MASTERSYMMETRIES FOR COMPLETELY

INTEGRABLE SYSTEMS IN STATISTICAL MECHANICS

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1. INTRODUCTION

For dynamical systems, mastersymmetries are a useful tool to construct conservation laws and symmetry groups. They have first been discovered in the case of the Benjamin-Ono equation[1] and the Kadomtsev-Petviashvili equation[2]. They exist for almost all the popular completely integrable systems[3] and recently they have been studied on a systematic basis [4,5]. We give the relevant definitions in the abstract case of an arbitrary Lie-algebra L, this is suitable for all the situations we have in mind. Let L_1 be a sub-Lie-algebra of L. Recall that a map $d : L_1 \to L$ is said to be a *derivation* (on L_1) if.

$$d[A,B] = [d(A),B] + [A,d(B)] \quad \text{for all} \quad A,B \in L_1 . \tag{1.1}$$

Special derivations are given by the adjoint $\hat{G}$ of elements $G \in L$ (i.e. $\hat{G}\,A \underset{\text{def}}{=} [G,A]$ for all $A \in L_1$). These derivations are called *inner*. A derivation on L_1 is said to be an L_1 - *mastersymmetry* if it maps L_1 into L_1. If such a derivation is inner, then the mastersymmetry is called inner. For every L_1 the inner mastersymmetries are constituting a sub-Lie-algebra. In case of inner derivations we can easily define *higher order* mastersymmetries: The adjoints of elements in L_1 are the mastersymmetries of degree 0 and an inner derivation $d = \hat{G}$ is called a *mastersymmetry of degree* $n+1$ if, for all $A \in L_1$, $d(A) = [G,A]$ is a mastersymmetry of degree n. The mastersymmetries of degree n include those of any lower degree (Jacobi identity). If $\hat{G}_1, \hat{G}_2$ are of degree n_1 and n_2, respectively, then the adjoint of $[G_1,G_2]$ is in general of degree $n_1 + n_2 - 1$.

Before going any further into technical details let me briefly explain what practical purpose a mastersymmetry can have. For example, if we fix $K \in L$ and put $L_1 = K^\perp = \{G \in L \mid [G,K] = 0\}$ to be the commutant of K then a $K^\perp$-mastersymmetry d has the property that $d(K), d^2(K), \ldots, d^n(K)$ are elements of $K^\perp$. Hence, we are eventually able to generate in a recursive way out of K infinitely many elements of $K^\perp$, maybe even all of $K^\perp$ (which in fact is the case for most of the popular completely integrable systems). Of course, such a generic construction of $K^\perp$ has important consequences. For example, in the case of:

a. Vectorfields

Let M be a C^∞- manifold, denote the variable on M by u and consider an evolution equation

$$u_t = K(u) , \tag{1.2}$$

where K is a C^∞- vectorfield on M. Recall that the C^∞- vectorfields are endowed with a Lie-algebra structure, namely the infinitesimal structure of the group of C^∞-diffeomorphisms on M. Therefore the construction of $K^\perp$ amounts to the construction of the infinitesimal generators of the one-parameter symmetry groups of (1.2). This way of construction works for all the popular completely integrable systems, like KdV, mKdV, SG, BO, KP etc.
Complete integrability in all these cases implies that $K^\perp$ is abelian, an aspect to which we come back. In addition to that, the $K^\perp$-mastersymmetries of, say, degree m have a direct meaning in terms of time-dependent symmetry groups. In order to explain this consider a time-dependent flow on M

$$u_t = G(u,t) \tag{1.3}$$

where $G(t) = G(\cdot,t)$ is a C^∞- family of C^∞- vectorfields on M. Let $u(t,u_o,\tau)$ be the solution of (1.3) fulfilling at time τ the initial condition $u(\tau,u_o,\tau) = u_o$. Then the diffeomorphisms given by $u_o \to u(t,u_o,\tau)$, $u_o \in M$, $\tau \in \mathbb{R}$ commute with the flow (1.2) if and only if

$$\frac{d}{dt} G(t) = [K,G(t)] . \tag{1.4}$$

For this reason we call $G(t)$ a *time-dependent* symmetry generator for (1.2) if (1.4) is fulfilled. A simple calculation [3] shows that, when G_o is a $K^{\perp}$- mastersymmetry of degree m then

$$G(t) = \exp(t\,\hat{K})G_o = \sum_{k=0}^{\infty} \frac{t^n}{n!}\ \hat{K}^n G_o \tag{1.5}$$

is a time-dependent symmetry generator. Observe, that then in (1.5) the "infinite" sum extends only up to m , the degree of G_o .

b. Conservation laws

Assume that (1.2) is a degenerate [6] or nondegenerate Hamiltonian system, i.e. the vector field $K = \Theta$ grad H is the image of a gradient field grad H (H the Hamiltonian) under an implectic [6] (inverse symplectic) map Θ. Then in the space of zero-forms (scalar quantities on M) one has a canonical Lie-algebra structure $\{,\}$ induced by Θ (Poisson brackets with respect to Θ) . Now, a zero form is a conserved quantity with respect to the flow (1.2) if and only if it commutes with H in the Lie algebra of Poisson brackets. Hence, we are able to construct out of H further conserved quantities via commutation with $H^{\perp}$- mastersymmetries. And, among the scalar quantities depending explicitly on time in a polynomial way, those which are invariant with respect to (1.2) correspond uniquely to $H^{\perp}$ - mastersymmetries. To be precise: The zero form G_o is an $H^{\perp}$- mastersymmetry if and only if

$$G(t) = \exp(t\,\hat{H})G_o = \sum_{n=0}^{\infty} \frac{t^n}{n!}\ \{H,\dots,\{H,G_o\}\dots\}\ (n\text{-times}) \tag{1.6}$$

is invariant under the flow (1.2) .

c. The quantum mechanical case

Here the Lie-algebra under consideration are the operators on a suitable Hilbert space. The Lie-product is given by the usual commutation of operators. Given an operator H then, via $H^{\perp}$-mastersymmetries, we are able to construct in a recursive way operators commuting with H . Eventually, we may be able to construct all of $H^{\perp}$. This would be most interesting, because if H is normal, then knowing a maximal abelian subalgebra of $H^{\perp}$ amounts to knowing the spectral resolution of H . Hence finding $H^{\perp}$-mastersymmetries constitutes a big step towards the diagonalisation of H . But at this point one gets disillusioned by looking at the spectral properties of, say, a mastersymmetry of degree 1. For example let H be self-adjoint and assume that the operator $H_1 = [T,H]$ commutes with H . Consider an eigenvector $|x\rangle$ of H with an eigenvalue of multiplicity 1 . Since H_1 commutes with H , $|x\rangle$ must again be an eigenvector of H_1 with eigenvalue 0 (because of $< x, H_1 | x > \; = \; < x, [T,H] | x > \; = 0$) . This implies that $T|x>$ is again an eigenvector of H . Hence $|x>$ must be an eigenvector of T . Therefore, at least in case of nondegenerate spectrum, we cannot expect any meaningful $H^{\perp}$- mastersymmetries which are inner. The way out of this is to consider either mastersymmetries of outer type or to consider operators with weird spectral properties (of course, only in the infinite dimensional case). We shall give meaningful examples for both cases.

d. A useful technical result

At this point I have to admit that I cheated over a very crucial point. I pretended that one can use $K^{\perp}$- mastersymmetries for the construction of $K^{\perp}$. But, looking at our definition for master-symmetries, on discovers that in order to check whether or not a quantity is a mastersymmetry,we have to try it out on all of $K^{\perp}$, which seems only possible if we know $K^{\perp}$ in advance. Fortunately, this is not so in abelian situations.

Consider $\mathcal{L}_1 \subset \mathcal{B} \subset \mathcal{L}$ where $\mathcal{L}_1$ and $\mathcal{B}$ are sub-Lie-algebras of $\mathcal{B}$ and $\mathcal{L}$, respectively. Fix $K \in \mathcal{L}_1$ such that $\mathcal{L}_1$ is equal to the commutant $K^{\perp}(\mathcal{B}) = \{A \in \mathcal{B} \mid [A,K] = 0\}$ of K in $\mathcal{B}$ (not in $\mathcal{L}$). Assume that $\mathcal{L}_1 = K^{\perp}(\mathcal{B})$ is abelian . Then an inner derivation $d : \mathcal{B} \to \mathcal{L}$ with $d(\mathcal{L}_1) \subset \mathcal{B}$ is a $K^{\perp}(\mathcal{B})$-mastersymmetry if and only if $d(K) \in K^{\perp}(\mathcal{B})$. Thus,we only have to try how d acts on K . This criterion can be generalized to higher degrees.

The proof of this simple fact is mainly based on a successive application of the Jacobi identity, it can easily be adapted from the considerations given in Ref. 4 (theorem 1).

And,even if we do not know the commutant $K^{\perp}(\mathcal{B})$, we are quite often able to decide beforehand whether it is abelian. For example, in the quantum mechanical case when the spectrum of the operator under consideration has multiplicity 1. In other situations we also succeed quite often by application of ad-hoc arguments. A systematic study of this point is contained in the dissertation of W. Oevel [5].

2. THE XYZ-MODEL

A couple of years ago much excitement was created by the discovery of the quantum inverse scattering method for the one-dimensional spin 1/2 anisotropic Heisenberg spin chain, the so called XYZ-model. This model has been proved to be equivalent to a vertex model for which R.J. Baxters ingenious method of solution was available. For details of these developments the reader is referred to the work of R.J. Baxter [7], Takhtadyan and Faddeev [8] and Sogo and Wadati [9].

The purpose of this section is to give a straight-forward method for the computation of the commutant of the Hamiltonian of the XYZ-model.

At each point n of the lattice $\mathbb{Z}$ (all integers) a spin operator $\vec{S}_n = (S_n^1, S_n^2, S_n^3)$ is given. These operators are assumed to be spin-1/2 operators, i.e.

$$S_n^j S_n^k = \delta_{jk} + i\, \varepsilon^{jkl} S_n^l \tag{2.1}$$

where ε^{jkl} is the cyclic totally antisymmetric tensor with $\varepsilon^{123} = 1$. We either consider the unbounded case, where no periodicity of the lattice is assumed, i.e. where all spin operators at different places commute

$$[S_n^j, S_m^k] = 0 \quad \text{for} \quad n \neq m \tag{2.2}$$

or we consider the periodic case where some N is given such that $S_n^k = \pm S_{n+N}^k$ and where (2.2) only holds for those $n \neq m$ which are different modulo N.

The Hamiltonian of the XYZ- model is

$$H = -\frac{1}{2} \sum_{n,k} J_k S_n^k S_{n-1}^k \tag{2.3}$$

where the sum either goes over all $n \in \mathbb{Z}$ (unbounded case) or from 1 to N (periodic case with periodicity N). In both cases the equations of motion are

$$\dot{\vec{S}}_n = i[H, \vec{S}_n], \tag{2.4}$$

or explicitly

$$\dot{S}_n^k = \sum_{l,r} \varepsilon^{klr} J_r S_n^l (S_{n-1}^r + S_{n+1}^r) \tag{2.5}$$

a. <u>The unbounded case</u>

Consider the operator

$$T = \sum_n n\, 2\, J_k S_n^k S_{n-1}^k \tag{2.6}$$

which is even more unbounded than H. Commutation with H yields the operator

$$H_1 = [T,H] = \sum_{l,r,n} J_l J_r\, \varepsilon^{lkr} S_{n+1}^l S_n^k S_{n-1}^r \tag{2.7}$$

which commutes with H . For the periodic case, when the sum goes only over $n = 1,\dots,N$ this is the conserved operator next to H found by Lüscher [10] .

From the Jacobi identity we conclude directly that $H_2 = [T,H_1]$ commutes with H . But at this moment we are not yet sure whether successive commutation

$$H_{M+1} = [T,H_M] \qquad , \quad M = 1,2,.. \tag{2.9}$$

generates operators which commute with H . If we had a simple spectrum for H , then this fact would follow right away from section 1.d. But Obviously,for general $\vec{J}$, the spectrum of H has multiplicity 2. In order to see this we define in the underlying Hilbert space a selfadjoint operator σ with $\sigma^2 = I$ via $\sigma S_n \sigma = S_{-n}$. An operator $\tilde{H}$ is said to be *odd* or *even* if $\sigma \tilde{H} \sigma = \tilde{H}$ or $\sigma \tilde{H} \sigma = -\tilde{H}$, respectively. Then T is odd and H is even, H_1 odd, H_2 even,and so on. Since the projections $P\pm = \frac{1}{\sqrt{2}}(I \pm \sigma)$ commute with H , the spectrum of H has at least multiplicity 2, and exactly two for general $\vec{J}$. Now, since H_2 is even and commutes with H one finds out that H_2 commutes with H_1 (consider suitable subspaces). Then the Jacobi identity shows that H_3 commutes with H , and by similar arguments one concludes that it also commutes with H_1,H_2 . In this way we proceed further and obtain that all H_M commute. Hence T is an $H^{\perp}$- mastersymmetry.

The essence of all these arguments is,that the simple ideas behind the statement in section 1.d carry over to cases where the spectrum is not simple but well behaved.

b. The periodic case

Because of the arguments which were presented in section 1.c we cannot expect an inner $H^{\perp}$- mastersymmetry in the periodic case since the underlying Hilbert space is finite dimensional. But, indeed there is an outer $H^{\perp}$- mastersymmetry. We regard the n , appearing in (2.6),as elements of the equivalence classes modulo N and we restrict the summation in n from 1 to N , i.e. we consider

$$T = \sum_{n\in\mathbb{Z}/N} n \; 2 \; J_k \; S_n^k \; S_{n-1}^k \quad . \tag{2.10}$$

Then T is certainly not an honest operator since it is undetermined up to a multiple of H . But as soon as it is commuted with an element of $H^{\perp}$ this amount of undetermination disappears completely. Hence T can be considered as an outer derivation on $H^{\perp}$. Now, all our arguments go through for this case. $H_1 = [T,H]$ is the conserved current next to Hamiltonian H given by Lüscher [10]. And a repetition of the arguments of section 2.b yields that the sequence of operators defined recursively by

$$H_{M+1} = [T,H_M] \quad , \qquad M = 1,2,\ldots \tag{2.11}$$

is in involution.

In principle, it is quite clear that from the knowledge of $H^{\perp}$ we are able to find a diagonalisation of H. Of course, one certainly encounters computational difficulties doing this. Therefore, carrying out such a program would go beyond the aim of this paper.

c. Concluding remarks

There is a rather simple way to obtain the commutativity of the Hamiltonians for the periodic case out of their commutativity for the unbounded case. One considers the equations of motion as a classical flow on a manifold M of operators. (Of course, this manifold is infinite-dimensional in the unbounded case). The submanifold M_{per} given by the requirements of periodicity is invariant under the sequence of flows

$$\dot{\vec{S}}_n = i[H_M,\vec{S}_n] \tag{2.12_M}$$

where the H_M are the Hamiltonians for the unbounded case (defined by (2.9)). Since the H_M commute the corresponding flows are commuting. Now, the restrictions of the flows (2.12_M) to the invariant submanifold M_{per} are exactly the flows given by the Hamiltonians for the periodic case. Since these flows are commuting the corresponding Hamiltonians have to commute. Hence, the Hamiltonians in the periodic case commute.

An interesting problem seems to be the question whether or not there are further mastersymmetries for the XYZ-model. In fact there are infinitely many. I obtained them by a horrible calculation which is much to involved to be presented here. Certainly, there must be a simpler way to find them.

3. ARBITRARY SPIN AND CONTINUOUS LIMITS

The continuous limit of the XYZ-model yields the Landau-Lifschitz equation. On the level of the inverse scattering method the details were carried out in the beautiful paper of Sklyanin [11].

Nevertheless, to me it was always a mystery how this worked, because, to my simple-minded understanding, for taking the continuous limit, one should first go over to higher order spin systems thus making the distribution of the spin-eigenvalues more and more dense. And, to my knowledge, there is not yet any satisfactory and completely integrable system known for spin higher than one half.

Let me present my philosophy in order to explain this puzzling situation. I believe it possible that there is no chain of completely integrable systems having the Landau-Lifschitz equation (LL) as limit. But certainly, there are spin systems being integrable in an approximative sense such that the spin-1/2 case is the XYZ-model and the continuous limit is the LL . Here, "approximation" can be expressed in terms of the distance of the points of the lattice under consideration.

In order to explain this I need some notation. We consider a lattice of points with distance δ , the points are again numbered by $\mathbb{Z}$. At each point n there sits an operator-valued vector $\vec{S}_n = (S_n^1, S_n^2, S_n^3)$. *No commutation relations are prescribed.* As operators we consider $P(S)$, the polynomials in S_n^k , $n \in \mathbb{Z}$, $k = 1,2,3$. The quotient of this vector space with respect to the subspace $Q(S) = \{AB - BA \mid A,B \in P(S)\}$ we call space of *densities*. Observe that we only factored out with respect to a vector space but not an algebra, i.e. we have $AB \equiv BA$ but not always $CAB \equiv CBA$.
This construction makes sense insofar as the trace-operation does not distinguish between different members of the same equivalence class. Furthermore we consider the space of operator-valued vectors $\vec{P}(S)$, i.e. functions $\vec{A}$ assigning to each $n \in \mathbb{Z}$ an operator-valued vector $\vec{A}_n = (A_n^1, A_n^2, A_n^3)$, where $A_n^k \in P(S)$. For these vectors we define a *density-valued inner product*

$$(\vec{A},\vec{B}) = \text{equivalence class of } \sum_{n,k} A_n^k B_n^k \,, \qquad \vec{A},\vec{B} \in \vec{P}(S). \tag{3.1}$$

All this is done for the definition of gradients and Poisson brackets. For $A = A(S_n^k;\, n \in \mathbb{Z}\,,\, k=1,2,3) \in P(S)$ and $\vec{B} \in \vec{P}(S)$ we consider the directional derivative

$$A'[\vec{B}] = \frac{\partial}{\partial \varepsilon} A(S_n^k + \varepsilon B_n^k;\, n \in \mathbb{Z},\, k = 1,2,3)_{|\varepsilon=0} \,. \tag{3.2}$$

Our notion of density was chosen in such a way that there is always a unique operator-valued vector ∇A such that

$$A'[\vec{B}] \equiv (\nabla A,\vec{B}) \qquad \text{for all } \vec{B} \in \vec{P}(S). \tag{3.3}$$

This quantity ∇A is called the *gradient* of A .
For example, the gradients of

$$A = \Sigma S_n^k S_n^k \tag{3.4}$$

$$H = -\frac{1}{2} \Sigma J_k S_n^k S_{n-1}^k \tag{3.5}$$

are

$$(\Delta A)_n^k = 2 S_n^k \tag{3.6}$$

$$(\nabla H)_n^k = -\frac{1}{2} J_k (S_{n-1}^k + S_{n+1}^k) . \tag{3.7}$$

We introduce a vector-product

$$(\vec{B} \times \vec{A})_n^k = \frac{1}{2} \sum_{rs} \varepsilon^{rsk}(B_n^r A_n^s - A_n^r B_n^s) . \tag{3.8}$$

Some examples

$$\vec{S} \times \vec{S} = 0 \tag{3.9}$$

$$(S \times \nabla H)_n^k = -\frac{1}{4} \Sigma \varepsilon^{rsk} J_s \{S_n^r(S_{n-1}^s + S_{n+1}^s) + + (S_{n-1}^s + S_{n+1}^s)S_n^r\} . \tag{3.10}$$

Observe that in case of the XYZ-model, where the S_n, S_m commute for different $n \neq m$, the equation of motion can be written as

$$\dot{\vec{S}} = -2 \ (\vec{S} \times \nabla H) . \tag{3.11}$$

Therefore we are going to study dynamical systems of the form

$$\dot{\vec{S}} = -2 \ (\vec{S} \times \nabla H) \tag{3.12}$$

on the submanifold $M(S^2 = c)$ of operators given by the constraints

$$[S_n^j, S_m^k] = 2\alpha i \,\delta_{n,m}\, \varepsilon^{jkr} S_n^r \quad ; \quad \sum_{k=1}^{3} S_n^k S_m^k = c \quad \text{for all} \quad n \in \mathbb{Z} .$$

Fortunately, all the flows (3.12) leave this manifold invariant. Particular attention will be given to the special case where $H = H_o$,

$$H_o = -\delta(2H + \lambda A) = \delta \sum_{n,k} (J_k S_n^k S_{n-1}^k - \lambda S_n^k S_n^k) , \tag{3.13}$$

yielding for special δ ,in the spin 1/2 case,the equation of motion of the XYZ-model. On $M(S^2 = c)$ we introduce a bracket for densities

$$\{G,H\} \underset{\text{def}}{=} -2 \ (\nabla G, S \times \nabla H) \tag{3.14}$$

which fulfills the Jacobi-identy on this special manifold. We call this Lie-structure the *Poisson brackets*. This makes sense since a density G is invariant under the flow (3.12) if and only if

$$G_t + \{G,H\} \equiv 0 , \tag{3.15}$$

therefore H is called the Hamiltonian of (3.12). Furthermore, the map $G \rightarrow -2\vec{S} \times \nabla G$ is a Lie-algebra homomorphism from the Poisson brackets into the vector fields. This is very much the same as in the continuous case [12]. Hence $\vec{S}\times$ maps gradients of conserved densities into infinitesimal generators of one-parameter symmetry groups.

Now, we are looking for mastersymmetries for the Hamiltonian H_o in the Lie-algebra of Poisson brackets. A good candidate seems to be

$$T = \sum_{n,k} \delta\, n S_n^k\, S_n^k \tag{3.16}$$

since it lead to success in the spin one-half case. But alas, it is not a mastersymmetry in the case of arbitrary spin. Fortunately, we can consider it as an approximative mastersymmetry. Let me explain what that means. Recall that δ was the distance of lattice points. Terms of the form δ^N we call of N-th *order*, furthermore differences like $S_n - S_{n-1}$ are called of first order, and so on. For other terms the δ-order is introduced by the requirement that it shall be multiplicative. A vectorfield or a density is said to be an approximative symmetry generator or conserved density if its commutant with H_o or $\vec{S} \times \nabla H_o$, respectively, is at least of first or higher order in δ. Now, even in the case that we put in (3.13)

$$J_k = \delta^{-2} + j_k \quad , \qquad \lambda = \delta^{-2} \tag{3.17}$$

the quantity (3.16) is an approximative mastersymmetry in the sense that $\{T,H_o\}$ is approximately conserved. Hence, we obtain (after some commutativity arguments) a mastersymmetry if we put $\delta \rightarrow 0$. Assuming that S then becomes a differentiable function we can replace sums by integrals and (δ^{-1}-times-differences) by derivatives. Then the Hamiltonian H_o goes over into [11]

$$H_{o_{Limit}} = \int \Big(\sum_k j^k\, S^k(x)\, S^k(x) + S^k(x)\, S^k_{xx}(x)\Big)\, dx \tag{3.18}$$

and, after a suitable rescaling, the equation of motion goes over in the well-known LL-equation.

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx} + \Sigma\, \vec{S} \times (\vec{j}\vec{S}) \quad \text{where} \quad (\vec{j}\vec{S})^k = j_e\, S^k \;. \tag{3.19}$$

Our limit procedure yields mastersymmetries in the Poisson brackets, as well as in the vector fields. These are exactly the mastersymmetries described in great detail in Ref. 12. Rather puzzling is the fact that the artificial mastersymmetry of the XYZ-model becomes an honest inner derivation by this procedure.

Still, we are allowed to interprete (3.19) as an evolution equation for operators or scalars. But in case of operators we have to keep in mind definition 3.8. Of course, the mastersymmetries given in Ref. 12 are only these for the scalar case. They have to be modified in the operator case. However, the operator case seems rather dubious to me,since it means that we have to look for a solution in a space of operator-valued distributions which can be multiplied. Maybe, that can be carried out in the context of almost-bounded distributions[13].

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