T_-) & ;\quad 2u_2(u_1^2T_+-T_-u_1^2) \\ \quad +u_1^2T_+-T_-u_1^2T_-)u_1 & \qquad +2(u_1^2T_+-T_-u_1^2)u_2 \end{bmatrix} \tag{3·17}$$

are local, the higher ones again involve Δ^{-1}. All the K_i's as well as the conserved functions f_i are local, which follows from the fact that all these objects can be constructed using the local τ_1 (note that this argument does not hold for the higher τ's and P's, as $[\tau_1,\,\tau_1]=0$ and $L_{\tau_1}P_2=0$).

Example 2: *The Volterra lattice*[3)]

The bi-hamiltonian formulation of the Volterra lattice[3)]

$$\frac{d}{dt}\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix} = K_1(u) = \begin{bmatrix} u_1(n)(u_2(n+1)-u_2(n)) \\ u_2(n)(u_1(n)-u_1(n-1)) \end{bmatrix} \tag{3·18}$$

is found to be $K_1 = P_0 \nabla f_1 = P_1 \nabla f_0$ with

$$P_0(u) = \begin{bmatrix} 0 & ; & u_1(T_+-1)u_2 \\ u_2(1-T_-)u_1 & ; & 0 \end{bmatrix}, \quad f_1 = \sum_{n\in Z}(u_1(n)+u_2(n)),$$

$$P_1(u) = \begin{bmatrix} u_1(T_+u_2-u_2T_-)u_1 & ; & u_1(u_1(T_+-1)+(T_+-1)u_2)u_2 \\ u_2((1-T_-)u_1+u_2(1-T_-))u_1 & ; & u_2(u_1T_+-T_-u_1)u_2 \end{bmatrix},$$

$$f_0 = \sum_{n\in Z}\ln(u_1(n)). \tag{3·19}$$

The Poisson operators are homogeneous expressions in the field variable u, so we again have a conformal symmetry transformation $u \to \exp(\varepsilon)u$ generated by $\tau_0(u)=u$, one finds:

$$L_{\tau_0}P_0 = 0, \quad L_{\tau_0}P_1 = P_1, \quad \nabla\langle \nabla f_0, \tau_0\rangle = 0. \tag{3·20}$$

With

$$P_0^{-1}(u) = \begin{bmatrix} 0 & ; & \frac{1}{u_1}\Delta^{-1}(1+T_+)\frac{1}{u_2} \\ \frac{1}{u_2}\Delta^{-1}(1+T_-)\frac{1}{u_1} & ; & 0 \end{bmatrix} \tag{3·21}$$

one finds the recursion operator $\Phi = P_1P_0^{-1}$ and the first nontrivial mastersymmetry

$$\tau_1(u) = 2[u]K_1(u) + \begin{bmatrix} u_1^2+2u_1u_2+u_1T_+u_2 \\ 3u_2T_-u_1+u_2^2 \end{bmatrix}. \tag{3·22}$$

Verifying $L_{\tau_1}P_0 = -P_1$, one checks the compatibility of the hamiltonian pair P_0 and P_1. Hence all the assumptions of the theorem are satisfied and we can construct all the higher invariants of the Volterra lattice using Φ or τ_1. The function f_0 turns out to be a Casimir of P_0, an additional conservation law exists for the integrable system with $C(u) = \sum(\ln(u_1(n)) - \ln(u_2(n)))$, which is a Casimir for both P_0 and P_1. The scaling field τ_0 turns out to be hamiltonian: $\tau_0 = P_0\nabla\sum n(\ln(u_1(n)) + \ln(u_2(n)))$. Again τ_0 and τ_1 are the only local mastersymmetries, P_0 and P_1 are the only local operators among the P_i's.

Example 3: *The Lumped Network system*[4)]

Performing the transformation

$$\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix} \to \begin{bmatrix} u_1(n)+\lambda \\ \lambda u_2(n) \end{bmatrix} \tag{3·23}$$

in the Volterra system (3·18) and then taking the limit $\lambda\to\infty$, one obtains the Lumped Network equation

$$\frac{d}{dt}\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix} = K_1(u) = \begin{bmatrix} u_2(n+1)-u_2(n) \\ u_2(n)(u_1(n)-u_1(n-1)) \end{bmatrix}. \tag{3·24}$$

A bi-hamiltonian formulation $K_1=P_0\nabla f_1=P_1\nabla f_0$ for (3·24) is found with

$$P_0(u)=\begin{bmatrix} 0 & ; & (T_+-1)u_2 \\ u_2(1-T_-)\,; & & 0 \end{bmatrix}, \quad f_1=\sum_{n\in Z}\left(\frac{1}{2}u_1^2(n)+u_2(n)\right),$$

$$P_1(u)=\begin{bmatrix} T_+u_2-u_2T_-\,; & u_1(T_+-1)u_2 \\ u_2(1-T_-)u_1\,; & u_2(T_+-T_-)u_2 \end{bmatrix}, \quad f_0=\sum_{n\in Z}u_1(n). \tag{3·25}$$

Again, the dynamical system as well as the Poisson operators are homogeneous expressions in the field variable u, so we again have a conformal symmetry transformation generated by τ_0, one finds:

$$\tau_0(u)=\begin{bmatrix} u_1(n) \\ 2u_2(n) \end{bmatrix}, \quad L_{\tau_0}P_0=-P_0, \quad L_{\tau_0}P_1=0, \quad \langle\nabla f_0, \tau_0\rangle=f_0. \tag{3·26}$$

With

$$P_0^{-1}(u)=\begin{bmatrix} 0 & ;\, \Delta^{-1}(1+T_+)\dfrac{1}{u_2} \\ \dfrac{1}{u_2}\Delta^{-1}(1+T_-)\,; & 0 \end{bmatrix} \tag{3·27}$$

one finds the recursion operator $\Phi=P_1P_0^{-1}$ and the first nontrivial mastersymmetry

$$\tau_1(u)=2[n]K_1(u)+\begin{bmatrix} u_1^2+2u_2+2T_+u_2 \\ u_1u_2+3u_2T_-u_1 \end{bmatrix}. \tag{3·28}$$

Verifying $L_{\tau_1}P_0=-2P_1$, one checks the compatibility of the hamiltonian pair P_0 and P_1. Hence all the assumptions of the theorem are satisfied and we can construct all the higher invariants of the Lumped Network lattice using Φ or τ_1. As before, the function f_0 is a Casimir for P_0, an additional conservation law is found with $C(u)=\sum\ln(u_2(n))$ (a Casimir of both P_0 and P_1). The scaling field τ_0 turns out to be hamiltonian: $\tau_0=P_1\nabla\sum n\ln(u_2(n))$. As before, τ_0 and τ_1 are the only local mastersymmetries. As $P_2=-L_{\tau_1}P_1$ is generated via the local τ_1 and P_1 we find a third local hamiltonian operator

$$P_2(u)=\begin{bmatrix} u_1(T_+u_2-u_2T_-) & ; \ (u_1^2(T_+-1)+(T_+-1)u_2 \\ \quad +(T_+u_2-u_2T_-)u_1 & \quad -u_2T_-+T_+u_2T_+)u_2 \\ u_2((1-T_-)u_1^2+u_2(1-T_-) & ; \ 2u_2(u_1T_+-T_-u_1)u_2 \\ \quad +T_+u_2-T_-u_2T_-) & \end{bmatrix} \tag{3·29}$$

for the above hierarchy. An additional mastersymmetry ("Galilean invariance") $\tau_{-1}(u)=[1,0]^T$ is found by performing the above limit from the scaling field of the Volterra lattice. One finds $\Phi\tau_{-1}=\tau_0$ and τ_{-1} fits into the algebraic relations of the theorem.

Example 4: A class of Kac-Moerbeke-Langmuir lattices[5)]

For the lattice equation

$$\frac{d}{dt}u(n)=K_1(u)=u(n)\Delta u^\varepsilon(n) \tag{3·30}$$

one finds the bi-hamiltonian formulation $K_1=P_0\nabla f_1=P_1\nabla f_0$ with

$$P_0(u)=u\Delta u\,,\quad f_1=\sum_{n\in Z}\frac{1}{\varepsilon}u^\varepsilon(n)\,,$$

$$P_1(u)=u(u^\varepsilon\Delta+\Delta u^\varepsilon+T_+u^\varepsilon T_+-T_-u^\varepsilon T_-)u\,,\quad f_0=\sum_{n\in Z}\frac{1}{2}\ln(u(n))\,, \tag{3·31}$$

where $\varepsilon\neq 0$ is an arbitrary parameter. A conformal symmetry is generated by $\tau_0(u)=u$, one finds:

$$L_{\tau_0}P_0=0\,,\quad L_{\tau_0}P_1=\varepsilon P_1\,,\quad \nabla\langle\nabla f_0,\tau_0\rangle=0\,. \tag{3·32}$$

With $P_0^{-1}(u)=(1/u)\Delta^{-1}(1/u)$ one finds the recursion operator $\Phi=P_1P_0^{-1}$ and the first nontrivial mastersymmetry

$$\tau_1(u)=[n]K_1(u)+u_1^{\varepsilon+1}+u_1T_+u_1^\varepsilon+2u_1T_-u_1^\varepsilon\,. \tag{3·33}$$

Verifying $L_{\tau_1}P_0=-\varepsilon P_1$, one checks the compatibility of the hamiltonian pair P_0 and P_1. Hence all the assumptions of the theorem are satisfied and we can construct all the higher invariants of the above lattices using Φ or τ_1. The scaling field is hamiltonian: $\tau_0=P_0\nabla\sum(n/2)\ln(u(n))$. For $\varepsilon=1$ Eq. (3·30) reduces to the Langmuir lattice,[5a)] for $\varepsilon=2$ one finds the Kac-Moerbeke lattice.[5b)] We remark that all the equations (3·30) for different ε are related by the simple transformation $v(n)=u^\alpha(n)$ sending $u_t=u\Delta u^\varepsilon$ to $v_t=\alpha v\Delta v^{\varepsilon/\alpha}$.

Example 5: A class of Network equations[3),6)]

We consider the equation

$$\frac{d}{dt}\begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix}=K_1(n)=\begin{bmatrix} S_1(n)(u_2(n)-u_2(n-1)) \\ S_2(n)(u_1(n+1)-u_1(n)) \end{bmatrix}, \tag{3·34}$$

where

$$S_1 := \varepsilon_1 + \varepsilon u_1^2 , \quad S_2 := \varepsilon_2 + \varepsilon u_2^2 \tag{3·35}$$

with arbitrary parameters ε, ε_1 and ε_2. A hamiltonian formulation is given by $K_1 = P_0 \nabla f_1$ with

$$P_0(u) = \begin{bmatrix} 0 & ; & S_1(1-T_-)S_2 \\ S_2(T_+-1)S_1 ; & & 0 \end{bmatrix} , \quad f_1 = \frac{1}{2\varepsilon} \sum_{n\in Z} \ln(S_1(n)S_2(n)) . \tag{3·36}$$

A first mastersymmetry $\tau_1 = [n]K_1 + Z_1$ was found using the computer algorithms briefly described in the next section. Here the translation invariant part is found to be

$$Z_1(u) = \begin{bmatrix} \frac{1}{2} S_1(3T_- u_2 - u_2) \\ S_2 u_1 \end{bmatrix} . \tag{3·37}$$

The essential property of this vectorfield is that

$$K_2 := [\tau_1, K_1] = \begin{bmatrix} \frac{1}{2} S_1(1-T_-)S_2(1+T_+)u_1 \\ \frac{1}{2} S_2(T_+-1)S_1(1+T_-)u_2 \end{bmatrix} \tag{3·38}$$

is a symmetry of (3·34). From the Jacobi identity of the vectorfield commutator one concludes that also $K_3 := [\tau_1, K_2]$ is a further symmetry of (3·34). Assuming the commutant $K_1^{(0)} = \{K, [K, K_1] = 0\}$ of K_1 to be abelian one concludes that the above τ_1 maps $K_1^{(0)}$ into itself, i.e., the commutator of τ_1 with a (time independent) symmetry of K_1 should yield another symmetry of (3·34). Hence the sequence of vectorfields defined by $K_{i+1} := [\tau_1, K_i]$ should yield a hierarchy of commuting symmetries of (3·34).

Looking for a recursion operator one calculates the Lie derivative $L_{\tau_1} P_0 = 0$. Hence we cannot construct a second hamiltonian operator using τ_1 and P_0, instead we find that τ_1 is hamiltonian w.r.t. P_0:

$$\tau_1 = P_0 \nabla \sum_{n\in Z} \frac{1}{4\varepsilon} (n 2 \ln(S_1(n)S_2(n)) - \ln(S_1^2(n)S_2(n))) . \tag{3·39}$$

As a consequence the above hierarchy of commuting symmetries of (3·34) consists of hamiltonian vectorfields $K_i = P_0 \nabla f_i$, where the functions in involution can be constructed from $f_{i+1} := \langle \nabla f_i, \tau_1 \rangle$. Further conservation laws are given by the 2 Casimir functions

$$C_1(u) = \sum_{n\in Z} \ln\left(\frac{\delta_1 - \delta u_1(n)}{\delta_1 + \delta u_1(n)} \right), \quad C_2(u) = \sum_{n\in Z} \ln\left(\frac{\delta_2 - \delta u_2(n)}{\delta_2 + \delta u_2(n)} \right), \tag{3·40}$$

where δ_1, δ_2 and δ have to satisfy $\delta_1^2 \varepsilon + \varepsilon_1 \delta^2 = 0 = \delta_2^2 \varepsilon + \varepsilon_2 \delta^2$.

We also found a second mastersymmetry $\tau_2 = [n]K_2 + Z_2$, where the translation invariant part is given by

$$Z_2(u)=\begin{bmatrix} S_1(\varepsilon u_1 T_- u_2^2 + T_- u_1 S_2) \\ \qquad +\frac{1}{2}\varepsilon (K_1)_1 \Delta^{-1}(1+T_+)(T_- u_1 u_2 + u_1 T_- u_2) \\ \frac{1}{4}S_2(3\varepsilon u_1^2 u_2 + 3S_1 T_- u_2 + \varepsilon u_2 T_+ u_1^2 + T_+ S_1 u_2) \\ \qquad +\frac{1}{2}\varepsilon (K_1)_2 \Delta^{-1}(1+T_+)(u_1 u_2 + u_1 T_- u_2) \end{bmatrix}. \tag{3·41}$$

This vectorfield is not hamiltonian w.r.t. P_0, we find a non-trivial second hamiltonian operator $P_2:=L_{\tau_2}P_0$, which is of such complicated form that we will not give it here explicitly. Using computer algebra it was checked that it is an invariant of (3·34) and in this sense is a second hamiltonian operator for (3·34). So a hereditary recursion operator $\Phi:=P_2(P_0)^{-1}$ is found, We propose that the hierarchies generated by the mastersymmetries coincide with the hierarchies generated by the recursion operator, e.g., we checked

$$[\tau_2, K_1]=\frac{1}{2}([\tau_1,[\tau_1, K_1]]-\varepsilon_1\varepsilon_2 K_1)=-\frac{1}{2}\Phi K_1\,. \tag{3·42}$$

§ 4. Computational aspects

In this section we briefly describe the simple ideas which lead to the algorithm by which the results of the preceeding sections have been obtained. The algorithm will be reported in full detail elsewhere and its implementation will then be discussed. Eventually, after serious testing, the computer algebra package containing the implementations of this algorithm will be made available.

We essentially rely on the concept of mastersymmetries, i.e., for a given dynamical system $u_t=K(u)$ we look for a vectorfield τ, say, having the property that $\tilde{S}:=[\tau, K](\neq 0)$ is a symmetry of K. Such mastersymmetries exist for most of the known integrable equations,[8),11),12)] they are an important tool to construct higher invariants for integrable systems by applying Lie derivatives to simple invariants.[12)] For example, the commutator of such a mastersymmetry with a symmetry yields a new symmetry and the Lie derivative of a hamiltonian operator yields a second hamiltonian formulation and hence a hereditary recursion operator for the considered equation. So, knowing a single (non-trivial) mastersymmetry gives immediate access to almost the entire algebraic structure of the equation. Actually, this was the way how the multi-hamiltonian formulations of the preceding examples have been found: Exploiting the computer algebra algorithms to be described below we found the first non-trivial mastersymmetry τ_1. Then, using a first hamiltonian formulation of the equation we constructed further hamiltonian operators by applying the Lie derivative into the direction of this mastersymmetry to the first hamiltonian operator. Once a second hamiltonian formulation and hence a recursion operator is constructed the results can be summarized by the theorem of § 3.

Hence the computational effort essentially consists of finding one mastersymmetry, i.e., for a given dynamical system $u_t=K(u)$ we have to "solve" the equation

$$[K,[K,\tau]]=0 \tag{4·1}$$

for τ. Before attacking this problem we have to mention some technical details: A crucial role is played by a "highest-order projection" for vectorfields. Recall that the vectorfields under consideration are polynomial such that if one evaluates the vectorfield at the place n of the lattice also field variables at other places do enter because there is some interaction between neighboring points. Projecting a vectorfield K, say, onto those terms where the interaction reaches farthest (highest distances with respect to lattice points) and then taking the highest polynomial degree of these projected terms constitutes the highest order projection. The result of this projection is denoted by $ho(K)$ and we are able to define a suitable degree function yielding these highest order terms. Now an important role in the program is played by an approximate solution of the division problem in the Lie algebra of vectorfields. By approximate we mean that given vectorfields K and R we are able to find a vectorfield X such that

$$ho[ho(K), X]=ho(R)\,. \tag{4·2}$$

This routine is called $CS(K, R)$ ("commutator solution") and this subroutine is the heart of the whole matter. The reason why such "commutator solutions" can be found lies in the fact that restricting the considerations to terms less than a fixed degree more or less simulates the situation of a finite dimennional Lie algebra.

Another important point that is essential to find solutions of (4·1) in an algorithmic way is the fact that the explicit form of the mastersymmetries is known to a certain extent; from experience we know that the typical form of these vectorfields is given by

$$\tau(u)=[n]S(u)+Z(u)\,, \tag{4·3}$$

where S and Z are translation invariant (i.e., do not explicitly depend on the lattice point n) and S is a symmetry of K. The following algorithms use this structure by starting with a "first approximation" $\tau=[n]$ *symmetry* for the wanted solution of (4·1).

For a given symmetry S of K we now attack (4·1) by splitting it into 2 parts: 1) First we determine the new symmetry $\tilde{S}=[\tau, K]$. On the basis of the observation that the highest order term of τ can be assumed to be given by the highest order term of $[n]S$, we know that the highest order term of $\tilde{S}=[\tau, K]$ is given by the highest order term of $[[n]S, K]$ and we can use the following algorithm SYM of successive approximation to find $\tilde{S}=[\tau, K]=:$SYM(K, S):

PROCEDURE SYM (K, S):

{*The procedure SYM determines that symmetry* $\tilde{S}$ *of* K *with* $ho(\tilde{S})=ho([[n]S, K])$.}

Step 0: *Put* $\tilde{S}:=[[n]S, K]$.

Step 1: *Put* $R:=[\tilde{S}, K]$. *If* $R=0$ *then* $RETURN(\tilde{S})$ *else GOTO Step* 2.

Step 2: *Determine* $\delta S:=CS(K, R)$, *where* $CS(\ )$ *is applied by restricting the considerations to terms of degree less than the degree of* $\tilde{S}$. *If there is no solution then RETURN ("There is no symmetry of this form") else GOTO Step* 3.

Step 3: *Put* $\tilde{S}:=\tilde{S}+\delta S$ *and GOTO Step* 1.

Obviously, in *Step* 0 the wanted new symmetry $\tilde{S}$ is computed correctly in its highest order and each run computes $\tilde{S}$ correctly up to one order less. Hence the algorithm either has to stop after a number of runs given by the degree of $\tilde{S}$ thus finally giving the correct symmetry $\tilde{S}$, or it stops before by telling us that for the given S there is no symmetry whose highest order is generated by $[[n]S, K]$. Although, as a language problem, this algorithm does not have finite length it terminates since all descending chains (with respect to degree) are finite. Of course, this algorithm is based on a symmetry S which has to be known already. But observe that one can always use $S=K$.
2) Now, once the new symmetry $\tilde{S}$ has been found we can use the following algorithm to solve $\tilde{S}=[\tau, K]$ for τ:

PROCEDURE MAS (K, N):

{*The procedure MAS determines the mastersymmetry* τ *with* $[\tau, K]=SYM(K, S)$.}

Step 0: *Put* $\tilde{S}:=SYM(K, S)$, $\tau:=[n]K$.

Step 1: *Put* $R:=[\tau, K]-\tilde{S}$. *If* $R=0$ *then* $RETURN(\tau)$ *else GOTO Step* 2.

Step 2: *Determine* $\delta\tau:=CS(K, R)$, *where* $CS(\ )$ *is applied by restricting the considerations to terms of degree less than the degree of* τ. *If there is no solution then RETURN ("There is no solution") else GOTO Step* 3.

Step 3: *Put* $\tau:=\tau+\delta\tau$ *and GOTO Step* 1.

The program package is implemented in MAPLE,[18] a formula manipulation system developed by the University of Waterloo. The choice for a formula manipulation system was mainly based on our desire for rapid prototyping and on the fact that for these systems many sophisticated algorithms are available.

§ 5. Conclusions

As can be seen from the theorem given in § 3, mastersymmetries provide all the information which can be obtained from the hereditary symmetries (recursion operators), since these can be recovered from the mastersymmetries (and the hamiltonian operator). Furthermore, they usually yield the recursive structure of completely integrable systems even when hereditary operators cannot be found.

From the computational viewpoint the approach to completely integrable systems via mastersymmetries seems to be more powerful compared to the attempt of finding hereditary recursion operators or Lax pairs. The obvious reason is that vectorfields (satisfying the required properties) are easier to detect than tensor fields of higher degree. So, mastersymmetries seem to be the ideal tools in order to design computer programs for the study of complete integrability.

It should be remarked that the development of computer programs for the determination of symmetry groups goes back to Schwarz.[19] However, there is an essential difference between his work and ours: We rely on the existence of a nontrivial (higher) mastersymmetry, which—translating these structures into the framework of differential equations—corresponds to a Lie-Baecklund symmetry, whereas Schwarz determines all the Lie-point symmetries of a given equation.

References

1) E. Fermi, J. Pasta and S. Ulam, Collected papers of Enrico Fermi (Univ. Chicago Press, Chicago, 1965), vol. II, p. 978.
2) M. Toda, *Theory of Nonlinear Lattices* (Springer, Berlin, 1981).
3) R. Hirota and J. Satsuma, J. Phys. Soc. Jpn. **40** (1976), 891.
4) R. Hirota, J. Phys. Soc. Jpn. **35** (1973), 286.
5a) S. Novikov et al., *Theory of Solitons, The Inverse Scattering Method* (Consultants Bureau, New York, 1984), p. 45.
b) M. Kac and P. van Moerbeke, Adv. in Math. **16** (1975), 160.
6) R. Hirota, J. Phys. Soc. Jpn. **35** (1973), 289.
7) W. Oevel, B. Fuchssteiner, H. Zhang and O. Ragnisco, Preprint, Paderborn 1988.
8) B. Fuchssteiner and U. Falck, *Symmetries and Nonlinear Phenomena, Proceedings of the International School on Applied Mathematics, Columbia, 1988,* ed. D. Levi and P. Winternitz (World Scientific Publ., 1988), p. 22.
9) B. Fuchssteiner, Nonlinear Analysis TMA **3** (1979), 849.
10) B. Fuchssteiner and A. S. Fokas, Physica **4D** (1981), 47.
11) B. Fuchssteiner, Prog. Theor. Phys. **70** (1983), 1508.
12) W. Oevel, Preprint, Paderborn, 1986; *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations,* ed. M. Ablowitz, B. Fuchssteiner and M. Kruskal (World Scientific Publ., Singapore, 1987), p. 108.
13) H. M. M. ten Eikelder, *Symmetries for Dynamical and Hamiltonian Systems,* CWI Tract **17** (Centrum voor Wiskunde en Informatica, Amsterdam, 1985).
14) P. M. Santini and A. S. Fokas, *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations,* ed. M. Ablowitz, B. Fuchssteiner and M. Kruskal (World Scientific Publ., Singapore, 1987), p. 1.
15) B. Fuchssteiner and G. Oevel, Preprint, Paderborn, 1988.
16) B. Fuchssteiner and R. N. Aiyer, J. of Phys. **20A** (1987), 375.
17) A. Weinstein, J. Differential Geometry **18** (1983), 523.
18) K. O. Geddes, G. H. Gonnet and B. W. Char, *MAPLE User's Manual,* 3rd edition (Waterloo, 1983).
19) F. Schwarz, J. Symbolic Computation **34** (1985), 91.