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Masur–Veech volumes and intersection theory on moduli spaces of Abelian differentials

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Abstract We show that the Masur–Veech volumes and area Siegel–Veech constants can be obtained using intersection theory on strata of Abelian differentials with prescribed orders of zeros. As applications, we evaluate their large genus limits and compute the saddle connection Siegel–Veech constants for all strata. We also show that the same results hold for the spin and hyperelliptic components of the strata.

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1 Introduction

Computing volumes of moduli spaces has significance in many fields. For instance, the Weil–Petersson volumes of moduli spaces of Riemann surfaces can be written as intersection numbers of tautological classes due to the work of Wolpert [47] and of Mirzakhani for hyperbolic bordered surfaces with geodesic boundaries [38]. In this paper we establish similar results for the Mazur–Veech volumes of moduli spaces of Abelian differentials.

Denote by $\Omega\mathcal{M}_{g,n}(\mu)$ the moduli spaces (or strata) of Abelian differentials (or flat surfaces) with labeled zeros of type $\mu = (m_1, \ldots, m_n)$, where $m_i \geq 0$ and where $\sum_{i=1}^n m_i = 2g - 2$. Masur [36] and Veech [44] showed that the hypersurface of flat surfaces of area one in $\Omega\mathcal{M}_{g,n}(\mu)$ has finite volume, called the Masur–Veech volume, and we denote it by $\mathrm{vol}\,(\Omega\mathcal{M}_{g,n}(\mu))$. The starting point of this paper is the following expression of Masur–Veech volumes in terms of intersection numbers on the incidence variety compactification $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ described in [6]. For $1 \leq i \leq n$ we define

$$\beta_i = \frac{1}{m_i + 1}\, \xi^{2g-2} \prod_{j \neq i} \psi_j \quad \in H^{2(2g-3+n)}(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu), \mathbb{Q}) \tag{1}$$

where ξ is the universal line bundle class of the projectivized Hodge bundle and ψ_j is the vertical cotangent line bundle class associated to the jth marked point (see Sect. 3 for a more precise definition of these tautological classes).

Theorem 1.1 *The Masur–Veech volumes can be computed as intersection numbers*

$$\mathrm{vol}\,(\Omega\mathcal{M}_{g,n}(m_1,\ldots,m_n)) = -\frac{2(2i\pi)^{2g}}{(2g-3+n)!}\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \xi^{2g-2}\cdot\prod_{i=1}^{n}\psi_i \tag{2}$$

$$= \frac{2(2i\pi)^{2g}}{(2g-3+n)!}\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \beta_i\cdot\xi \tag{3}$$

for each $1 \le i \le n$.

The equality of the two expressions on the right-hand side is a non-trivial claim about intersection numbers on $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$. Note that we follow the volume normalization in [20] that differs slightly from the one in [21] (see [12, Section 19] for the conversion).

Theorem 1.1 is the interpolation and generalization of [42, Proposition 1.3] (for the minimal strata) and [12, Theorem 4.3] (for the Hurwitz spaces of torus covers). In order to prove it, we show that both sides of Eq. (2) satisfy the same recursion formula. On the volume side, the recursion formula is expressed via an operator acting on Bloch and Okounkov's algebra of shifted symmetric functions (see Sect. 4). The recursion for intersection numbers is first proved at the numerical level using the techniques developed in [43] to compute the classes of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ (see Sects. 2, 3). Then we formally lift this relation to the algebra of shifted symmetric functions and show that it is equivalent to the previous one (see Sect. 5).

In particular, the recursion arising from intersection calculations provides the following useful formula. We define the rescaled volume

$$v(\mu) = (m_1+1)\cdots(m_n+1)\mathrm{vol}\,(\Omega\mathcal{M}_{g,n}(m_1,\ldots,m_n))\,. \tag{4}$$

For a partition μ, we denote by $n(\mu)$ the cardinality of μ and by $|\mu|$ the sum of its entries.

Theorem 1.2 *The rescaled volumes of the strata satisfy the recursion*

$$v(\mu) = \sum_{k\ge 1}\sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})\cdot\frac{\prod_{i=1}^{k}(2g_i-1+n(\mu_i))!\,v(\mu_i,p_i-1)}{2^{k-1}\,k!\,(2g-3+n)!} \tag{5}$$

where $\mathbf{g} = (g_1,\ldots,g_k)$ *is a partition of* g, $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_k)$ *is a* k*-tuple of multisets with* $(m_3,\ldots,m_n) = \mu_1\sqcup\cdots\sqcup\mu_k$, *and* $\mathbf{p} = (p_1,\ldots,p_k)$ *is defined* *by* $p_i = 2g_i - 1 - |\mu_i|$ *and required to satisfy* $p_i > 0$. *Here the Hurwitz number* $h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p})$ *is defined for any* $\mathbf{p}$ *by*

$$h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) = (k-1)![t_1^{m_1+1} t_2^{m_2+1}] \left(\prod_{i=1}^{k} t_1 t_2 \frac{(t_2^{p_i} - t_1^{p_i})}{t_2 - t_1} \right). \tag{6}$$

The relevant Hurwitz spaces of $\mathbb{P}^1$ covers will be introduced in Sect. 2. Note that $h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \neq 0$ only if $\sum_{i=1}^{k}(p_i + 1) = m_1 + m_2 + 2$. This implies that $k \leq \min(m_1 + 1, m_2 + 1)$ in the summation of the theorem.

For special μ, the strata $\Omega\mathcal{M}_{g,n}(\mu)$ can be disconnected. There are up to three connected components altogether, at most one of which is hyperelliptic, classified by Kontsevich and Zorich [33]. We show the refinements of Theorems 1.1 and 1.2 for the spin and hyperelliptic components respectively, given as Theorems 6.3 and 6.12 (conditional on a technical Assumption 6.1 in Sect. 6, which can be deduced from [7]).[1]

Equation (5) has a similar form compared to the recursion formula obtained by Eskin, Masur and Zorich [20] for computing saddle connection Siegel–Veech constants (joining two distinct zeros). Consider a generic flat surface with n labeled zeros of orders $\mu = (m_1, \ldots, m_n)$. The growth rate of the number of saddle connections of length at most L joining, say, the first two zeros is quadratic in L and the leading term of the asymptotics (up to a factor of π to ensure rationality) is called the saddle connection Siegel–Veech constant. Intuitively, the saddle connection Siegel–Veech constant should be proportional to the cone angles around the two concerned zeros. For quadratic differentials this is not correct as shown by Athreya, Eskin and Zorich [2]. Nevertheless as an application of our formulas, we show that for Abelian differentials the intuitive expectation indeed holds, if we use a minor modification $c_{1\leftrightarrow 2}^{\mathrm{hom}}(\mu)$ of the Siegel–Veech constant counting homologous saddle connections only once. An overview about the variants of Siegel–Veech constants is given in Sect. 7.

Theorem 1.3 *The saddle connection Siegel–Veech constant* $c_{1\leftrightarrow 2}^{\mathrm{hom}}(\mu)$ *joining the first and the second zeros on a generic flat surface of type* μ *is given by*

$$c_{1\leftrightarrow 2}^{\mathrm{hom}}(\mu) = (m_1 + 1)(m_2 + 1). \tag{7}$$

When the stratum is disconnected, we also show that the theorem holds for each connected component under Assumption 6.1. We remark that as an asymptotic equality as g tends to infinity, the formula (7) for the entire stratum was previously shown in the appendix by Zorich to [3] for saddle connections of multiplicity one and by Aggarwal [4] for all multiplicities.

[1] This assumption was later verified in [13].

Another important kind of Siegel–Veech constants is the area Siegel–Veech constant, which counts cylinders (weighted by the reciprocal of their areas) on flat surfaces and is related to the sum of Lyapunov exponents [15] (see Sect. 7 for the definition of area Siegel–Veech constants). We similarly establish an intersection formula for area Siegel–Veech constants.

Theorem 1.4 *The area Siegel–Veech constants of the strata can be evaluated as*

$$c_{\text{area}}(\mu) = \frac{-1}{4\pi^2} \frac{\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \beta_i \cdot \delta_0}{\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \beta_i \cdot \xi} \tag{8}$$

for each $1 \le i \le n$, where δ_0 is the divisor class of the locus of curves with a non-separating node.

Theorem 1.4 completes the investigation of area Siegel–Veech constants begun in [12, Section 4] (for the principal strata) and [42, Equation (2)] (for the minimal strata). It also justifies a speculation of Kontsevich [32, Section 7] about the existence of such a β-class for computing area Siegel–Veech constants and sums of Lyapunov exponents of the strata.

Another application of the volume recursion is a geometric proof of the large genus limit conjecture by Eskin and Zorich [23] for the volumes of the strata and area Siegel–Veech constants. A proof using direct combinatorial arguments was given by Aggarwal [3,4]. Our proof, in addition, gives a uniform expression for the second order term as conjectured in [42] (see Sect. 11).

Theorem 1.5 [23, Main Conjectures] *Consider the strata $\Omega\mathcal{M}_{g,n}(\mu)$ such that all the entries of μ are positive. Then*

$$\begin{aligned} v(\mu) &= 4 - \frac{2\pi^2}{3 \cdot \sum_{i=1}^n (m_i + 1)} + O(1/g^2), \\ c_{\text{area}}(\mu) &= \frac{1}{2} - \frac{1}{2\sum_{i=1}^n (m_i + 1)} + O(1/g^2), \end{aligned}$$

where the implied constants are independent of μ and g.

Finally we settle another conjecture of Eskin and Zorich on the asymptotic comparison of spin components.

Theorem 1.6 ([23, Conjecture 2]) *The volumes and area Siegel–Veech constants of odd and even spin components are comparable for large values of g.*

More precisely,

$$\frac{v(\mu)^{\mathrm{odd}}}{v(\mu)^{\mathrm{even}}} = 1 + O(1/g),$$
$$\frac{c_{\mathrm{area}}(\mu)^{\mathrm{odd}}}{c_{\mathrm{area}}(\mu)^{\mathrm{even}}} = 1 + O(1/g),$$

where the implied constants are independent of μ *and* g.

Further directions

Our work opens an avenue to study a series of related questions. First, we point out an interesting comparison with the proofs by Mirzakhani [38], by Kontsevich [31], and by Okounkov–Pandharipande [41] of Witten's conjecture: the generating function of ψ-class intersections on moduli spaces of curves is a solution of the KdV hierarchy of partial differential equations. Mirzakhani considered the Weil–Petersson volumes of moduli spaces of hyperbolic surfaces and analyzed geodesics that bound pairs of pants, while we consider the Masur–Veech volumes of moduli spaces of flat surfaces and analyze geodesics that join two zeros (i.e. saddle connections). Kontsevich interpreted ψ-classes as associated to certain polygon bundles, while we have the interpretation of Abelian differentials as polygons. Okounkov and Pandharipande used Hurwitz spaces of $\mathbb{P}^1$ covers, while we rely on Hurwitz numbers of torus covers. Therefore, we speculate that generating functions of Masur–Veech volumes and area Siegel–Veech constants should also satisfy a certain interesting hierarchy as in Witten's conjecture.

In another direction, one can consider saddle connections joining a zero to itself (see [20, Part 2]) or impose other specific configurations to refine the Siegel–Veech counting (see e.g. the appendix by Zorich to [3]). From the viewpoint of intersection theory, such a refinement should pick up the corresponding part of the principal boundary when flat surfaces degenerate along the configuration, hence we expect that the resulting Siegel–Veech constant can be described similarly by a recursion formula involving intersection numbers.

One can also investigate volumes and Siegel–Veech constants for affine invariant manifolds (i.e. $\mathrm{SL}_2(\mathbb{R})$-orbit closures in the strata). It is thus natural to seek intersection theoretic interpretations of these invariants for affine invariant manifolds, e.g. the strata of quadratic differentials (see [5,11,14] for interesting related results in the case of the principal strata). We plan to treat these questions in future work.

Organization of the paper

In Sect. 2 we introduce relevant intersection numbers on Hurwitz spaces of $\mathbb{P}^1$ covers that will appear as coefficients in the volume recursion. In Sect. 3 we prove that the expression of volumes by intersection numbers satisfies the recursion in (5), thus showing the equivalence of Theorems 1.1 and 1.2. In Sect. 4 we exhibit another recursion of volumes by using the algebra of shifted symmetric functions and cumulants. In Sect. 5 we show that the two recursions are equivalent by interpreting them as the same summation over certain oriented graphs, thus completing the proof of Theorems 1.1 and 1.2. In Sect. 6 we refine the results for the spin and hyperelliptic components of the strata. In Sects. 7, 8 and 9 we respectively review the definitions of various Siegel–Veech constants, prove Theorem 1.3 regarding saddle connection Siegel–Veech constants and interpret the result from the perspective of Hurwitz spaces of torus covers. In Sect. 10 we establish similar intersection and recursion formulas for area Siegel–Veech constants, thus proving Theorem 1.4. Finally in Sect. 11 we apply our results to evaluate large genus limits of volumes and area Siegel–Veech constants, proving Theorems 1.5 and 1.6.

2 Hurwitz spaces of $\mathbb{P}^1$ covers

In this section we recall the definition of the moduli space of admissible covers of [27] as a compactification of the classical Hurwitz space (see also [28]), and prove formulas to compute recursively intersection numbers of ψ-classes on these moduli spaces. These intersection numbers will appear as coefficients and multiplicities in the volume recursion. Along the way we introduce basic notions on stable graphs and level functions.

2.1 Hurwitz spaces and admissible covers

Let d, g, and g' be non-negative integers. Let $\Pi = (\mu^{(1)}, \cdots, \mu^{(n)})$ be a *ramification profile* consisting of n partitions. We define the *Hurwitz space* $H_{d,g,g'}(\Pi)$ to be the moduli space parametrizing branched covers of smooth connected curves $p\colon X \to Y$ of degree d with profile Π and such that the genera of X and Y are given by g and g' respectively. That is, p is ramified over n points and over the ith branch point the sheets coming together form the partition $\mu^{(i)}$ (completed by singletons if $|\mu^{(i)}| < \deg(p)$).

The Hurwitz space $H_{d,g,g'}(\Pi)$ has a natural compactification $\overline{H}_{d,g,g'}(\Pi)$ parametrizing *admissible covers*. An admissible cover $p\colon X \to Y$ is a finite morphism of connected nodal curves such that

(i) the smooth locus of X maps to the smooth locus of Y and the nodes of X map to the nodes of Y,
(ii) at each node of X the two branches have the same ramification order, and
(iii) the target curve Y marked with the branch points is stable.

The space $\overline{H}_{d,g,g'}(\Pi)$ is equipped with two forgetful maps

$$\begin{array}{ccc} & \overline{H}_{d,g,g'}(\Pi) & \\ {}^{f_S}\swarrow & & \searrow^{f_T} \\ \overline{\mathcal{M}}_{g,m} & & \overline{\mathcal{M}}_{g',n} \end{array}$$

obtained by mapping an admissible cover to the stabilization of the source or the target. Here n denotes the number of branch points or equivalently the length of Π and m denotes the number of ramification points or equivalently the number of parts (of length > 1) of all the $\mu^{(i)}$. The *Hurwitz number* $N^{\circ}_{d,g,g'}(\Pi)$ is the degree of the map f_T, or equivalently the number of connected covers $p\colon X \to Y$ of degree d with profile Π and the location of the branch points fixed in Y. We also denote by $N_{d,g,g'}(\Pi)$ the Hurwitz number of covers without requiring X to be connected. We remark that each cover is counted with weight given by the reciprocal of the order of its automorphism group, as is standard for the Hurwitz counting problem.

2.2 Intersection of ψ-classes on Hurwitz spaces

From now on in this section we will consider the special case $g' = 0$. Let $\mu[0] = (m_1, \dots, m_n)$ be a list of non-negative integers and $\mu[\infty] = (p_1, \dots, p_k)$ a list of positive integers. We consider the Hurwitz space with profile Π given by $\mu^{(i)} = (m_i + 1)$ for $i \le n$ and $\mu^{(n+1)} = (p_1, \dots, p_k)$ such that $d = \sum_{i=1}^{k} p_i$, i.e. we consider

$$\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty]) = \overline{H}_{d,g,0}((m_1+1), \dots, (m_n+1), (p_1, \dots, p_k))\,.$$

By the Riemann–Hurwitz formula, the genus g of the covering surfaces satisfies that

$$2 - 2g = k + d - \sum_{i=1}^{n} m_i = k + \sum_{i=1}^{k} p_i - \sum_{i=1}^{n} m_i\,.$$

The forgetful map f_S goes from $\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty])$ to $\overline{\mathcal{M}}_{g,n+k}$, where we assume that the first n marked points are the first n ramification points and the preimages of the last branch point are the k last marked points. Since there are $n+1$ branch points in the target surface of genus zero, we conclude that $\dim \overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty]) = \dim \overline{\mathcal{M}}_{0,n+1} = n-2$.

For $g=0$, define the following intersection numbers on the Hurwitz spaces

$$h_{\mathbb{P}^1}(\mu[0], \mu[\infty]) = \int_{\overline{H}_{\mathbb{P}^1}(\mu[0],\mu[\infty])} f_S^* \left(\prod_{i=3}^{n} \psi_i \right). \tag{9}$$

The definition of ψ-classes will be recalled in Sect. 3. If $n=2$, then $\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty])$ is of dimension zero, and hence the intersection number on the right is just the number of points of the Hurwitz space, i.e.

$$h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty]) = N^{\circ}_{d,0,0}((m_1+1), (m_2+1), (p_1, \ldots, p_k)).$$

Again we emphasize that the Hurwitz number on the right-hand side is counted with weight $1/|\mathrm{Aut}|$ for each cover. Correspondingly the intersection numbers are computed on the Hurwitz space treated as a stack. Our goal for the rest of the section is to show the following result.

Proposition 2.1 *For $n=2$, $h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty])$ can be computed by the coefficient extraction*

$$h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty]) = (k-1)![t^{m_1+1}] \prod_{i=1}^{k} \frac{t - t^{p_i+1}}{1-t},$$

and for $n \geq 3$, $h_{\mathbb{P}^1}(\mu[0], \mu[\infty])$ can be computed recursively by the sum

$$h_{\mathbb{P}^1}(\mu[0], \mu[\infty]) = \sum_{\Gamma \in \mathrm{RT}(\mu[0],\mu[\infty])_{1,2}} h(\Gamma)$$

over rooted trees.

The definitions of rooted trees and the local contributions $h(\Gamma)$ are given in Sects. 2.3 and 2.4 respectively. The above formula for the Hurwitz number obviously agrees with (6).

Proof of Proposition 2.1, case $n=2$. Let S_d be the symmetric group acting on $[\![1, d]\!] = \{1, \ldots, d\}$, where $d = p_1 + \cdots + p_k = m_1 + m_2 + 2 - k$. Define the set of Hurwitz tuples

$$A(m_1, m_2, \mu[\infty]) = \{(\sigma_1, \sigma_2, \sigma_\infty)\} \subset S_d \times S_d \times S_d$$

such that

- the permutation σ_∞ is in the conjugacy class of $(p_1, \ldots, p_k)$ and we fix a bijection of its cycles with $[\![1, k]\!]$ such that the ith cycle has length p_i,
- the partitions σ_1 and σ_2 are cycles of order $m_1 + 1$ and $m_2 + 1$ respectively,
- the relation $\sigma_1 \circ \sigma_2 = \sigma_\infty$ holds, and
- the group generated by σ_1, σ_2 and σ_∞ acts transitively on $[\![1, d]\!]$.

Then the (weighted) Hurwitz number $h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty]) = |A(m_1, m_2, \mu[\infty])|/d!$.

The second and third conditions above imply that the union of the supports of the cycles σ_1 and σ_2 is $[\![1, d]\!]$. Therefore, σ_1 and σ_2 contain exactly $(m_1 + 1) + (m_2 + 1) - d = k$ common elements. We can write

$$\sigma_1 = (a_1, \ldots, a_{i_1-1}, c_1; a_{i_1+1}, \ldots, a_{i_2-1}, c_2; \ldots; a_{i_{k-1}+1}, \ldots, a_{i_k-1}, c_k),$$

where $1 \le i_1 < i_2 < \cdots < i_k = m_1 + 1$ and $c_1, \ldots, c_k$ are the common elements of σ_1 and σ_2. Since $\sigma_1 \circ \sigma_2 = \sigma_\infty$ is of conjugacy type $(p_1, \ldots, p_k)$, it is easy to see that σ_2 must be of the form

$$\sigma_2 = (b_1, \ldots, b_{j_1-1}, c_k; b_{j_1+1}, \ldots, b_{j_2-1}, c_{k-1}; \ldots; b_{j_{k-1}+1}, \ldots, b_{j_k-1}, c_1),$$

for certain b_i, such that

$$\begin{aligned} &\{j_1 + i_1, (j_2 - j_1) + (i_k - i_{k-1}), \ldots, (j_k - j_{k-1}) + (i_2 - i_1)\} \\ &\quad = \{p_1 + 1, p_2 + 1, \ldots, p_k + 1\}. \end{aligned}$$

If τ is a permutation on $[\![1, k]\!]$ such that $(j_{k+1-\ell} - j_{k-\ell}) + (i_{\ell+1} - i_\ell) = p_{\tau(\ell)} + 1$, then we have $1 \le i_{\ell+1} - i_\ell \le p_{\tau(\ell)}$. Conversely, such τ and i-indices determine the j-indices.

There are $m_1!\binom{d}{m_1+1}$ choices for σ_1. Fixing σ_1, to construct σ_2 we first choose

- a permutation $\tau \in S_k$, and then
- a partition $(i_1, i_2 - i_1, \ldots, i_k - i_{k-1})$ of $m_1 + 1$ such that $1 \le i_{\ell+1} - i_\ell \le p_{\tau(\ell)}$ for all ℓ.

This gives $k![t^{m_1+1}] \prod_{i=1}^k (t + \cdots + t^{p_i})$ choices. Choose c_1 out of the elements in σ_1, which gives $m_1 + 1$ choices. Along with the i-indices this determines the elements $c_2, \ldots, c_k$ as well as the set of b's as the complement of the union of a's and c's. Finally σ_2 is determined by arranging the b's, which gives $(m_2 + 1 - k)!$ choices. Note that in this process only the cyclic order of $(c_1, \ldots, c_k)$ matters and we cannot actually determine which one is the first c, hence we need to divide the final count by k.

In summary, we conclude that

$$|A(m_1, m_2, \mu[\infty])|$$

$$= (m_1 + 1)!(m_2 + 1 - k)!\binom{d}{m_1 + 1} \cdot (k - 1)![t^{m_1+1}]\prod_{i=1}^{k}(t + \cdots + t^{p_i}).$$

Since $d = m_1 + m_2 + 2 - k$, we obtain

$$\begin{aligned} &h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty]) \\ &\quad = |A(m_1, m_2, \mu[\infty])|/d! = (k - 1)![t^{m_1+1}]\prod_{i=1}^{k}\frac{t(1 - t^{p_i})}{1 - t} \end{aligned}$$

using that $(m_1 + 1)!\,(m_2 + 1 - k)!\,\binom{d}{m_1+1} = d!$. □

We remark that the above Hurwitz counting problem can also be interpreted by the angular data of the configurations of saddle connections joining two zeros z_1 and z_2 of order m_1 and m_2 respectively in the setting of [20]. Suppose $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a branched cover parameterized in the Hurwitz space $H_{d,0,0}((m_1+1), (m_2+1), (p_1, \ldots, p_k))$, where we treat f as a meromorphic function with k poles of order $p_1, \ldots, p_k$. Then the meromorphic differential $\eta = df$ has two zeros of order m_1 and m_2 as well as k poles of order $p_1 + 1, \ldots, p_k + 1$ with no residue. Conversely given such η, integrating η gives rise to a desired branched cover f. Such η can be constructed using flat geometry as in [10, Section 2.4]. In particular, it is determined by the angles $2\pi(a'_i + 1)$ between the saddle connections (clockwise) at z_1 and the angles $2\pi(a''_i + 1)$ (counterclockwise) at z_2, such that $\sum_{i=1}^{k}(a'_i + 1) = m_1 + 1$, $\sum_{i=1}^{k}(a''_i + 1) = m_2 + 1$ and $a'_i + a''_i + 2 = p_i + 1$. We see again that the choices involve a partition $(a'_1 + 1, \ldots, a'_k + 1)$ of $m_1 + 1$ such that $a'_i + 1 \le p_i$ for all i.

2.3 Level graphs and rooted trees

The boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ is naturally stratified by the topological types of the stable marked surfaces. These boundary strata are in one-to-one correspondence with stable graphs, whose definition we recall below. The boundary strata of Hurwitz spaces and of moduli spaces of Abelian differentials are encoded by adding level structures and twists to stable graphs.

Definition 2.2 A *stable graph* is the datum of

$$\Gamma = (V, H, g\colon V \to \mathbb{N}, a\colon H \to V, i\colon H \to H, E, L \simeq [\![1, n]\!])$$

satisfying the following properties:

- V is a vertex set with a genus function g;
- H is a half-edge set equipped with a vertex assignment a and an involution i (and we let $n(v) = |a^{-1}(v)|$);
- E, the edge set, is defined as the set of length-2 orbits of i in H (self-edges at vertices are permitted);
- (V, E) define a connected graph;
- L is the set of fixed points of i, called *legs* or *markings*, and is identified with $[\![1, n]\!]$;
- for each vertex v, the stability condition $2g(v) - 2 + n(v) > 0$ holds.

Let $v(\Gamma)$ and $e(\Gamma)$ denote the cardinalities of V and E respectively. The genus of Γ is defined by $\sum_{v\in V(\Gamma)} g(v) + e(\Gamma) - v(\Gamma) + 1$.

We denote by $\mathrm{Stab}(g, n)$ the set of stable graphs of genus g and with n legs. A stable graph is said of *compact type* if $h^1(\Gamma) = 0$, i.e. if the graph has no loops, which is thus a *tree*.

We will use two extra structures on stable graphs, called level functions and twists. As in [6] we define a *level graph* to be a stable graph Γ together with a level function $\ell\colon V(\Gamma) \to \mathbb{R}_{\leq 0}$. An edge with the same starting and ending level is called a *horizontal edge*. A *bi-colored graph* is a level graph with two levels (in which case we normalize the level function to take values in $\{0, -1\}$) that has no horizontal edges. We denote the set of bi-colored graphs by $\mathrm{Bic}(g, n)$.

Recall the notation $\mu[0] = (m_1, \ldots, m_n)$ and $\mu[\infty] = (p_1, \ldots, p_k)$ where $m_i \geq 0$ and $p_j > 0$ for all i and j.

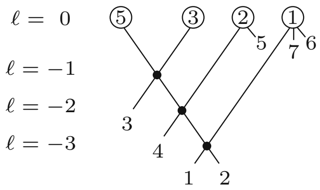
Definition 2.3 Let Γ be a stable graph in $\mathrm{Stab}(g, n+k)$. A *twist assignment* on Γ of type $(\mu[0], \mu[\infty])$ is a function $\mathbf{p}\colon H(\Gamma) \to \mathbb{Z}$ satisfying the following conditions:

- If (h, h') is an edge, then $\mathbf{p}(h) + \mathbf{p}(h') = 0$.
- For all $1 \leq i \leq n$, the twist of the ith leg is $m_i + 1$ and for all $1 \leq i \leq k$ the twist of the $(n + i)$th leg is $-p_i$.
- For all vertices v of Γ

$$2g(v) - 2 + n(v) = \sum_{h\in L,\, a(h)=v} \mathbf{p}(h)\,.$$

Suppose the graph Γ comes with a level structure ℓ. We say that a twist p is *compatible* with the level structure if for all edges (h, h') the condition $\mathbf{p}(h) > 0$ implies that $\ell(a(h)) > \ell(a(h'))$, and respectively for the cases $<$ and $=$. In this case we call the triple $(\Gamma, \ell, \mathbf{p})$ a *twisted level graph*. For the reader familiar with related results of compactifications of strata of Abelian differentials, the above definition characterizes twisted differentials (or canonical divisors) in [6] and [24] (regarding the m_i as the zero orders and $p_j + 1$ as the pole orders

Fig. 1 A rooted tree of genus eleven with three vertices of genus zero (black) and seven legs



of twisted differentials on irreducible components of the corresponding stable curves). In particular, every level graph has only finitely many compatible twists. For a graph of compact type, there exists a unique twist $\mathbf{p}$ if the entries of $(\mu[0], \mu[\infty])$ satisfy the condition that $\sum_{i=1}^{n} m_i - \sum_{j=1}^{k}(p_j+1) = 2g-2$.

Definition 2.4 Let $1 \le i < j \le n$. A *stable rooted tree* (or simply a *rooted tree*) is a twisted level graph $(\Gamma, \ell, \mathbf{p})$ of compact type satisfying the following conditions:

(i) One vertex v_j carries the ith and jth legs and no other of the first n legs; (The vertex v_j is called the *root*, a vertex on the path from v to the root is called an *ancestor* of v, and a vertex whose ancestors contain v is called a *descendant* of v.)
(ii) There are no horizontal edges;
(iii) A vertex v is on level 0 if and only if v is a leaf. If v is not a leaf, then $\ell(v) = \min\{\ell(v') \mid v' \text{ is a descendant of } v\} - 1$;
(iv) All vertices of positive genus are leaves and hence on level 0;
(v) Each vertex of genus zero other than v_j carries exactly one of the first n legs.

Since the root is an ancestor of any other vertex, by definition it is the unique vertex lying on the bottom level, hence it has genus zero. Moreover, it is easy to see from the definition that any path towards the root is strictly going down (Fig. 1).

We denote by $\mathrm{RT}(g, \mu[0], \mu[\infty])_{i,j}$ the set of such rooted trees, and sometimes simply by $\mathrm{RT}(\mu[0], \mu[\infty])_{i,j}$ if $g = 0$.

2.4 The sum over rooted trees

Now we assume that $g = 0$. Below we define the local contributions from rooted trees in Proposition 2.1 and complete its proof. Consider a graph $\Gamma \in \mathrm{RT}(\mu[0], \mu[\infty])_{1,2}$. Since by assumption every vertex of Γ has genus zero, condition v) implies that Γ has exactly $n-1$ vertices and $n-2$ edges. Denote by $v_2, \dots, v_n$ the vertices of Γ such that v_i carries the ith leg h_i for $3 \le i \le n$ and v_2 carries the first two legs. This convention is consistent with our previous notation for the root. We denote by $\mu[\infty]_i$ the list of negative twists at half-edges adjacent to v_i. These half-edges are either part of the whole edges joining

v_i to its descendants (as adjacent vertices to v_i on higher level) or part of the k last legs (corresponding to the k marked poles).

If $i \neq 2$, then there is a unique (non-leg) half-edge $\widetilde{h}_i \neq h_i$ adjacent to v_i such that $\widetilde{m}_i := \mathbf{p}(\widetilde{h}_i) - 1 \geq 0$. Namely, this half-edge is part of the whole edge joining v_i to its ancestor (as the adjacent vertex to v_i on lower level). With this notation we define the contribution of the rooted tree Γ as

$$h(\Gamma) = h_{\mathbb{P}^1}((m_1, m_2), \mu[\infty]_2) \cdot \prod_{i=3}^{n} h_{\mathbb{P}^1}((m_i, \widetilde{m}_i), \mu[\infty]_i)\,. \tag{10}$$

Let $J \subset [\![1, n+k]\!]$ be a subset such that the cardinalities of J and J^c are at least two. Denote by δ_J the class of the boundary divisor of $\overline{\mathcal{M}}_{0,n+k}$ parameterizing curves that consist of a component with the markings in J union a component with the markings in J^c. We need the following classical result (see e.g. [1, Lemma 7.4]).

Lemma 2.5 *For all $3 \leq i \leq n+k$, the following relation of divisor classes holds on $\overline{\mathcal{M}}_{0,n+k}$:*

$$\psi_i = \sum_{i \in J \subset [\![3,n+k]\!]} \delta_J\,.$$

If J is a subset of $[\![3, n]\!]$, then we denote by $\widetilde{\delta}_J = \sum_{J' \subset [\![n+1,n+k]\!]} \delta_{J \cup J'}$. For $3 \leq i \leq n$, the above lemma implies that

$$\psi_i = \sum_{i \in J \subset [\![3,n]\!]} \widetilde{\delta}_J\,. \tag{11}$$

We also need the following result about the boundary divisors of the Hurwitz spaces $\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty])$.

Lemma 2.6 *There is a bijection between the boundary divisors of $\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty])$ and the corresponding bi-colored graphs (i.e. with two levels only). Moreover, a boundary divisor drops dimension under the source map f_S if its bi-colored graph has more than two vertices.*

Proof The first part of the claim follows from the same argument as in [43, Proposition 7.1]. Here the vertices of level 0 in the bi-colored graphs correspond to the stable components of the admissible covers that contain the marked poles. For the other part, suppose that a generic point of a boundary divisor has at least two vertices on level 0 (or on level -1). Then one can scale one of the two functions that induce the covers on the two vertices such that the domain marked curve is fixed while the admissible covers vary. It implies that f_S restricted to this boundary divisor has positive dimensional fibers. □

Proof of Proposition 2.1, *case* $n \geq 3$. We will prove the result by induction on n. The initial case $n = 2$ follows from the definition of $h(\Gamma)$ in (10) and we have also described it explicitly in Sect. 2.2. The strategy of the induction for higher n is by successively replacing the ψ_i in (9) with the sum over boundary divisors as in the preceding lemma, starting with $i = 3$. To simplify notation, we write $\overline{H}(\mu[0], \mu[\infty])$ instead of $\overline{H}_{\mathbb{P}^1}(\mu[0], \mu[\infty])$. We also simply write ψ and δ as classes in the Hurwitz space for their pullbacks via f_S.

Consider a boundary divisor δ_J of $\overline{\mathcal{M}}_{0,n+k}$ pulled back to $\overline{H}(\mu[0], \mu[\infty])$, which is a union of certain boundary divisors of $\overline{H}(\mu[0], \mu[\infty])$. We would like to compute the intersection number $\delta_J \cdot \prod_{i=4}^{n} \psi_i$ on $\overline{H}(\mu[0], \mu[\infty])$. By Lemma 2.6 and the projection formula, the only possible non-zero contribution is from the loci in δ_J whose bicolored graphs have a unique edge $e = (h, h')$ connecting two vertices v_0 and v_{-1} on level 0 and level -1 respectively, such that the last k markings (i.e. the k marked poles) are contained in v_0. In this case we can assume that $\mathbf{p}(h) > 0$ (and hence $\mathbf{p}(h') < 0$ as $\mathbf{p}(h) + \mathbf{p}(h') = 0$ by definition). The admissible covers restricted to v_0 and to v_{-1} belong to Hurwitz spaces of similar type, where at the node (i.e. the edge e) the ramification order of the restricted maps is given by $\mathbf{p}(h) - 1 = -\mathbf{p}(h') - 1$. It implies that the locus of such admissible covers can be identified with $\overline{H}(\mu[0]^0, \mu[\infty]^0) \times \overline{H}(\mu[0]^{-1}, \mu[\infty]^{-1})$, where $\mu[0]^0$ is the part of $\mu[0]$ contained in v_0 union with $\mathbf{p}(h) - 1$, $\mu[\infty]^0 = \mu[\infty]$, $\mu[0]^{-1}$ is the part of $\mu[0]$ contained in v_{-1}, and $\mu[\infty]^{-1}$ has a single entry $\mathbf{p}(h)$.

By Eq. (11) we have $\psi_3 = \sum_{3 \in J \subset [\![3, n+k]\!]} \delta_J$. Fix a subset $J \subset [\![3, n+k]\!]$ such that $3 \in J$. If the third marking belongs to v_{-1}, we claim that

$$\overline{H}(\mu[0], \mu[\infty]) \cdot \delta_J \cdot \prod_{i=4}^{n} \psi_i = 0\,.$$

To see this, note that the dimension of $\overline{H}(\mu[0]^{-1}, \mu[\infty]^{-1})$ is equal to $n(\mu[0]^{-1}) - 2$, as $\mu[\infty]^{-1}$ has only one entry. However there are $n(\mu[0]^{-1}) - 1$ markings with label ≥ 4 on v_{-1}. Consequently the intersection with the product of those ψ_i vanishes on $\overline{H}(\mu[0]^{-1}, \mu[\infty]^{-1})$.

Therefore, we only need to consider the case when v_0 contains the third marking (hence all markings labeled by J), and consequently v_{-1} contains the first and second markings (hence all markings labeled by J^c). In this case we obtain that

$$\overline{H}(\mu[0], \mu[\infty]) \cdot \delta_J \cdot \prod_{i=4}^{n} \psi_i = h_{\mathbb{P}^1}(\mu[0]^0, \mu[\infty]^0) \cdot h_{\mathbb{P}^1}(\mu[0]^{-1}, \mu[\infty]^{-1}),$$

where $h_{\mathbb{P}^1}$ is defined in (9), where in the first factor on the right-hand side the ψ-product skips the third marking and the marking from the half-edge of v_0, and where in the second factor the ψ-product skips the first and second markings.

Now we use the induction hypothesis to decompose the factors $h_{\mathbb{P}_1}(\mu[0]^a, \mu[\infty]^a)$ for $a = 0$ and $a = -1$. It leads to a sum over all possible pairs of rooted trees, where the two rooted trees in each pair generate a new rooted tree. More precisely, one rooted tree in the pair contains the markings of $J^c \cup \{h'\}$ whose root v_2 carries the first and second markings, the other rooted tree contains the markings in $J \cup \{h\}$ whose root v_3 carries the third marking and h, and they generate a new rooted tree by gluing the legs h and h' as a whole edge and by using v_2 as the new root.

Therefore, if J is a subset of $[\![3, n]\!]$ such that $3 \in J$, then we obtain that

$$\overline{H}(\mu[0], \mu[\infty]) \cdot \widetilde{\delta}_J \cdot \prod_{i=4}^{n} \psi_i = \sum_{\substack{\Gamma \in \mathrm{RT}(\mu[0],\mu[\infty])_{1,2}, \\ j \in J \Leftrightarrow \ell(v_3) \leq \ell(v_j)}} h(\Gamma),$$

where the sum is over all rooted trees Γ such that the descendants of v_3 are exactly the vertices v_j for $j \in J \backslash \{3\}$.

In summary if we write $\psi_3 = \sum_{3 \in J \subset [\![3,n]\!]} \widetilde{\delta}_J$, then by the above analysis we thus conclude that $\overline{H}(\mu[0], \mu[\infty]) \cdot \psi_3 \cdot \prod_{i=4}^{n} \psi_i$ is equal to the sum of the contributions $h(\Gamma)$ over all rooted trees Γ. □

3 Volume recursion via intersection theory

In this section we show that the two main theorems of the introduction, Theorems 1.1 and 1.2 are equivalent. This section does not yet provide a direct proof of either of them.

We first show that the intersection numbers in Theorem 1.1 are given by a recursion formula of the same shape as in Theorem 1.2. Together with an agreement on the minimal strata this proves the equivalence of the two theorems. Along the way we introduce special classes of stable graphs that are used for recursions throughout the paper.

3.1 Intersection numbers on the projectivized Hodge bundle

Fix g and n such that $2g - 2 + n > 0$. We denote by $f: \mathcal{X} \to \overline{\mathcal{M}}_{g,n}$ the universal curve and by $\omega_{\mathcal{X}/\overline{\mathcal{M}}_{g,n}}$ the relative dualizing line bundle. We will use the following cohomology classes:

- Let $1 \leq i \leq n$. We denote by $\sigma_i: \overline{\mathcal{M}}_{g,n} \to \mathcal{X}$ the section of f corresponding to the ith marked point and by $\mathcal{L}_i = \sigma_i^* \omega_{\mathcal{X}/\overline{\mathcal{M}}_{g,n}}$ the cotangent line at the ith marked point. With this notation, we define $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

- For $1 \leq i \leq g$, we denote by $\lambda_i = c_i(\overline{\Omega\mathcal{M}}_{g,n}) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ the ith Chern class of the Hodge bundle. (We use the same notation for a vector bundle and its total space.)
- We denote by $\delta_0 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ the Poincaré-dual class of the divisor parameterizing marked curves with at least one non-separating node.
- The projectivized Hodge bundle $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}$ comes with the universal line bundle class $\xi = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q})$.

Unless otherwise specified, we denote by the same symbol a class in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and its pull-back via the projection $p\colon \mathbb{P}\overline{\Omega\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Recall that the splitting principle implies that the structure of the cohomology ring of the projectivized Hodge bundle is given by

$$H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q}) \simeq H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})[\xi]/(\xi^g + \lambda_1\xi^{g-1} + \cdots + \lambda_g)\,.$$

Let $\mu = (m_1, \ldots, m_n)$ be a partition of $2g-2$. We denote by $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ the closure of the projectivized stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ inside the total space of the projectivized Hodge bundle $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}$. This space is called the *(ordered) incidence variety compactification*[2].

In this section we study the intersection numbers

$$a_i(\mu) = \int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \beta_i \cdot \xi = \frac{1}{m_i+1} \int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)} \xi^{2g-1} \cdot \prod_{j\neq i} \psi_j \qquad (12)$$

for all $1 \leq i \leq n$. The reader should think of the $a_i(\mu)$ as certain normalization of volumes. In fact, Theorem 1.1 can be reformulated as

$$\mathrm{vol}(\Omega\mathcal{M}_{g,n}(\mu)) = \frac{2(2\pi)^{2g}(-1)^g}{(2g-3+n)!}\, a_i(\mu), \qquad (13)$$

implying in particular that $a_i(\mu)$ is independent of i.

We prove a collection of properties defining recursively the $a_i(\mu)$ as the coefficients of some formal series. As the base case for $n = 1$, i.e. $\mu = (2g-2)$, define the formal series

$$\mathcal{A}(t) = \frac{1}{t} + \sum_{g\geq 1} (2g-1)^2\, a_1(2g-2)\, t^{2g-1} \quad \in \frac{1}{t}\mathbb{Q}[[t]] \qquad (14)$$

[2] In [6] the notation $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathrm{inc}}_{g,n}(\mu)$ is used. Here we drop the superscript "inc" for simplicity. In [43] this space is denoted by $\mathbb{P}\overline{\mathcal{H}}_{g,n}(\mu)$.

and set

$$B(z) := \frac{z/2}{\sinh(z/2)} =: \sum_{j\geq 0} b_j z^j = 1 - \frac{z^2}{24} + \frac{7z^4}{5760} + \cdots . \tag{15}$$

For a partition μ, recall that $n(\mu)$ denotes the number of its entries and $|\mu|$ denotes the sum of the entries.

Theorem 3.1 *The generating function $\mathcal{A}$ of the minimal stratum intersection numbers $a_i(2g-2)$ is determined by the coefficient extraction identity*

$$[t^0]\frac{1}{j!}\mathcal{A}(t)^j = b_j, \tag{16}$$

while the intersection numbers $a(\mu) = a_i(\mu)$ with $n(\mu) \geq 2$ are given recursively by

$$\begin{aligned}(m_1+1)(m_2+1)a(m_1,\ldots,m_n) \\ = \sum_{k=1}^{\min} \frac{1}{k!} \sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p}) \prod_{j=1}^{k} (2g_j - 1 + n(\mu_j)) p_j\, a(p_j - 1, \mu_j),\end{aligned} \tag{17}$$

where $\min = \min(m_1+1, m_2+1)$, with the same summation conventions as in Theorem 1.2.

The first identity (16) was proved in [42] and gives

$$\mathcal{A}(t) = \frac{1}{t} - \frac{1}{24}t + \frac{3}{640}t^3 - \frac{1525}{580608}t^5 + \frac{615881}{199065600}t^8 - \cdots .$$

By Lagrange inversion, this formula can be written equivalently as

$$\mathcal{A}(t) = \frac{1}{Q^{-1}(t)}, \quad \text{where} \quad Q(u) = u \exp\Bigl(\sum_{k\geq 1}(k-1)!b_k u^k\Bigr)$$

and will in fact be proved in this form in Sect. 4.4. We observe in passing that $Q(u)$ is the asymptotic expansion of $\psi(u^{-1}+\frac{1}{2})$ as $u \to 0$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. The proof of the second identity (17) will be completed by the end of Sect. 3.5.

In the course of proving Theorem 3.1 we will prove the following complementary result, justifying the implicitly used fact that $a_i(\mu)$ is independent of i.

Proposition 3.2 *For all* $1 \leq i \leq n$*, we have*

$$a_i(\mu) = -\int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)} \xi^{2g-2} \cdot \prod_{j=1}^{n} \psi_j \,.$$

3.2 Boundary components of moduli spaces of Abelian differentials

In Sect. 2 we introduced several families of stable graphs to describe the boundary of Hurwitz spaces. Here we show how these graphs encode relevant parts of the boundary of moduli spaces of Abelian differentials.

The recursions in Theorems 1.2 and 3.1 can be phrased as sums over a small subset of twisted level graphs, with only two levels and more constraints, that we call (rational) backbone graphs, inspired by Fig. 2.

Recall that a bi-colored graph is a level graph with two levels $\{0, -1\}$ that has no horizontal edges.

Definition 3.3 An *almost backbone graph* is a bi-colored graph with only one vertex at level -1. For such a graph to be a *(rational) backbone graph* we require moreover that it is of compact type and that the vertex at level -1 has genus zero.

We denote by $\mathrm{BB}(g, n) \subset \mathrm{ABB}(g, n) \subset \mathrm{Bic}(g, n)$ the sets of backbone, almost backbone and bi-colored graphs. We denote by $\mathrm{BB}(g, n)_{1,2} \subset \mathrm{BB}(g, n)$ the set of backbone graphs such that the first and second legs are adjacent to the vertex of level -1. Moreover, let $\mathrm{BB}(g, n)^{\star}_{1,2} \subset \mathrm{BB}(g, n)_{1,2}$ be the subset where precisely the first two legs are adjacent to the lower level vertex. Similarly, we define $\mathrm{ABB}(g, n)_{1,2}$ and $\mathrm{ABB}(g, n)^{\star}_{1,2}$ and drop (g, n) if there is no source of confusion.

We remark that the backbone graphs will play an important role here, while the graphs in $\mathrm{ABB}(g, n)$ appear only in the Hurwitz space interlude in Sect. 9. We fix some notations for these graphs, used throughout in the sequel. For $\Gamma \in \mathrm{BB}(g, n)$ we denote by v_{-1} the vertex of level -1. Given a partition $\mu = (m_1, \ldots, m_n)$ of $2g - 2$, let $\mathbf{p}$ be the unique twist of type $(\mu, \emptyset)$ for Γ (see

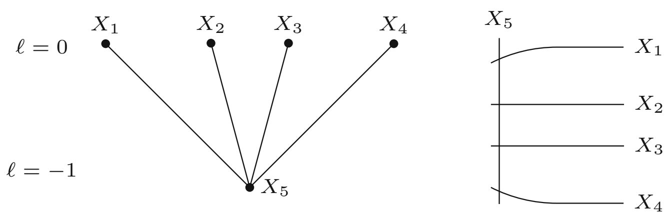


Fig. 2 A backbone graph and the corresponding stable curve

Definition 2.3). We denote by $\mu[0]_{-1}$ the list of m_i for all legs i at level -1 and with a slight abuse of notation we denote by $\mathbf{p} = (p_1, \ldots, p_k)$ the list of $\mathbf{p}(h)$ for half-edges h that are adjacent to the k vertices of level 0. Said differently, the restriction of the twist to level -1 provides v_{-1} with a twist of type $(\mu[0]_{-1}, \mu[\infty]_{-1} = \mathbf{p})$. Finally if v is a vertex of level 0, we denote by μ_v the list of $\mathbf{p}(h) - 1$ for all half-edges adjacent to v.

The goal in the remainder of the section is to introduce the classes $\alpha_{\Gamma,\ell,p}$ in (19) below that will be used in Proposition 3.11 to compute intersection numbers on $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)$. A stable graph $\Gamma \in \mathrm{Stab}(g, n)$ determines the moduli space

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$$

and comes with a natural morphism $\zeta_\Gamma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$. Let ℓ be a level function on Γ such that (Γ, ℓ) is a bi-colored graph with two levels $\{0, -1\}$. We define the following vector bundle

$$\overline{\Omega\mathcal{M}}_{\Gamma,\ell} = \left(\prod_{v \in V(\Gamma), \ell(v)=0} \overline{\Omega\mathcal{M}}_{g(v),n(v)} \right) \times \left(\prod_{v \in V(\Gamma), \ell(v)=-1} \overline{\mathcal{M}}_{g(v),n(v)} \right)$$

over $\overline{\mathcal{M}}_\Gamma$. This space comes with a natural morphism $\zeta^\#_{\Gamma,\ell} : \overline{\Omega\mathcal{M}}_{\Gamma,\ell} \to \overline{\Omega\mathcal{M}}_{g,n}$, defined by the composition $\overline{\Omega\mathcal{M}}_{\Gamma,\ell} \to \zeta^*_\Gamma \left(\overline{\Omega\mathcal{M}}_{g,n}\right) \to \overline{\Omega\mathcal{M}}_{g,n}$ where the first arrow is the inclusion of a vector sub-bundle and the second is the map on the Hodge bundles induced from ζ_Γ by pull-back. The morphism $\zeta^\#_{\Gamma,\ell}$ determines a morphism (denoted by the same symbol) on the projectivized Hodge bundles $\zeta^\#_{\Gamma,\ell} : \mathbb{P}\overline{\Omega\mathcal{M}}_{\Gamma,\ell} \to \mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}$. The image of $\zeta^\#_{\Gamma,\ell}$ is the closure of the locus of differentials supported on curves with dual graph Γ such that the differentials vanish identically on components of level -1. In the sequel we will need the following lemma (see [43, Proposition 5.9]).

Lemma 3.4 *The Poincaré-dual class of $\zeta^\#_{\Gamma,\ell}(\mathbb{P}\overline{\Omega\mathcal{M}}_{\Gamma,\ell})$ is divisible by $\xi^{h^1(\Gamma)}$ in $H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q})$.*

Take a bi-colored graph (Γ, ℓ) and a partition μ of $2g-2$. Now we consider a twist $\mathbf{p}$ of type $(\mu[0] = \mu, \mu[\infty] = \emptyset)$ compatible with ℓ and construct a subspace $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell} \subset \mathbb{P}\overline{\Omega\mathcal{M}}_{\Gamma,\ell}$ such that $\zeta^\#_{\Gamma,\ell}(\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell})$ lies in the boundary of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$. Let

$$\Omega\mathcal{M}_0 \subset \prod_{v \in V(\Gamma), \ell(v)=0} \Omega\mathcal{M}_{g(v),n(v)},$$

$$\mathcal{M}_{-1} \subset \prod_{v \in V(\Gamma), \ell(v)=-1} \mathcal{M}_{g(v),n(v)} \tag{18}$$

be the loci defined by the following three conditions:

(i) A differential in $\Omega\mathcal{M}_0$ has zeros of orders m_i at the relevant marked points and of orders $\mathbf{p}(h) - 1$ at the relevant branches of the nodes.
(ii) For each v of level -1 there exists a non-zero (meromorphic) differential ω_v on the component X_v corresponding to v that has zeros at the relevant marked points of orders prescribed by μ and poles at the relevant branches of the nodes of orders prescribed by $\mathbf{p}$, i.e. such that the canonical divisor class of X_v is given by $\sum_{h \in H, a(h)=v} (\mathbf{p}(h) - 1)x_h$, where $x_h \in X_v$ is the marked point or the node corresponding to the half-edge h.
(iii) There exist complex numbers $k_v \neq 0$ for all vertices v of level -1 such that $\omega = \sum_{\ell(v)=-1} k_v\omega_v$ satisfies the global residue condition of [6].

In particular for a backbone graph Γ, since it is of compact type with a unique vertex v_{-1} of level -1, we have the identification $\Omega\mathcal{M}_0 = \prod_{v \neq v_{-1}} \Omega\mathcal{M}_{g(v),n(v)}(\mu_v)$.

We define $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}$ as the Zariski closure of $\mathbb{P}\Omega\mathcal{M}_0 \times \mathcal{M}_{-1}$ in $\mathbb{P}\overline{\Omega\mathcal{M}}_{\Gamma,\ell}$ and define

$$\alpha_{\Gamma,\ell,\mathbf{p}} = \begin{cases} \zeta^{\#}_{\Gamma,\ell*}[\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}] & \text{if } \dim(\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}) = \dim(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)) - 1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

as the corresponding class in $H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q})$. By [43, Proposition 5.9], we can describe $\alpha_{\Gamma,\ell,\mathbf{p}}$ with the following lemma.

Lemma 3.5 *If* $(\Gamma, \ell, \mathbf{p})$ *is a bi-colored graph of compact type, then* $\alpha_{\Gamma,\ell,\mathbf{p}} \neq 0$ *if and only if there is a unique vertex* v_{-1} *of level* -1*, and in this case* $\alpha_{\Gamma,\ell,\mathbf{p}}$ *is divisible by*

$$\xi^{g(v_{-1})} + \zeta_{\Gamma*}(\lambda_{v_{-1},1})\xi^{g(v_{-1})-1} + \cdots + \zeta_{\Gamma*}(\lambda_{v_{-1},g(v_{-1})}),$$

where

$$\lambda_{v_{-1},i} = (\lambda_i, 1, \ldots, 1) \in H^*(\overline{\mathcal{M}}_{g(v_{-1}),n(v_{-1})}, \mathbb{Q}) \bigotimes_{v \in V(\Gamma), v \neq v_{-1}} H^*(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{Q})$$
$$\simeq H^*(\overline{\mathcal{M}}_\Gamma, \mathbb{Q}).$$

3.3 A first reduction of the computation

Recall the (marked and projectivized) Hodge bundle projection $p\colon \mathbb{P}\overline{\Omega\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. As before we usually denote by the same symbol a class in $\overline{\mathcal{M}}_{g,n}$ and its pullback via p. In this section we show that many p-push forwards of intersections of $\alpha_{\Gamma,\ell,p}$ with tautological classes vanish or can be computed recursively. The starting point is the following important lemma proved by Mumford in [40, Equation (5.4)].

Lemma 3.6 *The Segre class of the Hodge bundle is the Chern class of the dual of the Hodge bundle, i.e.*

$$c_*(\overline{\Omega\mathcal{M}}_{g,n}) \cdot c_*(\overline{\Omega\mathcal{M}}_{g,n}^\vee) = 1\,.$$

In particular, we have $\lambda_g^2 = 0 \in H^{4g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Together with the definition of Segre class, this lemma implies that

$$p_*(\xi^k\gamma) = s_{k-g+1}(\overline{\Omega\mathcal{M}}_{g,n})\gamma = (-1)^{k-g+1}\lambda_{k-g+1}\gamma$$

for all $\gamma \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and all $k \geq g-1$. Another important lemma is the following (see e.g. [1, Section 13, Equation (4.31)]).

Lemma 3.7 *Let $1 \leq k \leq g$ and let Γ be a stable graph. Then*

$$\zeta_\Gamma^*\lambda_k = \sum_{\substack{(k_v)_{v\in V}\in\mathbb{N}^V,\\ |(k_v)|=k}} \prod_{v\in V} \lambda_{k_v},$$

where the sum is over all partitions of k into non-negative integers k_v assigned to each vertex $v \in V = V(\Gamma)$.

In particular if $h^1(\Gamma) > g-k$, then $\sum_{v\in V} g(v) = g - h^1(\Gamma) < k$, hence the above lemma implies that $\zeta_\Gamma^*\lambda_k = 0$ as there exists some $k_v > g(v)$ for any partition $(k_v)_{v\in V}$ of k.

As a consequence of the above discussion, we obtain the following result.

Lemma 3.8 *Let $\alpha = \sum_{i\geq 0}\xi^i\alpha_i$ be a class in $H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q})$ where the classes α_i are pull-backs from $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Then we have*

$$\begin{aligned} p_*(\xi^{2g-1}\alpha) &= (-1)^g\alpha_0\lambda_g, \\ p_*(\xi^{2g-2}\alpha) &= (-1)^g\alpha_1\lambda_g + (-1)^{g-1}\alpha_0\lambda_{g-1}\,, \\ p_*(\xi^{2g-2}\delta_0\alpha) &= (-1)^{g-1}\alpha_0\delta_0\lambda_{g-1}\,. \end{aligned}$$

Recall the expressions of the intersection numbers $a_i(\mu)$ in (12) and in Proposition 3.2. In order to compute $a_i(\mu)$, by Lemma 3.8 we only need to consider the ξ-degree zero and one parts of the class $[\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)]$ in $H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}, \mathbb{Q})$.

Combining Lemmas 3.7 and 3.8 together with Lemmas 3.4 and 3.5 of the previous section, we can already prove the following vanishing result for classes associated with some bi-colored graphs.

Proposition 3.9 *If* $(\Gamma, \ell, \mathbf{p})$ *is not a backbone graph, then*

$$p_*\left(\xi^{2g-2}\alpha_{\Gamma,\ell,\mathbf{p}}\right) = 0 \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

where $\alpha_{\Gamma,\ell,\mathbf{p}}$ *is defined in* (19).

Proof For simplicity we write $\alpha = \alpha_{\Gamma,\ell,\mathbf{p}}$ in the proof. We assume first that Γ is not of compact type, i.e. $h^1(\Gamma) > 0$. Then by Lemma 3.4, the class α is divisible by ξ. Note that the cohomology ring of a projective bundle is generated by the universal line bundle class with the classes pulled back from the base. Therefore, we can write

$$\xi^{2g-2}\alpha = \sum_{i \geq 0} \xi^{2g-1+i}\alpha'_i,$$

where α'_i is a pullback from $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ that is supported on $\zeta_\Gamma(\overline{\mathcal{M}}_\Gamma)$ for all $i \geq 0$. Thus by Lemma 3.8, $p_*(\xi^{2g-2}\alpha) = (-1)^g\alpha'_0\lambda_g = 0$, because Γ is not of compact type.

Now we assume that Γ is of compact type. By Lemma 3.5, we only need to consider the case when there is a unique vertex v_1 of level -1. Since Γ is not a backbone graph, v_1 has positive genus g_1. Still by Lemma 3.5 and simplifying the notation $\zeta_{\Gamma*}(\lambda_{v_1,i})$ by $\lambda_{v_1,i}$, the class α is divisible by $\xi^{g_1} + \xi^{g_1-1}\lambda_{v_1,1} + \cdots + \lambda_{v_1,g_1}$. Consequently we can write

$$\alpha = (\xi\lambda_{v_1,g_1-1} + \lambda_{v_1,g_1})\gamma_0 + \xi\lambda_{v_1,g_1}\gamma_1 \; + \; O(\xi^2),$$

where γ_0 and γ_1 are pullbacks from $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and the $O(\xi^2)$ term stands for a class divisible by ξ^2. By Lemma 3.8, we obtain that

$$p_*(\xi^{2g-2}\alpha) = (-1)^g(\lambda_{v_1,g_1-1}\lambda_g - \lambda_{v_1,g_1}\lambda_{g-1})\gamma_0 + (-1)^g\lambda_{v_1,g_1}\lambda_g\gamma_1 \,.$$

Using Lemma 3.7, we also obtain that

$$\zeta_\Gamma^*(\lambda_g) = \bigotimes_{v\in V}\lambda_{g_v} \quad \text{and} \quad \zeta_\Gamma^*(\lambda_{g-1}) = \sum_{v\in V}\Big(\lambda_{g_v-1}\bigotimes_{v'\neq v}\lambda_{g_{v'}}\Big).$$

From the projection formula we deduce that

$$\lambda_{v_1,g_1} \cdot \lambda_g = \zeta_{\Gamma *}(\lambda_{v_1,g_1} \cdot \zeta_\Gamma^*(\lambda_g)) = \zeta_{\Gamma_*}\Big(\lambda_{g_1}^2 \bigotimes_{v' \neq v_1} \lambda_{g_{v'}}\Big) = 0,$$

because $\lambda_{g_1}^2 = 0 \in H^*(\overline{\mathcal{M}}_{g(v_1),n(v_1)}, \mathbb{Q})$ by Lemma 3.6. Once again the same lemma implies that

$$\begin{aligned}
\lambda_{v_1,g_1-1} \cdot \lambda_g &= \zeta_{\Gamma_*}\Big(\lambda_{g_1}\lambda_{g_1-1} \bigotimes_{v' \neq v_1} \lambda_{g_{v'}}\Big), \\
\lambda_{v_1,g_1} \cdot \lambda_{g-1} &= \zeta_{\Gamma_*}\Big(\lambda_{g_1}\lambda_{g_1-1} \bigotimes_{v' \neq v_1} \lambda_{g_{v'}}\Big) + \sum_{v \neq v_1} \sum_{v' \neq v, v_1} \Big(\lambda_{g_1}^2 \otimes \lambda_{g_v - 1} \bigotimes_{v' \neq v} \lambda_{g_{v'}}\Big) \\
&= \lambda_{v_1,g_1-1} \cdot \lambda_g \,.
\end{aligned}$$

Putting everything together, we thus conclude that $p_*(\xi^{2g-2}\alpha) = 0$. □

We define the *multiplicity* of a twist $\mathbf{p}$ to be

$$m(\mathbf{p}) = \prod_{(h,h') \in E(\Gamma)} \sqrt{-\mathbf{p}(h)\mathbf{p}(h')}\,. \tag{20}$$

Proposition 3.10 *If* $(\Gamma, \ell, \mathbf{p})$ *is a backbone graph in* $\mathrm{BB}(g,n)_{1,2}$*, then*

$$\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}} \alpha_{\Gamma,\ell,\mathbf{p}} \cdot \xi^{2g-1} \cdot \prod_{i=3}^{n} \psi_i = m(\mathbf{p}) \cdot h_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p}) \cdot \prod_{\substack{v \in V(\Gamma) \\ \ell(v)=0}} a_1(p_v - 1, \mu_v),$$

where p_v *is the entry of* $\mathbf{p}$ *corresponding to the twist on the unique edge of each vertex* v *of level* 0 *and* μ_{-1} *is the list of entries in* μ *whose corresponding legs are adjacent to the vertex of level* -1.

As a preparation for the proof we relate the space $\mathcal{M}_{-1}$ defined in (18) to the Hurwitz space for backbone graphs. The idea behind this relation was already mentioned in the last paragraph of Sect. 2.2. If (Γ, ℓ) is a backbone graph, then we claim that $H_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p}) \cong \mathcal{M}_{-1}$, where the isomorphism is provided by the source map f_S that marks the critical points of the branched covers. To verify the claim, let ω be the meromorphic differential on the unique vertex of Γ of level -1 as in part iii) of the definition for $\mathcal{M}_{-1}$. Since Γ is of compact type, the global residue condition in [6] imposed to ω implies that all residues of ω vanish. Therefore, a point in $\mathcal{M}_{-1}$ can be identified with such a meromorphic differential ω (up to scale) on $\mathbb{P}^1$ without residues, such that ω has zeros of order m_i for $m_i \in \mu_{-1}$ at the corresponding markings and has poles of order $p_j + 1$

for $p_j \in \mathbf{p}$ at the corresponding nodes. In particular, ω is an exact differential and integrating it on $\mathbb{P}^1$ provides a meromorphic function that can be regarded as a branched cover f parameterized in $H_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p})$. Conversely given f in $H_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p})$, we can treat f as a meromorphic function and taking df gives rise to such ω. We thus conclude that $H_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p}) \cong \mathcal{M}_{-1}$. Consequently for $\Gamma \in \mathrm{BB}(g, n)_{1,2}$, we have

$$h_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p}) = \int_{\overline{H}_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p})} f_S^*\Bigg(\prod_{\substack{3 \le i \le n \\ i \mapsto v_{-1}}} \psi_i\Bigg) = \int_{\overline{\mathcal{M}}_{-1}} \prod_{\substack{3 \le i \le n \\ i \mapsto v_{-1}}} \psi_i, \qquad (21)$$

where $i \mapsto v_{-1}$ means that the ith marking belongs to the vertex of level -1.

Now we can proceed with the proof of Proposition 3.10.

Proof We write $\alpha_{\Gamma,\ell,\mathbf{p}} = \sum_{i \ge 0} \alpha^i_{\Gamma,\ell,\mathbf{p}} \xi^i$ where $\alpha^i_{\Gamma,\ell,\mathbf{p}}$ is a pull-back from $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. By Lemma 3.8 we deduce that

$$p_*(\xi^{2g-1} \alpha_{\Gamma,\ell,p}) = (-1)^g \lambda_g \alpha^0_{\Gamma,\ell,\mathbf{p}}\,. \qquad (22)$$

Therefore we only need to consider the ξ-degree zero part of $\alpha_{\Gamma,\ell,\mathbf{p}}$, which is given by

$$\zeta_{\Gamma *}\left([\overline{\mathcal{M}}_{-1}] \bigotimes_{v \in V(\Gamma), \ell(v)=0} [\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v - 1, \mu_v)]^0 \right),$$

where $[\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v - 1, \mu_v)]^0$ is the degree zero part of the Poincaré-dual class of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v - 1, \mu_v)$ in $\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}$. Therefore, we have

$$\lambda_g \cdot \alpha^0_{\Gamma,\ell,\mathbf{p}} = \zeta_{\Gamma *}\left([\overline{\mathcal{M}}_{-1}] \bigotimes_{v \in V(\Gamma), \ell(v)=0} \left(\lambda_{g_v} \cdot [\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v - 1, \mu_v)]^0\right) \right).$$

Multiplying this expression by $\prod_{i=3}^n \psi_i$, we obtain that

$$\lambda_g \cdot \alpha^0_{\Gamma,\ell,\mathbf{p}} \cdot \prod_{i=3}^n \psi_i = \zeta_{\Gamma *}\Bigg(\Bigg([\overline{\mathcal{M}}_{-1}] \cdot \prod_{i \mapsto v_{-1}, i \ge 3} \psi_i \Bigg) \bigotimes_{v \in V(\Gamma), \ell(v)=0} \Bigg(\lambda_{g_v} \cdot [\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v - 1, \mu_v)]^0 \cdot \prod_{i \mapsto v} \psi_i \Bigg) \Bigg).$$

For the first factor on the right-hand side, equality (21) implies that

$$[\overline{\mathcal{M}}_{-1}] \cdot \prod_{i\mapsto v_{-1}, i\geq 3} \psi_i = h_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p})\,.$$

Moreover for all v of level 0, we have

$$\begin{aligned} p_v\, a_1(p_v-1,\mu_v) &= \int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g_v,n_v}(p_v-1,\mu_v)} \xi^{2g_v-1} \prod_{i\mapsto v} \psi_i \\ &= (-1)^{g_v} \int_{\overline{\mathcal{M}}_{g_v,n_v}} \lambda_{g_v} \cdot \prod_{i\mapsto v} \psi_i \cdot [\mathbb{P}\Omega\overline{\mathcal{M}}_{g_v,n_v}(p_v-1,\mu_v)]^0 \end{aligned}$$

where the second identity follows from Lemma 3.8. Since p_v is the (positive) twist value assigned to the edge of v, the product of p_v over all vertices of level 0 equals $m(\mathbf{p})$ defined in (20). In addition, the sum of g_v over all vertices of level 0 equals the total genus g, because Γ is of compact type and v_{-1} has genus zero. Putting everything together we thus obtain that

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \cdot \alpha^0_{\Gamma,\ell,\mathbf{p}} \cdot \prod_{i=3}^{n} \psi_i = m(\mathbf{p}) \cdot h_{\mathbb{P}^1}(\mu_{-1}, \mathbf{p}) \cdot \prod_{v\in V(\Gamma), \ell(v)=0} a_1(p_v-1, \mu_v),$$

which is the desired statement. □

3.4 The induction formula for cohomology classes

The main tool of the section is the induction formula in [43, Theorem 6 (1)] which we recall now.

Proposition 3.11 *For all $1 \leq i \leq n$, the relation that*

$$(\xi + (m_i+1)\psi_i)\,[\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)] = \sum_{\substack{(\Gamma,\ell,p)\\ i\mapsto v, \ell(v)=-1}} \frac{m(\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|} \alpha_{\Gamma,\ell,\mathbf{p}} \tag{23}$$

holds in $H^*(\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, where the sum is over all twisted bi-colored graphs such that the ith leg is carried by a vertex of level -1.

There are two ways of using Eq. (23). First one can compute the Poincaré-dual class of $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)$ in $H^*(\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ in terms of the ψ, λ, ξ classes and boundary classes associated to stable graphs. This strategy is used in [42] to deduce the first formula in Theorem 3.1.

Alternatively, one can compute relations in the Picard group of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ to deduce relations between intersection numbers on $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ and intersection numbers on boundary strata associated to twisted graphs. This is the strategy that we will use here. We will use this proposition with $i \in \{1,2\}$ and multiply the formula by $\xi^{2g-1}\prod_{i=3}^{n}\psi_i$ to obtain $a_1(\mu)$ on the left-hand side. Then we will use Propositions 3.9 and 3.10 to compute the right-hand side. A first application of this strategy gives a proof of the complementary proposition.

Proof of Proposition 3.2 We use Proposition 3.11 with $i = 1$. Multiplying formula (23) by $\xi^{2g-2}\cdot\prod_{i=2}^{n}\psi_i$, we obtain that

$$(m_1+1)\left(a_1(\mu)+\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)}\xi^{2g-2}\cdot\prod_{i=1}^{n}\psi_i\right)$$
$$=\sum_{\substack{(\Gamma,\ell,\mathbf{p})\\ 1\mapsto v,\ell(v)=-1}}\frac{m(\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}}\alpha_{\Gamma,\ell,\mathbf{p}}\cdot\xi^{2g-2}\cdot\prod_{i=2}^{n}\psi_i\,.$$

It suffices to check that each summand in the right-hand side vanishes. Proposition 3.9 implies that if $(\Gamma,\ell,\mathbf{p})$ is not a backbone graph, then the corresponding summand vanishes. If $(\Gamma,\ell,\mathbf{p})$ is a backbone graph, then we have seen (in the paragraph below Proposition 3.10) that $\overline{\mathcal{M}}_{-1}$ is birational to a Hurwitz space of admissible covers of dimension $n_{-1}-2$, where n_{-1} is the number of legs adjacent to the vertex of level -1. Since the ψ-product restricted to level -1 contains $n_{-1}-1$ terms (i.e. it misses ψ_1 only), which is bigger than $\dim\overline{\mathcal{M}}_{-1}$, it implies that the intersection of $\alpha_{\Gamma,\ell,\mathbf{p}}$ with $\prod_{i=2}^{n}\psi_i$ vanishes. □

Now we know that $a_i(\mu)$ is independent of the choice of $1\le i\le n$ and hence we can drop the subscript i. The second use of the strategy presented above leads to the following induction formula.

Lemma 3.12 *The intersection numbers $a(\mu)$ satisfy the recursion*

$$(m_1+1)(m_2+1)a(\mu)=\sum_{k\ge 1}\sum_{\mathbf{g},\boldsymbol{\mu}}h_{\mathbb{P}^1}((m_1,m_2,\mu_0),\mathbf{p})\cdot\frac{1}{k!}\cdot\prod_{i=1}^{k}p_i^2a(p_i-1,\mu_i)$$

where $\mathbf{g}=(g_1,\ldots,g_k)$ is a partition of g, $\boldsymbol{\mu}=(\mu_0,\mu_1,\ldots,\mu_k)$ is a $(k+1)$-tuple of multisets with $(m_3,\ldots,m_n)=\mu_0\sqcup\cdots\sqcup\mu_k$ and $\mathbf{p}=(p_1,\ldots,p_k)$ has entries $p_i=2g_i-1-|\mu_i|>0$.

We remark that this induction formula is not quite the same as the induction formula of Theorem 3.1, e.g. the sums in the two formulas do not run over the same set. Theorem 3.1 will follow further from a combination of Lemma 3.12 and Proposition 2.1 of the previous section.

Proof We apply the induction formula of Proposition 3.11 with $i = 2$:

$$(\xi + (m_2+1)\psi_2)\,[\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)] = \sum_{\substack{(\Gamma,\ell,\mathbf{p})\\ 2\mapsto v,\ell(v)=-1}} \frac{m(\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\alpha_{\Gamma,\ell,\mathbf{p}}\,.$$

We multiply this expression by $\xi^{2g-1}\prod_{i=3}^n \psi_i$ and apply p_*. Since Lemma 3.6 gives $p_*(\xi^{2g}[\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)]) = 0$, the above equality implies that

$$(m_1+1)(m_2+1)a(\mu) = \sum_{\substack{(\Gamma,\ell,\mathbf{p})\\ 2\mapsto v,\ell(v)=-1}} \frac{m(\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|} p_*\left(\xi^{2g-1}\cdot\prod_{i=3}^n \psi_i\cdot\alpha_{\Gamma,\ell,\mathbf{p}}\right).$$

By Proposition 3.9 a term in the sum of the right-hand side vanishes if (Γ,ℓ,p) is not a backbone graph. Suppose $(\Gamma,\ell,\mathbf{p})$ is a backbone graph such that the first leg does not belong to the vertex of level -1 (which contains n_{-1} legs). Then on level -1 the product of ψ-classes contains $n_{-1}-1$ terms (i.e. this product misses ψ_2 only), which exceeds the dimension of $\overline{\mathcal{M}}_{-1}$ (being $n_{-1}-2$), hence the corresponding term in the sum also vanishes.

Now we only need to consider the case when $(\Gamma,\ell,\mathbf{p})$ is a backbone graph in $\mathrm{BB}(g,n)_{1,2}$, i.e. the vertex of level -1 carries both the first and second legs. Then the intersection number $\xi^{2g-1}\cdot\prod_{i=3}^n \psi_i\cdot\alpha_{\Gamma,\ell,\mathbf{p}}$ is given by Proposition 3.10. We thus conclude that

$$\begin{aligned}
&(m_1+1)(m_2+1)a(\mu)\\
&\quad= \sum_{(\Gamma,\ell,\mathbf{p})\in \mathrm{BB}_{1,2}} \frac{h_{\mathbb{P}^1}(\mu_{-1},\mathbf{p})\,m(\mathbf{p})^2}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\cdot \prod_{v\in V(\Gamma),\ell(v)=0} a(p_v-1,\mu_v)\\
&\quad= \sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} m(\mathbf{p})^2 h_{\mathbb{P}^1}((m_1,m_2,\mu_0),\mathbf{p})\cdot\frac{1}{k!}\cdot\prod_{i=1}^k a(p_i-1,\mu_i).
\end{aligned}$$

The last equality comes from the fact that $g_1+\cdots+g_k = g$ where $g_i\geq 1$ and $(m_3,\ldots,m_n) = \mu_0\sqcup\mu_1\sqcup\cdots\sqcup\mu_k$ determines uniquely a graph $(\Gamma,\ell,\mathbf{p})$ in $\mathrm{BB}(g,n)_{1,2}$ and an automorphism of the backbone graph is determined by a permutation in S_k that preserves both the partition of g and the sets $\mu_1,\ldots,\mu_k$. □

3.5 Sums over rooted trees

The purpose of this section is to combine the preceding Lemma 3.12 with Proposition 2.1 that describes the computation of intersection numbers on Hurwitz spaces. We will show that the numbers $a(\mu)$ can be expressed as sums over rooted trees in a similar way as we did for intersection numbers on Hurwitz spaces in Sect. 2.4.

Let $2 \leq i \leq n$ and $(\Gamma, \ell, \mathbf{p})$ be a rooted tree in $\mathrm{RT}(g, \mu)_{1,i}$ (here $\mu[\infty]$ is empty). Since there is no marked pole, it implies that any vertex of genus zero has at least one edge with a negative twist, hence it is an internal vertex of Γ and lies on a negative level. Denote by $\mu[\infty]_0$ the list obtained by taking the (positive) entries $\mathbf{p}(h)$ for all half-edges h adjacent to a vertex of level 0. Denote by $\mu[0]_0$ the list of entries of μ from those legs carried by the internal vertices (of genus zero). With this notation we define the rooted tree $(\Gamma_0, \ell_0, \mathbf{p}_0)$ in $\mathrm{RT}(0, \mu[0]_0, \mu[\infty]_0)_{1,i}$ obtained by removing the leaves of Γ (i.e. vertices of positive genus and hence on level 0). We also define the multiplicity $m_0(\mathbf{p})$ of $(\Gamma_0, \ell_0, \mathbf{p}_0)$ to be the product of entries of $\mu[\infty]_0$. Now we define the a-contribution of the rooted tree $(\Gamma, \ell, \mathbf{p})$ as

$$a(\Gamma, \ell, \mathbf{p}) = m_0(\mathbf{p})^2 h(\Gamma_0, \ell_0, \mathbf{p}_0) \prod_{v \in V(\Gamma), \ell(v)=0} a(p_v - 1, \mu_v), \tag{24}$$

where $h(\Gamma_0, \ell_0, \mathbf{p}_0)$ is the contribution of the rooted tree defined in (10).

Lemma 3.13 *The following equality holds:*

$$(m_1 + 1)(m_2 + 1)a(\mu) = \sum_{(\Gamma, \ell, \mathbf{p}) \in \mathrm{RT}(g, \mu)_{1,2}} \frac{a(\Gamma, \ell, \mathbf{p})}{|\mathrm{Aut}(\Gamma, \ell, \mathbf{p})|}.$$

Proof Removing the leaves of a rooted tree induces a bijection between $\mathrm{RT}(g, \mu)_{1,2}$ and the set

$$\bigcup_{(\Gamma, \ell, \mathbf{p}) \in \mathrm{BB}_{1,2}} \mathrm{RT}(0, \mu[0]_0, \mu[\infty]_0)_{1,2}$$

which is a partition of $\mathrm{RT}(g, \mu)_{1,2}$ over all possible decorations of the leaves of the rooted trees (i.e. each decoration is induced by a graph in $\mathrm{BB}_{1,2}$). Moreover, an automorphism of a rooted tree in $\mathrm{RT}(g, \mu)_{1,2}$ is determined by an automorphism of the backbone graph in $\mathrm{BB}(g, n)_{1,2}$, because all internal vertices of the rooted tree (i.e. those of genus zero and hence on negative levels) have marked legs by Definition 2.4. Then we can first use Lemma 3.12 to write $a(\mu)$ as a sum over backbone graphs in $\mathrm{BB}(g, n)_{1,2}$ and then use Proposition 2.1 to express it as the desired sum over the set

$$\bigcup_{(\Gamma,\ell,\mathbf{p})\in \mathrm{BB}_{1,2}} \mathrm{RT}(0,\mu[0]_0,\mu[\infty]_0)_{1,2} \simeq \mathrm{RT}(g,\mu)_{1,2}$$

as claimed in the lemma. □

We define

$$\mathrm{RT}(g,\mu)_1 = \{\text{trivial graph}\} \cup \bigcup_{i=2}^{n} \mathrm{RT}(g,\mu)_{1,i}$$

and the a-contribution of the trivial graph • as $a(\bullet,\ell,\mathbf{p}) = (m_1+1)^2\, a(\mu)$.

End of the proof of Theorem 3.1 We will prove for $n \geq 2$ the equality that

$$\sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})\,\frac{1}{k!}\prod_{j=1}^{k}(2g_j-1+n(\mu_j))p_j a(p_j-1,\mu_j)$$
$$= \sum_{(\Gamma,\ell,\mathbf{p})\in \mathrm{RT}(g,\mu)_{1,2}} \frac{a(\Gamma,\ell,\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\,. \tag{25}$$

This formula together with Lemma 3.13 thus implies Theorem 3.1. Since by definition $\sum_{i=1}^{n}(m_i+1) = 2g-2+n$, Lemma 3.13 implies that

$$\begin{aligned}(2g-2+n)(m_1+1)a(\mu) &= (m_1+1)^2 a(\mu) + \sum_{i=2}^{n}(m_i+1)(m_1+1)a(\mu)\\ &= (m_1+1)^2 a(\mu) + \sum_{i=2}^{n}\sum_{\substack{(\Gamma,\ell,\mathbf{p})\in\\ \mathrm{RT}(g,\mu)_{1,i}}} \frac{a(\Gamma,\ell,\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\\ &= \sum_{\substack{(\Gamma,\ell,\mathbf{p})\in\\ \mathrm{RT}(g,\mu)_1}} \frac{a(\Gamma,\ell,\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}\,.\end{aligned}$$

Therefore, the left-hand side of (25) can be rewritten as

$$\sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})\frac{1}{k!}\cdot\prod_{j=1}^{k}\sum_{\substack{(\Gamma,\ell,\mathbf{p})\in\\ \mathrm{RT}(g_j,(p_j-1,\mu_j))_1}} \frac{a(\Gamma,\ell,\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|}$$

$$= \sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})\,\frac{1}{k!}\cdot \sum_{\substack{(\Gamma_j,\ell_j,\mathbf{p}_j)\in\\ \mathrm{RT}(g_j,(p_j-1,\mu_j))_1}} \prod_{j=1}^{k}\frac{a(\Gamma_j,\ell_j,\mathbf{p}_j)}{|\mathrm{Aut}(\Gamma_j,\ell_j,\mathbf{p}_j)|}\,.$$

We claim that there is a bijection

$$\mathrm{RT}(g,\mu)_{1,2}\simeq \bigcup_{(\Gamma',\ell',\mathbf{p}')\in \mathrm{BB}^{\star}_{1,2}}\ \prod_{v\in V(\Gamma'),\ell(v)=0} \mathrm{RT}(g_v,(p_v-1,\mu_v))_1\,.$$

Indeed given a rooted tree $(\Gamma,\ell,\mathbf{p})$ in $\mathrm{RT}(g,\mu)_{1,2}$ we can construct $(\Gamma',\ell',\mathbf{p}')\in \mathrm{BB}(g,n)^{\star}_{1,2}$ by contracting all edges except those adjacent to the root, and the rooted trees $(\Gamma_v,\ell_v,\mathbf{p}_v)\in \mathrm{RT}(g_v,(p_v-1,\mu_v))_1$ are the connected components of the graph obtained from $(\Gamma,\ell,\mathbf{p})$ by deleting the root. Moreover for a rooted tree $(\Gamma,\ell,\mathbf{p})$, by Eq. (10) we have

$$h(\Gamma_0,\ell_0,\mathbf{p}_0)=h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p}')\cdot\prod_{j=1}^{k} h(\Gamma_{j0},\ell_{j0},\mathbf{p}_{j0}),$$

where as before Γ_0 is obtained from Γ by removing the leaves and the Γ_{j0} are the connected components after removing the root of Γ_0. Together with the definition of the a-contribution in (24), it implies that

$$a(\Gamma,\ell,\mathbf{p})=h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p}')\cdot \prod_{v\in V(\Gamma'),\ell(v)=0} a(\Gamma_v,\ell_v,\mathbf{p}_v)\,.$$

Note also that

$$\mathrm{Aut}(\Gamma,\ell,\mathbf{p})=\mathrm{Aut}(\Gamma',\ell',\mathbf{p}')\times \prod_{v\in V(\Gamma'),\ell(v)=0} \mathrm{Aut}(\Gamma_v,\ell_v,\mathbf{p}_v)\,.$$

Combining the above we thus conclude that equality (25) holds. □

Proof of the equivalence of Theorems 1.1 and 1.2 We first assume that Theorem 1.2 holds. By Theorem 3.1, the quantities

$$\mathrm{vol}(\Omega\mathcal{M}_{g,n}(m_1,\ldots,m_n))\quad\text{and}\quad \frac{2(2\pi i)^{2g}}{(2g-3+n)!}\,a(m_1,\ldots,m_n)$$

satisfy the same induction relation that determines both collections of these numbers starting from the case $n=1$. The base case (i.e. the minimal strata) that

$$\mathrm{vol}(\Omega\mathcal{M}_{g,1}(2g-2)) = \frac{2(2\pi i)^{2g}}{(2g-2)!}\, a(2g-2) \tag{26}$$

was proved in [42] under a mild assumption of regularity of a natural Hermitian metric on $\mathcal{O}(-1)$, and we will give an alternative (unconditional) proof in Sect. 4.4. Consequently we conclude that Theorem 1.2 implies Theorem 1.1. The converse implication follows similarly. □

4 Volume recursion via q-brackets

In this section we define recursively polynomials in the ring $R = \mathbb{Q}[h_1, h_2, \ldots]$ and show that they compute volumes of the strata after a suitable specialization. The method of proof relies on lifting the E_2-derivative via the Bloch–Okounkov q-bracket and expressing cumulants in terms of this lift. This recursion looks quite different from the recursion given in Theorem 1.2, since it is only defined on the level of polynomials in the variables h_i and requires h_i-derivatives.

To define the substitution, first recall the numbers b_j introduced in (15). We let

$$P_B(u) = \exp\Big(-\sum_{j\geq 1} j!\, b_{j+1} u^{j+1}\Big) \quad \text{and} \quad \alpha_\ell = \big[u^\ell\big]\frac{1}{(u/P_B(u))^{-1}}, \tag{27}$$

where the denominator denotes the inverse function of $u/P_B(u)$. For the recursion we define for a finite set $I = \{i_1, \ldots, i_n\}$ of positive integers the formal series $\mathcal{H}_I \in R[[z_{i_1}, \ldots, z_{i_n}]]$ if $|I| \geq 2$ and $\mathcal{H}_{\{i\}} \in \frac{1}{z_i} R[[z_i]]$ by

$$\begin{aligned}
\mathcal{H}_{\{i\}} &= \frac{1}{z_i} + \sum_{\ell\geq 1} h_\ell z_i^\ell,\\
\mathcal{H}_{\{i,j\}} &= \frac{z_i\mathcal{H}'(z_i) - z_j\mathcal{H}'(z_j)}{\mathcal{H}(z_j) - \mathcal{H}(z_i)} - 1\\
&= 2h_1 z_i z_j + h_2(3z_i^2 z_j + 3z_i z_j^2) + 4h_3 z_i^3 z_j + (2h_1^2 + 4h_3) z_i^2 z_j^2\\
&\quad + 4h_3 z_i z_j^3 + \cdots,\\
\mathcal{H}_I &= \frac{1}{2(n-1)} \sum_{\substack{I = I' \sqcup I''\\ I', I'' \neq \emptyset}} D_2(\mathcal{H}_{I'}, \mathcal{H}_{I''}),
\end{aligned} \tag{28}$$

where we abbreviate $\mathcal{H}_n = \mathcal{H}_{[\![1,n]\!]}$, $\mathcal{H} = \mathcal{H}_1$ and $h_{\ell_1,\ldots,\ell_n} = [z_1^{\ell_1} \ldots z_n^{\ell_n}]\mathcal{H}_n$ and where the symmetric bi-differential operator D_2 is defined by

$$D_2(f, g) = \sum_{\ell_1,\ell_2\geq 1} h_{\ell_1,\ell_2} \frac{\partial f}{\partial h_{\ell_1}} \frac{\partial g}{\partial h_{\ell_2}} . \tag{29}$$

Theorem 4.1 *The rescaled volume of the stratum with signature $\mu = (m_1, \ldots, m_n)$ can be computed as*

$$v(\mu) = \frac{(2\pi i)^{2g}}{(2g-2+n)!} h_{m_1+1,\ldots,m_n+1}\big|_{h_\ell \mapsto \alpha_\ell}$$

using the recursion (28) *and the values of the α_ℓ in* (27).

4.1 Three sets of generators for the algebra of shifted symmetric functions

We let Λ^* be the algebra of shifted symmetric functions (see e.g. [21,49] or [12]) and recall the standard generators

$$p_\ell(\lambda) = \sum_{i=1}^{\infty} \Big((\lambda_i - i + \tfrac{1}{2})^\ell - (-i + \tfrac{1}{2})^\ell \Big) + (1 - 2^{-\ell})\, \zeta(-\ell) . \tag{30}$$

Note that $(1-2^{-\ell})\,\zeta(-\ell) = \ell! b_{\ell+1}$. The algebra Λ^* is provided with a grading where each p_ℓ has weight $\ell + 1$. For Hurwitz numbers the geometrically interesting generators are

$$f_\ell(\lambda) = z_\ell \chi^\lambda(\ell)/\dim \chi^\lambda, \tag{31}$$

where z_ℓ is the size of the conjugacy class of the cycle of length ℓ, completed by singletons. The first few of these functions are

$$f_1 = p_1 + \frac{1}{24}, \quad f_2 = \frac{1}{2}p_2, \quad f_3 = \frac{1}{3}p_3 - \frac{1}{2}p_1^2 + \frac{3}{8}p_1 + \frac{9}{640} .$$

The third set of generators, defined implicitly by Eskin and Okounkov, will serve as top term approximations of f_ℓ. We define $h_\ell \in \Lambda^*$ by

$$h_\ell = \frac{-1}{\ell}[u^{\ell+1}]P(u)^\ell \quad \text{where} \quad P(u) = \exp\Big(-\sum_{s\geq 1} u^{s+1} p_s\Big) . \tag{32}$$

Observe that by definition h_ℓ has pure weight $\ell+1$. The first few of these functions are

$$h_1 = p_1, \quad h_2 = p_2, \quad h_3 = p_3 - \frac{3}{2}p_1^2\,.$$

Proposition 4.2 [21, Theorem 5.5] *The difference $f_\ell - h_\ell/\ell$ has weight strictly less than $\ell+1$.*

We abuse the notation h_ℓ for generators of R and for elements in Λ^*. This is intentional and should not lead to confusion, since the map $h_\ell \mapsto h_\ell$ induces an isomorphism of algebras $R \cong \Lambda^*$, by the preceding proposition.

4.2 The lift of the evaluation map to the Bloch–Okounkov ring

Let $f\colon \mathbf{P} \to \mathbb{Q}$ be an arbitrary function on the set $\mathbf{P}$ of all partitions. Bloch and Okounkov [9] associated to f the formal power series

$$\langle f\rangle_q = \frac{\sum_{\lambda\in\mathbf{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathbf{P}} q^{|\lambda|}} \quad \in \; \mathbb{Q}[[q]], \tag{33}$$

which we call the *q-bracket*, and proved that this q-bracket is a quasimodular form of weight k whenever f belongs to the subspace of Λ^* of weight k (see [9], and [49] or [26] for alternative proofs).

In [12, Section 8] we studied in detail an evaluation map Ev (implicitly defined in [21]) on the ring of quasimodular forms that measures the growth rate of the coefficients of quasimodular forms, or equivalently, their asymptotics as $\tau \to 0$ along the imaginary axis [12, Proposition 9.3]. The purpose of this section is to lift this evaluation map to the Bloch–Okounkov ring and to express it in terms of the generators h_i introduced in the previous section.

The map Ev is the algebra homomorphism from the ring of quasimodular forms $\widetilde{M}_* = \mathbb{Q}[E_2, E_4, E_6]$ to $\mathbb{Q}[X]$, sending the Eisenstein series E_2 (normalized to have constant coefficient one) to $X+12$, E_4 to X^2, and E_6 to X^3. In this way, the larger the degree of $\mathrm{Ev}(f)$, the larger the (polynomial) growth of the coefficients of f, see [12, Proposition 9.4] for the precise statement. It is also convenient to work with the evaluation map[3]

$$\mathrm{ev}[F](\hbar) = \frac{1}{\hbar^{k/2}}\,\mathrm{Ev}[F]\Big|_{X\mapsto\frac{1}{\hbar}} \;\in\; \mathbb{Q}[1/\hbar] \quad \text{for} \quad F\in \widetilde{M}_k\,. \tag{34}$$

3 This would be $(2\pi i)^k \mathrm{ev}[F](-4\pi^2\hbar)$ in the notation of [12, Equation (85)].

We also use the brackets $\langle f\rangle_X := \mathrm{Ev}\big[\langle f\rangle_q\big](X)$ and $\langle f\rangle_\hbar := \mathrm{ev}\big[\langle f\rangle_q\big](\hbar)$ for $f \in \Lambda^*$ as abbreviation. Note that Λ^* admits a natural ring homomorphism to $\mathbb{Q}$, the evaluation at the emptyset, explicitly given by the map $p_\ell \mapsto \ell! b_{\ell+1}$.

Proposition 4.3 *There is a second order differential operator $\Delta\colon \Lambda^* \to \Lambda^*$ of degree -2 and a derivation $\partial\colon \Lambda^* \to \Lambda^*$ of degree -1 such that*

$$\big\langle f\big\rangle_\hbar = \frac{1}{\hbar^k}\big(e^{\hbar(\Delta-\partial^2-\partial/\partial p_1)/2} f\big)\,(\emptyset) \tag{35}$$

for $f \in \Lambda^*_k$ homogeneous of weight k.

The differential operators are given in terms of the generators p_ℓ by

$$\partial(f) = \sum_{i\ge 2} i p_{i-1}\frac{\partial}{\partial p_i} \quad \text{and} \quad \Delta(f) = \sum_{k,\,\ell\ge 1} (k+\ell)\, p_{k+\ell-1}\,\frac{\partial^2}{\partial p_k\,\partial p_\ell}. \tag{36}$$

Proof From the definition and [12, Proposition 9.2] we deduce that the evaluation map can be computed for any $F \in \widetilde{M}_k$ as

$$\mathrm{ev}[F](\hbar) = \frac{1}{\hbar^k} a_0\big(e^{\hbar\mathfrak{d}} F\big) \tag{37}$$

where $\mathfrak{d} = 12\partial/\partial E_2$ and where $a_0\colon F \mapsto F(\infty)$ is the constant term map from $\widetilde{M}_*$ to $\mathbb{Q}$. From [12, Proposition 8.3] we deduce (note that differentiation with respect to Q_i in loc. cit. gives the extra p_1-derivative here) that the differential operators defined above have the property that

$$\mathfrak{d}\,\langle f\rangle_q = \big\langle \tfrac{1}{2}\big(\Delta - \partial^2 - \partial/\partial p_1\big)\, f\big\rangle_q \qquad (f \in \Lambda^*)\,. \tag{38}$$

Since the constant term of the q-bracket of f is in Λ^*, the claim follows from these two equations. □

To motivate the next section, we recall the notion of cumulants. Let R and R' be two commutative $\mathbb{Q}$-algebras with unit and $\langle\ \rangle\colon R \to R'$ a linear map sending 1 to 1. (Of course the cases of interest to us will be when R is the Bloch–Okounkov ring Λ^* and $\langle\ \rangle$ is the q-, X-, or $\hbar$-bracket to $R' = \widetilde{M}_*$, $\mathbb{Q}[X]$, or $\mathbb{Q}[\pi^2][\hbar]$, respectively.) Then we extend $\langle\ \rangle$ to a multi-linear map $R^{\otimes n} \to R'$ for every $n \ge 1$, called *connected brackets*, the image of $g_1 \otimes \cdots \otimes g_n$ being denoted by $\langle g_1|\cdots|g_n\rangle$, that we define by

$$\langle g_1|\cdots|g_n\rangle = \sum_{\alpha\in\mathcal{P}(n)} (-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)!\,\prod_{A\in\alpha}\Big\langle\prod_{a\in A} g_a\Big\rangle. \tag{39}$$

The most important property of connected brackets is their appearance in the logarithm of the original bracket applied to an exponential:

$$\begin{aligned}\log\bigl(\langle e^{g_1+g_2+g_3+\cdots}\rangle\bigr) &= \log\Bigl(1+\sum_i\langle g_i\rangle+\frac{1}{2!}\sum_{i,j}\langle g_i g_j\rangle+\frac{1}{3!}\sum_{i,j,k}\langle g_i g_j g_k\rangle+\cdots\Bigr)\\ &= \sum_i\langle g_i\rangle+\frac{1}{2!}\sum_{i,j}\langle g_i|g_j\rangle+\frac{1}{3!}\sum_{i,j,k}\langle g_i|g_j|g_k\rangle+\cdots\,.\end{aligned}$$

We specialize to the Bloch–Okounkov ring Λ^* and we want to compute the leading terms of the connected brackets associated with the $\langle\cdot\rangle_X$- or $\langle\cdot\rangle_\hbar$-brackets. Recall from [12, Proposition 11.1]:

Proposition 4.4 *Let $g_i \in \Lambda^*_{\leq k_i}$ ($i = 1, \ldots, n$) be elements of weight less than or equal to k_i and let $g_i^\top \in \Lambda^*_{k_i}$ be their top weight components. Let $k = k_1 + \cdots + k_n$ be the total weight. Then* $\deg(\langle g_1|\cdots|g_n\rangle_X) \leq 1-n+k/2$ *and*

$$[X^{1-n+k/2}]\,\langle g_1|\cdots|g_n\rangle_X = [X^{1-n+k/2}]\,\langle g_1^\top|\cdots|g_n^\top\rangle_X\,. \tag{40}$$

The leading terms of the brackets are consequently

$$\begin{aligned}\langle g_1|\cdots|g_n\rangle_L &= [X^{1-n+k/2}]\,\langle g_1|\cdots|g_n\rangle_X = \lim_{X\to\infty}\frac{\mathrm{Ev}[\langle g_1|\cdots|g_n\rangle_q](X)}{X^{1-n+k/2}}\\ &= [\hbar^{-k-1+n}]\,\langle g_1|\cdots|g_n\rangle_\hbar = \lim_{\hbar\to 0}\hbar^{k+1-n}\,\mathrm{ev}[\langle g_1|\cdots|g_n\rangle_q](\hbar)\,.\end{aligned} \tag{41}$$

We call them *rational cumulants*.

4.3 The cumulant recursion

In this section we prove a formula for computing the connected brackets associated with the $\langle\cdot\rangle_q$- or rather the $\langle\cdot\rangle_\hbar$-brackets. The core mechanism for their computation is summarized in the following purely algebraic property.

Let R be an $\mathbb{N}$-graded commutative $\mathbb{Q}$-algebra with $R_0 = \mathbb{Q}$, complete with respect to the maximal ideal $\mathfrak{m} = R_{>0}$. The following statement gives a general recursion for expressions that appear in cumulants. We will specialize R to the Bloch–Okounkov ring subsequently.

Key Lemma 4.5 *Suppose that $D\colon R \to R$ is a linear map. Then the following statements are equivalent:*

(1) *We have $D(x^3) - 3x\,D(x^2) + 3x^2 D(x) = 0$ for all $x \in R$.*

(2) *For all* $x \in R$ *and all* $n \geq 2$

$$D(x^n) = \binom{n}{2} D(x^2)x^{n-2} - n(n-2)D(x)x^{n-1}.$$

(3) *For all* $x, y, z \in R$

$$D(xyz) = xD(yz) + yD(xz) + zD(xy) - xyD(z) - xzD(y) - yzD(x).$$

(4) *If we denote by* $D_2\colon R^2 \to R$ *the symmetric bilinear form*

$$D_2(x, y) = D(xy) - xD(y) - yD(x)$$

then

$$\begin{aligned} D(x_1 \cdots x_n) &= \sum_{i=1}^{n} D(x_i)x_1 \cdots \widehat{x_i} \cdots x_n \\ &\quad + \sum_{1 \leq i < j \leq n} D_2(x_i, x_j)x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n \end{aligned}$$

for all $x_1, \ldots, x_n \in R$.

(5) *For any fixed* $x \in R$, *the bilinear form* $D_2(x, y)$ *is a derivation in* y.

(6) *The map* $D \in \mathrm{Sym}^2(\mathrm{Der}(R))$, *i.e.* D *is a second order differential operator without constant term.*

(7) *For all* $X \in \mathfrak{m}$ *there exists* $\mathcal{L}(X) \in R$ *such that*

$$\log(e^{\hbar D}(e^{X/\hbar})) = \frac{1}{\hbar}\mathcal{L}(X) + O(1) \quad (\hbar \to 0). \tag{42}$$

If any of these statements holds, the leading term of (42) *is given by* $\mathcal{L}(X) = L(1)$, *where*

$$L(0) = X, \quad L'(t) = \frac{1}{2}D_2(L(t), L(t)). \tag{43}$$

Proof The implication (1) $\Rightarrow$ (3) follows by passing to the polarization. The implication (3) $\Rightarrow$ (4) can be proved by induction (replace x_1 by x_0x_1). The implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1), (2), (3) follow by specialization. The equivalence (5) $\Leftrightarrow$ (3) follows by direct computation. To show (6) $\Leftrightarrow$ (5) think deeply. To prove (7) $\Rightarrow$ (1) it suffices to consider the cubic term: the coefficient of $1/\hbar^2$ is the expression in (1).

To prove (6) $\Rightarrow$ (7) and the final formula for $\mathcal{L}(X)$ we write

$$e^{\hbar D}(e^x) = e^{y(\hbar)}. \tag{44}$$

Then $y(0) = x$. Note that (2) implies that

$$D(e^x) = \frac{1}{2} D(x^2) e^x + D(x)(1-x)e^x \,.$$

Differentiation of (44) with respect to $\hbar$ implies that $y' = D(y) + \frac{1}{2} D_2(y, y)$. Equivalently, writing $y = \sum_{n \geq 0} f_n(x) \hbar^n$, then the initial condition is that $f_0(x) = x$, and

$$(n+1) f_{n+1}(x) = D(f_n) + \frac{1}{2} \sum_{m=0}^{n} D_2(f_m(x), f_{n-m}(x)) \,.$$

Recursively this implies that $f_n(X/\hbar) = \mathcal{L}_n(X) \hbar^{-n-1} + O(\hbar^{-n})$ with $\mathcal{L}_0(X) = X$ and

$$(n+1) \mathcal{L}_{n+1}(X) = \frac{1}{2} \sum_{m=0}^{n} D_2(\mathcal{L}_m(X), \mathcal{L}_{n-m}(X)) \,.$$

We now let $L(t) = \sum_{n \geq 0} \mathcal{L}_n(X) t^n$ and the claims follow. □

4.4 Application to volume computations

We now return to the proof of the main theorem of this section. Recall the main idea from [21] that the volume of a stratum is given by the growth rate of the number of connected torus covers and thus to the leading terms of cumulants of the f_ℓ. More precisely for $2g - 2 = \sum_{i=1}^{n} m_i$, the same argument as in [12, Proposition 19.1] gives that

$$\operatorname{vol}(\Omega\mathcal{M}_{g,n}(m_1, \ldots, m_n)) = (2\pi i)^{2g} \frac{\langle f_{m_1+1} | \cdots | f_{m_n+1} \rangle_L}{(2g-2+n)!} \,. \tag{45}$$

Proof of Theorem 4.1, one variable case First, $P_B(u) = P(u)|_{p_\ell \mapsto \ell! b_\ell} = P(u)(\emptyset)$. Next, recall that Lagrange inversion for a power series $F \in u\mathbb{C}[[u]]$ with non-zero linear term and inverse $G(z)$ states that $k[z^k]G^n = n[u^{-n}]F^{-k}$ for $k, n \neq 0$. We apply this to $F = u/P_B(u)$ and to $k = 2g-1$ and $n = -1$ to obtain that

$$\begin{aligned} \frac{(2g-1)!}{(2\pi i)^{2g}} v(2g-2) &= (2g-1) \langle f_{2g-1} \rangle_L = \langle h_{2g-1} \rangle_L \\ &= \frac{-1}{2g-1} [u] (P_B(u)/u)^{2g-1} = \left[u^{2g-1}\right] \frac{1}{(u/P_B)^{-1}} \end{aligned} \tag{46}$$

using Proposition 4.2, (32) and Lagrange inversion. □

We pause for a moment to check the initial condition of the theorem in the previous section independently of the Hermitian metric extension problem along the boundary of the strata.

Proof of (26) using (46) We want to show that $(2g-1)^2\, a(2g-2) = \langle h_{2g-1}\rangle_L$. Recall that a version of Lagrange inversion (see e.g. [25, Formula (2.2.8)]), in fact the case $k=0$ excluded in the version of the previous proof, states that if $F \in z + z^2\mathbb{C}[[z]]$ with composition inverse $G(u)$, then for any Laurent series $\phi(z)$

$$[z^0]\phi(F) = [u^0]\phi(u) + [u^{-1}]\phi'(u)\log(G/u)\,. \tag{47}$$

If we let $\widetilde{\mathcal{A}}(z) = 1/z + \sum_{g\geq 1}\langle h_{2g-1}\rangle_L\, z^{2g-1}$, then we need to show that $\widetilde{\mathcal{A}}(z) = \mathcal{A}(z)$. We apply Lagrange inversion to $\phi(z) = z^{-2g}$ and $F = 1/\widetilde{\mathcal{A}}(z)$ to obtain that

$$\begin{aligned}\frac{1}{2g!}[z^0]\widetilde{\mathcal{A}}^{2g} &= \frac{1}{2g!}[z^0]\phi(1/\widetilde{\mathcal{A}}(z)) = \frac{-1}{(2g-1)!}[u^{2g}]\log(1/P_B)\\ &= \frac{-1}{(2g-1)!}[u^{2g}]\sum_{s\geq 1} s!b_{s+1}u^{s+1} = [u^{2g}]B(u) = \frac{1}{(2g)!}[z^0]\mathcal{A}^{2g}\end{aligned}$$

using (46) and (16). This implies the claim. □

For the general case of the theorem, we apply Sect. 4.3 to the differential operator

$$D = \tfrac{1}{2}(\Delta - \partial^2 - \partial/\partial p_1)\,. \tag{48}$$

Proposition 4.6 *The bilinear differential operator D_2 defined in (29) is the polarization of D, namely, $D_2(f,g) = D(fg) - fD(g) - gD(f)$ for all f and g.*

Proof In terms of the p_ℓ-generators the polarization is given by

$$D_2(f,g) = \sum_{k,\ell\geq 1}\Big((k+\ell)p_{k+\ell-1} - k\ell p_{k-1}p_{\ell-1}\Big)\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial p_\ell}\,. \tag{49}$$

The definition (32) of h_ℓ in terms of p_ℓ implies that

$$\frac{\partial \mathcal{H}(z)}{\partial p_k} = -\frac{z\mathcal{H}'(z)}{\mathcal{H}(z)^{k+1}} \quad \text{and} \quad \sum_{n\geq 2} n p_{n-1} \mathcal{H}^{-n}(z) = -\mathcal{H}(z)/z\mathcal{H}'(z) - 1. \tag{50}$$

We compute that

$$\begin{aligned}
&\sum_{k,\ell\geq 1} \Big((k+\ell)p_{k+\ell-1} - k\ell p_{k-1}p_{\ell-1}\Big) \frac{z_1\mathcal{H}'(z_1)}{\mathcal{H}(z_1)^{k+1}} \cdot \frac{z_2\mathcal{H}'(z_2)}{\mathcal{H}(z_2)^{\ell+1}} \\
&\quad = \frac{z_1\mathcal{H}'(z_1)z_2\mathcal{H}'(z_2)}{\mathcal{H}(z_1)\mathcal{H}(z_2)} \Big(\sum_{n\geq 2} n p_{n-1} \frac{\mathcal{H}(z_1)^{1-n} - \mathcal{H}(z_2)^{1-n}}{\mathcal{H}(z_2) - \mathcal{H}(z_1)} \\
&\qquad - \Big(1 + \frac{\mathcal{H}(z_1)}{z_1\mathcal{H}'(z_1)}\Big)\Big(1 + \frac{\mathcal{H}(z_2)}{z_2\mathcal{H}'(z_2)}\Big)\Big) \\
&\quad = \mathcal{H}_2(z_1, z_2),
\end{aligned}$$

and this implies the claim by the chain rule. □

We now define the partition function of h-brackets

$$\begin{aligned}
\Phi^H(\mathbf{u})_q &= \Big\langle \exp\Big(\sum_{\ell\geq 1} h_\ell u_\ell\Big)\Big\rangle_q = \sum_{\mathbf{n}\geq 0} \langle \underbrace{h_1 \cdots h_1}_{n_1} \underbrace{h_2 \cdots h_2}_{n_2} \cdots \rangle_q \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\ell_1,\ldots,\ell_n\geq 1} \langle h_{\ell_1} \cdots h_{\ell_n} \rangle_q \, u_{\ell_1} \cdots u_{\ell_n}
\end{aligned} \tag{51}$$

in the h_ℓ-variables. Then the partition function of the rational cumulants for the h_ℓ-generators

$$\Psi^H(\mathbf{u})_q = \sum_{\mathbf{n}\geq 0} \langle \underbrace{h_1| \cdots |h_1}_{n_1} | \underbrace{h_2| \cdots |h_2}_{n_2} | \cdots \rangle_q \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} = \log \Phi^H(\mathbf{u})_q \tag{52}$$

is simply the logarithm of Φ^H.

Proof of Theorem 4.1, general case We first show that the pieces of Φ^H sorted by total degree in $\mathbf{u}$ can be recursively computed using the D_2-operator. For this purpose we let $\widetilde{h}_i = \hbar^{-i} h_i$. From the definition of cumulants, Eq. (37) and $a_0(\langle g \rangle_q) = g(\emptyset)$, we obtain that

$$\sum_{\mathbf{n}\geq 0}\langle\underbrace{\widetilde{h}_1|\cdots|\widetilde{h}_1}_{n_1}|\underbrace{\widetilde{h}_2|\cdots|\widetilde{h}_2}_{n_2}|\cdots\rangle_\hbar \frac{\mathbf{u}^\mathbf{n}}{\hbar^{|\mathbf{n}|}\,\mathbf{n}!} = \log\Big(\langle\exp(\frac{1}{\hbar}\sum_{i\geq 1}\widetilde{h}_i u_i)\rangle_\hbar\Big)$$
$$= \log\Big(e^{\hbar D}\exp\Big(\tfrac{1}{\hbar}\sum_{i\geq 1}\widetilde{h}_i u_i\Big)\Big)(\emptyset)\,. \tag{53}$$

By applying the Key Lemma with $X = \sum_{i\geq 1}\widetilde{h}_i u_i$ and undoing the rescaling of the h_i using (41) we obtain that

$$\sum_{\mathbf{n}\geq 0}\langle\underbrace{h_1|\cdots|h_1}_{n_1}|\underbrace{h_2|\cdots|h_2}_{n_2}|\cdots\rangle_L \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} = \Big(\sum_{n=0}^{\infty}\mathcal{L}_n\Big)(\emptyset) \tag{54}$$

with

$$\mathcal{L}_0 = \sum_{n\geq 1} h_i u_i \quad \text{and} \quad \mathcal{L}_n = \frac{1}{2n}\sum_{r+s=n-1} D_2(\mathcal{L}_r, \mathcal{L}_s) \tag{55}$$

for $n > 0$. Now define a linear map $\mho_n : \mathbb{Q}[\mathbf{u}] \to \mathbb{Q}[\mathbf{z}]$ by

$$\mho_n(u_{\ell_1}\cdots u_{\ell_n}) = \mathrm{Symm}(z_1^{\ell_1}\cdots z_n^{\ell_n})$$

and zero for monomials of length different from n, where Symm denotes symmetrization with respect to the S_n action on the variables z_i. In this notation $\mathcal{H}_1 = \mho_n\mathcal{L}_{n-1}$ and

$$\mathcal{H}_n = \mho_n\Big(\sum_{\mathbf{n}\geq 0}\langle\underbrace{h_1|\cdots|h_1}_{n_1}|\underbrace{h_2|\cdots|h_2}_{n_2}|\cdots\rangle_L \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}\Big)\,.$$

Consequently, (55) and (28) together with (45) and Proposition 4.2 imply the claim. □

5 Equivalence of volume recursions

In this section we introduce another "averaged volume" recursion that interpolates between the D_2-recursion introduced in Sect. 4 and the volume recursion in Theorem 1.2. We will show that the averaged volume recursion and the D_2-recursion give the same generating functions, and then Theorem 1.1 will follow from it.

Recall from (28) the definition of $\mathcal{H}_{\{i,j\}} \in R[[z_i, z_j]]$ for $i \neq j$, where $R = \mathbb{Q}[h_1, h_2, \ldots]$. For any list of positive integers $\mathbf{p} = (p_1, \ldots, p_k)$ we define

$$\mathcal{H}^{\mathbf{p}}_{\{i,j\}} = \frac{\partial^k}{\partial h_{p_1}\cdots\partial h_{p_k}}\mathcal{H}_{\{i,j\}}\,.$$

For a finite set $I = \{i_1, \ldots, i_n\}$ of positive integers we define the formal series $\mathcal{A}_I \in R[[z_{i_1}, z_{i_2}, \ldots]]$ inductively by

$$\mathcal{A}_I = \mathcal{H}_I \in \frac{1}{z_i}R[[z_i]] \qquad \text{if } n = 1, \text{ and otherwise}$$

$$\mathcal{A}_I = \frac{1}{n-1}\sum_{1\le r<s\le n}\sum_{k\ge 0}\sum_{\substack{\mathbf{p}=(p_1,\ldots,p_k)\\ I=\{i_r,i_s\}\sqcup I_1\sqcup\cdots\sqcup I_k}}\frac{1}{k!}\mathcal{H}^{\mathbf{p}}_{i_r,i_s}\cdot\prod_{j=1}^{k}\mathcal{A}^{[p_j]}_{I_j}, \tag{56}$$

where $\mathcal{A}^{[p]}_I := [z_i^p]\mathcal{A}_{I\cup\{i\}}$ for any $i \notin I$. We set $\mathcal{A}_n = \mathcal{A}_{[\![1,n]\!]}$. Note that $\mathcal{A}_I = \mathcal{H}_I$ by definition if $|I| = 2$. We have chosen to sum in definition of $\mathcal{A}_I$ over all $\mathbf{p}$ rather than partitions $g = \sum_{i=1}^k g_i$ as e.g. in Theorem 1.2. The two summations are equivalent, since g_i and p_i determine each other once I has been partitioned. Our goal here is to show the following result.

Theorem 5.1 *For all non-empty sets of positive integers I, we have $\mathcal{A}_I = \mathcal{H}_I$.*

For the proof of this theorem we will show that both $\mathcal{A}_I$ and $\mathcal{H}_I$ can be written as a sum which ranges over certain oriented trees (see Sect. 5.2). The two recursions can then be viewed as stemming from cutting the trees at a local maximum (a "top") or a local minimum (a "bottom") respectively.

5.1 Proof of Theorems 1.2 and 1.1

We assume in this section that Theorem 5.1 holds and finish the proof of Theorems 1.2 and 1.1 under this assumption. We abbreviate $\lambda = (\ell_1, \ldots, \ell_n)$ and recall from Sect. 4 that we denoted the coefficients of $\mathcal{A}_n$ for $n \ge 2$ by

$$\mathcal{A}_n = \mathcal{H}_n = \sum_{\ell_1,\ldots,\ell_n\ge 1} h_\lambda\, z_{i_1}^{\ell_1}\cdots z_{i_n}^{\ell_n}\,.$$

Proposition 5.2 *The coefficients h_λ are uniquely determined by the recursion*

$$h_\lambda = \sum_{1\le r<s\le n}\frac{\ell_r+\ell_s}{n-1}\sum_{k\ge 1}\frac{1}{k!}\sum_{\mathbf{g},\mu} h_{\mathbb{P}^1}(\ell_r-1,\ell_s-1,\mathbf{p})\cdot\prod_{i=1}^{k}h_{\lambda_i,p_i} \tag{57}$$

for $n \ge 2$, where the summation is as in Theorem 1.2, except that $\lambda_1\sqcup\cdots\sqcup\lambda_k = [\![1,n]\!]\smallsetminus\{r,s\}$.

Proof We begin by showing the formula for λ with two entries, which in view of (6) is equivalent to show that

$$\sum_{\ell_1,\ell_2\geq 1} \frac{h_{\ell_1,\ell_2}}{\ell_1+\ell_2} z_1^{\ell_1} z_2^{\ell_2} = \sum_{k\geq 1} \frac{1}{k} \left(\sum_{p\geq 1} h_p z_1 z_2 \frac{z_1^p - z_2^p}{z_1 - z_2} \right)^k$$
$$= -\log\left(\frac{z_1 z_2}{z_1 - z_2} \left(\mathcal{H}_1(z_2) - \mathcal{H}_1(z_1)\right)\right).$$

Since applying $(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2})$ to the left-hand side above gives $\mathcal{A}_2$, and since neither side has a constant term, this in turn follows from

$$\mathcal{H}_2 = \mathcal{A}_2 = -1 + \frac{z_1\mathcal{H}_1'(z_1) - z_2\mathcal{H}_1'(z_2)}{\mathcal{H}_1(z_2) - \mathcal{H}_1(z_1)}$$
$$= -\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right) \log\left(\frac{z_1 z_2}{z_1 - z_2} \left(\mathcal{H}_1(z_2) - \mathcal{H}_1(z_1)\right)\right).$$

For λ with more entries, we deduce (for all $\ell_r, \ell_s \geq 1$ and all $\mathbf{p}$) from the preceding calculation that

$$[z_r^{\ell_r} z_s^{\ell_s}]\mathcal{H}_{\{r,s\}} = \sum_{k'>0} \frac{\ell_r+\ell_s}{k'!} \sum_{\mathbf{p}'} h_{\mathbb{P}^1}(\ell_r-1, \ell_s-1, \mathbf{p}') \prod_{i=1}^{k'} h_{p_i'}$$

and $$[z_r^{\ell_r} z_s^{\ell_s}]\mathcal{H}^{\mathbf{p}}_{\{r,s\}} = \sum_{k'>0} \frac{\ell_r+\ell_s}{k'!} \sum_{\mathbf{p}'} h_{\mathbb{P}^1}(\ell_r-1, \ell_s-1, \mathbf{p}\cup\mathbf{p}') \prod_{i=1}^{k'} h_{p_i'},$$

Besides, the recursion formula (56) defining $\mathcal{A}_I$ can be translated for λ with n parts into

$$h_\lambda = \sum_{1\leq r<s\leq n} \frac{1}{n-1} \sum_{\substack{k\geq 1, g_i, \lambda_i \\ |\lambda_i|>0}} \frac{1}{k!} [z_r^{\ell_r} z_s^{\ell_s}]\mathcal{H}^{\mathbf{p}}_{\{r,s\}} \prod_{i=1}^{k} h_{\lambda,p_i}$$
$$= \sum_{1\leq r<s\leq n} \frac{\ell_r+\ell_s}{n-1} \sum_{\substack{k\geq 1, g_i, |\lambda_i|>0 \\ k'>0, \mathbf{p}'}} \frac{1}{k!k'!} h_{\mathbb{P}^1}(\ell_r-1, \ell_s-1, \mathbf{p}\cup\mathbf{p}') \prod_{i=1}^{k} h_{\lambda,p_i} \prod_{j=1}^{k'} h_{p_j'}, \tag{58}$$

where $\lambda_1 \sqcup \cdots \sqcup \lambda_k$ partitions $\lambda\backslash\{\ell_r, \ell_s\}$. Next we remark that the interior sum of (57) is over all backbone graphs with the two markings labelled with r and s at the lower level. In the preceding formula (58) the contribution of vertices with at least one marking is separated from the vertices with no markings. This choice results in a binomial coefficient $\binom{k+k'}{k}$ and transforms $\frac{1}{k!k'!}$ into $\frac{1}{(k+k')!}$, thus showing that the two recursive formulas (58) and (57) are equivalent. □

Proof of Theorems 1.2 and 1.1 For $\mu = (m_1, \ldots, m_n)$ consider the intersection numbers $a(\mu)$ that satisfy the recursion in Theorem 3.1, and recall that $a(\mu) = a_i(\mu)$ is independent of the index i by Proposition 3.2. In particular the $a(\mu)$ satisfy the recursion (17) for any distinguished pair of indices, and hence satisfy every weighted average of these recursions. We use the weighted average where the recursion with (i, j) distinguished is taken with weight $\prod_{k \notin \{i,j\}}(m_k + 1)$. Conversely, the $a(\mu)$ are uniquely determined by this weighted average and the initial values for μ of length one given in (16).

On the other hand, the collection of $(2g - 2 + n) \prod_{i=1}^{n}(m_i + 1)a(\mu)$ and the collection of $h_{m_1+1,\ldots,m_n+1}$ both satisfy the recursion (57), by observing that

$$(2g - 2 + n)a(\mu) \prod_{i=1}^{n}(m_i + 1) = \sum_{1 \le r < s \le n} \frac{(m_r + 1 + m_s + 1)}{n - 1} a(\mu) \prod_{i=1}^{n}(m_i + 1). \tag{59}$$

Note that $\mathcal{A}_1|_{h_\ell \mapsto \alpha_\ell} = \mathcal{A}$ by Theorem 4.1 and since we already checked (see (46) and the subsequent proof) that the one-variable rescaled volumes $v(2g - 2)$ and $a(2g - 2)$ agree (see (26)) up to the factor $(2\pi i)^{2g}/(2g - 1)!$. This implies that

$$a(\mu) = \frac{h_{m_1+1,\ldots,m_n+1}|_{h_\ell \mapsto \alpha_\ell}}{(2g - 2 + n) \prod_{i=1}^{n}(m_i + 1)}. \tag{60}$$

The claim now follows from Theorems 4.1, 5.1 and the conversion (13) of volumes to the $a(\mu)$. □

5.2 Oriented trees

We now start preparing for the proof of Theorem 5.1. An *oriented tree* is the datum of a graph $G = (V, E \subset V \times V)$ whose underlying graph of (V, E) is a tree. In particular it is required to be connected. If $(v, v') \in E$, we will denote $v > v'$. Moreover, a vertex $v \in V$ is called a *bottom* (respectively a *top*) if there exists no $v' \in V$ such that $v > v'$ (respectively $v < v'$). We will denote by $B(G)$ and $T(G)$ the sets of bottoms and tops of G.

For any oriented tree G with n vertices, we define the rational number

$$f^{\#}(G) := \frac{\operatorname{Card}\left\{\sigma : V \xrightarrow{\sim} [\![1, n]\!],\ \text{s.t.} \forall (v, v') \in E, \sigma(v) > \sigma(v')\right\}}{n!}, \tag{61}$$

whose numerator is the number of total orderings on the set of vertices compatible with the orientation of G.

Lemma 5.3 *The function $f^{\#}$ can be expressed as*

$$f^{\#}(G) = \frac{1}{n} \cdot \sum_{v \in B(G)} \left(\prod_{G'} f^{\#}(G') \right) = \frac{1}{n} \cdot \sum_{v \in T(G)} \left(\prod_{G'} f^{\#}(G') \right),$$

where in both cases the product is over all connected components G' of the oriented graph obtained by deleting the vertex v.

Proof In order to define a total ordering on V compatible with the orientation of G, we begin by choosing a minimal element $v \in V$. This element is necessarily a bottom. Let us fix such a choice and denote by $(G_1, \ldots, G_k)$ the connected components of $G \backslash \{v\}$. Let n_i be the number of vertices of G_i for $1 \leq i \leq k$. A total ordering on V with minimal element v is equivalent to choosing a total ordering on the vertices of G_i for all $1 \leq i \leq k$ and a partition of $[\![1, n-1]\!]$ into k sets of size $(n_1, \ldots, n_k)$. Such an ordering on V is compatible with the orientation of G if and only if each ordering on the vertices of G_i is compatible with the orientation of G_i for all $1 \leq i \leq k$. This implies that the number of total orderings on V with minimal element v is equal to

$$\binom{n-1}{n_1 \ \cdots \ n_k} \cdot \prod_{i=1}^{k} (n_i! f^{\#}(G_i)) = (n-1)! \cdot \prod_{i=1}^{k} f^{\#}(G_i).$$

Summing over all possible choices of a minimal element, the number of total orderings on vertices of G compatible with the orientation of G is equal to

$$(n-1)! \cdot \sum_{v \in B(G)} \left(\prod_{G'} f^{\#}(G') \right),$$

which completes the proof. □

5.3 Decorations of oriented trees

Let I be a non-empty finite set of positive integers. An *I-decoration* of an oriented tree $\Gamma = (V, E)$ is the datum of a function $\mathrm{dec}\colon I \to V$ such that for each vertex v the number of outgoing edges plus the number of decorations is equal to two, i.e.

$$\#(\mathrm{dec}^{-1}(v)) + \#(E \cap (\{v\} \times V)) = 2$$

for all $v \in V$. If I has cardinality greater than one, we denote by $\mathrm{OT}(I)$ the set of I-decorated oriented trees. One can easily check that the following two properties hold:

- if I has cardinality $n \geq 2$, then Γ has $n - 1$ vertices;
- a vertex of a decorated tree is a bottom if and only if it has exactly two markings.

We denote by $\mathrm{OT}(I)^v$, $\mathrm{OT}(I)^b$ and $\mathrm{OT}(I)^t$ the sets of decorated trees with a choice of an arbitrary vertex, a choice of a bottom and a choice of a top, respectively. If $I = \{i\}$ has only one element, we define $\mathrm{OT}(\{i\})^v = \{i\}$ as a trivial graph decorated by i.

Lemma 5.4 *If I has cardinality greater than one, then there is a bijection*

$$\varphi^t : \mathrm{OT}(I)^t \to \left(\bigcup_{I' \subset I} \mathrm{OT}(I')^v \times \mathrm{OT}(I \backslash I')^v \right) \Big/ (I' \sim I \backslash I') \tag{62}$$

given by cutting at a top vertex, were the union is over all non-empty proper subsets. Similarly, there is a bijection

$$\varphi^b : \mathrm{OT}(I)^b \to \bigcup_{\{i_1,i_2\} \subset I, k>0} \left(\bigcup_{I = \{i_1,i_2\} \sqcup I_1 \sqcup \cdots \sqcup I_k} \prod_{j=1}^{k} \mathrm{OT}(I_j \cup \{e_j\}) \right) \Big/ S_k \tag{63}$$

given by cutting at a bottom vertex, were the union is over all partitions of I into $k+1$ non-empty sets such that the first distinguished set has precisely two elements and where the element $e_j = \max(I) + j$ for all $1 \leq j \leq k$.

We denote by ψ^t and ψ^b the inverses of φ^t and φ^b, respectively.

Proof Given a decorated tree with a chosen top vertex v, we define its image under φ^t as follows:

- If there are two markings on v, then v has no outgoing edges, hence the graph has v as a unique vertex and I has only two elements. It follows that $\mathrm{OT}(I)^v$ has only one element (and so does the right-hand side of (62)).
- If there is only one marking $i \in I$ on v and one outgoing edge to a vertex $v' < v$, then $I' = \{i\}$ and the corresponding element in $\mathrm{OT}(I \backslash I')^v$ is the graph obtained by deleting v and choosing v' as the distinguished vertex.
- If there are two outgoing edges to vertices $v' < v$ and $v'' < v$, then v has no I-markings and the graph obtained by deleting v has two connected components. We define I' to be the set of markings on the component containing v' and define the corresponding elements of $\mathrm{OT}(I')^v$ and $\mathrm{OT}(I \backslash I')^v$ to be the connected components containing v' and v'' as chosen vertices, respectively.

The inverse of φ^t in the first two cases is clear, and in the last case is given by adding a top vertex adjacent to the two chosen vertices.

Given a decorated tree with a chosen bottom vertex v, in the same spirit we define the function φ^b as follows. Since v is a bottom, it has no outgoing edges, hence it has exactly two I-markings i_1 and i_2, and the corresponding k graphs on the right-hand side of (63) are the k connected components of the graph obtained by removing v. The inverse of φ^b is given by gluing these k graphs back to v along the vertices marked by $e_1, \ldots, e_k$. □

Now we fix a ring R' and a function $g\colon \mathrm{OT}(I) \to R'$. By slight abuse of notation, we write $f^{\#}\colon \mathrm{OT}(I) \to \mathbb{Q}$ for the composition of $f^{\#}$ defined in (61) with the forgetful map of the decorations. As a consequence of the two preceding lemmas we see that the sum

$$S(g) = \sum_{\Gamma \in \mathrm{OT}(I)} f^{\#}(\Gamma) g(\Gamma)$$

can be rewritten in two different ways, namely

$$S(g) = \frac{1}{2(n-1)} \sum_{I' \subset I} \sum_{\substack{(\Gamma', v') \in \mathrm{OT}(I')^v \\ (\Gamma'', v'') \in \mathrm{OT}(I \backslash I')^v}} f^{\#}(\Gamma') f^{\#}(\Gamma'')\, g\big(\psi^t(\Gamma', v', \Gamma'', v'')\big) \tag{64}$$

and

$$S(g) = \frac{1}{n-1} \sum_{\substack{\{i_1, i_2\} \subset I, k > 0 \\ I = \{i_1, i_2\} \sqcup I_1 \sqcup \cdots \sqcup I_k}} \frac{1}{k!} \sum_{(\Gamma_j)_{j=1}^k} \left(\prod_{j=1}^k f^{\#}(\Gamma_j) \right) \cdot g\left(\psi^b \left(\prod_{j=1}^k \Gamma_j \right) \right), \tag{65}$$

where n is the cardinality of I (i.e. $\Gamma \in \mathrm{OT}(I)$ has $n-1$ vertices as remarked before).

5.4 Explicit expansions over decorated trees

In order to finish the proof of Theorem 5.1, we will show that both $\mathcal{A}_I$ and $\mathcal{H}_I$ are equal to a generating series $\mathcal{S}_I$ that is directly defined as a sum over $\mathrm{OT}(I)$.

Let $\Gamma = (V, E, I \to V)$ be an oriented tree with decoration by I. A *twist assignment* on Γ is a function $\mathbf{p}\colon E \to \mathbb{Z}_{>0}$. We work over the ring $R[[(z_i)_{i\in I}, (z_e)_{e\in E}]]$ of formal series in variables indexed by $I \cup E$. Given a twisted decorated oriented tree, we define the contribution of a vertex as

$$\mathcal{H}_v = \mathcal{H}_2^{\mathbf{p}_v}(z_{v,1}, z_{v,2}) \in R[[(z_i)_{i\in I}, (z_e)_{e\in E}]],$$

where $\mathbf{p}_v$ is the list of twists associated to all vertices v' with $v' > v$ and where $(z_{v,1}, z_{v,2})$ are the variables attached to either the markings of v or the outgoing edges from v to vertices v' with $v' < v$. Then we define the contribution of the oriented tree Γ as

$$\mathrm{cont}(\Gamma) = \sum_{\mathbf{p}: E\to\mathbb{Z}_{>0}} \Big[\prod_{e\in E} z_e^{\mathbf{p}(e)}\Big] \prod_{v\in V} \mathcal{H}_v$$

if Γ is non-trivial, and define $\mathrm{cont}(\Gamma) = \mathcal{H}_v$ for the trivial graph Γ with a unique vertex v and no edges. Finally, we set $\mathcal{S}_{\{i\}} = \mathcal{H}_{\{i\}} = \mathcal{A}_{\{i\}}$ and for $|I| \geq 2$

$$\mathcal{S}_I = \sum_{\Gamma\in\mathrm{OT}(I)} f^{\#}(\Gamma)\,\mathrm{cont}(\Gamma)\,.$$

End of the proof of Theorem 5.1 We will show that $\mathcal{S}_I = \mathcal{H}_I$ and $\mathcal{S}_I = \mathcal{A}_I$ for all sets of positive integers I with $n = \mathrm{Card}(I) > 2$. The equalities in the case $n = 2$ are obvious from the definition. We assume now that $n \geq 3$ and that $\mathcal{S}_{I'} = \mathcal{H}_{I'} = \mathcal{A}_{I'}$ for all I' such that $\mathrm{Card}(I') < n$.

We first prove that $\mathcal{S}_I = \mathcal{H}_I$. We begin by rewriting the defining Eq. (28) with two auxiliary "edge" variables $z_{e'}$ and $z_{e''}$ for distinct indices $e', e'' \in \mathbb{N}\backslash I$ as

$$\mathcal{H}_I = \frac{1}{2(n-1)} \sum_{I'\subset I} \sum_{p_{e'}, p_{e''}>0} \Big([z_{e'}^{p_{e'}} z_{e''}^{p_{e''}}]\mathcal{H}_{\{e',e''\}}\Big) \frac{\partial \mathcal{H}_{I'}}{\partial h_{p_{e'}}} \frac{\partial \mathcal{H}_{I\backslash I'}}{\partial h_{p_{e''}}}\,.$$

To evaluate the derivative of $\mathcal{H}_{I'}$, there are two cases to consider, depending on the cardinality of I'. If $I' = \{i\}$, then $\frac{\partial \mathcal{H}_{I'}}{\partial h_p} = z_i^p$ for all $p > 0$. Otherwise, we use the induction hypothesis to compute that

$$\begin{aligned}\frac{\partial \mathcal{H}_{I'}}{\partial h_p} = \frac{\partial \mathcal{S}_{I'}}{\partial h_p} &= \sum_{\Gamma \in \mathrm{OT}(I')} f^{\#}(\Gamma) \frac{\partial\, \mathrm{cont}(\Gamma)}{\partial h_p} \\ &= \sum_{\Gamma \in \mathrm{OT}(I')} f^{\#}(\Gamma) \sum_{\mathbf{p}: E \to \mathbb{Z}_{>0}} \left[\prod_{e \in E} z_e^{\mathbf{p}(e)}\right] \sum_{v \in V} \frac{\partial \mathcal{H}_v}{\partial h_p} \left(\prod_{\hat{v} \neq v} \mathcal{H}_{\hat{v}}\right) \\ &= \sum_{(\Gamma, v) \in \mathrm{OT}(I')^v} f^{\#}(\Gamma) \sum_{\mathbf{p}: E \to \mathbb{Z}_{>0}} \left[\prod_{e \in E} z_e^{\mathbf{p}(e)}\right] \frac{\partial \mathcal{H}_v}{\partial h_p} \left(\prod_{\hat{v} \neq v} \mathcal{H}_{\hat{v}}\right).\end{aligned}$$

Now we assume that both I' and $I \backslash I'$ have at least two elements. Take two oriented trees $(\Gamma' = (E', V', I' \to V'), v') \in \mathrm{OT}(I')^v$ and $\Gamma'' = (E'', V'', I \backslash I' \to V''), v'') \in \mathrm{OT}(I'')^v$. Let $(\Gamma = (V, E, I \to V), v) = \psi^t(\Gamma', \Gamma'')$ be the combined graph in $\mathrm{OT}(I)^t$ as described in Lemma 5.4.

The datum of two twist assignments $\mathbf{p}'$ and $\mathbf{p}''$ on Γ' and Γ'' respectively together with a pair of positive integers $(p_{e'}, p_{e''})$ is equivalent to the datum of a twist assignment $\mathbf{p}\colon V \to \mathbb{Z}_{>0}$ on the graph Γ. Moreover, the contributions of the vertices of Γ with the twist assignment $\mathbf{p}$ are given by

- $\mathcal{H}_{\{e', e''\}}$ for the top vertex v,
- $\frac{\partial \mathcal{H}_{v'}}{\partial h_{p_{e'}}}$ and $\frac{\partial \mathcal{H}_{v''}}{\partial h_{p_{e''}}}$ for the two distinguished vertices of Γ' and Γ'', and
- $\mathcal{H}_{\hat{v}}$ for all other vertices $\hat{v}$.

One checks that this is still true if one of the graphs Γ' or Γ'' has only one marking. In summary we obtain that

$$\begin{aligned}\mathcal{H}_I &= \frac{1}{2(n-1)} \sum_{I' \subset I} \sum_{\substack{(\Gamma', v') \in \mathrm{OT}(I')^v \\ (\Gamma'', v'') \in \mathrm{OT}(I \backslash I')^v}} f^{\#}(\Gamma') f^{\#}(\Gamma'') \sum_{\mathbf{p}\colon E \to \mathbb{Z}_{>0}} \left[z_e^{\mathbf{p}(e)}\right] \left(\prod_{\hat{v} \in V} \mathcal{H}_{\hat{v}}\right) \\ &= \frac{1}{2(n-1)} \sum_{I' \subset I} \sum_{\substack{(\Gamma', v') \in \mathrm{OT}(I')^v \\ (\Gamma'', v'') \in \mathrm{OT}(I \backslash I')^v}} f^{\#}(\Gamma') f^{\#}(\Gamma'') \mathrm{cont}(\psi^t(\Gamma', v', \Gamma'', v'')) \\ &= \mathcal{S}_I,\end{aligned}$$

where we use (64) to pass from the above second line to the third.

Finally, we prove that $\mathcal{S}_I = \mathcal{A}_I$ by a similar argument. It suffices to prove for the case $I = [[1, n]]$. Using the induction hypothesis that $\mathcal{A}_{I_j \cup \{n+j\}} = \mathcal{S}_{I_j \cup \{n+j\}}$, we can rewrite the inductive definition (56) of $\mathcal{A}_I$ as

$$\mathcal{A}_n = \frac{1}{(n-1)} \sum_{\substack{k>0, [\![1,n]\!]=\{i_1,i_2\}\sqcup I_1\sqcup\cdots\sqcup I_k \\ \Gamma_j\in \mathrm{OT}(I_j\cup\{n+j\})}} \frac{\prod_{j=1}^k f^{\#}(\Gamma_j)}{k!}$$

$$\cdot \sum_{\substack{\mathbf{p}=(p_1,\ldots,p_k) \\ \mathbf{p}^{(j)}:\, E_j\to\mathbb{Z}_{>0}}} \left([z_{n+1}^{p_1}\cdots z_{n+k}^{p_k}]\,\mathcal{H}_{i_1,i_2}^{\mathbf{p}}\right)\cdot \left[\prod_{j=1}^{k}\prod_{e\in E_j} z_e^{\mathbf{p}^{(j)}(e)}\right] \prod_{v\in V_1\cup\cdots\cup V_k} \mathcal{H}_v\,.$$

The datum of $\mathbf{p}$ together with the twist assignments $\mathbf{p}^{(j)}$ for the split graphs Γ_j for $1 \le j \le k$ is equivalent to a twist assignment for the combined graph $\Gamma = \psi^b(\Gamma_1,\ldots,\Gamma_k)$ defined in Lemma 5.4. Moreover, given such a twist assignment the contribution of the vertex carrying i_1 and i_2 is $\mathcal{H}_{i_1,i_2}^{\mathbf{p}}$. Thus we obtain that

$$\begin{aligned}
\mathcal{A}_n &= \frac{1}{(n-1)} \sum_{\substack{k>0,\\ [\![1,n]\!]=\{i_1,i_2\}\sqcup I_1\sqcup\cdots\sqcup I_k \\ \Gamma_j\in \mathrm{OT}(I_j\cup\{n+j\})}} \frac{\prod_{j=1}^k f^{\#}(\Gamma_j)}{k!}\cdot \sum_{\mathbf{p}:\,E(\Gamma)\to\mathbb{Z}_{>0}} \left[\prod_{e\in E(\Gamma)} z_e^{\mathbf{p}(e)}\right]\prod_{v\in V(\Gamma)}\mathcal{H}_v \\
&= \frac{1}{(n-1)} \sum_{\substack{k>0,\\ [\![1,n]\!]=\{i_1,i_2\}\sqcup I_1\sqcup\cdots\sqcup I_k \\ \Gamma_j\in \mathrm{OT}(I_j\cup\{n+j\})}} \frac{\prod_{j=1}^k f^{\#}(\Gamma_j)}{k!}\cdot \mathrm{cont}(\psi^b(\Gamma_1,\ldots,\Gamma_k)) \\
&= \mathcal{S}(I),
\end{aligned}$$

where we use (65) to pass from the above second line to the third. □

6 Spin and hyperelliptic components

In this section we prove a refinement of Theorems 1.2 and 1.1 with spin structures taken into account. We also prove the corresponding refinement for hyperelliptic components in Sect. 6.5. Along the way we revisit the counting problem for torus covers with sign given by the spin parity and complete the proof of Eskin, Okounkov and Pandharipande [22] that the generating function is a quasimodular form of the expected weight. We then show that the D_2-recursion has a perfect analog when counting with spin parity and use the techniques of Sect. 5 to convert this into the recursion for intersection numbers.

In this section we assume that all entries of $\mu = (m_1,\ldots,m_n)$ are even. The *spin parity* of a flat surface $(X, x_1,\ldots,x_n,\omega) \in \Omega\mathcal{M}_{g,n}(\mu)$ is defined as

$$\phi(X,\omega) = h^0\Big(X, \sum_{i=1}^{n} \frac{m_i}{2} x_i\Big) \mod 2\,.$$

The parity is constant in a connected family of flat surfaces by [39]. We will denote by $\Omega\mathcal{M}_{g,n}(\mu)^{\bullet}$ with $\bullet \in \{\text{odd}, \text{even}\}$ the moduli spaces of flat surfaces with a fixed odd or even spin parity and simply call them *spin (sub)spaces* of the corresponding strata. Moreover, we will denote by $\overline{\Omega\mathcal{M}}_{g,n}(\mu)^{\bullet}$ their incidence variety compactification and will similarly use this symbol, e.g. in the form $v(\mu)^{\bullet}$, $c_{1\leftrightarrow 2}(\mu,\mathcal{C})^{\bullet}$, and $c_{\text{area}}(\mu)^{\bullet}$ for volumes and Siegel–Veech constants.

We remark that with our choice of notation the spin spaces $\overline{\Omega\mathcal{M}}_{g,n}(\mu)^{\bullet}$ are not necessarily connected. Indeed, for $\mu = (2g-2)$ with $g \geq 4$ and $\mu = (g-1, g-1)$ with $g \geq 5$ odd, one of the two spin spaces is disconnected, since it contains an extra hyperelliptic component (see [33, Theorem 2]). Nevertheless, the hyperelliptic components will be treated separately in Sect. 6.5. Taking the difference of the volumes thus gives a formula for the volume of each connected component.

To state the refined version of the volume recursion we need a generalization of the spin parity. Let $(\Gamma, \ell, \mathbf{p})$ be a backbone graph. A *spin assignment* is a function

$$\phi\colon \{v \in V(\Gamma), \ell(v) = 0\} \to \{0, 1\}.$$

The *parity of the spin assignment* is defined as

$$\phi(\Gamma) := \sum_{v \in V(\Gamma), \ell(v)=0} \phi(v) \mod 2. \tag{66}$$

Our goal is the following refinement of Theorems 1.2 and 1.1 under a mild assumption. Recall that the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ has a natural hermitian metric given by the area form $h(X,\omega) = \frac{i}{2}\int_X \omega \wedge \overline{\omega}$.

Assumption 6.1 There exists a desingularization $f\colon Y \to \mathbb{P}\overline{\Omega\mathcal{M}}_{g,1}(2g-2)$ such that f^*h extends to a good hermitian metric on $f^*\mathcal{O}(-1)$.

This assumption was already present in [42] and can be deduced from the smooth strata compactification constructed in [7].[4] Note that we do not need this assumption for Theorem 1.2, as it is stated for the entire stratum whose cohomology class was computed recursively in [43]. However, currently we do not know the cohomology class of each individual spin subspace.

[4] Assumption 6.1 was recently verified in [13], to which we refer the reader for more details about the area form being a good metric. See also [11] for a similar application to the volumes of strata of quadratic differentials with odd zeros.

Theorem 6.2 *If $n \geq 2$, then the rescaled volumes satisfy the recursion*

$$v(\mu)^{\mathrm{odd}} = \sum_{k\geq 1} \sum_{\mathbf{g},\boldsymbol{\mu},\,\phi\,\mathrm{odd}} h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \cdot \frac{\prod_{i=1}^{k}(2g_i - 1 + n(\mu_i))!\,v(\mu_i, p_i - 1)^{\phi(i)}}{2^{k-1}k!(2g - 3 + n)!},$$

where the summation conventions for $\mathbf{g}$, $\boldsymbol{\mu}$ *and* $\mathbf{p}$ *are as in Theorem 1.2 and the superscript* $\phi(i)$ *indicates the corresponding spin subspace.*

We remark that the same formula holds when replacing "odd" by "even" in the theorem, which follows simply by subtracting the formula in Theorem 6.2 from that in Theorem 1.2.

Theorem 6.3 *Let* $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)^{\bullet}$ *with* $\bullet \in \{\mathrm{odd}, \mathrm{even}\}$ *be the subspaces of* $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ *with a fixed spin parity. Then the volume can be computed as an intersection number*

$$\mathrm{vol}(\Omega\mathcal{M}_{g,n}(\mu)^{\bullet}) = -\frac{2(2i\pi)^{2g}}{(2g-3+n)!}\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^{\bullet}} \xi^{2g-2} \cdot \prod_{i=1}^{n} \psi_i \,.$$

We first show in Sect. 6.1 that the intersection numbers on the right-hand side of Theorem 6.3 satisfy a recursion as in Theorem 6.2. This is parallel to Sect. 3. We then complete in Sect. 6.2 properties of the strict brackets introduced by [22]. The volume recursion in Sect. 6.3 is parallel to Sect. 4 and allows efficient computations of volume differences of the spin subspaces. We do not need to prove an analog of Sect. 5 but can rather apply the results, since the structures of the two recursions are exactly the same as before. Only in Sect. 6.4 we need Assumption 6.1 to prove the initial case of Theorem 6.2, i.e. the case of the minimal strata.

6.1 Intersection theory on spin subspaces and hyperelliptic components

With a view toward Sect. 6.5 for the hyperelliptic components, we allow here also the profile $\mu = (g-1, g-1)$ (with $g-1$ not necessarily even) and $\bullet \in \{\mathrm{odd}, \mathrm{even}, \mathrm{hyp}\}$, and study the corresponding union of connected components $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)^{\bullet}$.

Let $(\Gamma, \ell, \mathbf{p})$ be a twisted bi-colored graph and D be an irreducible component of the boundary $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}$. We recall from Sect. 3 that $\zeta^{\#}_{\Gamma,\ell}(D)$ is a divisor of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$ if and only if $\dim(\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}) = \dim(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)) - 1$. Hence in this case we define $\alpha(D) = \zeta^{\#}_{\Gamma,\ell *}(D) \in H^*(\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n})$, and define $\alpha(D) = 0$ otherwise.

We will denote by $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\bullet}_{\Gamma,\ell}$ the union of the irreducible components of $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}$ that are mapped to $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^{\bullet}$.

Proposition 6.4 *For* $1 \leq i \leq n$ *and each irreducible component* D *of* $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\bullet}_{\Gamma,\ell}$*, there exist constants* $m_i^\bullet(D) \in \mathbb{Q}$ *such that*

$$(\xi + (m_i + 1)\psi_i)[\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^\bullet] = \sum_{\substack{(\Gamma,\ell,\mathbf{p}) \\ i\mapsto v,\ell(v)=-1}} \sum_{D\subset \mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\bullet}_{\Gamma,\ell}} m_i^\bullet(D)\,\alpha(D),$$

where the sum is over all twisted bi-colored graphs $(\Gamma, \ell, \mathbf{p})$ *with the* i*th marking in the lower level. Moreover, if* $D \subset \mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}$ *and* $(\Gamma, \ell, \mathbf{p})$ *is a backbone graph, then* $m_i^\bullet(D) = m(\mathbf{p})$ *is the multiplicity defined in* (20).

Proof We follow the same strategy as in [43, Theorem 5]. We consider the line bundle $\mathcal{O}(1) \otimes \mathcal{L}_i^{\otimes(m_i+1)}$ on $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^\bullet$. It has a global section s defined by mapping a differential to its $(i+1)$-st order at the marked point x_i. The vanishing locus of this section is exactly the union of the boundary components $\zeta^\#_{\Gamma,\ell}(\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell})$, thus proving the first part of the proposition.

If $(\Gamma, \ell, \mathbf{p})$ is a backbone graph, then each irreducible component D of $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\bullet}_{\Gamma,\ell}$ is contained in the boundary of exactly one connected component of the stratum (see e.g. [10, Corollary 4.4]). Thus the neighborhood of a generic point of D in $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)^\bullet$ is given by [43, Lemma 5.6 and the subsequent formula]. In particular the multiplicity of D in the vanishing locus of s is the same as that of the entire boundary stratum, which implies that $m_i^\bullet(D) = m(\mathbf{p})$. □

By the same arguments as in Sect. 3.3 (see Proposition 3.9) one can show that $D \cdot \xi^{2g-2} = 0$ unless D is an irreducible component of $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p}}_{\Gamma,\ell}$ with $(\Gamma, \ell, \mathbf{p})$ a backbone graph. If $(\Gamma, \ell, \mathbf{p})$ is a backbone graph, then we let $\alpha^\bullet_{\Gamma,\ell,\mathbf{p}} = \zeta^\#_{\Gamma,\ell *}[\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\bullet}_{\Gamma,\ell}]$. Besides, we let $a_i(\mu)^\bullet = \int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^\bullet} \beta_i \cdot \xi$ for $1 \leq i \leq n$.

Proposition 6.5 *If* $n \geq 2$*, then the values of* $a_i(\mu)^\bullet$ *are the same for all* $1 \leq i \leq n$*, denoted by* $a(\mu)^\bullet$*, and can be computed as*

$$(m_1 + 1)(m_2 + 1)a(\mu)^\bullet = \sum_{(\Gamma,\ell,\mathbf{p})\in \mathrm{BB}_{1,2}} \frac{m(\mathbf{p})}{|\mathrm{Aut}(\Gamma, \ell, \mathbf{p})|} \int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}} \alpha^\bullet_{\Gamma,\ell,\mathbf{p}} \cdot \xi^{2g-1} \cdot \prod_{i>2} \psi_i\,.$$

Proof This follows from the same argument as in the proof of Lemma 3.12. □

Proposition 6.6 *For μ of length bigger than one and with even entries, we have*

$$(m_1+1)(m_2+1)a(\mu)^{\mathrm{odd}} = \sum_{\substack{(\Gamma,\ell,\mathbf{p},\phi),\\ \phi \text{ odd}}} \frac{h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p},\phi)|} \cdot \prod_{v\in V(\Gamma),\, \ell(v)=0} p_v(2g_v-1+n(\mu_v))a(\mu_v,p_v-1)^{\phi(v)},$$

where the sum is over all choices of backbone graphs with only the first two marked points in the lower level component.

This proposition is a refined combination of Lemma 3.13 and Eq. (25). Again we remark that the same formula holds when replacing "odd" by "even" in the proposition, which simply follows from subtracting the above from the corresponding formula for the entire stratum.

Proof We apply Proposition 6.5 to $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^{\mathrm{odd}}$. The proposition then follows from the description of the boundary divisors of connected components of $\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)$.

Let $(L \to \mathcal{X} \to \Delta)$ be a one-parameter family of theta characteristics, i.e. L is a line bundle such that $L|_X^{\otimes 2} \simeq \omega_X$ for every fiber curve X parametrized by a complex disk (centered at the origin) such that

- the restriction of $\mathcal{X}$ to $\Delta\backslash 0$ is a family of smooth curves;
- the central fiber X_0 is of compact type.

The second condition above is due to the fact that the graphs in $\mathrm{BB}(g,n)_{1,2}$ are of compact type. We assume that L is odd, i.e. L restricted to every smooth fiber is an odd theta characteristic. The restriction of L to each irreducible component of X_0 (minus the nodes) is a theta characteristic of that component. Since X_0 is of compact type, the parity of $L|_{X_0}$ equals the sum of the parities over all irreducible components of X_0 (see e.g. [10, Proposition 4.1]), which implies that the number of components of X_0 with an odd theta characteristic is odd.

Let $(\Gamma,\ell,\mathbf{p}) \in \mathrm{BB}(g,n)_{1,2}$. From the above description we deduce that $\mathbb{P}\overline{\Omega\mathcal{M}}^{\mathbf{p},\mathrm{odd}}_{\Gamma,\ell}$ can be written as

$$\bigcup_{\phi \text{ odd}} \overline{\mathcal{M}}_{-1} \times \prod_{v\in V(\Gamma)} \mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v-1,\mu_v)^{\phi(v)}$$

where $\overline{\mathcal{M}}_{-1}$ and $\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v-1,\mu_v)$ are defined as in Sect. 3.2. The arguments in the proof of Lemma 3.12 imply that

$$(m_1+1)(m_2+1)a(\mu)^{\mathrm{odd}} = \sum_{\substack{(\Gamma,\ell,\mathbf{p})\in \mathrm{BB}_{1,2}\\ \phi \text{ odd}}} \frac{h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p},\phi)|} \cdot \prod_{v\in V(\Gamma),\,\ell(v)=0} p_v^2\, a(\mu_v,p_v-1)^{\phi(v)}\,.$$

Then by the same line of arguments as in Sect. 3.5 (expansions over rooted trees), we get the desired expression. □

6.2 Strict brackets and Hurwitz numbers with spin parity

Let $f\colon \mathrm{SP}\to\mathbb{Q}$ be any function on the set of strict partitions (i.e. partitions with strictly decreasing part lengths). The replacement of the q-bracket in the context of spin-weighted counting is the *strict bracket* defined by

$$\langle f\rangle_{\mathrm{str}} = \frac{1}{(q)_\infty}\sum_{\lambda\in\mathrm{SP}}(-1)^{\ell(\lambda)}f(\lambda)q^{|\lambda|},\ \Big((q)_\infty=\prod_{n\geq 1}(1-q^n)=\sum_{\lambda\in\mathrm{SP}}(-1)^{\ell(\lambda)}q^{|\lambda|}\Big)\,.$$

The analog of the algebra Λ^* is the algebra $\mathbf{\Lambda}^*=\mathbb{Q}[\mathbf{p}_1,\mathbf{p}_3,\mathbf{p}_5,\ldots]$ of *super-symmetric functions*, where for odd ℓ the functions $\mathbf{p}_\ell$ are defined by

$$\mathbf{p}_\ell(\lambda)=\sum_{i=1}^{\infty}\lambda_i^\ell-\frac{\zeta(-\ell)}{2}\,.$$

Note the modification of the constant term and the absence of the shift in comparison to (30). We provide $\mathbf{\Lambda}^*$ with the *weight grading* by declaring $\mathbf{p}_\ell$ to have weight $\ell+1$. On the other hand, [22] used characters of the modified Sergeev group $C(d)=S(d)\ltimes \mathrm{Cliff}(d)$ to produce elements in $\mathbf{\Lambda}^*$. Here $\mathrm{Cliff}(d)$ is generated by involutions $\xi_1,\ldots,\xi_d$ and a central involution ε with the relation $\xi_i\xi_j=\varepsilon\xi_j\xi_i$. Irreducible representations of $C(d)$ are V^λ indexed by $\lambda\in\mathrm{SP}$. We denote by $\mathbf{f}_\mu(\lambda)$ the central character of the action of a permutation $g_\mu\in S(d)\subset C(d)$ of cycle type μ on V^λ by conjugation. The analog of the Burnside formula is [22, Theorem 2] stating that for a fixed profile $\Pi=(\mu_1,\ldots,\mu_n)$

$$\sum_p \frac{(-1)^{\phi(p)}q^{\deg(p)}}{|\mathrm{Aut}(p)|} = 2^{\sum_{i=1}^n(\ell(\mu_i)-|\mu_i|)/2}\langle \mathbf{f}_{\mu_1}\mathbf{f}_{\mu_2}\cdots\mathbf{f}_{\mu_n}\rangle_{\mathrm{str}}, \tag{67}$$

where the sum is over all covers $p\colon X \to E$ of a fixed base curve and profile Π.

Theorem 6.7 *If we define*

$$\mathbf{h}_\ell = \frac{-1}{\ell}[u^{\ell+1}]\mathbf{P}(u)^\ell \quad \textit{where} \quad \mathbf{P}(u) = \exp\Big(-\sum_{s\geq 1, s \text{ odd}} u^{s+1}\mathbf{p}_s\Big), \tag{68}$$

then the difference $\mathbf{f}_\ell - \ell\mathbf{h}_\ell$ *has weight strictly less than* $\ell+1$. *In particular* $\mathbf{f}_\ell$ *belongs to the subspace* $\mathbf{\Lambda}^*_{\leq \ell+1}$ *of weight less than or equal to* $\ell+1$. *More precisely,*

$$\mathbf{f}_{(\ell)} = \frac{-1}{2\ell}[t^{\ell+1}]\bigg(\prod_{j=1}^{\ell-1}(1-jt)\cdot\exp\Big(\sum_{j \text{ odd}}\frac{2\mathbf{p}_j t^j}{j}(1-(1-\ell t)^{-j})\Big)\bigg). \tag{69}$$

This statement was missing in the proof of the following corollary, one of the main theorems of [22].

Corollary 6.8 *The strict bracket* $\langle \mathbf{f}_{\ell_1}\mathbf{f}_{\ell_2}\cdots\mathbf{f}_{\ell_n}\rangle_{\text{str}}$ *is a quasimodular form of mixed weight less than or equal to* $\sum_{i=1}^n(\ell_i+1)$.

We now prepare for the proof of Theorem 6.7 and prove the corollary along with more precise statements on strict brackets in the next subsection. From [30, Definition 6.3 and Proposition 6.4] we know that the central characters are given by

$$\mathbf{f}_\rho = \sum_{\mu\in\text{SP}} X^\rho_\mu P^\downarrow_\mu \tag{70}$$

where the objects on the right-hand side are defined as follows. We define for any partition λ the Hall-Littlewood symmetric polynomials

$$P_\lambda(x_1,\ldots,x_m;t) = \sum_{\sigma\in S_n}\sigma\Big(\prod_{i=1}^n x_i^{\lambda_i}\prod_{i<j,\, i<\ell(\lambda)}\frac{x_i - tx_j}{x_i - x_j}\Big).$$

These polynomials have cousins where the powers are replaced by falling factorials. That is, writing $n^{\downarrow k} = n(n-1)(n-2)\cdots(n-k+1)$, we define

$$P^\downarrow_\lambda(x_1,\ldots,x_m;t) = \sum_{\sigma\in S_n}\sigma\Big(\prod_{i=1}^n x_i^{\downarrow\lambda_i}\prod_{i<j,\, i<\ell(\lambda)}\frac{x_i - tx_j}{x_i - x_j}\Big).$$

Next, we define $X_\bullet^\bullet(t)$ to be the base change matrix from the basis of $\mathbf{p}_\rho$ to the basis $P_\lambda(x_1, \ldots, x_m; t)$, that is, we define them by

$$\mathbf{p}_\rho = \sum_{\lambda \vdash |\rho|} X_\lambda^\rho(t) P_\lambda(\,\cdot\,; t)\,. \tag{71}$$

The existence and the fact that the $X_\bullet^\bullet(t)$ are polynomials in t is shown in [35, Section III.7]. We abbreviate $X_\lambda^\rho = X_\lambda^\rho(-1)$ and similarly $P_\lambda = P_\lambda(\,\cdot\,; -1)$ and $P_\lambda^\downarrow = P_\lambda^\downarrow(\,\cdot\,; -1)$.

Proof of Theorem 6.7 We need to prove (69). From there one can then derive (68) by expanding the exponential function (just as in [29] Proposition 3.5 is derived from Proposition 3.3). We use that for $\rho = (\ell)$ a cycle, the coefficients X_ρ^λ in (70) are supported on λ with at most two parts. More precisely, by [35, Example III.7.2] we know that

$$\begin{aligned} \mathbf{f}_{(\ell)} &= P_{(\ell)}^\downarrow + 2 \sum_{i=1}^{\lfloor \ell/2 \rfloor} (-i)^i P_{(\ell-i,i)}^\downarrow \\ &= \sum_{\substack{1 \le a, b \le \ell(\lambda) \\ a \ne b}} \sum_{i=0}^{\ell} (-1)^i \lambda_a^{\downarrow \ell - i} \lambda_b^{\downarrow i} \frac{\lambda_a + \lambda_b}{\lambda_a - \lambda_b} \prod_{i \ne a,b} \frac{\lambda_a + \lambda_i}{\lambda_a - \lambda_i} \frac{\lambda_b + \lambda_i}{\lambda_b - \lambda_i}\,. \end{aligned} \tag{72}$$

Using

$$-\sum_{j \in \mathbb{N}} \frac{1 - (-1)^j \mathbf{p}_j(\lambda) t^j}{j} (1 - \ell t)^{-j} = \log\Bigl(\prod_{i=1}^{\ell(\lambda)} \frac{1 - (\lambda_i + \ell)t}{1 + (\lambda_i - \ell)t}\Bigr)$$

and the specialization of this formula for $\ell = 0$, our goal is to show that

$$\begin{aligned} \mathbf{f}_{(\ell)}(\lambda) &= \frac{-1}{2\ell}[t^1]\Bigl(\prod_{j=0}^{\ell-1}(t^{-1} - j) \prod_{i=1}^{\ell(\lambda)} \frac{t^{-1} + \lambda_i}{t^{-1} - \lambda_i} \frac{t^{-1} - (\lambda_i + \ell)}{t^{-1} + (\lambda_i - \ell)}\Bigr) \\ &= \frac{-1}{2\ell}[z^{-1}]\Bigl(\prod_{j=0}^{\ell-1}(z - j) \prod_{i=1}^{\ell(\lambda)} \frac{z + \lambda_i}{z - \lambda_i} \frac{z - (\lambda_i + \ell)}{z + (\lambda_i - \ell)}\Bigr) \\ &= \sum_{a=1}^{\ell(\lambda)} \Bigl(\prod_{j=0}^{\ell-1}(\lambda_a - j) \prod_{i \ne a} \frac{\lambda_a + \lambda_i}{\lambda_a - \lambda_i} \frac{\lambda_a - (\lambda_i + \ell)}{\lambda_a + (\lambda_i - \ell)}\Bigr)\,. \end{aligned}$$

Using that for ℓ odd

$$\sum_{i=0}^{\ell}(-1)^i\left(\lambda_a^{\downarrow\ell-i}\lambda_b^{\downarrow i} - \lambda_b^{\downarrow\ell-i}\lambda_a^{\downarrow i}\right) = \frac{\lambda_a^{\downarrow\ell}(\lambda_a-\lambda_b-\ell)-\lambda_b^{\downarrow\ell}(\lambda_b-\lambda_a-\ell)}{\lambda_a+\lambda_b-\ell},$$

we see that our goal and the known (72) agree. □

6.3 Volume computations via cumulants for strict brackets

We denote by an upper index Δ the difference of the even and odd spin related quantities, e.g. $v(\mu)^\Delta = v(\mu)^{\text{even}} - v(\mu)^{\text{odd}}$. Cumulants for strict brackets are defined by the same formula (39) as for q-brackets. We are interested in cumulants for the same reason as we were for the case of the strata as in (45).

Proposition 6.9 *The difference of the volumes of the even and odd spin subspaces of $\Omega\mathcal{M}_{g,n}(\mu)$ can be computed in terms of cumulants by*

$$\text{vol}(\Omega\mathcal{M}_{g,n}(\mu))^\Delta = \frac{(2\pi i)^{2g}}{(2g-2+n)!}\left\langle \mathbf{f}_{(m_1+1)}|\cdots|\mathbf{f}_{(m_n+1)}\right\rangle_{\text{str},L},$$

and thus, in combination with Theorem 6.7 we have

$$v(\mu)^\Delta = \frac{(2\pi i)^{2g}}{(2g-2+n)!}\left\langle \mathbf{h}_{(m_1+1)}|\cdots|\mathbf{h}_{(m_n+1)}\right\rangle_{\text{str},L}.$$

Here the subscript L refers to the leading term

$$\begin{aligned}\langle g_1|\cdots|g_n\rangle_{\text{str},L} &= [\hbar^{-k-1+n}]\,\langle g_1|\cdots|g_n\rangle_{\text{str},\hbar}\\ &= \lim_{h\to 0}\hbar^{k+1-n}\,\text{ev}[\langle g_1|\cdots|g_n\rangle_{\text{str},q}](\hbar)\end{aligned}$$

for g_i homogeneous of weight k_i and $k=\sum_{i=1}^n k_i$.

This proposition was certainly the motivation of [22], which stops short of this step. To derive the proposition from (67), we need one more tool, the analog of the degree drop in Proposition 4.4. We use the fact that for strict brackets we have a closed formula (rather than only a recursion as for q-brackets), proved in [22, Section 3.2.2] and in more detail in [9, Section 13]. First,

$$(-1)\cdot\left\langle \mathbf{p}_\ell\right\rangle_{\text{str}} = G_{\ell+1} := \frac{\zeta(-\ell)}{2} + \sum_{n\ge 1}\sigma_\ell(n)q^n \tag{73}$$

and for the more general statement we define the "oddification" of the Eisenstein series to be

$$\mathcal{G}^{\mathrm{odd}}(z_1,\ldots,z_n) = -\sum_{r=1}^{\infty} D_q^{(n-1)} G_{2r} \sum_{s_1+\cdots+s_n=r+n-1} \frac{z_1^{2s_1-1}\cdots z_n^{2s_n-1}}{(2s_1-1)!\cdots(2s_n-1)!},$$

where $D_q = q\partial/\partial q$. Then by Proposition 13.3 in loc. cit. the n-point function is given by

$$\sum_{\ell_i\geq 1,\ \ell_i \text{ odd}} \big\langle \mathbf{p}_{\ell_1}\mathbf{p}_{\ell_2}\cdots\mathbf{p}_{\ell_n}\big\rangle_{\mathrm{str}} \frac{z_1^{\ell_1}\cdots z_n^{\ell_n}}{\ell_1!\cdots\ell_n!} = \sum_{\alpha\in\mathcal{P}(n)} \prod_{A\in\alpha} \mathcal{G}^{\mathrm{odd}}\Big(\{z_a\}_{a\in A}\Big). \tag{74}$$

Consequently, the cumulants are simply given by

$$\big\langle \mathbf{p}_{\ell_1}|\mathbf{p}_{\ell_2}|\cdots|\mathbf{p}_{\ell_n}\big\rangle_{\mathrm{str}} = \Big[\frac{z_1^{\ell_1}\cdots z_n^{\ell_n}}{\ell_1!\cdots\ell_n!}\Big]\mathcal{G}^{\mathrm{odd}}(z_1,\ldots,z_n). \tag{75}$$

Proof of Proposition 6.9 Recall the evaluation map Ev used in [12, Section 8] and in Sect. 4.2. We have $\deg\big(\mathrm{Ev}\langle\mathbf{p}_{\ell_1}\mathbf{p}_{\ell_2}\cdots\mathbf{p}_{\ell_n}\rangle_{\mathrm{str}}\big) = \frac{1}{2}\sum_{i=1}^n \ell_i$, the highest term being contributed by the partition into singletons. From (75) we deduce that $\deg\big(\mathrm{Ev}\langle\mathbf{p}_{\ell_1}|\mathbf{p}_{\ell_2}|\cdots|\mathbf{p}_{\ell_n}\rangle_{\mathrm{str}}\big) = \frac{1}{2}\sum_{i=1}^n \ell_i - (n-1)$, and thus obtain the expected degree drop. The claim now follows from the usual approximation of Masur–Veech volumes by counting torus covers [21] and [12, Proposition 19.1]. □

While (75) provides an easy and efficient way to compute cumulants of strict brackets, we show that the more complicated way via lifting of differential operators to $\mathbf{\Lambda}^*$ and the Key Lemma 4.5 also works here. The analog of Proposition 4.3 is the following result.

Proposition 6.10 *With* $\Delta(f) = \sum_{\ell_1,\,\ell_2\geq 1}(\ell_1+\ell_2)\,\mathbf{p}_{\ell_1+\ell_2-1}\frac{\partial^2}{\partial\mathbf{p}_{\ell_1}\partial\mathbf{p}_{\ell_2}}$ *we have*

$$\langle f\rangle_{\mathrm{str},\hbar} = \frac{1}{\hbar^k}(e^{\hbar(\Delta-\partial/\partial\mathbf{p}_1)/2}f)\,(\emptyset), \tag{76}$$

where the evaluation at the empty set is explicitly given by $\mathbf{p}_\ell \mapsto -\frac{\zeta(-\ell)}{2}$.

Note that we can regard the differential operator $(\Delta-\partial/\partial\mathbf{p}_1)/2$ appearing in the exponent the same as the operator D defined in (48) when viewing $\mathbf{\Lambda}^*$ as a quotient algebra of Λ^* with all the even p_ℓ set to zero, since the differential operator ∂ sending p_ℓ to a multiple of $p_{\ell-1}$ is zero on this quotient.

Proof Using the description (37) of the $\hbar$-evaluation we need to show that $\mathfrak{d}\langle f\rangle_{\mathrm{str},\hbar} = \langle(\Delta - \partial/\partial\mathbf{p}_1)f\rangle_{\mathrm{str},\hbar}$. Contrary to the case of q-brackets we will actually show the stronger statement that $\mathfrak{d}\langle f\rangle_{\mathrm{str}} = \langle(\Delta - \partial/\partial\mathbf{p}_1)f\rangle_{\mathrm{str}}$. It suffices to check this for all the n-point functions. For $n = 1$ this can be checked directly from (73). For general n, we write $W(z) = \sum_{s\geq 1} z^{2s-1}/(2s-1)!$. Using (74) and that the commutator $[\mathfrak{d}, D_q]$ is multiplication by the weight, we compute that

$$\mathfrak{d}\Big\langle \prod_{i=1}^{n} W(z_i)\Big\rangle_{\mathrm{str}} = \sum_{\alpha\in\mathcal{P}(n)} \sum_{\substack{A_1\in\alpha,\\ |A_1|\geq 2}} \Big(\mathfrak{d}\mathcal{G}^{\mathrm{odd}}\big(\{z_a\}_{a\in A_1}\big)\cdot \prod_{A\in\alpha\setminus\{A_1\}} \mathcal{G}^{\mathrm{odd}}\big(\{z_a\}_{a\in A}\big)\Big)$$

$$- \frac{1}{2}\sum_{i=1}^{n} z_i \cdot \Big(\sum_{\alpha\in\mathcal{P}(\{1,\ldots,n\}\setminus\{i\})} \prod_{A\in\alpha} \mathcal{G}^{\mathrm{odd}}\big(\{z_a\}_{a\in A}\big)\Big) \tag{77}$$

where for the factor in the summand with $|A_1| \geq 2$

$$\mathfrak{d}\mathcal{G}^{\mathrm{odd}}\big(\{z_a\}_{a\in A_1}\big) = -\frac{1}{2}\sum_{r\geq 1} 2r\, D_q^{|A_1|-2} G_{2r} \cdot \sum_{\substack{s_a\geq 1,\\ \sum s_a = r+|A_1|-1}} \prod_{a\in A_1} \frac{z_a^{2s_a-1}}{(2s_a-1)!}, \tag{78}$$

and where the summation is over all tuples $(s_a)_{a\in A_1}$. Since $\frac{1}{2}\partial/\partial\mathbf{p}_1(\prod_{i=1}^{n} W(z_i)) = \frac{1}{2}\sum_{i=1}^{n} z_i \prod_{j\neq i} W(z_j)$, the strict bracket of this expression is precisely the second line on the right-hand side of (77). Since

$$\frac{1}{2}\Delta\Big(\prod_{i=1}^{n} W(z_i)\Big) = \sum_{1\leq i\neq j\leq n} (z_i + z_j) W(z_i + z_j) \prod_{k\in\{1,\ldots,n\}\setminus\{i,j\}} W(z_k),$$

its strict bracket matches the first line on the right-hand side of (77), and the part containing the variable for $W(z_i + z_j)$ produces of course the special factor (78). □

This proposition provides an efficient algorithm to compute the differences of volumes of the spin subspaces. The definitions below are completely analogous to the beginning of Sect. 4, except that objects with even indices have disappeared and they are written in boldface letters for distinction. For the substitution, we define

$$\mathbf{P}_Z(u) = \exp\Big(\sum_{j\geq 1}\Big(\frac{1}{2}\Big)^{\frac{j+1}{2}} \zeta(-j)\, u^{j+1}\Big) \quad \text{and} \quad \boldsymbol{\alpha}_\ell = \big[u^\ell\big]\frac{1}{(u/\mathbf{P}_Z(u))^{-1}}\,.$$

We let $\mathbf{R} = \mathbb{Q}[\mathbf{h}_1, \mathbf{h}_3, \ldots]$ and define for a finite set $I = \{i_1, \ldots, i_n\}$ of positive integers the formal series $\mathcal{H}_I \in \mathbf{R}[[z_{i_1}, \ldots, z_{i_n}]]$ by

$$\mathcal{H}_{\{i\}} = \frac{1}{z_i} + \sum_{\ell \geq 1} \mathbf{h}_\ell z_i^\ell, \quad \mathcal{H}_{\{i,j\}} = \frac{z_i \mathcal{H}'(z_i) - z_j \mathcal{H}'(z_j)}{\mathcal{H}(z_j) - \mathcal{H}(z_i)} - 1,$$
$$\mathcal{H}_I = \frac{1}{2(n-1)} \sum_{I = I' \sqcup I''} D_2(\mathcal{H}_{I'}, \mathcal{H}_{I''}), \tag{79}$$

with

$$D_2(f, g) = \sum_{\ell_1, \ell_2 \geq 1, \text{ odd}} [z_1^{\ell_1} z_2^{\ell_2}] \mathcal{H}_{\{1,2\}} \frac{\partial f}{\partial \mathbf{h}_{\ell_1}} \frac{\partial g}{\partial \mathbf{h}_{\ell_2}}.$$

We still set $\mathcal{H}_n = \mathcal{H}_{[\![1,n]\!]}$ and $\mathbf{h}_{\ell_1,\ldots,\ell_n} = [z_1^{\ell_1} \cdots z_n^{\ell_n}]\mathcal{H}_n$.

Corollary 6.11 *The even-odd volume differences of the stratum with signature $\mu = (m_1, \ldots, m_n)$ can be computed as*

$$v(\mu)^\Delta = \frac{(2\pi i)^{2g}}{(2g-2+n)!} \mathbf{h}_{m_1+1,\ldots,m_n+1}\big|_{\mathbf{h}_\ell \mapsto \boldsymbol{\alpha}_\ell}$$

using the recursion (79).

Proof Thanks to Proposition 6.10 and the subsequent remark, the proof of Theorem 4.1 can be copied verbatim here. The extra factor $2^{-\ell/2}$ in the definition of $\mathbf{P}_Z$ in comparison to the constant term $-\zeta(-\ell)/2$ of the evaluation of $\mathbf{p}_\ell$ compensates for the fact that the strict bracket of the $\mathbf{f}_\ell$ gives the counting function in (67) up to a power of two. □

6.4 Conclusion of the proofs for spin subspaces

Proof of Theorems 6.3 and 6.2 Theorem 6.2 is a consequence of Corrolary 6.11. Indeed the arguments of Sect. 5 adapted to the series $\mathcal{H}_n$ show that the recursion in Theorem 6.2 is a consequence of the recursion in (79).

To prove Theorem 6.3, we consider first the case $n = 1$, i.e. $\mu = (2g-2)$. Let $\overline{\nu}_\mu$ be the push-forward of the Masur–Veech volume ν_μ form to $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$. Assumption 6.1 implies by the same argument as in [42, Lemma 2.1] that $\frac{2(2i\pi)^{2g}}{(2g-1)!}\xi^{2g-1}$ can be represented by a meromorphic differential form (of Poincaré growth at the boundary), whose restriction to $\mathbb{P}\Omega\mathcal{M}_{g,1}(2g-2)$ is equal to $\overline{\nu}_\mu$. This implies that

$$\mathrm{vol}(2g-2)^{\bullet} = \frac{2(2i\pi)^{2g}}{(2g-1)!}\int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,1}(2g-2)^{\bullet}} \xi^{2g-1}\,.$$

Now for the case $n \geq 2$ Theorem 6.2 and Proposition 6.6 determine $a(\mu)^{\bullet}$ and $v(\mu)^{\bullet}$ by the equivalent recursive formulas, and hence they coincide up to the obvious normalizing factors. □

6.5 Volume recursion for hyperelliptic components

In this subsection we prove the volume recursion for hyperelliptic components, which is analogous to but not quite the same as the recursion in Theorem 1.2. It is a consequence of the work of Athreya, Eskin and Zorich [2] on volumes of the strata of quadratic differentials in genus zero.

Recall that only the strata $\Omega\mathcal{M}_g(g-1, g-1)$ and $\Omega\mathcal{M}_g(2g-2)$ have hyperelliptic components. For the hyperelliptic components we still have an interpretation of their volumes as intersection numbers as well as a volume recursion as follows.

Theorem 6.12 *For the hyperelliptic components we have*

$$\mathrm{vol}(\Omega\mathcal{M}_{g,1}(2g-2)^{\mathrm{hyp}}) = \frac{2(2i\pi)^{2g}}{(2g-1)!}\int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,1}(2g-2)^{\mathrm{hyp}}} \xi^{2g-1}$$

and

$$\mathrm{vol}(\Omega\mathcal{M}_{g,2}(g-1, g-1)^{\mathrm{hyp}}) = \frac{2(2i\pi)^{2g}}{g(2g-1)!}\int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,2}(g-1,g-1)^{\mathrm{hyp}}} \xi^{2g-1}\psi_2,$$

provided that Assumption 6.1 holds.

As before we set

$$v(2g-2)^{\mathrm{hyp}} = (2g-1)\mathrm{vol}\,(\Omega\mathcal{M}_{g,1}(2g-2)^{\mathrm{hyp}}),$$
$$v(g-1, g-1)^{\mathrm{hyp}} = g^2\,\mathrm{vol}\,(\Omega\mathcal{M}_{g,2}(g-1, g-1)^{\mathrm{hyp}})\,.$$

Proposition 6.13 *The Masur-Veech volumes of the hyperelliptic components of $\Omega\mathcal{M}_{g,2}(g-1, g-1)$ satisfy the recursion*

$$v(g-1, g-1)^{\mathrm{hyp}} = v(2g-2)^{\mathrm{hyp}} + \sum_{\ell=1}^{g-1} \frac{(2\ell-1)!v(2\ell-2)^{\mathrm{hyp}}\,(2(g-\ell)-1)!v(2g-2\ell-2)^{\mathrm{hyp}}}{4(2g-1)!}\,.$$

Note in comparison to Theorem 1.2 that only the terms $k = 1$ and $k = 2$ appear and that the Hurwitz number $h_{\mathbb{P}}$ is identically one here. As a preparation for the proof recall that the canonical double cover construction provides isomorphisms

$$\mathcal{Q}_g(2g-3, (-1)^{2g-3}) \cong \Omega\mathcal{M}_{g,1}(2g-2)^{\text{hyp}},$$
$$\mathcal{Q}_g(2g-2, (-1)^{2g-2}) \cong \Omega\mathcal{M}_{g,2}(g-1, g-1)^{\text{hyp}}$$

that preserve the Masur–Veech volume and the $\mathrm{SL}_2(\mathbb{R})$-action. Taking into account the factorials for labeling zeros and poles the main result of [2] can be translated as

$$\mathrm{vol}\,(\Omega\mathcal{M}_{g,1}(2g-2)^{\text{hyp}}) = \frac{2}{(2g+1)!}\frac{(2g-3)!!}{(2g-2)!!}\pi^{2g},$$
$$\mathrm{vol}\,(\Omega\mathcal{M}_{g,2}(g-1, g-1)^{\text{hyp}}) = \frac{8}{(2g+2)!}\frac{(2g-2)!!}{(2g-1)!!}\pi^{2g}$$

where the double factorial notation means $(2k)!! = 2^k k!$ and $(2k-1)!! = (2k)!/2^k k!$.

Proof Expanding the definition of the double factorials and including the summand $v(2g-2)^{\text{hyp}}$ as the two boundary terms of the sum (i.e. $\ell = 0$ and $\ell = g$), we need to show that

$$\sum_{\ell=0}^{g} \frac{1}{2\ell+1}\binom{2\ell}{\ell}\frac{1}{2g-2\ell+1}\binom{2(g-\ell)}{g-\ell} = 2\frac{16^g}{(g+1)^2}\binom{2g+2}{g+1}^{-1}. \tag{80}$$

For this purpose it suffices to prove the following two identities of generating series

$$\sum_{\ell\geq 0} \frac{1}{2\ell+1}\binom{2\ell}{\ell}x^{2\ell} = \frac{1}{2x}\arctan\left(\frac{2x}{\sqrt{1-4x^2}}\right) \tag{81}$$

and

$$2\sum_{g\geq 0} \frac{16^g}{g^2}\binom{2g}{g}^{-1}x^{2g} = \frac{1}{4x^2}\arctan\left(\frac{2x}{\sqrt{1-4x^2}}\right)^2, \tag{82}$$

so that we can take the square of the first series and compare the x^{2g}-terms. To prove (81) we multiply it by x, differentiate, and are then left with showing that

$\sum_{\ell\geq 0}\binom{2\ell}{\ell}x^{2\ell} = 1/\sqrt{1-4x^2}$, which follows from the binomial theorem. To prove (82) we differentiate and are then left with the identity which is already proved in [34, p. 452, Equation (9)]. □

The last ingredient is the following straightforward consequence of Proposition 6.5 (analogous to the case of spin subspaces in Proposition 6.6).

Proposition 6.14 *For* $\mu = (g-1, g-1)$, *we have*

$$\begin{aligned} g^2 a(g-1, g-1)^{\mathrm{hyp}} \\ &= (2g-1)^2 a(2g-2)^{\mathrm{hyp}} \\ &\quad + \frac{1}{2}\sum_{g_1=1}^{g-1}\Big(h_{\mathbb{P}^1}\big((g-1, g-1), (2g_1-1, 2g-2g_1-1)\big) \\ &\quad \cdot (2g_1-1)^2 a(2g_1-2)^{\mathrm{hyp}}(2g-2g_1-1)^2 a(2g-2g_1-2)^{\mathrm{hyp}}\Big). \end{aligned} \tag{83}$$

Proof of Theorem 6.12 Since the proof of Theorem 6.12 for the case $\mu = (2g-2)$ was already given along with the proof of Theorem 6.3, it remains to show that $v(g-1, g-1)^{\mathrm{hyp}}$ and $a(g-1, g-1)^{\mathrm{hyp}} = \int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,2}(g-1,g-1)^{\mathrm{hyp}}} \beta_i \cdot \xi$ satisfy the same recursion. It is elementary to check that $h_{\mathbb{P}^1}\big((g-1, g-1), (2g_1-1, 2g-2g_1-1)\big) = 1$. Then the desired conclusion thus follows from Propositions 6.13 and 6.14. □

7 An overview of Siegel–Veech constants

Let (X, ω) be a flat surface, consisting of a Riemann surface X and an Abelian differential ω on X. Siegel–Veech constants measure the asymptotic growth rate of the number of saddle connections (abbreviated s.c.) or cylinders with bounded length (of the waist curve) in (X, ω). There are many variants that we now introduce and compare.

7.1 Saddle connection and area Siegel–Veech constants

For each pair of zeros (z_1, z_2) of ω we let

$$A^{\mathrm{phy}}_{1\leftrightarrow 2}(T) = |\{\gamma \subset X \text{ a saddle connection joining } z_1 \text{ and } z_2, \left|\int_\gamma \omega\right| \leq T\}| \tag{84}$$

be the counting function. The upper index emphasizes that we count all physically distinct saddle connections. It should be distinguished from the version

$$A^{\mathrm{hom}}_{1\leftrightarrow 2}(T) = |\{\gamma \subset X \text{ a homology class of s.c. joining } z_1 \text{ and } z_2, \left|\int_\gamma \omega\right| \le T\}|, \tag{85}$$

where a collection of homologous saddle connections just counts for one. Quadratic upper and lower bounds for such counting functions were established by Masur [37]. Fundamental works of Veech [45] and Eskin-Masur [16] showed that for almost every flat surface (X, ω) in the sense of the Masur–Veech measure (see [36] and [44]) there is a quadratic asymptotic, i.e. that

$$A^{\mathrm{phy}}_{1\leftrightarrow 2}(T) \sim c^{\mathrm{phy}}_{1\leftrightarrow 2}(X,\omega)\,\pi T^2, \qquad A^{\mathrm{hom}}_{1\leftrightarrow 2}(T) \sim c^{\mathrm{hom}}_{1\leftrightarrow 2}(X,\omega)\,\pi T^2 . \tag{86}$$

The constants $c^{\mathrm{phy}}_{1\leftrightarrow 2}(X, \omega)$ and $c^{\mathrm{hom}}_{1\leftrightarrow 2}(X, \omega)$ are the first type of Siegel–Veech constants we study here, called the *saddle connection Siegel–Veech constants*. The difference between these two Siegel–Veech constants becomes negligible as the genus of X tends to infinity, which follows from the results of Aggarwal and Zorich (see [4, Remark 1.1]).

The second type of Siegel–Veech constants counts homotopy classes of closed geodesics, or equivalently flat cylinders. Again, there are two variants, the naive count and the count where each cylinder is weighted by its relative area. As above, the most important counting function with good properties (see e.g. [12]) and connection to Lyapunov exponents [15] is the second variant. For the precise definition we consider

$$A_{\mathrm{cyl}}(T) = \sum_{\substack{Z\subset X \text{cylinder}\\ w(Z)\le T}} 1, \qquad A_{\mathrm{area}}(T) = \sum_{\substack{Z\subset X \text{cylinder}\\ w(Z)\le T}} \frac{\mathrm{area}(Z)}{\mathrm{area}(X)}, \tag{87}$$

where $w(Z)$ denotes the width of Z, i.e. the length of its core curve. We then define the *cylinder Siegel–Veech constant* and the *area Siegel–Veech constant* by the asymptotic equalities

$$A_{\mathrm{cyl}}(T) \sim c_{\mathrm{cyl}}(X,\omega)\pi T^2, \qquad A_{\mathrm{area}}(T) \sim c_{\mathrm{area}}(X,\omega)\pi T^2 . \tag{88}$$

There is a natural action of $\mathrm{GL}_2(\mathbb{R})$ on the moduli space of flat surfaces $\Omega\mathcal{M}_g$ and the orbit closures are nice submanifolds, in fact linear in period coordinates by the fundamental work of Eskin, Mirzakhani and Mohammadi [17,18]. We refer to them as *affine invariant manifolds*, using typically the letter $\mathcal{M}$. The intersection with the hypersurface of area one flat surfaces (denoted by the same symbol $\mathcal{M}$) comes with a finite $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic measure $\nu_{\mathcal{M}}$ with support $\mathcal{M}$. This measure is unique up to scale and for affine invariant manifolds defined over $\mathbb{Q}$ there are natural choices of the scaling.

It follows from the Siegel–Veech axioms (see [16]) that Siegel–Veech constants for almost all flat surfaces (X, ω) in an $\mathrm{SL}_2(\mathbb{R})$-orbit closure $\mathcal{M}$ agree. We call these surfaces *generic* (for $\mathcal{M}$) and write e.g. $c^{\star}_{1\leftrightarrow 2}(\mathcal{M}) = c^{\star}_{1\leftrightarrow 2}(X, \omega)$ for (X, ω) generic.

The relevant orbit closures in this paper are the connected components of the strata of Abelian differentials and certain Hurwitz spaces inside the strata. We usually abbreviate by $c^{\star}_{1\leftrightarrow 2}(\mu) = c^{\star}_{1\leftrightarrow 2}(\Omega\mathcal{M}_{g,n}(\mu))$ the Siegel–Veech constants for the strata with signature μ.

7.2 Configurations and the principal boundary

One of the main insights of [20] is that Siegel–Veech constants can be computed separately according to topological types, called configurations. We formalize their notion of configurations briefly so that it also applies to Hurwitz spaces, and in fact to all $\mathrm{SL}_2(\mathbb{R})$-orbit closures $\mathcal{M}$ provided with the generalization of the Masur–Veech measure $\nu_{\mathcal{M}}$. The concept of configurations will be used for showing the equivalence between Theorems 1.3 and 1.2 in Sect. 8.

Let $(X, z_1, \ldots, z_n)$ be a pointed topological surface. A *configuration* $\mathcal{C}$ *of saddle connections joining* z_1 *and* z_2 for $\mathcal{M}$ is a set of simple non-intersecting arcs from z_1 to z_2 up to homotopy preserving the cyclic ordering of the arcs both at z_1 and z_2. The last condition implies that the tubular neighborhood of the configuration is a well-defined subsurface of X, in fact a *ribbon graph* $R(\mathcal{C})$ associated with the configuration. (Here the subsurfaces of X after removing a ribbon graph are allowed to have positive genus.) The number of arcs in the configuration is called the *multiplicity* of the configuration.

We say that a subset of the saddle connections joining z_1 and z_2 on a flat surface (X, ω) *belongs* to the configuration $\mathcal{C}$, if this subset is homotopic to $\mathcal{C}$. Each configuration gives rise to a counting function $A^{\star}_{1\leftrightarrow 2}(T, \mathcal{C})$ for sets of saddle connections belonging to the configuration and having individual length $\leq T$. From the counting function we deduce the corresponding Siegel–Veech constant $c^{\star}_{1\leftrightarrow 2}(\mathcal{M}, \mathcal{C})$, where $\star \in \{\mathrm{phy}, \mathrm{hom}\}$ respectively. A configuration $\mathcal{C}$ is *relevant* if $c^{\star}_{1\leftrightarrow 2}(\mathcal{M}, \mathcal{C}) > 0$.

A *full set of saddle connection configurations* for an affine invariant manifold $\mathcal{M}$ is a finite set of saddle connection configurations $\mathcal{C}_i$, with $i \in I$ such that the contributions of the configurations $\mathcal{C}_i$ sum up to the full Siegel–Veech constant, i.e. such that

$$\sum_{i\in I} c^{\star}_{1\leftrightarrow 2}(\mathcal{M}, \mathcal{C}_i) = c^{\star}_{1\leftrightarrow 2}(\mathcal{M}) \tag{89}$$

for $\star \in \{\text{phy}, \text{hom}\}$ respectively.

Note that [20, Section 3.2] in their definition of configurations made a further subdivision of the notion by adding metric data, i.e. specifying angles between saddle connections. In that context, Eskin, Masur and Zorich determined a full set of saddle connection configurations for the strata and used the Siegel–Veech transform to connect the computation to volume computations. The following statement summarizes Proposition 3.3, Corollary 7.2 and Lemma 8.1 of [20].

Proposition 7.1 *For any stratum $\Omega\mathcal{M}_{g,n}(\mu)$ a full set of saddle connection configurations is the set of collections of pairwise homologous simple disjoint arcs joining z_1 and z_2 (up to homotopy).*

In this proposition, several configurations are irrelevant, for example those with a connected component of genus zero after removing the saddle connections in the configuration.

The general strategy to compute Siegel–Veech constants is the following relation to volumes, where the submanifold $\mathcal{M}$ is in a stratum with labeled zeros.

Proposition 7.2 *The saddle connection Siegel–Veech constants of an affine invariant manifold $\mathcal{M}$ can be computed as*

$$c^{\star}_{1\leftrightarrow 2}(\mathcal{M}) = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \sum_{\mathcal{C}} m^{\star}(\mathcal{C}) \frac{\nu_{\mathcal{M}}(\mathcal{M}^{\varepsilon}(\mathcal{C}))}{\nu_{\mathcal{M}}(\mathcal{M})}, \tag{90}$$

where the sum runs over the full set of saddle connection configurations and where $m^{\text{hom}}(\mathcal{C}) = 1$ for all $\mathcal{C}$ while $m^{\text{phy}}(\mathcal{C})$ is equal to the number of arcs in $\mathcal{C}$.

Proof This is a direct consequence of the Siegel–Veech transform applied to the characteristic function of a disc of radius ε, see [20, Lemma 7.3] together with the Eskin-Masur bound on the number of short saddle connections (Theorem on p. 84 of [20]). □

We conclude with remarks on Siegel–Veech constants for general affine invariant manifolds to put the digression on Hurwitz spaces (Sect. 9) in context. There is another variant, besides (84) and (85), of counting saddle connections. Given an affine invariant manifold $\mathcal{M}$ we say that two saddle connections on $(X, \omega) \in \mathcal{M}$ are *$\mathcal{M}$-parallel* if they are parallel and stay parallel in a neighborhood of (X, ω) in $\mathcal{M}$. (The terminology is completely analogous to the notion of *$\mathcal{M}$-parallel cylinders* introduced in [48].) We thus define the counting function $A^{\mathcal{M}-\text{p}}_{1\leftrightarrow 2}(T)$ and the Siegel–Veech constant $c^{\mathcal{M}-\text{p}}_{1\leftrightarrow 2}(\mathcal{M})$ in analogy to (85) and (86), counting once every $\mathcal{M}$-parallel class of cylinders. [20, Proposition 3.1] can now be restated as $c^{\mathcal{M}-\text{p}}_{1\leftrightarrow 2}(\mathcal{M}) = c^{\text{hom}}_{1\leftrightarrow 2}(\mathcal{M})$ if $\mathcal{M}$ is a connected component of a stratum. For Hurwitz spaces the two values can be different, but we will see (Proposition 9.3) that their difference becomes negligible as the degree of the covers tends to infinity.

In the first part of [20] on recursive computations of Siegel–Veech constants, Eskin, Masur and Zorich called the locus of degenerate surfaces that contribute to the Siegel–Veech counting the *principal boundary*. At that time the notion of principal boundary was used only as a partial topological compactification. Presently, we dispose of a complete and geometric compactification for the strata [6] and for Hurwitz spaces (by admissible covers), and we can then identify the principal boundary as part of the compactification (see [10] for the case of the strata and Sect. 9 for the case of Hurwitz spaces). The reader should keep in mind that the locus "principal boundary" depends on the type of saddle connections under consideration.

Finally we remark that there is a zoo of possibilities of associating weights with saddle connections and cylinders and to define Siegel–Veech constants accordingly. This started with [46], and see also [8] for computations and conversions of Siegel–Veech constants.

8 Saddle connection Siegel–Veech constants

In this section we deduce from the volume recursion and its refinement for spin and hyperelliptic components a proof of Theorem 1.3. Almost all we need here has been proven already in [20]. We start with two more auxiliary statements.

Proposition 8.1 *The full set of saddle connection configurations for the strata given in Proposition 7.1 is in bijection with (possibly unstable) backbone graphs and a cyclic ordering of its vertices at level zero. The subset of relevant configurations is in bijection with stable backbone graphs.*

Although not needed in the sequel, we relate for the convenience of the reader our notion of twists and the angle information that [20] recorded. Let $\{\gamma_i\}_{i=1}^k$ be the arcs of a configuration realized by (X, ω) labeled cyclically and let a_i' and a_i'' be the angles between γ_i and γ_{i+1} at z_1 and z_2 respectively. If an edge e is separated by the loop formed by γ_i and γ_{i+1}, then the twist is

$$\mathbf{p}(e) = \frac{1}{2\pi}(a_i' + a_i'') - 1 \,. \tag{91}$$

The above proposition can be seen as follows. Given a collection of k homologous short saddle connections there is a sphere (with z_1 as its south pole and z_2 as its north pole, see [20, Figure 5]) supporting the k saddle connections. This sphere is the source of the backbone of the graph. The components at level zero are bounded by the arcs γ_i and γ_{i+1}. The converse is obvious, given that all the edges of a (stable) backbone graph are separating by definition. Finally formula (91) is just a restatement of the Gauss-Bonnet theorem.

Recall that a backbone graph (being of compact type) is compatible with a unique twist $\mathbf{p}(\cdot)$. If the vertices at level zero are labeled as $1, \ldots, k$ as usual and if $(h_j, i(h_j))$ is the edge connecting the jth vertex to level -1, we write $p_j = |\mathbf{p}(h_j)|$ as we did in Sect. 3.

Proposition 8.2 *For each configuration $\mathcal{C}$ corresponding to a backbone graph Γ, the volume of the subspace $\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})$ satisfies*

$$\sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} \pi\varepsilon^2 \frac{h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p})}{k!\,(2g-3+n)!} \cdot \prod_{i=1}^{k} (2g_i - 1 + n_i)!\, p_i \;\operatorname{vol}(\Omega\mathcal{M}_{g_i,n_i+1}(\mu_i, p_i - 1))$$
$$= 2^{1-k} \operatorname{vol}(\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})) + o(\varepsilon^2)$$

as $\varepsilon \to 0$, where the summation conventions and $\mathbf{p}$ *are as in Theorem 1.2 and where $n_i = n(\mu_i)$.*

Proof This is mainly contained in [20, Corollary 7.2, Formulas 8.1 and 8.2], stating that the volume of the locus with an ε-short configuration is $\pi\varepsilon^2$ times the volume of the corresponding boundary. We now explain the combinatorial factors that appear. First, the factor of two and the factorials result from the passage of the boundary volume element in the ambient stratum to the product of the volume elements of the components at the boundary, as explained in detail in [20, Section 6]. The $1/k!$ stems from labelling the level zero vertices. Second, we need to count the ways to obtain a surface in $\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})$ by gluing a collection of surfaces (X_i, ω_i) in $\Omega\mathcal{M}_{g_i,n_i+1}(\mu_i, p_i - 1)$.

Suppose we are given a branched cover $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ that has ramification profile $\Pi = ((m_1 + 1), (m_2 + 1), (p_1, \ldots, p_k))$ over the points 0, 1 and ∞. We provide the domain with the differential $\omega_{-1} = b^*dz$. Since this differential has no residues we can glue $t \cdot \omega_{-1}$ with the surfaces (X_i, ω_i) by cutting the pole of order $p_i + 1$ and gluing it to an annular neighborhood of the zero of order $p_i - 1$ of ω_i. For $t \leq \varepsilon$ this provides a surface in $\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})$, see e.g. [6, Section 4] for details of the construction. The plumbing construction also depends on the choice of a p_ith root of unity at each pole (from the choice of a horizontal slit at a zero of order $p_i - 1$). In total there are $h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \cdot \prod_{i=1}^{k} p_i$ possibilities involved in the construction, thus justifying the remaining combinatorial factors in the formula.

We claim that this construction provides a collection of maps to $\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})$ that are almost everywhere injections if none of the surfaces (X_i, ω_i) has a period of length smaller than ε. In fact, if two such plumbed surfaces are isomorphic, this isomorphism restricts to an automorphism of $(\mathbb{P}^1, \omega_{-1})$ (see [20, Lemma 8.1] for more details) and this happens only on a measure zero set. The locus where one of the (X_i, ω_i) has a short period is subsumed in the

$o(\varepsilon^2)$ [20, Lemma 7.1]. Conversely, for each surface (X, ω) in $\Omega\mathcal{M}^{\varepsilon}_{g,n}(\mu, \mathcal{C})$ we can cut a ribbon graph around the configuration $\mathcal{C}$. The restriction of ω has no periods since the boundary curves are homologous by definition of $\mathcal{C}$. It can thus be integrated and completed to a map $b\colon \mathbb{P}^1 \to \mathbb{P}^1$ with ramification profile as above. □

Proof of Theorem 1.3 We first focus on the case that $\Omega\mathcal{M}_{g,n}(\mu)$ is connected. A decomposition $g = \sum_{i=1}^{k} g_i$ and $(m_3, \ldots, m_n) = \mu_1 \sqcup \cdots \sqcup \mu_k$ (as in Eq. (5) we proved) determines uniquely a configuration and the converse is true up to the labeling of the k vertices at level zero by Proposition 8.1. The configuration is relevant if and only if the volumes on the right-hand side of (5) are non-zero. Since the saddle connection Siegel–Veech constant is the sum of ratios of the boundary volumes over the total volume (by Proposition (7.2)), comparing the formula in Proposition 8.2 with Eq. (5) thus implies Theorem 1.3. More precisely, note that the rescaled volume $v(\mu)$ defined in (4) involves a product of all $(m_i + 1)$, while on the right-hand side $(m_1 + 1)(m_2 + 1)$ is missing and this factor gives the right hand side of the desired formula in Theorem 1.3. The factor of π in Proposition 8.2 cancels with the one in Proposition 8.1.

For disconnected strata with components parameterized by $S \subseteq$ {odd, even, hyp} the same proof gives the averaged version that

$$\frac{1}{v(\mu)} \sum_{\bullet \in S} v(\mu)^{\bullet}\, c^{\mathrm{hom}}_{1\leftrightarrow 2}(\mu)^{\bullet} = (m_1 + 1)(m_2 + 1)$$

by rewriting Eq. (90) as a sum over the connected components. It remains to prove the statement for the odd spin subspaces (that may be disconnected for some strata in our notation, in which case there is an extra hyperelliptic component with the same spin parity) as well as for the hyperelliptic components. Then one obtains the desired result for each connected component by taking suitable differences.

For the odd spin subspaces we now focus on Theorem 6.2 instead of (5) and note that each decomposition of g and μ still determines uniquely a configuration. Moreover, since the Arf-invariant on stable curves of compact type is additive (see [10, Proposition 4.6] and [20, Lemma 10.1]) and since by definition the spin assignment is additive on the vertices, the configurations appearing on the right-hand side in Theorem 6.2 are precisely the configurations that contribute (in the sense of Proposition 7.2) to the Siegel–Veech constant of the odd spin subspaces. We thus obtain the analogous statement of Theorem 1.3 for the odd spin subspaces by comparing Proposition 8.2 to Theorem 6.2.

For the hyperelliptic components we use Proposition 6.13 and note that the summands on the right-hand side there correspond exactly to the configurations on hyperelliptic components, as explained in [10, Proposition 4.3] and [20, Lemma 10.3]. □

9 The viewpoint of Hurwitz spaces

This section is a digression on how to interpret the volume recursion and the saddle connection Siegel–Veech constant from the viewpoint of Hurwitz spaces. The results in this section are not needed for proving any of the theorems stated in the introduction. We will rather explain and motivate

- why the homologous count of saddle connections is more natural than the physical count from the viewpoint of intersection theory,
- how to heuristically deduce the value of the saddle connection Siegel–Veech constant in Theorem 1.3 from an equidistribution of cycles in Hurwitz tuples, and
- why backbone graphs correspond to configurations.

As usual $\mu = (m_1, \ldots, m_n)$ is a partition of $2g-2$ and the ramification profile Π consists of n cycles $\mu^{(i)}$ of length $m_i + 1$ unless stated otherwise. Let $r(\mu) = 2g + n(\mu) - 1$ be the dimension of the (unprojectivized) stratum $\Omega\mathcal{M}_g(\mu)$.

Theorem 9.1 *There exists a constant $M(\mu)$ such that the Hurwitz numbers $N_d^\circ(\Pi)$ for connected torus covers of profile Π can be approximated as*

$$M(\mu)\, N_d^\circ(\Pi) = \sum_{\Gamma \in \mathrm{BB}^\star_{1,2}} \frac{d}{k!} h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \sum_{d_1+\cdots+d_k=d} \prod_{i=1}^{k} p_i N_{d_i}^\circ(\Pi_i, (p_i)) + o(d^{r(\mu)-1}), \tag{92}$$

where $\Pi \backslash \{\mu^{(1)}, \mu^{(2)}\} = \Pi_1 \sqcup \cdots \sqcup \Pi_k$ is the decomposition of the profile according to the leg assignment in Γ and where $\mathbf{p} = (p_1, \ldots p_k)$ is the unique twist compatible with Γ.

At the end of the section we will show by combining Theorem 1.2 together with Theorem 9.1 the following result.

Proposition 9.2 *The constant $M(\mu)$ in Theorem 9.1 is $(m_1+1)(m_2+1)$. In particular, $M(\mu)$ depends only on the first two zero orders of μ.*

Indeed an independent proof of Proposition 9.2 would provide an alternative proof of Theorem 1.2 (and hence Theorem 1.1) that would bypass the complicated combinatorics in Sects. 4 and 5.

The strategy to prove Theorem 9.1 consists of comparing the Hurwitz number $N_d^\circ(\Pi)$, that is the fiber cardinality of the forgetful map $f_T\colon H_d(\Pi) \to \mathcal{M}_{1,n}$ to the target curve with the fiber cardinality of the extension of f_T to the space of admissible covers $\overline{H}_d(\Pi) = \overline{H}_{d,g,1}(\Pi)$ over degenerate targets of the following type.

9.1 Admissible torus covers

Let $E_{0,\{1,2\}}$ be the stable curve of genus one consisting of a $\mathbb{P}^1$-component carrying precisely the first two marked points and of an elliptic curve E carrying the remaining marked points, joint at a node q_E. If $p\colon X \to E_{0,\{1,2\}}$ is an admissible cover, we denote by X_0 and X_{-1} the (possibly reducible) curves mapping to E and to $\mathbb{P}^1$ respectively, both deprived of their unramified $\mathbb{P}^1$-components. See Fig. 3 for examples of such admissible covers.

The admissible covers of $E_{0,\{1,2\}}$ come in two types. One possibility is that the first two branch points are in the same (hence the unique) component of X_{-1}. The stable dual graph of the cover is thus a graph $\Gamma \in \mathrm{ABB}^\star_{1,2}$ (recall the definition of ABB graphs in Sect. 3.2) and we denote by $N_d^\circ(\Pi, E_{0,\{1,2\}}, \Gamma)$ the number of such covers. The second possibility is that each of the two branch points is on its own component $X_{-1}^{(i)}$ for $i = 1, 2$. Consequently, the map $p|_{X_{-1}^{(i)}}$ is a cyclic cover of degree $(m_i + 1)$. By contracting the components over $\mathbb{P}^1$ we see that such covers are in bijective correspondence (up to the automorphism group of size $|\mathrm{Aut}(X_{-1}/\mathbb{P}^1)| = (m_1 + 1)(m_2 + 1)$) with covers of E with the profile $\Pi_{(12)} = ((\mu^{(1)}, \mu^{(2)}), \mu^{(3)}, \ldots, \mu^{(n)})$, where the first two ramification points are piled over the same branch point.

Proposition 9.3 *For* $\Gamma \in \mathrm{ABB}^\star_{1,2}$ *the bound* $N_d^\circ(\Pi, E_{0,\{1,2\}}, \Gamma) = O(d^{r(\mu)-2})$ *holds as* $d \to \infty$. *The upper bound is attained, i.e. there exists* $C > 0$ *such that* $N_d^\circ(\Pi, E_{0,\{1,2\}}, \Gamma) \ge C d^{r(\mu)-2}$, *if and only if* $\Gamma \in \mathrm{BB}^\star_{1,2}$. *Moreover in this case*

$$N_d^\circ(\Pi, E_{0,\{1,2\}}, \Gamma) = \frac{1}{k!} h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \sum_{d=d_1+\cdots+d_k} \prod_{i=1}^{k} N_{d_i}^\circ(\Pi_i, (p_i)), \tag{93}$$

where Π_i *and* $\mathbf{p}$ *are associated with* Γ *as in Theorem* 9.1.

Proof Recall from [21] or the proof of [12, Proposition 9.4] that if Π is the profile for a cover π with $\pi^*\omega \in \Omega\mathcal{M}_{g,n}(\mu)$, then there exist $C_1, C_2 \neq 0$ such that $C_1 \cdot d^{r(\mu)-1} \le N_d^\circ(\Pi) \le C_2 \cdot d^{r(\mu)-1}$ as $d \to \infty$. Suppose that Γ has k components on level 0, each of genus g_i and with n_i marked points or nodes. Let g_0 be the genus of the component on level -1, and let $b = h^1(\Gamma)$. Then

$$b + \sum_{i=0}^{k} g_i = g \quad \text{and} \quad \sum_{i=1}^{k} n_i = n - 2 + k + b\,.$$

The cover of the $\mathbb{P}^1$-component has finitely many choices independent of d. Over the elliptic component of $E_{0,\{1,2\}}$, the number of choices of covers has asymptotic growth given by $B \sum_{d_1+\cdots+d_k=d} d_i^{2g_i-2+n_i}$ for some constant B independent of d. This quantity is a polynomial of degree

$$\Big(\sum_{i=1}^{k}(2g_i - 2 + n_i)\Big) + (k-1) = 2g - 2g_0 - b + n - 3\,.$$

We thus conclude that the total number of admissible covers $N_d^0(\Pi, E_{0,\{1,2\}}, \Gamma)$ has asymptotic growth given by a polynomial of degree $r(\mu) - 2 - b - 2g_0 \le r(\mu) - 2$, with equality attained if and only if $b = g_0 = 0$, i.e. if and only if $\Gamma \in \mathrm{BB}^{\star}_{1,2}$.

To justify Eq. (93) we refer to the computation of the Hurwitz numbers in Proposition 2.1 and divide by $k!$ to account for our auxiliary labeling of the k components of X_0. □

We remark that Proposition 9.3 is not used in any other proofs in the paper, but it reveals the geometric reason for the homologous count of saddle connections behind the recursions in Sects. 3–5. The factor $h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p})$ (possibly with $1/k!$ if all branches are labeled) appears in the direct count of admissible covers and in the count of configurations. There is no extra factor k in (93), which corresponds to our setting of the coefficient $m^{\mathrm{hom}}(\mathcal{C}) = 1$ (instead of k) in (90) for homologous count of saddle connections (instead of physical count).

Proof of Theorem 9.1 We first show that

$$\begin{aligned} &N_d^{\circ}(\Pi) - N_d^{\circ}(\Pi_{(12)}) \\ &\quad = \sum_{\Gamma \in \mathrm{BB}^{\star}_{1,2}} \frac{1}{k!} h_{\mathbb{P}^1}((m_1, m_2), \mathbf{p}) \sum_{d_1+\cdots+d_k=d} \prod_{i=1}^{k} p_i N_{d_i}^{\circ}(\Pi_i, (p_i)) + O(d^{r(\mu)-2})\,. \end{aligned} \tag{94}$$

To see this, note that the ramification order of f_T over $E_{0,\{1,2\}}$ at the branch through a cover $\pi : X \to E$ is equal to the product of ramification orders at the nodes of X. This results in the factors p_i inside the product of the right-hand side and cancels the factor $1/|\mathrm{Aut}(X_{-1}/\mathbb{P}^1)|$ when counting $E_{0,\{1,2\}}$-covers instead of counting $N_d^\circ(\Pi_{(12)})$.

On the other hand, since the volume of the stratum can be approximated by counting covers of profile $\Pi_{(12)}$ and since the generating function of counting these covers is a quasimodular form, arguments as in [12, Proposition 9.4] imply the existence of a constant $M(\mu)$ such that

$$\frac{N_d^\circ(\Pi_{(12)})}{N_d^\circ(\Pi)} = \frac{d - M(\mu)}{d} + o(1/d)\,. \tag{95}$$

The combination of Eqs. (94) and (95) thus implies the desired formula (92). □

We now address the equidistribution heuristics for saddle connection Siegel–Veech constants. Recall that $N^\circ(\Pi)$ is the number (weighted by $|\mathrm{Aut}(p)|$) of transitive Hurwitz tuples $(\alpha, \beta, (\gamma_i)_{i=1}^n) \in S_d^{n+2}$ with $[\alpha, \beta] = \prod_{i=1}^n \gamma_i$ and γ_i of type $\mu^{(i)}$.

Proposition 9.4 *If the pairs (γ_1, γ_2) appearing in the Hurwitz tuples of profile Π equidistribute among pairs of $(m_i + 1)$-cycles in S_d^2 as $d \to \infty$, then $M(\mu) = (m_1 + 1)(m_2 + 1)$.*

Proof If the non-trivial cycles in γ_1 and γ_2 have no letter in common, then taking $(\alpha, \beta, \gamma_1 \circ \gamma_2, \gamma_3, \ldots, \gamma_n)$ is a Hurwitz tuple of profile $\Pi_{(12)}$. Assuming equidistribution and comparing to the total number of Hurwitz tuples, the number of Hurwitz tuples with γ_1 and γ_2 having two letters in common is negligible, while the ratio of those cycles having one letter in common is $(m_1 + 1)(m_2 + 1)/d + o(1/d)$. □

Example 9.5 For the reader's convenience we illustrate the contributions to the right-hand side of (94) for the stratum $\Omega\mathcal{M}_2(1, 1)$ explicitly in Fig. 3.

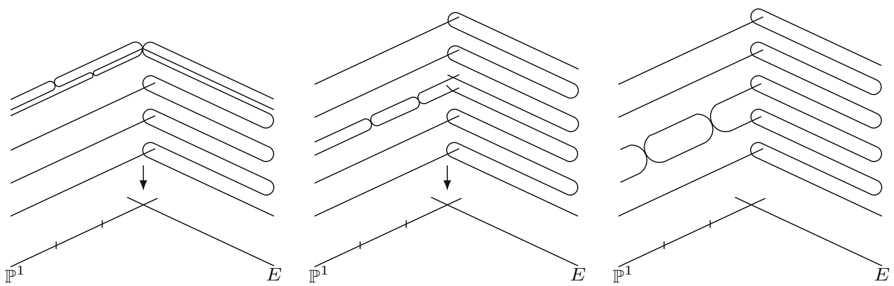


Fig. 3 Configurations for Hurwitz spaces in $\Omega\mathcal{M}_2(1, 1)$

The picture on the left gives stable graphs in $\mathrm{BB}^{\star}_{1,2}$, while the pictures in the middle and on the right give graphs in $\mathrm{ABB}^{\star}_{1,2} \setminus \mathrm{BB}^{\star}_{1,2}$. The preimages of E in the middle and on the right are unramified and thus again are elliptic curves, while on the left the preimage of E is a curve of genus two.

The saddle connection Siegel–Veech counting in this case was carried out in [19] in a similar way as summarized in Theorem 9.1, despite that only primitive torus covers were considered.

9.2 The principal boundary of Hurwitz spaces

We focus on saddle connections joining the first two marked zeros and determine a full set of configurations and the corresponding principal boundary of the Hurwitz spaces. We say that $\Gamma \in \mathrm{ABB}^{\star}_{1,2}$ is *realizable* in $\overline{H}_d(\Pi)$ if there is an admissible cover $p\colon X \to E_{0,\{1,2\}}$ whose stable graph is Γ and such that the vertices with $\ell(v) = 0$ correspond bijectively to the components of X_0. Recall also the definition of ribbon graphs associated to configurations in Sect. 7.2.

Proposition 9.6 *Associating with a configuration $\mathcal{C}$ the boundary curves of the ribbon graph $R(\mathcal{C})$ induces a map $\varphi\colon \mathcal{C} \to \Gamma(\mathcal{C})$ from a full set of saddle connection configurations onto the subset of* $\mathrm{ABB}^{\star}_{1,2}$ *that is realizable in* $\overline{H}_d(\Pi)$. *The image of φ is independent of d for d large enough. The fibers of φ are finite with cardinality bounded independently of d.*

Moreover if a graph $\Gamma \in \mathrm{BB}^{\star}_{1,2}$ is realizable, then the configurations in $\varphi^{-1}(\Gamma)$ are in bijection with cyclic orderings of the components at level 0.

Proof To define φ, we pinch the boundary curves of $R(\mathcal{C})$ to obtain a pointed nodal curve. The configuration $\mathcal{C}$ of saddle connections remains in one component of the curve that contains z_1 and z_2. We provide the dual graph of the curve with the level structure such that the component containing z_1 and z_2 is the unique one at level -1 and all the other components are on level 0. This way we thus obtain a graph $\varphi(\mathcal{C}) \in \mathrm{ABB}^{\star}_{1,2}$. We leave the straightforward verification of the other statements to the reader. □

An application of the Riemann–Hurwitz formula shows that any configuration in $\varphi^{-1}(\Gamma)$ has multiplicity $|E(\Gamma)| + 2g(X_{-1})$. In the special case $\Gamma \in \mathrm{BB}^{\star}_{1,2}$ (i.e. if $g(X_{-1}) = 0$ and Γ is of compact type), the configuration consists of $k = |E(\Gamma)|$ pairwise homologous arcs. However, the cover on the right-hand side of Fig. 3 shows that graphs in $\Gamma \in \mathrm{ABB}^{\star}_{1,2} \setminus \mathrm{BB}^{\star}_{1,2}$ also contribute. It is not hard to give an example that the fiber cardinality of φ over a target graph with $g(X_{-1}) > 0$ can indeed be larger than one, and we leave it to the reader since it is irrelevant to our applications.

Finally we address that Theorem 9.1 and an a priori knowledge that $M(\mu) = (m_1 + 1)(m_2 + 1)$ would give an alternative proof of Theorem 1.2. In terms of our volume normalization, [12, Proposition 9.4] says that

$$\sum_{d=1}^{D} N_d^\circ(\Pi) = \frac{v(\mu)}{2r\prod_{i=1}^{n}(m_i+1)}\, D^r + O(D^{r-1}\log D) \tag{96}$$

as $D \to \infty$, where $r = 2g + n - 1$. To sum the right-hand side of (92) we let $S_D(\Pi_i) = \sum_{d=1}^{D} N_d^\circ(\Pi_i)$. The Euler integral $\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{(a-1)!(b-1)!}{(a+b-1)!}$ used recursively implies the following result.

Lemma 9.7 *Suppose that $S_D(\Pi_i) = v_i D^{r_i} + O(D^{r_i-1}\log D)$ as $D \to \infty$ and that there exists a constant C depending on Π_i only such that $N_d^\circ(\Pi_i) < Cd^{r_i-1}$ for $i = 1, \dots, k$. Then*

$$\lim_{D\to\infty} \sum_{d_1+\cdots+d_k\le D} \frac{(d_1+\cdots+d_k)\, N_{d_1}^\circ(\Pi_1)\cdots N_{d_k}^\circ(\Pi_k)}{D^{r_1+\cdots+r_k+1}} = \frac{\prod_{i=1}^{k}(r_i! v_k)\sum_{i=1}^{k} r_i}{(r_1+\cdots+r_k+1)!}. \tag{97}$$

Alternative proof of Theorem 9.1 (assuming $M(\mu) = (m_1+1)(m_2+1)$). With the abbreviation $r_i = 2g_i + n(\mu_i)$ we obtain from (96) that

$$\sum_{d_i=1}^{D} N_{d_i}^\circ(\Pi_i,(p_i)) = \frac{v(\mu_i, p_i-1)}{2r_i p_i \prod_{m_i\in\mu_i}(m_i+1)} D^{r_i} + O(D^{r_i-1}\log D).$$

Since $r_1 + \cdots + r_k = 2g + n - 2$, Lemma 9.7 implies that

$$\begin{aligned}\sum_{d_1+\cdots+d_n\le D} (d_1+\cdots+d_n)\prod_{i=1}^{k} p_i\, N_{d_i}^\circ(\Pi_i,(p_i))\\ = D^{2g+n-1}\frac{\prod_{i=1}^{k}(2g_i+n(\mu_i)-1)!\,v(\mu_i,p_i-1)}{2^k(2g+n-1)(2g+n-3)!\prod_{i=3}^{n}(m_i+1)}.\end{aligned} \tag{98}$$

Summing over all backbone graphs and taking the limit after dividing by D^{2g+n-1} thus implies the desired formula (5). □

Proof of Proposition 9.2 Conversely, the above argument shows that the mere knowledge of Theorem 9.1 gives the recursion in Theorem 1.2 with $M(\mu)$ on the left-hand side that replaces $(m_1+1)(m_2+1)$, and hence the two theorems taken together thus determine the value $M(\mu) = (m_1+1)(m_2+1)$ as claimed in Proposition 9.2. □

10 Area Siegel–Veech constants

The goal of this section is to show that area Siegel–Veech constants are ratios of intersection numbers, i.e. to prove Theorem 1.4. For this purpose we introduce

$$d_i(\mu) = \int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)} \beta_i \cdot \delta_0 = \frac{1}{m_i+1}\int_{\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}(\mu)} \xi^{2g-2} \cdot \delta_0 \cdot \prod_{j\neq i} \psi_j, \tag{99}$$

and then Theorem 1.4 can be reformulated as

$$c_{\mathrm{area}}(\mu) = \frac{-1}{4\pi^2}\frac{d_i(\mu)}{a_i(\mu)}. \tag{100}$$

The proof proceeds similarly to the proof of Theorem 1.1 by showing a recursive formula for both the intersection numbers and the area Siegel–Veech constants. The difference in the formulas is that one vertex at level zero of the backbone graphs is distinguished by carrying the Siegel–Veech weight. We remark that in this section area Siegel–Veech constants for disconnected strata are volume-weighted averages of the constants for the individual components.

The intersection number recursion leads to the remarkable formula

$$(m_1+1)(m_2+1)c_{\mathrm{area}}(\mu) = \sum_{\substack{\mathcal{C} \text{ saddle connection} \\ \text{configuration}}} c^{\mathrm{hom}}_{1\leftrightarrow 2}(\mathcal{C})c_{\mathrm{area}}(\mathcal{C}), \tag{101}$$

where $c_{\mathrm{area}}(\mathcal{C})$ is defined to be the sum of the area Siegel–Veech constants of the splitting pieces induced by the saddle connection configuration. The other recursion (via q-brackets as in Sect. 4) leads to a very efficient way to compute area Siegel–Veech constants, given in Theorem 10.6.

10.1 A recursion for the $d_i(\mu)$ via intersection theory

We have seen that the values of $a_i(\mu)$ do not depend on the index i. Similarly for $d_i(\mu)$ it suffices to focus on the case $i = 1$. To state the recursion, we introduce the generating series

$$\Delta(t) = \sum_{g\geq 1}(2g-1)^2 d_1(2g-2)t^{2g} = \frac{1}{2}t^2 - \frac{1}{16}t^4 + \frac{91}{2304}t^6 - \frac{4173}{829440}t^8 + \cdots,$$

whose coefficients are determined using the following proposition. Recall the generating function $\mathcal{A}$ in (14) and the numbers b_j defined in (15).

Proposition 10.1 *The generating function $\Delta(t)$ of the intersection numbers $d_1(2g-2)$ is determined by the coefficient extraction identity*

$$\frac{2}{j!}[t^1]\left(\Delta(t)\mathcal{A}(t)^j\right) = b_{j-1} \tag{102}$$

while the intersection numbers $d(\mu) = d_i(\mu)$ with $n(\mu) \geq 2$ are given recursively by

$$\begin{aligned}(m_1+1)(m_2+1)d_1(\mu) \\ = \sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} \frac{h_{\mathbb{P}^1}((m_1,m_2),\mathbf{p})}{(k-1)!}\,\frac{d_1(p_1-1,\mu_1)}{a_1(p_1-1,\mu_1)}\prod_{i=1}^{k}(2g_i-1+n(\mu_i))p_i a_1(p_i-1,\mu_i)\end{aligned} \tag{103}$$

for $n = n(\mu) \geq 2$, with the usual summation conventions as in Theorems 1.2 and 3.1.

The first identity (102) was proved in [42]. The proof of the second identity (103) will be completed by the end of this subsection. This identity together with the conversions in Sect. 8 implies (101) immediately. We start the proof with the following analog of Proposition 3.10.

Proposition 10.2 *If $(\Gamma, \ell, \mathbf{p})$ is a backbone graph in $\mathrm{BB}(g,n)_{1,2}$, then*

$$\begin{aligned}\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}} \alpha_{\Gamma,\ell,\mathbf{p}}\cdot\xi^{2g-2}\cdot\delta_0\cdot\prod_{i=3}^{n}\psi_i &= m(\mathbf{p})\cdot h_{\mathbb{P}_1}(\mu_{-1},(p_v)_{v\in V(\Gamma),\ell(v)=0}) \\ &\cdot\sum_{v\in V(\Gamma),\ell(v)=0} d_1(p_v-1,\mu_v)\prod_{v'\in V(\Gamma)\setminus\{v\},\ell(v')=0} a_1(p_{v'}-1,\mu_{v'})\end{aligned}$$

with the conventions for p_v as in Proposition 3.10.

Proof We have the equality that

$$\zeta_\Gamma^*(\delta_0) = \sum_{v\in V(\Gamma),\ell(v)=0} \delta_0^v,$$

where $\delta_0^v = \delta_0 \otimes 1 \in H^*(\overline{\mathcal{M}}_{g_v,n_v},\mathbb{Q}) \otimes \left(\bigotimes_{v'\neq v} H^*(\overline{\mathcal{M}}_{g_{v'},n_{v'}},\mathbb{Q})\right) \simeq H^*(\overline{\mathcal{M}}_\Gamma,\mathbb{Q})$. Combining with the fact that $\delta_0\lambda_g = 0$, it implies that

$$\zeta_\Gamma^*(\delta_0\lambda_{g-1}) = \sum_{v\in V(\Gamma),\ell(v)=0}\left(\delta_0^v\lambda_{v,g_v-1}\bigotimes_{v'\neq v,\ell(v')=0}\lambda_{v',g_{v'}}\right).$$

Therefore, we obtain that

$$\lambda_{g-1}\cdot\delta_0\cdot\alpha^0_{\Gamma,\ell,\mathbf{p}} = \sum_{v\in V(\Gamma),\ell(V)=0} \zeta_{\Gamma*}\Big([\overline{\mathcal{M}}_{-1}]\otimes\big(\lambda_{v,g_v-1}\cdot\delta_0\cdot[\mathbb{P}\overline{\Omega\mathcal{M}}_{g_v,n_v}(p_v-1,\mu_v)]^0\big) \bigotimes_{v'\neq v,\ell(v')=0} \lambda_{v',g_{v'}}[\mathbb{P}\overline{\Omega\mathcal{M}}_{g_{v'},n_{v'}}(p_{v'}-1,\mu_{v'})]^0\Big).$$

Using the last formula in Lemma 3.8, the rest of the proof then follows from the same argument as in Proposition 3.10. □

We also need the following analog of Lemma 3.12.

Lemma 10.3 *The values of $d_1(\mu)$ satisfy the recursion*

$$(m_1+1)(m_2+1)d_1(\mu) = \sum_{k\geq 1}\sum_{\mathbf{g},\boldsymbol{\mu}} h_{\mathbb{P}^1}((m_1,m_2,\mu_0),\mathbf{p})\cdot\frac{d_1(p_1-1,\mu_1)}{(k-1)!}\cdot\left(\prod_{i=2}^{k}p_i^2 a(p_i-1,\mu_i)\right) \tag{104}$$

with summation conventions as in Lemma 3.12.

Proof We use the formula in Proposition 3.11 for $i=2$, multiply by $\xi^{2g-2}\delta_0\prod_{i=3}^n\psi_i$ and apply p_*. The left-hand side then evaluates (by Lemma 3.8 and the fact that $\delta_0\lambda_g=0$) to the left-hand side of (104). The right-hand side evaluates (by Proposition 3.9) to the weighted sum over all $(\Gamma,\ell,\mathbf{p})\in \mathrm{BB}(g,n)_{1,2}$ of the expression in Proposition 10.2. To prove the lemma we interpret as usual a backbone graph as a decomposition of g and the marked points. The factor $(k-1)!$ (instead of $k!$ in Lemma 3.12) comes from the fact that one of the top level vertices of the backbone graph is distinguished. □

With the same notation as in Sect. 3.5 we now define the d-contribution of a rooted tree to be

$$d(\Gamma,\ell,\mathbf{p}) = m_0(\mathbf{p})^2 h(\Gamma_0,\ell_0,\mathbf{p}_0)\sum_{\substack{v\in V(\Gamma),\\ \ell(v)=0}} d_1(p_v-1,\mu_v)\prod_{\substack{v'\in V(\Gamma)\setminus\{v\},\\ \ell(v')=0}} a(p_{v'}-1,\mu_{v'})\,.$$

Then we can rewrite Lemma 10.3 as

$$(m_1+1)(m_2+1)d_1(\mu) = \sum_{(\Gamma,\ell,\mathbf{p})\in \mathrm{RT}(g,\mu)_{1,2}} \frac{d(\Gamma,\ell,\mathbf{p})}{|\mathrm{Aut}(\Gamma,\ell,\mathbf{p})|},$$

which is the analog of Lemma 3.13.

End of the proof of Proposition 10.1 Now the proof of the proposition can be completed similarly to the end of the proof of Theorem 3.1 at the end of Sect. 3. □

As a consequence, $d_i(\mu)$ does not depend on i and we simply write $d(\mu)$ from now on.

10.2 A recursion for area Siegel–Veech numerators via weighted counting of covers

We recall the main steps of [12] that reduce the computation of area Siegel–Veech constants to a statement about cumulants. An application of the Siegel–Veech formula [12, Proposition 17.1] gives the quantity we want to compute by

$$c_{\mathrm{area}}\left(\Omega\mathcal{M}_g(m_1,\ldots,m_n)\right) = \lim_{D\to\infty} \frac{3}{\pi^2} \frac{\sum_{d=1}^{D} c_{-1}^{\circ}(d,\Pi)}{\sum_{d=1}^{D} N_d^{\circ}(\Pi)}, \tag{105}$$

where $N_d^{\circ}(\Pi)$ is the number of connected torus covers of degree d with ramification profile $\Pi = (m_1+1,\ldots,m_n+1)$ and where $c_{-1}^{\circ}(d,\Pi)$ is the sum over those covers with -1st Siegel–Veech weight (see [12, Section 3]). The relation of the sum of Fourier coefficients to the growth polynomial [12, Proposition 9.4] and a rewriting of the Siegel–Veech weighted counting [12, Corollary 13.2] translate this into

$$\begin{aligned} c_{\mathrm{area}}\left(\Omega\mathcal{M}_g(m_1,\ldots,m_n)\right) &= \frac{3}{\pi^2}\frac{\left\langle T_{-1}|f_{m_1+1}|\cdots|f_{m_n+1}\right\rangle_L}{\left\langle f_{m_1+1}|\cdots|f_{m_n+1}\right\rangle_L} \\ &= \frac{3}{\pi^2}\frac{\left\langle T_{-1}|h_{m_1+1}|\cdots|h_{m_n+1}\right\rangle_L}{\left\langle h_{m_1+1}|\cdots|h_{m_n+1}\right\rangle_L}, \end{aligned} \tag{106}$$

where T_{-1} is a hook-length moment function on partitions (but not an element of the ring Λ^*, see [12, Section 13]). Here we use Proposition 4.2 and (40) in the second equality for the denominator. Before passing to the leading term, the numerator $\langle T_{-1}|f_{m_1+1}|\cdots|f_{m_n+1}\rangle$ is a linear combination of differences of the form $\left\langle T_{-1}f\right\rangle_q - \left\langle T_{-1}\right\rangle_q\left\langle f\right\rangle_q$. By [12, Theorem 6.1] these differences can be written as explicit linear combinations of derivatives of Eisenstein series and homogeneous differential operators. We can perform the same rewriting with $\langle T_{-1}|h_{m_1+1}|\cdots|h_{m_n+1}\rangle$. Again thanks to (40) the difference is a cumulant with each entry being of strictly smaller weight. Consequently, it does not contribute to the leading order term.

We have seen in Sect. 4 how to compute the denominator and related it in Sect. 5 to $a(\mu)$. Now we take care of the numerator. Recall that we defined $\Phi^H(\mathbf{u})_q$ in (51) in Sect. 4. Define now

$$C'_{-1}(\mathbf{u})_q = \sum_{\mathbf{n}>0} \Big\langle T_{-1} \cdot \prod_{\ell \geq 1} h_\ell^{n_\ell} \Big\rangle_q \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}\,.$$

By definition of cumulants (or by [12, Proposition 6.2]) we are interested in the leading term of the quotient

$$C^\circ_{-1}(\mathbf{u})_q \;:=\; \frac{C'_{-1}(\mathbf{u})_q}{\Phi^H(\mathbf{u})_q} = \sum_{\mathbf{n}\geq 0} \langle T_{-1}|\underbrace{h_1|\cdots|h_1}_{n_1}|\underbrace{h_2|\cdots|h_2}_{n_2}|\cdots\rangle_q \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}\,.$$

To evaluate the numerator of this fraction, recall from [12, Section 16] the definition of the modified q-bracket

$$\langle f\rangle_q^\star = \langle T_{-1}\, f\rangle_q \;-\; \langle T_{-1}\rangle_q\, \langle f\rangle_q \;-\; \frac{1}{24}\,\langle \partial_2(f)\rangle_q, \tag{107}$$

where ∂_2 is the differential operator

$$\partial_2 \colon \frac{\partial}{\partial p_1} + \sum_{\ell\geq 2} \ell(\ell-1)\, p_{\ell-2}\frac{\partial}{\partial p_\ell}\,.$$

This bracket is useful, since its effect can be computed by differential operators acting (contrary to T_{-1}) within the Bloch–Okounkov algebra. In fact, [12, Theorem 16.1] states that

$$\langle f\rangle_q^\star = \sum_{j\geq 1} G_2^{(j-1)}\,\langle \rho_{0,j}(f)\rangle_q \;+ \sum_{i\geq 2,\; j\geq 0} G_i^{(j)}\,\langle \rho_{i,j}(f)\rangle_q,$$

where $\rho_{i,j}$ are differential operators of degree j that shift the weight by $-i-2j$, whose definition we recall in (110) below. Motivated by the action of these operators we define

$$\mathcal{C}'_{-1}(\mathbf{u}) = \sum_{\mathbf{n}>0} \sum_{j\geq 1} j!\, \rho_{0,j} \Big(\prod_{\ell\geq 1} h_\ell^{n_\ell} \Big) \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \tag{108}$$

and we let $\Phi^H(\mathbf{u}) = \exp(\sum_{\ell\geq 1} h_\ell u_\ell)$ such that $\Phi^H(\mathbf{u})_q = \langle \Phi^H(\mathbf{u})\rangle_q$.

Lemma 10.4 *The leading terms of*

$$C^\circ_{-1}(\mathbf{u})_q \quad \textit{and} \quad \mathcal{C}^\circ_{-1}(\mathbf{u})_q := \frac{-1}{24}\frac{\langle \mathcal{C}'_{-1}(\mathbf{u})\rangle_q}{\Phi^H(\mathbf{u})_q}$$

agree.

Proof First note that $\mathrm{Ev}(G_2^{(j-1)})(X) = \frac{-1}{24}(j!X + (j-1)!)$ by the defining formulas in [12, Section 9]. This is the reason for the factor $j!$ in (108). From the non-vanishing of the area Siegel–Veech constant, we know that the leading degree contribution is as in (41). Lower weight terms before passing to the cumulant quotient will contribute to lower order in the growth polynomial. Since ∂_2 is of degree -2, its contribution in (107) is negligible and we can work with the star-brackets. For the same reason, the terms with $i > 0$ in the definition $\langle f\rangle_q^\star$ are dominated by the corresponding term with $i = 0$ and can be neglected. □

Our goal is to compute the h-evalutaion of $\mathcal{C}^\circ_{-1}(\mathbf{u})$ and its leading term using Proposition 4.3.

Lemma 10.5 *The commutation relation*

$$\partial_2 \circ e^D(f) = e^D\Big(\sum_{j\geq 1} j!\,\rho_{0,j}(f)\Big) \tag{109}$$

holds for every $f \in \Lambda^*$.

Proof We will check the relation on the n-point function for every n. Since we will recall formulas from [12] we use the rescalings $Q_k = p_{k-1}/(k-1)!$ of the generators of Λ^*, where $Q_0 = 1$ and $Q_1 = 0$. The following identities even hold on the polynomial ring $R = \mathbb{Q}[Q_0, Q_1, Q_2, \ldots]$ mapping to Λ^*. We set $W(z) = \sum_{k\geq 0} Q_k z^{k-1}$. We recall from [12, Theorem 14.2] the action of the operators $\rho_{0,j}$, namely

$$\rho_{0,j}\big(W(z_1)\cdots W(z_n)\big) = \sum_{\substack{J\subset N\\ |J|=j}} W(z_J)\, z_J\Big(\prod_{j\in J} z_j\Big) \prod_{\nu\in N\smallsetminus J} W(z_\nu) \tag{110}$$

where $z_J = \sum_{j\in J} z_j$ and $N = \{1,\ldots,n\}$. On the other hand, in terms of the Q_i, the operator D defined in Sect. 4 is just $D = \frac{1}{2}(\Delta - \partial^2)$, where ∂ is the differential operator sending Q_i to Q_{i-1}. From [12, Proposition 10.5] we know that

$$e^D\left(W(z_1)\cdots W(z_n)\right) = e^{-z_N^2/2}\sum_{\alpha\in\mathcal{P}(n)}\Bigl(\prod_{A\in\alpha} z_A^{|A|-1}W(z_A)\Bigr).$$

Using these identities we can evaluate both sides of (109) to be of the form

$$e^{-z_N^2/2}\sum_{\alpha\in\mathcal{P}(n)}\Bigl(\prod_{A\in\alpha} W(z_A)R(\{z_a\}_{a\in A})\Bigr)$$

where $R(\{z_a\})$ are polynomials that are visibly different on the two sides, but in fact agree by using the identity

$$z_A^{n+1} = \sum_{\emptyset\neq J\subset A} |J|!\, z_J\Bigl(\prod_{\nu\in J} z_\nu\Bigr) z_A^{n-|J|}.$$

To verify this expression, let $e_i = (-1)^i[x^{n-i}]\prod_{a\in A}(x-z_a)$ be the elementary symmetric functions in the z_a. Then the contribution with $|J| = j$ to the right-hand side is $e_1^{n-j}(e_1e_j - (j+1)e_{j+1})$. This means that the right-hand side is a telescoping sum where only the first term remains after summation. □

The preceding Lemma 10.5, Proposition 4.3 for the computation of the h-brackets, Lemma 10.4 and (106) now imply immediately our goal:

Theorem 10.6 *The area Siegel–Veech constants can be computed as*

$$c_{\text{area}}(m_1,\ldots,m_n) = \frac{-1}{8\pi^2}\frac{[z_1^{m_1+1}\cdots z_n^{m_n+1}]\,\partial_2(\mathcal{H}_n)}{[z_1^{m_1+1}\cdots z_n^{m_n+1}]\,\mathcal{H}_n}\bigg|_{h_\ell\mapsto\alpha_\ell},$$

where $\mathcal{H}_n(z_1,\ldots,z_n)$ is recursively defined as in Sect. 4.

10.3 Proof of Theorem 1.4

We start with an explicit formula for the ∂_2-derivative used in Theorem 10.6 in the case of the minimal strata.

Proposition 10.7 *The area Siegel–Veech constants for the minimal strata are*

$$c_{\text{area}}(\Omega\mathcal{M}_g(2g-2)) = \frac{-1}{8\pi^2}\frac{[u^{2g-1}]\,\mathcal{D}(u)}{[u^{2g-1}]\,\mathcal{A}(u)}, \tag{111}$$

where

$$\mathcal{D}(u) = (\mathcal{A}'(u)+u\mathcal{A}''(u))/u\mathcal{A}'(u)^2 = t - \frac{1}{18}t^3 + \frac{91}{2304}t^5 - \cdots.$$

Proof Differentiating (50) gives

$$\sum_{n\geq 2} n^2 p_{n-1}\mathcal{H}^{-n}(z) = \frac{\mathcal{H}(z)}{\mathcal{H}'(z)}\Big(\frac{1}{z} - \frac{\mathcal{H}(z)}{z^2\mathcal{H}'(z)} - \frac{\mathcal{H}(z)\mathcal{H}''(z)}{z\mathcal{H}'(z)^2}\Big).$$

Combining these two equalities gives $\partial_2(\mathcal{H}_1(u)) = (\mathcal{H}'(u)+u\mathcal{H}''(u))/u\mathcal{H}'(u)^2$ and the claim by substituting $h_\ell \mapsto \alpha_\ell$. □

Proof of Theorem 1.4 We start with the case of a single zero. Comparing (111) and (102) we need to show that $\mathcal{D}(u) = 2\Delta(u)/u$, i.e. in view of (16) we need to show that

$$[u^0](\mathcal{D}(u)\mathcal{A}(u)^{2g-1}) = (2g-1)![u^{2g-2}]B(u) = (2g-1)[u^0]\mathcal{A}(u)^{2g-2}.$$

This equality can be implied by showing that

$$[u^{-1}]\mathcal{A}(u)^{2g-1}\frac{\mathcal{A}'(u)+u\mathcal{A}''(u)}{(2g-1)u^2\mathcal{A}'(u)^2} = [u^{-1}]\frac{\mathcal{A}(u)^{2g-2}}{u},$$

which in turn follows since the derivative

$$\Big(\frac{\mathcal{A}(u)^{2g-1}+u\mathcal{A}'(u)}{(2g-1)u\mathcal{A}'(u)}\Big)' = \mathcal{A}^{2g-1}\frac{\mathcal{A}'(u)+u\mathcal{A}''(u)}{(2g-1)u^2\mathcal{A}'(u)^2} - \frac{\mathcal{A}(u)^{2g-2}}{u}$$

has no (-1)-term. Finally, to deal with the case of multiple zeros, we recall from (60) that $a_i(\mu) = [z_1^{m_1+1}\cdots z_n^{m_n+1}]/(2g-2+n)\prod_{j=1}^n(m_j+1)\,\mathcal{H}_n$ and hence we need to show that

$$d(\mu) = \frac{[z_1^{m_1+1}\ldots z_n^{m_n+1}]\partial_2(\mathcal{H}_n)}{(2g-2+n)\prod_{i=1}^n(m_i+1)}\Big|_{h_\ell\mapsto\alpha_\ell}$$

after knowing that this is true for the case of $n(\mu)=1$. This follows immediately from differentiating (57), since after substituting $h_\ell \mapsto \alpha_\ell$ this is exactly the sum of the recursions (103) (known to hold for the $d(\mu)$), averaging over all pairs (m_r, m_s) of the entries of μ, as in (59). □

Given Theorem 1.4 for the area Siegel–Veech computation of the strata on one hand, and the refined Theorem 6.3 for the volume computation of the spin components on the other hand, it is natural to suspect that area Siegel–Veech constants for the spin components can also be computed as ratios of intersection numbers

$$c_{\text{area}}(\mu)^\bullet = \frac{-1}{4\pi^2}\frac{\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^\bullet}\beta_i\cdot\delta_0}{\int_{\mathbb{P}\overline{\Omega\mathcal{M}}_{g,n}(\mu)^\bullet}\beta_i\cdot\xi} \tag{112}$$

for all $1 \leq i \leq n$, where $\bullet \in \{\text{even}, \text{odd}\}$. Using Assumption 6.1 to deal with the case of the minimal strata, the validity of (112) is equivalent to the validity of

$$c_{\text{area}}(\mu)^{\text{odd}} = \frac{-1}{16\pi^2}[z_1^{m_1+1} \cdots z_n^{m_n+1}] \left(\frac{(2\pi i)^{2g}(\partial_2(\mathcal{H}_n) - \partial_2^{\Delta}(\boldsymbol{\mathcal{H}}_n))}{(2g-2+n)!v(\mu)^{\text{odd}}} \right) \Bigg|_{\substack{h_\ell \mapsto \alpha_\ell \\ \mathbf{h}_\ell \mapsto \boldsymbol{\alpha}_\ell}}$$

where $\mathcal{H}_n$ and $\boldsymbol{\mathcal{H}}_n$ are recursively defined as in Sects. 4 and 6, and where

$$\partial_2^{\Delta} = \frac{\partial}{\partial \mathbf{p}_1} + \sum_{\ell \geq 1}^{\infty} 2\ell(2\ell+1)\mathbf{p}_{2\ell-1} \frac{\partial}{\partial \mathbf{p}_{2\ell+1}}$$

is the analog of the differential operator ∂_2 on the algebra of super-symmetric functions. There is a clear strategy towards this goal:

- Show that there are operators like the T_p for $p \geq -1$ odd as in [12, Section 12], whose strict brackets compute Siegel–Veech weighted and spin-weighted Hurwitz numbers.
- Show that the action of T_p inside strict brackets can be encoded by differential operators like the ρ_{ij} in [12, Section 14].
- Show that these operators satisfy a commutation relation as in Lemma 10.5, with ∂ replaced by ∂_2^{Δ}.
- Conclude by comparing the recursions as in the preceding proofs.

Given the length of this paper, we do not attempt to provide details here.

11 Large genus asymptotics

In this section we study the large genus asymptotics of Masur–Veech volumes and area Siegel–Veech constants and prove the conjectures of Eskin and Zorich in [23] by using our previous results.

11.1 Volume asymptotics

We recall from [12, Theorems 12.1 and 19.1] and [42, Theorem 1.9] that the asymptotic expansions of $v(2g-2)$ and $v(1, \ldots, 1)$ can be computed using the mechanisms of very rapidly divergent series [12, Appendix] as

$$v(1^{2g-2}) \sim 4\Big(1 - \frac{\pi^2}{24g} - \frac{60\pi^2 - \pi^4}{1152g^2} - \cdots\Big),$$
$$v(2g-2) \sim 4\Big(1 - \frac{\pi^2}{12g} - \frac{24\pi^2 - \pi^4}{288g^2} - \cdots\Big).$$

Let $\mu = (m_1, \dots, m_n)$ be a partition of $2g - 2$ into n positive integers with $n \geq 2$. We write $\mu' = (m_1 + m_2, m_3, \dots, m_n)$ and $\mu'' = (m_1 + m_2 - 2, m_3, \dots, m_n)$. We use Theorem 1.2 and the two obvious backbone graphs, the one with a single top level component of genus g (i.e. $k = 1$) and the one with two top level components (i.e. $k = 2$) of genus 1 and $g - 1$ respectively, to deduce the inequality

$$v(\mu) \geq v(\mu') + \frac{\pi^2(2g-5+n)!}{6(2g-3+n)!} v(\mu''), \tag{113}$$

where we use $h_{\mathbb{P}^1}((m_1, m_2), (m_1+m_2+2)) = h_{\mathbb{P}^1}((m_1, m_2), (m_1+m_2, 1)) = 1$ for $m_1, m_2 > 0$ and $v(0) = \pi^2/6$. In particular this inequality implies that $v(\mu) \geq v(\mu')$ and thus

$$v(2g-2) \ \leq \ v(\mu) \ \leq \ v(1, \dots, 1)$$

for all μ. Consequently, there exists a constant $C > 0$ such that for all μ we have the inequality $|v(\mu) - 4| < C/g$. Now we introduce the notation

$$\widehat{v}(\mu) := v(\mu) - 4 + \frac{2\pi^2}{3(2g-3+n)} \,.$$

Then the inequality (113) implies that

$$\begin{aligned} \widehat{v}(\mu) - \widehat{v}(\mu') &\geq -\frac{2\pi^2}{3(2g-3+n)(2g-4+n)} + \frac{\pi^2(2g-5+n)!}{6(2g-3+n)!} v(\mu'') \\ &\geq -\frac{C\pi^2}{6g(2g-3+n)(2g-4+n)} \,. \end{aligned}$$

In particular for all μ we have

$$\widehat{v}(2g-2) - \frac{C\pi^2(n-1)}{6g(2g-1)^2} \leq \widehat{v}(\mu) \leq \widehat{v}(1, \dots, 1) + \frac{C\pi^2(2g-2-n)}{6g(2g-1)^2} \,.$$

Since n is bounded by $2g - 2$, there exists a constant C' such that $|\widehat{v}(\mu)| \leq C'/g^2$ for all μ. Thus the first part of Theorem 1.5 holds.

11.2 Asymptotics of Siegel–Veech constants

We apply the same strategy to control the asymptotic behavior of area Siegel–Veech constants. We denote by

$$\widetilde{d}(\mu) := c_{\mathrm{area}}(\mu)v(\mu) = \frac{1}{4\pi^2}\frac{2(2\pi)^2 g}{(2g-3+n)!}\left(\prod_{i=1}^{n}(m_i+1)\right)\cdot|d(\mu)|$$

where $d(\mu)$ is defined in (99) and where the second equality stems from (100). For $\mu = (m_1, \ldots, m_n)$ with $n \geq 2$, we write $\mu' = (m_1 + m_2, m_3, \ldots, m_n)$ and $\mu'' = (m_1 + m_2 - 2, m_3, \ldots, m_n)$ as before. Then from Proposition 10.1 we have $\widetilde{d}(0) = 1$ and we obtain the inequality

$$\widetilde{d}(\mu) \geq \widetilde{d}(\mu') + \frac{\pi^2(2g-5+n)!}{6(2g-3+n)!}\widetilde{d}(\mu'') + \frac{(2g-5+n)!}{2(2g-3+n)!}v(\mu'').$$

In particular this inequality implies that $\widetilde{d}(\mu) \geq \widetilde{d}(\mu')$. Moreover, we know the asymptotic expansions

$$\widetilde{d}(2g-2) \sim 2 - \frac{3+\pi^2}{6g} + \cdots \quad \text{and} \quad \widetilde{d}(1,\ldots,1) \sim 2 - \frac{3+\pi^2}{12g} + \cdots$$

from [12, Theorem 19.4] and [42, Theorem 1.9]. Consequently, there exists a constant C such that $|\widetilde{d}(\mu) - 2| < C/g$ for all μ. Then by the same argument as above we can show that there exists a constant C' such that

$$\left|\widetilde{d}(\mu) - 2 + \frac{3+\pi^2}{3(2g-3+n)}\right| < C'/g^2.$$

Therefore, using the fact that $c_{\mathrm{area}}(\mu) = \widetilde{d}(\mu)/v(\mu)$ we thus deduce the second part of Theorem 1.5.

11.3 Spin asymptotics

The goal here is to prove that the volumes of the odd and even spin components are asymptotically equal. This is the content of Theorem 1.6 in the introduction that refines the conjecture of Eskin and Zorich [23, Conjecture 2]. Recall that we defined in Sect. 6.3

$$\mathbf{P}_Z(u) = \exp\Big(\sum_{j\geq 1}\Big(\frac{1}{2}\Big)^{j/2}\frac{\zeta(-j)}{2}u^{j+1}\Big). \tag{114}$$

Proposition 11.1 *The difference* $v(2g-2)^{\Delta} = v(2g-2)^{\text{even}} - v(2g-2)^{\text{odd}}$ *can be computed as the coefficient extraction*

$$v(2g-2)^{\Delta} = \frac{2(2\pi i)^{2g}}{(2g-1)!}[u^{2g-1}]\frac{1}{(u/\mathbf{P}_Z)^{-1}}. \tag{115}$$

Moreover, it has the asymptotics

$$v(2g-2)^{\Delta} \sim \left(\frac{-1}{2}\right)^{g-2}\left(1+\frac{2\pi^2}{3g}+\frac{12\pi^2+\pi^4}{18g^2}+\cdots\right) \tag{116}$$

as $g \to \infty$.

For the reader's convenience we give a table for low genus values:

g	1	2	3	4	5
$v(2g-2)^{\Delta}$	$\frac{-1}{3}$	$\frac{1}{40}$	$\frac{-143}{108864}$	$\frac{15697}{279936000}$	$\frac{-2561}{1103872000}$

Proof of Proposition 11.1 The first statement is a reformulation of a special case of Corollary 6.11.

The power series $\mathbf{P}_Z(u)$ is a very rapidly divergent series, just as $P_B(u)$ is, since the coefficients $\ell! b_\ell$ and $\zeta(-\ell)/2 = \ell! b_\ell \cdot \sqrt{2}(2^\ell - 2^{-\ell})$ differ by a factor that grows only geometrically. The asymptotic statement thus follows from the method of very rapidly divergent series. □

Proof of Theorem 1.6 Proposition 11.1 together with Theorem 1.5 implies that there exists a constant C' such that for all $g \geq 1$

$$|v(2g-2)^{\text{odd}} - v(2g-2)^{\text{even}}| \leq C'/g\,.$$

Repeated application of Theorem 1.5 implies that $v(\mu)^{\bullet} \geq v(2g-2)^{\bullet}$ for all μ with $|\mu| = 2g-2$. Theorem 1.5 moreover implies that there exists a constant C'' such that for all μ with $|\mu| = 2g-2$ the inequality

$$|v(\mu) - 4| \leq C''/g$$

holds. Thus for all μ with $|\mu| = 2g-2$ we have

$$\begin{aligned} v(2g-2)^{\text{odd}} \leq v(\mu)^{\text{odd}} = v(\mu) - v(\mu)^{\text{even}} &\leq v(\mu) - v(2g-2)^{\text{even}} \\ &\leq 2 + (C' + 3C'')/g\,. \end{aligned}$$

It follows that $|v(\mu)^{\text{odd}} - 2| \leq (C' + 3C'')/g$ and the same holds for $v(\mu)^{\text{even}}$. This implies the claim for the volume comparison.

For the comparison of area Siegel–Veech constants, since we have shown that the two spin components have the same volume asymptotic, combining with [4, Propositions 4.3 and 4.4] it suffices to compare the odd and even one-cylinder contributions in the part of the principal boundary where the corresponding configurations have either a figure-eight boundary or a pair of holes boundary. Using again the same volume asymptotics of the spin components in genus $g-1$ and combining with [20, Lemmas 14.2 and 14.4], we see that the odd and even contributions are comparable for those two parts of the principal boundary, thus implying the claim. □

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